Submanifolds of Kaehlerian Manifold with a Metric Compound Structure of Rank 2 (*).

SEIICHI YAMAGUCHI - HIROAKI NEMOTO - JAE-BOK JUN

Summary. – *The metric compound structure of rank r is an abstructed structure of an induced structure on a real submanifold in an almost Hermitian manifold. In this paper we deal with a submanifold with metric compound structure of rank 2 in a Kaehlerian manifold and we classify it under some suitable conditions. Namely it is a standard sphere or neither Einstein nor conformally flat.*

0. – Introduction.

In [3] and [8], Y. TASHIRO and I. -B. KIM have introduced the notion of metric compound structure of rank r which is naturally induced on the submanifold of an almost Hermitian manifold.

In [8], they have investigated the case of rank 1. Also the case of rank 2 has been studied by I. -B. KIM [3]. Although I. -B. KIM [3] has applied his energies to the study of such submanifolds, there is plenty of room for improvement. For instance, since the dominator of (6.8) in [3] may take a value zero, we must be careful how we treat it. Also, Theorem 6.4, 6.9 and their corollaries in [3] will be improved.

The purpose of this paper is to sharpen the Kim's results of Paragraph 6 in [3], that is, we will prove the following Theorem.

THEOREM. – *Let M be an n-dimensional complete and connected submanifold with an induced metric compound structure of rank 2 in a Kaehlerian manifold $\tilde{M}$. Suppose that λ does not vanish identically, α and β are umbilical sections on M and the sum of squared mean curvature ν^2 does not vanish on M and*

$$M' = \{p \in M: 1 - \lambda^2 \neq 0 \ \ at\ p\} \qquad is\ connected\,.$$

Then we have the following:

(1) *M is isometric with a sphere,*

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Indirizzo degli AA.: Department of Mathematics, Faculty of Science, Science University of Tokyo, Tokyo, Japan, 162.

or

(2) *each λ-hypersurface is AS-homothetic to a Sasakian manifold $\bar{M}$ and neither M is Einstein nor conformally flat.*

Although the majority of Kim's results in the Paragraph 6 are only to find the conditions that M is to be a space of constant curvature, by according to our above Theorem, we can determine the submanifold M as follows: M is isometric with a sphere or neither M is Einstein nor conformally flat and each λ-hypersurface is AS-homothetic to a Sasakian manifold $\bar{M}$.

We devote the first two sections for the preparation to a description of our Theorem. In § 1, we will recall the fundamental properties of submanifolds. In § 2, we will give a brief summary of the notion of the metric compound structure of rank 2 which is mainly developed by I. -B. KIM and Y. TASHIRO [3], [8]. The last two sections will be devoted to the proof of our Theorem.

Throughout this paper, we assume that manifolds and quantities are differentiable of class C^∞. Unless otherwise is stated, indices run over the following ranges

$$\varkappa, \lambda, \mu, \nu, \ldots = 1, 2, 3, \cdots\cdots, m,$$

$$h, i, j, k, \ldots = 1, 2, 3, \ldots, n,$$

$$p, q, r, s, \ldots = n+1, n+2, \ldots, m,$$

$$a, b, c, d, \ldots = 2, 3, \ldots, n,$$

respectively.

1. – Submanifolds.

Let $\tilde{M}$ be a Riemannian manifold of dimension m with Riemannian metric $\tilde{g} = (\tilde{g}_{\mu\lambda})$ and M be a submanifold of dimension n (> 2) of $\tilde{M}$ represented locally by the equation

$$y^\varkappa = y^\varkappa(x^h),$$

where $\{x^h\}$ are local coordinates of M and $\{y^\varkappa\}$ local coordinates of $\tilde{M}$.

If we put

$$B_i{}^\varkappa = \partial_i y^\varkappa, \qquad \partial_i = \partial/\partial x^i,$$

then $B_j = (B_j{}^\varkappa)$ $(j = 1, 2, \ldots, n)$ are linearly independent local vector fields tangent to M. The Riemannian metric tensor $g = (g_{ji})$ of M is given by

$$g_{ji} = \tilde{g}_{\mu\lambda} B_j{}^\mu B_i{}^\lambda. \tag{1.1}$$

We can locally choose $m-n$ mutually orthogonal unit normal vector fields $C_p=(C_p^\kappa)$ to M. Then the vectors B_i and C_p span the tangent space $T_x(\tilde{M})$ of $\tilde{M}$ at every point $x\in M$ and the matrix (B_i^κ, C_p^κ) is regular. We have

$$\tilde{g}_{\mu\lambda}B_i^\mu C_p^\lambda=0\,,\quad \tilde{g}_{\mu\lambda}C_p^\mu C_q^\lambda=\delta_{pq}\,. \tag{1.2}$$

