On Subtransversality (*).

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Summary. – *Let* (X, A) *and* (Y, B) *be pairs of manifolds. By means of a « completed » bijet bundle over the blow-up of* X *along* A *we give simple geometric interpretations of the notion of subtransversality of a smooth map* $f: (X, A) \to (Y, B)$ *along* A.

The notion of subtransversality is due to Aldo Andreotti and was introduced in [2]. (See also [1] for a recent contribution in the projective-geometric context.) In the present paper we show that subtransversality of $f: X \to Y$ to $B \subset Y$ along $A \subset X$, or rather subtransversality after blowing up A, has a simple geometric meaning in terms of ordinary transversality in tangent and normal bundles. Here (X, A) and (Y, B) are smooth manifolds—with—submanifolds and f is a smooth mapping sending A into B. The connection is made via a sort of « completed » bijet bundle $E = E(X, A; Y, B)$ over the blow-up $W = W(X, A)$ of X along A. The bundle space E contains a smooth « singularity » submanifold Z with two strata, and the subtransversality conditions are translated into transversality of the jet section to the strata of Z along the strata of W (proposition 2.1). From this one easily extracts the results (theorems 1.1 and 1.2).

1. – Preliminaries and statements.

We recall a few concepts from [2]. Let X and Y be smooth (i.e. C^∞-) manifolds, $\dim X > 0$, and let A and B be closed submanifolds of X and Y. We denote by $C^\infty(X, A; Y, B)$ the set of smooth maps $g: X \to Y$ such that $g(A) \subseteq B$. This is a closed subset of $C^\infty(X, Y)$ in the Whitney topology ($=$ the fine C^∞-topology).

Furthermore, denote by $C_a^\infty(X)$ the local ring of germs of smooth functions at $a \in X$. An ideal $I \subseteq C_a^\infty(X)$ is *regular* of codimension k if I has k generators $h_1, h_2, \ldots, h_k$ such that $dh_1 \wedge \ldots \wedge dh_k \neq 0$. This requires I to be a proper ideal of $C_a^\infty(X)$. In addition we consider $I = C_a^\infty(X)$ to be a regular ideal of codimension k for any integer k. Then $V(I) = \{x \in (X, a): h(x) = 0, \forall h \in I\}$ is the germ of a smooth submanifold of X at a of codimension k (empty if $I = C_a^\infty(X)$). Clearly a mapping

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$g\colon X \to Y$ is transverse to B at $a \in X$ if and only if $C_a^\infty(X)\cdot g^*I(B)_{g(a)}$ is a regular ideal of codimension k, where k is the codimension of B at $g(a)$ and $I(B)_{g(a)} \subseteq C_{g(a)}^\infty(Y)$ is the ideal of smooth germs at $g(a)$ vanishing on B.

Next, let $g \in C^\infty(X, A; Y, B)$ and let $a \in A$; then $C_a^\infty(X)\cdot g^*I(B)_{g(a)} \subseteq I(A)_a$. Consider the conductor ideal $c_g(I(A)_a, I(B)_{g(a)}) \subseteq C_a^\infty(X)$. By definition $h \in c_g(I(A)_a, I(B)_{g(a)})$ if and only if $h\cdot I(A)_a \subseteq C_a^\infty(X)\cdot g^*I(B)_{g(a)}$. We say that g is *subtransverse* to B at a if $c_g(I(A)_a, I(B)_{g(a)})$ is regular of codimension equal the codimension of B at $g(a)$, and *strongly subtransverse* to B at a if $c_g(I(A)_a, I(B)_{g(a)}) + I(A)_a$ is regular of codimension equal the sum of the codimensions of A and B at a and $g(a)$.

Finally, let $\tilde{X}$ be the blow-up of X along A and $\sigma\colon \tilde{X} \to X$ the collapse mapping. Then $\tilde{X}$ is canonically a smooth manifold with $\tilde{A} = \sigma^{-1}(A)$ a codimension one submanifold, cf. [3]. (Although the setting in [3] is complex analytic the methods work equally well in the C^∞ case.) A mapping $g \in C^\infty(X, A; Y, B)$ is (strongly) σ-subtransverse to B at a if $g\circ\sigma$ is (strongly) subtransverse to B at any point of $\sigma^{-1}\{a\}$.

The geometric content of these definitions is given by the following

Theorem 1.1. – *Let $g \in C^\infty(X, A; Y, B)$. Then the statements*

(i) *g is strongly σ-subtransverse to B at all points of A*

(ii) *νg is transverse to O_B outside O_A*

are equivalent.

Here $\nu g\colon \nu A \to \nu B$ is the normal bundle mapping, and O_A and O_B are the zero-sections of νA and νB. The theorem follows from proposition 2.1 and 2.2 of section 2.

We will consider in more detail the case where g is a product mapping $f\times f\colon N\times N \to P\times P$ and A and B are the diagonals Δ_N and Δ_P respectively. The normal bundles νA and νB can then be identified with the tangent bundles τN and τP. In this case we have the following sharper result.

Theorem 1.2. – *Let $f\colon N \to P$ be a smooth mapping. Then the statements*

(i) *$f\times f$ is σ-subtransverse to Δ_P at all points of Δ_N.*

(ii) *$f\times f$ is strongly σ-subtransverse to Δ_P at all points of Δ_N.*

(iii) *τf is transverse to O_P outside O_N*

are equivalent.

Here $\tau f\colon \tau N \to \tau P$ is the tangent bundle mapping and O_N and O_P are the zero-sections of τN and τP.

The theorem is a corollary of proposition 2.1 and 2.2 with addendum. It generalizes and elucidates the results of section 19 in [2]. In particular the genericity of smooth mappings f with $f\times f$ (strongly) σ-subtransverse is immediately explained (see remark 1 in section 3).

2. - Double points and residual singularities.

Let $W = W(X, A)$ be the blow-up of X along A. Thus W is obtained from X by suitably replacing A with $P\nu A$, the projectivized normal bundle of A, see for instance [3]. Set $X - A = W_0$ and $P\nu A = W_1$, so that $W = W_0 \cup W_1$.

We construct a smooth manifold $E = E(X, A; Y, B)$ over W depending functorially on (X, A) and (Y, B). First, set $E = E_0 \cup E_1$ where

$$E_0 = \{(x, y) : x \in X - A,\ y \in Y\}$$

$$E_1 = \{(x, l, y, \varphi) : x \in A,\ y \in B,\ l \in P\nu_x A,\ \varphi \in \mathrm{Hom}\,(l, \nu_y B)\}\ .$$

Then there is a natural projection π of E onto W defined by

$$\pi(x, y) = x \qquad (\text{on } E_0)$$

$$\pi(x, l, y, \varphi) = (x, l) \qquad (\text{on } E_1)\ .$$

Secondly, for every $g \in C^\infty(X, A; Y, B)$ there is an induced mapping $\hat{g}: W \to E$, which is a section of π, defined by

$$\hat{g}(x) = \big(x, g(x)\big) \qquad (\text{on } W_0)$$

$$\hat{g}(x, l) = \big(x, l, g(x), \nu g|l\big) \qquad (\text{on } W_1)\ .$$

When Y is a point and $B = Y$, then $E(X, A; Y, B) = W(X, A)$ (as a set), and π is the identity mapping.

We need a smooth structure on E. Set $\dim X = m$, $\dim A = r$ and $\dim Y = q$, $\dim B = s$. First notice that E_0 and E_1 are naturally smooth manifolds of dimensions $m + q$ and $(m - 1) + q$ over the smooth manifolds W_0 and W_1. In fact $E_0 = (X - A) \times Y$. As for E_1 let $L\nu A$ be the tautological line bundle over $P\nu A$, and $\mathrm{Hom}\,(L\nu A, \nu B)$ the corresponding vector bundle over $P\nu A \times B$; then $E_1 = \mathrm{Hom}\,(L\nu A, \nu B)$. We will show that $E = E(X, A; Y, B)$ has a canonical smooth structure compatible with that of E_0 and E_1, such that π is smooth and such that $\hat{g}$ is smooth for any smooth g. In particular $E(X, A; Y, B) = W(X, A)$ (as a manifold) when Y is a point and $B = Y$.

Consider first the case $X = \boldsymbol{R}^m$, $Y = \boldsymbol{R}^q$, $A = \boldsymbol{R}^r \times \{0\} \subset X$ and $B = \boldsymbol{R}^s \times \{0\} \subset Y$. Define $A_k \subset E$, $1 \leqslant k \leqslant m - r$, by $A_k = A_{k0} \cup A_{k1}$ where

$$A_{k0} = \{(x, y) \in E_0 : x_{r+k} \neq 0\}$$

$$A_{k1} = \{(x, l, y, \varphi) \in E_1 : l_k \neq 0\}$$

and $(l_1, \ldots, l_{m-r})$ are homogeneous coordinates for l. Evidently $E = A_1 \cup \ldots \cup A_{m-r}$.

Next, define mappings $\alpha_k\colon A_k \to \boldsymbol{R}^m \times \boldsymbol{P}^{m-r-1} \times \boldsymbol{R}^q$ $(1 \leqslant k \leqslant m-r)$ by

$$\alpha_k(x, y) = (x, \boldsymbol{R}x'', y', y''/x_{r+k}) \qquad (\text{on } A_{k0})$$

$$\alpha_k(x, l, y, \varphi) = \big(x, l, y', \varphi(l_{1k}, \ldots, l_{m-r,k})\big) \qquad (\text{on } A_{k1})$$

where $x = (x', x'') \in \boldsymbol{R}^r \times \boldsymbol{R}^{m-r}$, $y = (y', y'') \in \boldsymbol{R}^s \times \boldsymbol{R}^{q-s}$ and $l_{ik} = l_i/l_k$ for $1 \leqslant i \leqslant m-r$.