On a submanifold M of a Riemannian manifold $\tilde{M}$, the van der Waerden-Bortolotti covariant differentiation ∇_j is defined by

$$\nabla_j B_i^\kappa=\partial_j B_i^\kappa-B_h^\kappa\left\{\begin{matrix}h\\ j\ i\end{matrix}\right\}+\Gamma_{\mu\lambda}^\kappa B_j^\mu B_i^\lambda\,, \tag{1.3}$$

where $\left\{\begin{matrix}h\\ j\ i\end{matrix}\right\}$ and $\Gamma_{\mu\lambda}^\kappa$ are the Christoffel's symbols of M and $\tilde{M}$ respectively. Since $\nabla_j B_i^\kappa$ is normal to M for fixed i and j, we have the equation of Gauss

$$\nabla_j B_i^\kappa=h_{jip}C_p^\kappa\,, \tag{1.4}$$

where h_{jip} is the second fundamental tensor.

Throughout this paper, the summation convention is applied to the repeated indices on their own ranges. The equation of Weingarten is given by

$$\nabla_j C_q^\kappa=-h_j{}^i{}_q B_i^\kappa+l_{jqp}C_p^\kappa\,, \tag{1.5}$$

where we have put

$$\nabla_j C_q^\kappa=\partial_j C_q^\kappa+\Gamma_{\mu\lambda}^\kappa B_j^\mu C_q^\lambda\,,\qquad h_j{}^i{}_q=g^{ih}h_{jhq}\,, \tag{1.6}$$

and l_{jqp} is the so-called third fundamental tensor.

A normal vector field $N=(N^\kappa)=\Upsilon_p C_p^\kappa$ is called a normal section on M. The tensor $h_{jip}\Upsilon_p$ is said to be the second fundamental tensor belonging to the normal section N. The mean curvature vector of M in $\tilde{M}$ is given by

$$H=H_p C_p^\kappa\,,\qquad H_p=(1/n)g^{ji}h_{jip}\,.$$

If the relation

$$h_{jip}\Upsilon_p=\varrho g_{ji}$$

is satisfied with a function ϱ on M for a normal section N, then N is said to be an umbilical section on M or M is umbilical with respect to N. If N is a unit normal section, then the function $\varrho=H_i{}^i{}_p\Upsilon_p/n$ is called the mean curvature belonging to N. Moreover, if ϱ vanishes identically, then N is said to be a geodesic section on M.

2. – Submanifolds with a metric compound structure of rank 2 in a Kaehlerian manifold.

Let $\tilde{M}$ be a Kaehlerian manifold of dimension m with structure tensors $(\tilde{g}, J)$, where $\tilde{g}$ is the Hermitian metric tensor and J the complex structure one. We consider a real submanifold M of dimension n in $\tilde{M}$. For $X \in TM$ and $N \in T^{\perp}M$, we put

$$JX = fX + TX , \quad JN = -tN + f^{\perp}N ,$$

where fX (resp. $-tN$) denotes the tangential component of JX (resp. JN) and TX (resp. $f^{\perp}N$) the normal component of JX (resp. JN). Then f (resp. $f^{\perp}$) is an endomorphism on TM (resp. $T^{\perp}M$), T is a $T^{\perp}M$-valued homomorphism on TM and t is a TM-valued homomorphism on $T^{\perp}M$. Moreover the relation between T and t is given by

$$\tilde{g}(TX, N) = \tilde{g}(X, tN) .$$

If rank $(T) = r$ $\big(0 \leqq r \leqq \mathrm{Min}\,(n, m-n)\big)$ almost everywhere on M, then we say that M is a submanifold with compound metric structure of rank r. The phrase « almost everywhere on M » means « on the whole manifold M except a border subset of M ». If $r = 0$ on M, M is nothing but an invariant submanifold and hence M is also a Kaehlerian manifold. If $r = 1$ on M, M is to be an almost contact Riemannian manifold (see, [8]).