Clearly α_k is injective for all k. We topologize A_k so that α_k is a homeomorphism onto its image. Then $A_k \cap A_l$ is an open subset of A_k and A_l for each k and l, as is quickly checked, and the topology induced by A_k on $A_k \cap A_l$ coincides with the topology induced by A_l since the mappings $\alpha_l \circ \alpha_k^{-1}$ are continuous and therefore homeomorphisms. Consequently there is a unique topology on E such that each space A_k occurs as an open subspace of E. It is easy to see that E is a Hausdorff space.

We show that $\alpha_k(A_k)$ is a $(m+q)$-dimensional smooth submanifold of $\boldsymbol{R}^m \times \boldsymbol{P}^{m-r-1} \times \boldsymbol{R}^q$. Set $U_k = \boldsymbol{R}^m \times \boldsymbol{P}_k^{m-r-1} \times \boldsymbol{R}^q$ where $\boldsymbol{P}_k^{m-r-1}$ is the affine open coordinate set $\{L \in \boldsymbol{P}^{m-r-1}\colon L_k \neq 0\}$ in $\boldsymbol{P}^{m-r-1}$. Then $\alpha_k(A_k) \subset U_k$ for $k = 1, \ldots, m-r$; in fact (ξ, L, η) is in $\alpha_k(A_k)$ if and only if $L_k \neq 0$ and $\xi_{r+i} L_k = \xi_{r+k} L_i$ for $1 \leqslant i \leqslant m-r$.

Define $\theta_k\colon U_k \to \boldsymbol{R}^{m-r-1}$ by $\theta_k(\xi, L, \eta) = (\xi_{r+1} - L_{1k}\xi_{r+k}, \ldots, \xi_m - L_{m-r,k}\xi_{r+k})$ where the k-th component $(=0)$ is omitted. Then θ_k is a submersion onto $\boldsymbol{R}^{m-r-1}$. Since $\alpha_k(A_k) = \theta_k^{-1}\{0\}$, it follows that $\alpha_k(A_k)$ is a smooth submanifold of U_k, hence of $\boldsymbol{R}^m \times \boldsymbol{P}^{m-r-1} \times \boldsymbol{R}^q$, of codimension $m-r-1$.

By means of α_k we pull back the smooth structure on $\alpha_k(A_k)$ to A_k. We now need to show that A_k and A_l induce the same smooth structure on the open set $A_k \cap A_l$ for any two k and l. But this holds since the mappings $\alpha_l \circ \alpha_k^{-1}$ are smooth and therefore diffeomorphisms. Thus $E = A_1 \cup \ldots \cup A_{m-r}$ receives a smooth structure in which $A_1, \ldots, A_{m-r}$ are open submanifolds.

For $q = 0$, i.e. $B = Y = \{0\}$, we clearly get $E = W$. (Alternatively define the smooth structure on $W(\boldsymbol{R}^m, \boldsymbol{R}^r)$ as that of $E(\boldsymbol{R}^m, \boldsymbol{R}^r; 0, 0)$.) Throughout the paper we shall use primed letters $A_k', \alpha_k', \ldots$ in the particular case $E = W$, i.e. primed letters refer to W. Then we have a commutative diagram

$$\begin{array}{ccc} A_k & \xrightarrow{\ \alpha_k\ } & \boldsymbol{R}^m \times \boldsymbol{P}^{m-r-1} \times \boldsymbol{R}^q \\ \pi \big\downarrow & & \big\downarrow pr \\ A_k' & \xrightarrow{\ \alpha_k'\ } & \boldsymbol{R}^n \times \boldsymbol{P}^{m-r-1} \end{array}$$

showing that π is smooth on A_k, $1 \leqslant k \leqslant m-r$. Thus π is smooth (on E).

Finally we need to check that $\hat{g}\colon W \to E$ is smooth for smooth g. Obviously it suffices to check this at a point $(x, l) \in W_1$. Let k be such that $(x, l) \in A_k'$. We have $\hat{g}(A_k') \subset A_k$ and therefore a map $\tau_k\colon \alpha_k'(A_k') \to \alpha_k(A_k)$ defined by the commutative diagram

$$\begin{array}{ccc} A_k & \xrightarrow{\ \alpha_k\ } & \alpha_k(A_k) \\ \hat{g} \big\uparrow & & \big\uparrow \tau_k \\ A_k' & \xrightarrow{\ \alpha_k'\ } & \alpha_k'(A_k') \end{array}$$

Extend τ_k to a mapping $T_k\colon U'_k \to U_k$ in the following way: Write

$$g_{s+i}(\xi) = \sum_{j=1}^{m-r} \xi_{r+j}\, G_{ij}(\xi)\,, \qquad 1 \leqslant i \leqslant q-s\,,$$

with the

$$G_{ij}(\xi) = \int_0^1 \frac{\partial g_{s+i}}{\partial x_{r+j}}(\xi', t\xi'')\, dt \qquad \text{for } \xi = (\xi', \xi'') \in \boldsymbol{R}^r \times \boldsymbol{R}^{m-r}\,,$$

such that $G_{ij}(\xi) = (\partial g_{s+i}/\partial x_{r+j})(\xi)$ when $\xi'' = 0$. Now set

$$T_k(\xi, L) = \Big(\xi, L, g_1(\xi), \ldots, g_s(\xi)\,, \sum_{j=1}^{m-r} L_{jk} G_{1j}(\xi), \ldots, \sum_{j=1}^{m-r} L_{jk} G_{q-s,j}(\xi)\Big)\,.$$

Then T_k extends τ_k as claimed. Since T_k is smooth, so is τ_k. Consequently $\hat{g}$ is smooth.

This proves the claim in the affine case $X = \boldsymbol{R}^m$, $Y = \boldsymbol{R}^q$. The extension to the case where X and Y are diffeomorphic to $\boldsymbol{R}^m$ and $\boldsymbol{R}^q$ is by transport of structure; the result is easily seen to be independent of the choice of diffeomorphisms. The extension to the general case is then by patching over coordinate neighbourhoods in X and Y, thereby constructing the germ of E along E_1 compatible with E_0. The procedure is straightforward. We omit further details.

REMARKS. – 1) By construction E_0 and E_1 are built in as submanifolds of E. Since E_0 is an open submanifold, E_1 is a closed codimension one submanifold of E.

2) There is also a smooth projection $\pi_2\colon E \to Y$ defined by

$$\begin{aligned} \pi_2(x, y) &= y && (\text{on } E_0)\\ \pi_2(x, l, y, \varphi) &= y && (\text{on } E_1)\,. \end{aligned}$$

More symmetrically we have the smooth projections

$$X \xleftarrow{\;\pi_1\;} E \xrightarrow{\;\pi_2\;} Y$$

where $\pi_1 = \sigma \circ \pi$. Thus the extension $\hat{g}$ of g fits into the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\;\hat{g}\;} & E \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\pi_2} \\ X & \xrightarrow{\;g\;} & Y \end{array}$$

We next define a special submanifold Z of E. Let $Z = Z_0 \cup Z_1$, where

$$Z_0 = \{(x, y) \in E_0 \colon y \in B\}$$

$$Z_1 = \{(x, l, y, \varphi) \in E_1 \colon \varphi = 0\} .$$

Then $Z \subset E$; we claim that Z is a closed submanifold of E. First notice that $Z \cap E_0 = Z_0$ is certainly a closed submanifold of E_0. If $a \in E_1$ is in the closure of Z, then $a \in E(U, U \cap A; V, V \cap B)$ for suitable coordinate systems (U, φ) and (V, ψ) in X and Y such that $\varphi(U \cap A) = \boldsymbol{R}^r \times \{0\}$ and $\psi(V \cap B) = \boldsymbol{R}^s \times \{0\}$. Thus $a \in Z$ if $Z \cap E(U, U \cap A; V, V \cap B)$ is closed in $E(U, U \cap A; V, V \cap B)$. Moreover, Z is a submanifold of E locally around a if $Z \cap E(U, U \cap A; V, V \cap B)$ is a submanifold of $E(U, U \cap A; V, V \cap B)$. Consequently we are reduced to substantiating our claim in the affine case $X = \boldsymbol{R}^m$, $Y = \boldsymbol{R}^q$, $A = \boldsymbol{R}^r \times \{0\} \subset X$ and $B = \boldsymbol{R}^s \times \{0\} \subset Y$. Again, in the affine case it suffices to show that $Z \cap A_k$ is a closed submanifold of A_k for $k = 1, \ldots, m - r$. Let $\varrho \colon \boldsymbol{R}^m \times \boldsymbol{P}^{m-r-1} \times \boldsymbol{R}^q \to \boldsymbol{R}^{q-s}$ be the projection to the last $q - s$ coordinates. It is quickly checked that $\varrho | \alpha_k(A_k)$ has constant rank $q - s$, i.e. that $\varrho \circ \alpha_k$ has constant rank $q - s$. But $Z \cap A_k = (\varrho \circ \alpha_k)^{-1}\{0\}$, and so $Z \cap A_k$ is indeed a closed submanifold of A_k.

Notice that Z_1 is a closed codimension one submanifold of Z. This follows by the same arguments as above if we use the projection $\lambda_k \colon \boldsymbol{R}^m \times \boldsymbol{P}^{m-r-1} \times \boldsymbol{R}^q \to \boldsymbol{R}^{q-s+1}$ defined by

$$\lambda_k(\xi, l, \mu) = (\xi_{r+k}; \mu_{s+1}, \ldots, \mu_q)$$

instead of ϱ.