REMARK. – Even if $r = \dim M$, M is not necessary a totally real submanifold. Thus such submanifolds with metric compound structure of rank r are very wide class in all real submanifolds of a Kaehlerian manifold.

In the following, we assume that $r = 2$ almost everywhere on M. Then we can choose mutually orthogonal normal vector fields α and β such that

$$\begin{aligned} JX &= fX + u(X)\,\alpha + v(X)\beta , \\ JN &= -\alpha(N)\,U - \beta(N)\,V + f^{\perp}N , \end{aligned} \tag{2.1}$$

where α and β span image of T, $U = t\alpha$, $V = t\beta$, $u(X) = \tilde{g}(X, U)$, $v(X) = \tilde{g}(X, V)$, $\alpha(N) = \tilde{g}(\alpha, N)$ and $\beta(N) = \tilde{g}(\beta, N)$. In terms of local coordinates, they are expressed as

$$J_{\lambda}{}^{\varkappa} B_i{}^{\lambda} = f_i{}^h B_h{}^{\varkappa} + u_i \alpha_p C_p{}^{\varkappa} + v_i \beta_p C_p{}^{\varkappa} , \tag{2.2}$$

and

$$J_{\lambda}{}^{\varkappa} C_q{}^{\lambda} = -\alpha_q u^h B_h{}^{\varkappa} - \beta_q v^h B_h{}^{\varkappa} + f_{qp} C_p{}^{\varkappa} , \tag{2.3}$$

where we have set

$$f_{ji}=\tilde{g}(JB_j, B_i)\,,\qquad f_{qp}=\tilde{g}(f^{\perp}C_q, C_p)\qquad \text{and}\qquad f_j{}^i=g^{ih}f_{jh}\,.$$

Then, the tensor field $f_j{}^i$ and vector fields u^i, v^i, α_p and β_p satisfy the following equations

$$f_j{}^i f_i{}^h=-\delta_j{}^h+u_j u^h+v_j v^h\,, \tag{2.4}$$

$$f_j{}^i u_i=\lambda v_j\,,\qquad f_j{}^i v_i=-\lambda u_j\,, \tag{2.5}$$

$$u_i v^i=0\,,\qquad u_i u^i=v_i v^i=1-\lambda^2 \tag{2.6}$$

where we put $\lambda=f_{qp}\alpha_q\beta_p$. Therefore, the submanifold M has the so-called (f, g, u, v, λ)-structure.

Denote by τ and ϱ the mean curvature belonging to α_p and β_p respectively, namely

$$\tau=H_p\alpha_p\qquad \text{and}\qquad \varrho=H_p\beta_p\,.$$

Then, we put

$$\nu^2=\tau^2+\varrho^2\,,\qquad l_j=\beta_p\nabla_j^{\perp}\alpha_p\,, \tag{2.7}$$

where $\nabla_j^{\perp}\alpha_p=\partial_j\alpha_p+\alpha_q l_{jqp}$. We also define a vector field ξ on M by

$$\xi^h=\tau u^h+\varrho v^h\,, \tag{2.8}$$

and denote by η the associated 1-form of ξ. Suppose that both α and β are umbilical sections on M and one of them is not a geodesic section.

Consequently, from (2.4)-(2.6) and (2.8), we can get the following relations

$$f_j{}^i f_i{}^h=-\delta_j{}^h+\nu^{-2}(\lambda_j\lambda^h+\eta_j\xi^h)\,, \tag{2.9}$$

$$f_j{}^i\lambda_i=\lambda\eta_j\,,\qquad f_j{}^i\eta_i=-\lambda\lambda_j\,, \tag{2.10}$$

$$\lambda_i\xi^i=\eta_i\lambda^i=0\,,\qquad \lambda_i\lambda^i=\eta_i\xi^i=\nu^2(1-\lambda^2)\,, \tag{2.11}$$

where we put $\lambda_i=\nabla_i\lambda$.