The construction $Z \subset E \to W$ is hopefully justified by the following

PROPOSITION 2.1. – *Let $g \in C^\infty(X, A; Y, B)$. Then g is σ-subtransverse to B at all points of A if and only if $\hat{g}$ is transverse to Z on W_1 and strongly σ-subtransverse if and only if $\hat{g}$ is transverse to Z_1 on W_1.*

Apart from questions of subtransversality we have quite generally

PROPOSITION 2.2. – *Let $g \in C^\infty(X, A; Y, B)$. Then $\hat{g}$ is transverse to Z_1 on W_1 if and only if the normal bundle map $\nu g \colon \nu A \to \nu B$ is transverse to the zero-section $O_B \subset \nu B$ outside $O_A \subset \nu A$.*

REMARK. – If $\hat{g}$ is transverse to Z_1 on W_1, then it is transverse to Z_1 on all of W, since it cannot hit Z_1 outside W_1.

We lack a corresponding general interpretation of transversality of $\hat{g}$ to Z. However, in the situation where g equals $f \times f$ for a smooth mapping $f \colon N \to P$ and A and B are the diagonals in $N \times N$ and $P \times P$ the two cases coincide (over W_1).

ADDENDUM. – *If g is of the form $f \times f \colon N \times N \to P \times P$ and $A = \Delta_N$, $B = \Delta_P$, then on W_1 $\hat{g}$ is transverse to Z_1 if and only if it is transverse to Z.*

In this situation of course the normal bundles $\nu\Delta_N$ and $\nu\Delta_P$ can be identified with the tangent bundles τN and τP. For brevity we will denote the mapping $\hat{g}\colon W \to E$ by f^{Δ} when $g = f\times f$. Then we have

COROLLARY 2.3. – *The mapping $f^{\Delta}\colon W \to E$ is transverse to Z if and only if*

(i) *$f\times f$ is transverse to Δ_P outside Δ_N.*

(ii) *τf is transverse to $O_P \subset \tau P$ outside $O_N \subset \tau N$.*

In fact (i) *and* (ii) *are equivalent to the transversality of f^{Δ} to Z on W_0 and W_1, respectively.*

The proofs of proposition 2.1 and proposition 2.2 with addendum are given in section 3.

3. – Proofs. Complements.

We first prove proposition 2.2 and its addendum and then proposition 2.1. The symbol $\pitchfork$ will mean «transverse to».

Let $w = (a, l) \in W_1$. We will show that $\hat{g} \pitchfork Z_1$ at w if and only if $\nu g \pitchfork O_B$ at some (hence any) non-zero vector v in $l \subset \nu_a A$. Set $t = \operatorname{rank}(\nu g)_a$.

By restricting to suitable coordinate patches around a and $g(a)$, it suffices to consider the case $X = \boldsymbol{R}^m$, $Y = \boldsymbol{R}^q$, $A = \boldsymbol{R}^r\times\{0\} \subset X$, $B = \boldsymbol{R}^s\times\{0\} \subset Y$, $a = 0$, $g(a) = 0$. In fact we may assume the coordinatisation at a and $g(a)$ performed such that $g = (g_1, g_2)\colon \boldsymbol{R}^m \to \boldsymbol{R}^s\times\boldsymbol{R}^{q-s}$ with $g_1(0) = 0$ and

$$g_2(x) = \big(x_{r+1}, \dots, x_{r+t}, \psi(x)\big)\,,$$

where $\psi\colon \boldsymbol{R}^m \to \boldsymbol{R}^{q-s-t}$ is a smooth mapping such that $\psi(A) = \{0\}$ and $D\psi(0) = 0$.

Now, let $v = (v', v'') \in \boldsymbol{R}^t\times\boldsymbol{R}^{m-r-t}$ be a non-zero vector and $l \in \boldsymbol{P}^{m-r-1} = P\nu_0\boldsymbol{R}^{m-r}$ the line spanned by v. We have $\hat{g}(0, l) = \big(0, l, 0, \nu g(0)|l\big)$ with

$$\nu g(0) = \left[\begin{array}{c|c} I_t & 0 \\ \hline 0 & 0 \end{array}\right].$$

Thus $\nu g(0)v = v'$ and so

(i) $\hat{g}(0, l) \notin Z_1$ if and only if $v' \neq 0$.

Suppose $v' = 0$. With notations as before choose k such that $(0, l) \in A_k'$; then $\hat{g}(0, l) \in A_k$. Recall that $\lambda_k\circ\alpha_k\colon A_k \to \boldsymbol{R}^{q-s+1}$ is a submersion and that $Z_1 \cap A_k = (\lambda_k\circ\alpha_k)^{-1}\{0\}$. Thus

$$\begin{array}{lll} & \hat{g} \pitchfork Z_1 & \text{at} \quad (0, l) \\ \Leftrightarrow & \lambda_k\circ\alpha_k\circ\hat{g}\colon A_k' \to \boldsymbol{R}^{q-s+1} & \text{is submersive at } (0, l) \\ \Leftrightarrow & \lambda_k\circ\tau_k\colon \alpha_k'(A_k') \to \boldsymbol{R}^{q-s+1} & \text{is submersive at } \alpha_k'(0, l) \\ \Leftrightarrow & \lambda_k\circ T_k\circ i_k'\colon \alpha_k'(A_k') \to \boldsymbol{R}^{q-s+1} & \text{is submersive at } \alpha_k'(0, l)\,. \end{array}$$