The following Theorem A and B proved by Kim (Theorem 6.2 and 6.3 in [3]) play important roles in this paper:

THEOREM A. – *Let M be a submanifold of dimension n (>2) with an induced metric compound structure of rank 2 in a Kaehlerian manifold. Assume that λ does not vanish almost everywhere on M, α and β are umbilical sections on M and one of them is not a* *geodesic section. Then we have the equations*

(2.12) $$\nabla_k f_{ji} = \eta_j g_{ki} - \eta_i g_{kj} ,$$

(2.13) $$\nabla_j \lambda_i = \psi(\lambda_j \lambda_i + \eta_j \eta_i) - \lambda \nu^2 g_{ji} ,$$

(2.14) $$\nabla_j \eta_i = \psi(\lambda_j \eta_i - \eta_j \lambda_i) + \nu^2 f_{ji} ,$$

(2.15) $$\nabla_j \nu^2 = 2\nu^2 \psi \lambda_j ,$$

where $\psi = A/\nu^2$, $A = u^i(\varrho_i + \tau l_i)/(1 - \lambda^2)$ *and* $\varrho_i = \nabla_i \varrho$.

THEOREM B. – *Under the same assumptions of Theorem* A, *each* λ*-hypersurface is AS-homothetic to a Sasakian manifold* $\bar{M}$.

3. – Fundamental Lemmas.

At first, we prepare some equations. We restrict our calculations on M' in this section. Differentiating (2.15) covariantly and making use of (2.15) again, we have

$$\nabla_i \nabla_j \nu^2 = 2\nu^2(2\psi^2 \lambda_i \lambda_j + \psi_i \lambda_j + \psi \nabla_i \lambda_j) ,$$

where $\psi_i = \nabla_i \psi$. If we take a skew-symmetric part of the above equation, then we obtain

$$2\nu^2(\psi_i \lambda_j - \psi_j \lambda_i) = 0 ,$$

which means that ψ_i and λ_i are proportional to each other, that is,

(3.1) $$\psi_i = a\lambda_i ,$$

where a is a proportional factor and we have used $\nu^2 \neq 0$ on M.

Operating ∇_k to (2.14), (2.13) and (2.12) respectively and taking account of (2.10)-(2.15) and (3.1), we find the following equations

(3.2) $$\nabla_k \nabla_j \eta_i = b\lambda_k U_{ji} + \nu^2(1 + \lambda\psi)(\eta_j g_{ki} - \eta_i g_{kj}) + \nu^2 \psi(\lambda_j f_{ki} - \lambda_i f_{kj} + 2\lambda_k f_{ji}) ,$$

(3.3) $$\nabla_k \nabla_j \lambda_i = b\lambda_k(\lambda_j \lambda_i + \eta_j \eta_i) - \lambda \nu^2 \psi(g_{kj} \lambda_i + g_{ki} \lambda_j) + \nu^2 \psi(f_{kj} \eta_i + f_{ki} \eta_j) - \nu^2(1 + 2\lambda\psi) g_{ji} \lambda_k ,$$

(3.4) $$\nabla_l \nabla_k f_{ji} = (\psi U_{lj} + \nu^2 f_{lj}) g_{ki} - (\psi U_{li} + \nu^2 f_{li}) g_{kj} ,$$

where we have put $b = a + 2\psi^2$ and $U_{ji} = \lambda_j \eta_i - \lambda_i \eta_j$.

Therefore, by virtue of the Ricci's identity and the above equations, it follows that

$$R_{kji}{}^{r}\eta_r = b\lambda_i U_{kj} - \nu^2(1+\lambda\psi)(\eta_j g_{ki} - \eta_k g_{ji}) - \nu^2\psi(\lambda_k f_{ji} - \lambda_j f_{ki} - 2\lambda_i f_{kj}) , \tag{3.5}$$

$$R_{kji}{}^{r}\lambda_r = -\big(b\eta_i U_{kj} + \nu^2\psi(2\eta_i f_{kj} + \eta_j f_{ki} - \eta_k f_{ji}) + \nu^2(1+\lambda\psi)(\lambda_j g_{ki} - \lambda_k g_{ji})\big) , \tag{3.6}$$

$$\begin{aligned} -R_{lkj}{}^{r}f_{ri} - R_{lki}{}^{r}f_{jr} = {} & (\psi U_{lj} + \nu^2 f_{lj})g_{ki} - (\psi U_{li} + \nu^2 f_{li})g_{kj} \\ & - (\psi U_{kj} + \nu^2 f_{kj})g_{li} + (\psi U_{ki} + \nu^2 f_{ki})g_{lj} . \end{aligned} \tag{3.7}$$

Taking the cyclic sum of (3.7) with respect to the indices l, k and j and making use of the first Bianchi's identity, we get

$$\begin{aligned} -R_{lki}{}^{r}f_{jr} - R_{kji}{}^{r}f_{lr} - R_{jli}{}^{r}f_{kr} = {} & 2\big((\psi U_{lj} + \nu^2 f_{lj})g_{ki} + (\psi U_{kl} + \nu^2 f_{kl})g_{ji} \\ & + (\psi U_{jk} + \nu^2 f_{jk})g_{li}\big) . \end{aligned} \tag{3.8}$$

Let us show the following.

LEMMA 3.1. – *Under the same assumptions of Theorem* A, *we obtain*

$$\psi(1+\lambda\psi) = 0 \quad on\ M' . \tag{3.9}$$

PROOF. – Operating ∇_l to (3.5) and regarding to (2.12)-(2.15), we get

$$\begin{aligned} -\nabla_l R_{kji}{}^{r}\eta_r - R_{kji}{}^{r}(\psi U_{lr} + \nu^2 f_{lr}) = {} & (b_l\lambda_i + b\nabla_l\lambda_i)U_{jk} \\ & + b\lambda_i(\nabla_l\lambda_j\eta_k + \lambda_j\nabla_l\eta_k - \nabla_l\lambda_k\eta_j - \lambda_k\nabla_l\eta_j) + c\nu^2\lambda_l(\eta_j g_{ki} - \eta_k g_{ji}) \\ & + \nu^2(1+\lambda\psi)(\nabla_l\eta_j g_{ki} - \nabla_l\eta_k g_{ji}) + b\nu^2\lambda_l(\lambda_k f_{ji} - \lambda_j f_{ki} - 2\lambda_i f_{kj}) \\ & + \nu^2\psi(\nabla_l\lambda_k f_{ji} + \lambda_k\nabla_l f_{ji} - \nabla_l\lambda_j f_{ki} - \lambda_j\nabla_l f_{ki} - 2\nabla_l\lambda_i f_{kj} - 2\lambda_i\nabla_l f_{kj}) , \end{aligned}$$

where we have put $b_l = \nabla_l b$ and $c = 3\psi + 2\lambda\psi^2 + a\lambda$. Taking the cyclic sum of the above equation with respect to the indices l, k and j and taking account of (2.12)-(2.14), (3.5), (3.6), (3.8) and the second Bianchi's identity, we can find that

$$\lambda_i(b_l U_{jk} + b_k U_{lj} + b_j U_{kl}) + 2\nu^2\psi(1+\lambda\psi)(g_{li}U_{jk} + g_{ki}U_{lj} + g_{ji}U_{kl}) = 0 . \tag{3.10}$$

These calculations are simple but lengthy, so we omit the calculations. By contraction of this equation with λ^i, it is clear that

$$\lambda_r\lambda^r(b_l U_{jk} + b_k U_{lj} + b_j U_{kl}) = 0 .$$

Since $\lambda_r\lambda^r \neq 0$, we have

$$b_l U_{jk} + b_k U_{lj} + b_j U_{kl} = 0 \quad \text{on } M' .$$

Hence, it follows from (3.10) that

$$2\nu^2\psi(1+\lambda\psi)(g_{li}U_{jk}+g_{ki}U_{lj}+g_{ji}U_{kl})=0\ ,$$

which implies that

$$2(n-2)\nu^2\psi(1+\lambda\psi)=0\ .$$

Therefore, by virtue of the assumptions $n>2$ and $\nu^2\neq 0$, we have

$$\psi(1+\lambda\psi)=0\ .$$

This completes the proof.

Now, we put

$$U_1=\{p\in M'\colon \psi=0 \text{ at } p\}$$

and

$$U_2=\{p\in M'\colon 1+\lambda\psi=0 \text{ at } p\}\ .$$

Then U_1 and U_2 are closed subsets of M', $M'=U_1\cup U_2$ and $U_1\cap U_2=\emptyset$. If M' is connected, we can see that $M'=U_1$ or $M'=U_2$. So we can get the following.

LEMMA 3.2. – *Under the same assumptions of Theorem* A *and if M' is connected, then we find*

$$M'=U_1 \quad \textit{or} \quad M'=U_2\ .$$

4. – The case of $M'=U_1$.

Let us prove the following.