Here $i_k': \alpha_k'(A_k') \to U_k'$ is the inclusion mapping,

$$\begin{array}{ccccccc} \alpha_k'(A_k') & \xrightarrow{i_k'} & U_k' & \xrightarrow{T_k} & U_k & \xrightarrow{\lambda_k} & \mathbf{R}^{q-s+1} \\ & & \downarrow{\scriptstyle \theta_k'} & & & & \\ & & \mathbf{R}^{m-r-1} & & & & \end{array}$$

Consequently we want to determine the range of $D(\lambda_k \circ T_k \circ i_k')(\alpha_k'(0, l))$. Since $i_k'(\alpha_k'(A_k')) = \theta_k'^{-1}\{0\}$, we have range $Di_k'(\alpha_k'(0, l)) = \ker D\theta_k'(\alpha_k'(0, l))$, with $\alpha_k'(0, l) = (0, (v_1, \ldots, v_{m-r}))$. Now $D\theta_k'(\alpha_k'(0, l))$ has the matrix block form

$$\begin{bmatrix} 0 & I & -V_k' & 0 & 0 \\ 0 & 0 & -V_k'' & I & 0 \end{bmatrix}$$

where as usual I means an identity matrix and 0 a zero matrix. V_k' and V_k'' are the column matrices

$$\begin{bmatrix} v_{1,k} \\ \vdots \\ v_{k-1,k} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} v_{k+1,k} \\ \vdots \\ v_{m-r,k} \end{bmatrix},$$

where $v_{ik} = v_i/v_k$. (Recall that $v_k \neq 0$ since $(0, l) = (0, (v_1, \ldots, v_{m-r})) \in A_k'$.) In particular $v_{1,k} = \ldots = v_{t,k} = 0$ since $v' = 0$.

It now follows by straight forward computation that range $D(\lambda_k \circ T_k \circ i_k')(\alpha_k'(0, l))$ is spanned by the t standard basis vectors $e_2, \ldots, e_{t+1}$ in $\mathbf{R}^{q-s+1}$ together with the r vectors $(1 \leqslant i \leqslant r)$

$$\left(0, 0, \ldots, 0, \sum_{j=t+1}^{m-r} v_{jk} \frac{\partial^2 \psi_1}{\partial x_i \, \partial x_{r+j}}(0), \ldots, \sum_{j=t+1}^{m-r} v_{jk} \frac{\partial^2 \psi_{q-s-t}}{\partial x_i \, \partial x_{r+j}}(0)\right)$$

and the vector

$$\left(2, 0, \ldots, 0, \sum_{i=t+1}^{m-r} \sum_{j=t+1}^{m-r} v_{ik} v_{jk} \frac{\partial^2 \psi_1}{\partial x_{r+i} \, \partial x_{r+j}}(0), \ldots, \right.$$
$$\left. \sum_{i=t+1}^{m-r} \sum_{j=t+1}^{m-r} v_{ik} v_{jk} \frac{\partial^2 \psi_{q-s-t}}{\partial x_{r+i} \, \partial x_{r+j}}(0)\right).$$

We therefore have

(ii) $\hat{g}(0, l) \in Z_1$ and $\hat{g} \pitchfork Z_1$ at $(0, l)$ if and only if the vectors

$$\left(0, 0, \ldots, 0, \sum_{j=t+1}^{m-r} v_j \frac{\partial^2 \psi_1}{\partial x_i \, \partial x_{r+j}}(0), \ldots, \sum_{j=t+1}^{m-r} v_j \frac{\partial^2 \psi_{q-s-t}}{\partial x_i \partial x_{r+j}}(0)\right)$$

for $1 \leqslant i \leqslant r$ form a set of rank $q - s - t$ in $\mathbf{R}^{q-s+1}$.

To complete the proof of proposition 2.2 we now appeal to the following elementary

LEMMA 3.1. – *Let* $g \in C^\infty(\boldsymbol{R}^m, \boldsymbol{R}^r\times\{0\}; \boldsymbol{R}^q, \boldsymbol{R}^s\times\{0\})$ *be a mapping of the form* $g(x) = (g_1(x); x_{r+1}, \ldots, x_{r+t}, \psi(x))$ *with* $g_1\colon \boldsymbol{R}^m \to \boldsymbol{R}^s$, $\psi\colon \boldsymbol{R}^m \to \boldsymbol{R}^{q-s-t}$ *such that* $g_1(0) = 0$ *and* $\psi(\boldsymbol{R}^r\times\{0\}) = \{0\}$, $D\psi(0) = 0$. *Let* $(0, v) = (0, v', v'')$ *be a non-zero vector in the normal space* $\nu_0(\boldsymbol{R}^r\times\{0\}) = \{0\}\times\boldsymbol{R}^{m-r} = \{0\}\times\boldsymbol{R}^t\times\boldsymbol{R}^{m-r-t}$.