LEMMA 4.1. – *Let M be a complete and connected submanifold with the same assumptions of Theorem* A. *If $M'=U_1$, then M is isometric with a sphere of radius $1/\sqrt{\nu^2}$.*

PROOF. – We can find from (2.15) that ν^2 is constant on M because ν^2 is continuous on M and $M-M'$ has no interior points. Therefore, it is clear from (2.13) that

$$\nabla_j\nabla_i\lambda=-\nu^2\lambda g_{ji} \quad \text{on } M'\ .$$

Since λ is smooth on M, this equation holds on M. This implies that M is isometric with a sphere of radius $1/\sqrt{\nu^2}$ (cf. [6], [7] or [10]).

5. – The case of $M' = U_2$.

Let us recall some results on λ-hypersurfaces in [3]. Since points of M' are ordinary, we can choose a suitable local coordinate system (x^1, x^a) for the function λ in a neighborhood W, with respect to the components of metric tensor g of M' such that

$$g_{11} = 1 , \quad g_{b1} = g_{1a} = 0 .$$

The first local coordinate x^1 is the arc length of λ-curve R and (x^a) is the local coordinate of each λ-hypersurface M. Then M' is locally expressed as $M' = R \times \bar{M}$. In terms of such a coordinate system (x^1, x^a), we denote by prime the ordinary differentiation with respect to the arc length x^1 and it follows from (2.11) that

$$\eta_1 = \xi^1 = \lambda_a = 0 .$$

Let the $(n-1)$-dimensional manifold $\bar{M}$ be a Riemannian manifold endowed with the metric tensor $\bar{g}$ defined below. Then we have the relations

$$\bar{g}^{ba}\bar{\eta}_a = \bar{\xi}^b \quad \text{and} \quad \bar{\eta}_a\bar{\xi}^a = \bar{g}_{ab}\bar{\xi}^a\bar{\xi}^b = 1 , \tag{5.1}$$

where $\bar{g}^{ba}$ is given by

$$g^{ba} = (1 - \lambda^2)^{-1}\bar{g}^{ba} - \big((1 - \lambda^2)^{-1} - (\lambda')^{-2}\big)\bar{\xi}^a\bar{\xi}^b . \tag{5.2}$$

Moreover, we can see that $\bar{g}_{ab}$, $\bar{g}^{ab}$, $\bar{\eta}_a$ and $\bar{\xi}^a$ are independent of x^1. Now, we put

$$\bar{\nabla}_b\bar{\xi}^a = \bar{f}_b{}^a ,$$

where $\bar{\nabla}$ is the covariant differentiation with respect to $\bar{g}$ of $\bar{M}$. Since $\bar{\xi}$ is a unit Killing vector field of $\bar{M}$, we have

$$\bar{\xi}^e\bar{\nabla}_e\bar{\eta}_a = \bar{\eta}_a\bar{\nabla}_a\bar{\xi}^e = 0 .$$

In the following, we assume that $M' = U_2$, that is, $1 + \lambda\psi = 0$ on M'. We restrict our calculations on M' and note that $\lambda \neq 0$ on M'. Taking account of (2.15) and (2.11), we have

$$\psi = \lambda''/(\lambda')^2 + \lambda/(1 - \lambda^2) , \tag{5.3}$$

which yields that

$$\lambda''/(\lambda')^2 = -1/\lambda - \lambda/(1 - \lambda^2) .$$

Integrating the above equation, we can get

$$\lambda' = k\sqrt{1 - \lambda^2}/\lambda , \tag{5.4}$$

where k is nonzero constant. Also we can get from (5.4) that

$$\lambda'' = -k^2/\lambda^3 . \tag{5.5}$$

By the way, differentiating $1+\lambda\psi = 0$ covariantly and making use of (3.1), we get

$$a = 1/\lambda^2 ,$$

from which

$$b = a + 2\psi^2 = 3/\lambda^2 .$$

Consequently, the equation (3.5) can be rewritten as follows:

$$R_{kji}{}^r\eta_r = (3/\lambda^2)\lambda_i U_{kj} + (k^2/\lambda^3)(\lambda_k f_{ji} - \lambda_j f_{ki} - 2\lambda_i f_{kj}) .$$

Transvecting this equation with g^{ji}, we have

$$R_{kr}\eta^r = -(3k^2/\lambda^4)\eta_k . \tag{5.6}$$

LEMMA 5.1. – *Let M be a submanifold with the same assumptions of Theorem* A. *If* $1+\lambda\psi = 0$ *on* M', *then M is not an Einstein manifold.*

PROOF. – If M is an Einstein manifold, then the equation (5.6) implies that

$$S/n = -3k^2/\lambda^4 , \tag{5.7}$$

where S is the scalar curvature of M. Since the scalar curvature of Einstein manifold is constant, the equation (5.7) tells us that λ is constant. This contradicts to our assumptions. This completes the proof.