Then $\nu g \pitchfork O_{\boldsymbol{R}^s\times\{0\}}$ *at* $(0, v)$ *if and only if either*

(i) $v' \neq 0$ $(\textit{then } \nu g(0, v) \notin O_{\boldsymbol{R}^s\times\{0\}})$ *or*

(ii) $v' = 0$ $(\textit{then } \nu g(0, v) \in O_{\boldsymbol{R}^s\times\{0\}})$ *and the matrix*

$$\begin{bmatrix} \sum\limits_{j=t+1}^{m-r} v_j \dfrac{\partial^2 \psi_1}{\partial x_1\, \partial x_{r+j}}(0), \ldots, \sum\limits_{j=t+1}^{m-r} v_j \dfrac{\partial^2 \psi_1}{\partial x_r\, \partial x_{r+j}}(0) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \sum\limits_{j=t+1}^{m-r} v_j \dfrac{\partial^2 \psi_{q-s-t}}{\partial x_1\, \partial x_{r+j}}(0), \ldots, \sum\limits_{j=t+1}^{m-r} v_j \dfrac{\partial^2 \psi_{q-s-t}}{\partial x_r\, \partial x_{r+j}}(0) \end{bmatrix}$$

has rank $q - s - t$.

The proof of lemma 3.1 is left to the discretion of the reader.

Next we turn to the addendum. Again let $(a, l) \in W_1$ and assume that $f^\Delta(a, l) \in \boldsymbol{Z}_1$. By suitable coordinatisations we may assume $N = \boldsymbol{R}^n$, $a = 0$, $P = \boldsymbol{R}^p$, $f(a) = 0$. Using the diffeomorphism $\mu_n\colon \boldsymbol{R}^n\times\boldsymbol{R}^n \to \boldsymbol{R}^n\times\boldsymbol{R}^n$ defined by $\mu_n(x, y) = (x, y - x)$, we may further identify the diagonal $\Delta_{\boldsymbol{R}^n}$ with $\mu_n(\Delta_{\boldsymbol{R}^n}) = \boldsymbol{R}^n\times\{0\}$ and similarly $\Delta_{\boldsymbol{R}^p}$ with $\mu_p(\Delta_{\boldsymbol{R}^p}) = \boldsymbol{R}^p\times\{0\}$. The product mapping $f\times f\colon \boldsymbol{R}^n\times\boldsymbol{R}^n \to \boldsymbol{R}^p\times\boldsymbol{R}^p$ is then identified with $g = \mu_p\circ(f\times f)\circ\mu_n^{-1}$, which is given by $g(x, y) = (f(x), f(x + y) - f(x))$.

We know that $f^\Delta \pitchfork Z$ at $(0, l)$ if and only if $\varrho\circ T_k\circ i_k'\colon \alpha_k'(A_k') \to \boldsymbol{R}^p$ is a submersion at $\alpha_k'(0, l)$. Since $\varrho = pr_2\circ\lambda_k$ where $pr_2\colon \boldsymbol{R}\times\boldsymbol{R}^p \to \boldsymbol{R}^p$ is the projection, this is equivalent to $\lambda_k\circ T_k\circ i_k'$ being transverse to $K = \boldsymbol{R}\times\{0\} \subset \boldsymbol{R}\times\boldsymbol{R}^p$ at $\alpha_k'(0, l)$.

We show that $T_0K \subset \text{range } D(\lambda_k\circ T_k\circ i_k')(\alpha_k'(0, l))$. Thus if $f^\Delta \pitchfork Z$ at $(0, l)$, then $\lambda_k\circ T_k\circ i_k'$ is a submersion at $\alpha_k'(0, l)$ and so $f^\Delta \pitchfork Z_1$ at $(0, l)$. As usual let $(l_1, \ldots, l_n)$ be homogeneous coordinates for l and set $l_{jk} = l_j/l_k$ when $l_k \neq 0$, $j = 1, \ldots, n$. Define the smooth curve $c\colon \langle -\varepsilon, \varepsilon\rangle \to \alpha_k'(A_k')$ by $c(t) = (-tl_{1k}, \ldots, -tl_{nk}, 2tl_{1k}, \ldots, 2tl_{nk}, l)$; then $c(0) = \alpha_k'(0, l)$. Since

$$\lambda_k\circ T_k(\xi, L) = \left(\xi_k, \sum_{j=1}^{n} L_{jk}\int_0^1 \frac{\partial f}{\partial x_j}(\xi' + s\xi'')\, ds\right)$$

for $\xi = (\xi', \xi'') \in \boldsymbol{R}^n\times\boldsymbol{R}^n$, $L_k \neq 0$, we find

$$\lambda_k\circ T_k\circ i_k'\circ c(t) = \left(2t, \sum_{j=1}^{n} l_{jk}\int_0^1 \frac{\partial f}{\partial x_j}\big(t(2s-1)(l_{1k}, \ldots, l_{nk})\big)\, ds\right).$$

From this we get

$$\frac{d}{dt}(\lambda_k \circ T_k \circ i_k' \circ c)(0) = (2, 0, \ldots, 0) \in T_0 K$$

which confirms that T_0K sists in the range of $D(\lambda_k \circ T_k \circ i_k')(\alpha_k'(0, l))$. Thus $f^{\Delta} \pitchfork Z_1$ on W_1 if $f^{\Delta} \pitchfork Z$ on W_1. The converse is of course trivial.