As $M' = U_2$, we find the components of the Riemannian curvature tensor $R_{kji}{}^h$ of M as follows:

$$\tag{5.8} \left\{ \begin{aligned} & R_{111}{}^1 = R_{111}{}^a = R_{11b}{}^a = 0 , \\ & R_{dcb}{}^1 = -\frac{k(1-\lambda^2)^{\frac{3}{2}}}{\lambda^4}(\bar\eta_d \bar f_{cb} - \bar\eta_c \bar f_{db} - 2\bar\eta_b \bar f_{dc}) , \\ & R_{1cb}{}^a = -\frac{k^3(1-\lambda^2)^{\frac{1}{2}}}{\lambda^4}(2\bar\eta_c \bar f_b{}^a + \bar\eta_b \bar f_c{}^a) , \\ & R_{d1b}{}^1 = -\frac{3k^4(1-\lambda^2)}{\lambda^6}\bar\eta_d\bar\eta_b , \\ & R_{1c1}{}^a = \frac{3k^2}{\lambda^4}\bar\eta_c\bar\xi^a , \\ & R_{dcb}{}^a = \bar R_{dcb}{}^a - k^2(\delta_d{}^a \bar g_{cb} - \delta_c{}^a \bar g_{db}) + (k^2-1)(\delta_d{}^a\bar\eta_c\bar\eta_b - \delta_c{}^a\bar\eta_d\bar\eta_b - \bar g_{db}\bar\eta_c\bar\xi^a \\ & \qquad + \bar g_{cb}\bar\eta_d\bar\xi^a) + (k^2/\lambda^2 - 1)(2\bar f_{dc}\bar f_b{}^a + \bar f_{db}\bar f_c{}^a - \bar f_{cb}\bar f_d{}^a) , \end{aligned} \right.$$

where $\bar{R}_{dcb}{}^{a}$ is the Riemannian curvature tensor of $\bar{M}$. From these, we get the components of the Ricci tensor of M as follows:

$$(5.9)\quad \begin{cases} R_{11} = -3k^2/\lambda^4 , \\ R_{1a} = 0 , \\ R_{cb} = \bar{R}_{cb} - (k^2(n-3) - 3k^2/\lambda^2 + 2)\bar{g}_{cb} \\ \qquad + \left((k^2-1)(n-3) + 3(k^2/\lambda^2 - 1) - 3k^4(1-\lambda^2)/\lambda^6\right)\bar{\eta}_c\bar{\eta}_b , \end{cases}$$

where $\bar{R}_{cb}$ is the Ricci tensor of $\bar{M}$.

LEMMA 5.2. – *Let M be a submanifold with the same assumptions of theorem* A. *If $M' = U_2$, then M is not conformally flat.*

PROOF. – If M is conformally flat, then we have the equation

$$C_{kji}{}^{h} = R_{kji}{}^{h} + \frac{2}{n-2}(R_{ki}\delta_j^h - R_{ji}\delta_k^h + g_{ki}R_j{}^h - g_{ji}R_k{}^h) - \frac{S}{(n-1)(n-2)}(g_{ki}\delta_j^h - g_{ji}\delta_k^h) = 0 .$$

Putting $h = 1$, $i = b$, $j = c$ and $k = d$ in the above equation, we have from (5.9)

$$R_{dcb}{}^{1} = 0 .$$

Consequently, it follows from the second equation of (5.8) that

$$\frac{k^3(1-\lambda^2)^{\frac{3}{2}}}{\lambda^4}(\bar{\eta}_d\bar{f}_{cb} - \bar{\eta}_c\bar{f}_{db} - 2\bar{\eta}_b\bar{f}_{dc}) = 0 ,$$

from which we have $k = 0$. This contradicts to

$$\lambda' = k\sqrt{1-\lambda^2}/\lambda \neq 0 .$$

PROOF OF OUR THEOREM. – Summing up the Lemma 3.1, 3.2, 4.1, 5.1, 5.2 and Theorem B, we can prove the Theorem stated in § 0.

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