Finally we prove proposition 2.1. As always let $(a, l) \in W_1$ and $b = g(a)$. Again, by suitable coordinatisations we may assume that $X = \boldsymbol{R}^m$, $Y = \boldsymbol{R}^q$, $A = \boldsymbol{R}^r \times \{0\} \subset X$, $B = \boldsymbol{R}^s \times \{0\} \subset Y$ and that g is of the form $g(x) = (g_1(x); x_{r+1}, \ldots, x_{r+t}, \psi(x))$ with $g_1\colon \boldsymbol{R}^m \to \boldsymbol{R}^s$, $\psi\colon \boldsymbol{R}^m \to \boldsymbol{R}^{q-s-t}$ smooth mappings such that $g_1(0) = 0$, $\psi(A) = \{0\}$ and $D\psi(0) = 0$.

Let $(l_1, \ldots, l_{m-r})$ be homogeneous coordinates for l and assume $l_k \neq 0$, i.e. $l \in \boldsymbol{P}_k^{m-r-1}$. Define the projection $s_k\colon \boldsymbol{R}^m \times \boldsymbol{P}^{m-r-1} \times \boldsymbol{R}^q \to \boldsymbol{R}$ by $s_k(\xi, L, \mu) = \xi_{r+k}$ and let $s_k'\colon \boldsymbol{R}^m \times \boldsymbol{P}^{m-r-1} \to \boldsymbol{R}$ be equal s_k when $q = 0$. Then $s_k' \circ i_k' \circ \alpha_k'\colon A_k' \to \boldsymbol{R}$ is a submersion, and $W_1 \cap A_k' = (s_k' \circ i_k' \circ \alpha_k')^{-1}\{0\}$. Therefore $I(W_1)_{(0,l)}$ is the principal ideal generated by the germ of $s_k' \circ i_k' \circ \alpha_k'$ at $(0, l)$.

Now let $\varphi\colon \boldsymbol{R}^q \to \boldsymbol{R}^{q-s}$ and $\varrho\colon \boldsymbol{R}^m \times \boldsymbol{P}^{m-r-1} \times \boldsymbol{R}^q \to \boldsymbol{R}^{q-s}$ be projections to the last $q - s$ coordinates. Recall the commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{\hat{g}} & E \\ \sigma \big\downarrow & & \big\downarrow \pi_2 \\ X & \xrightarrow{g} & Y \end{array}$$

The ideal $I(B)_0$ is generated by the germs of $\varphi_1, \ldots, \varphi_{q-s}$ at 0. The pullback by the mapping $g \circ \sigma$ is therefore generated by the germs of $\varphi_j \circ \pi_2 \circ \hat{g}$ at $(0, l)$, $j = 1, \ldots, q - s$.

Let $r_k\colon \boldsymbol{R}^m \times \boldsymbol{P}^{m-r-1} \times \boldsymbol{R}^q \to \boldsymbol{R}^q$ be the mapping $r_k(\xi, l, \mu) = (\mu', \xi_{r+k}\mu'')$ for $\mu = (\mu', \mu'') \in \boldsymbol{R}^s \times \boldsymbol{R}^{q-s}$, $1 \leqslant k \leqslant m - r$. Since $\pi_2 | A_k = r_k \circ i_k \circ \alpha_k$, we have $\varphi \circ \pi_2 \circ \hat{g} | A_k' = (s_k \varrho) \circ i_k \circ \alpha_k \circ \hat{g} = (s_k \circ i_k' \circ \alpha_k')(\varrho \circ T_k \circ i_k' \circ \alpha_k')$ with T_k as before. The conductor $c_g(I(W_1)_{(0,l)}, I(B)_0)$ is therefore the ideal generated by the germs of $\varrho_j \circ T_k \circ i_k' \circ \alpha_k'$ at $(0, l)$, $j = 1, \ldots, q - s$.

Finally, let $\lambda_k\colon \boldsymbol{R}^m \times \boldsymbol{P}^{m-r-1} \times \boldsymbol{R}^q \to \boldsymbol{R}^{q-s+1}$ be the projection $\lambda_k(\xi, l, \mu) = (\xi_{r+k}, \mu'')$ for $\mu = (\mu', \mu'') \in \boldsymbol{R}^s \times \boldsymbol{R}^{q-s}$. Then $c_g(I(W_1)_{(0,l)}, I(B)_0) + I(W_1)_{(0,l)}$ is the ideal generated by the germs of $\lambda_{kj} \circ T_k \circ i_k' \circ \alpha_k'$ at $(0, l)$, $j = 1, \ldots, q - s + 1$.

For the first part of the theorem: Suppose $l_k \neq 0$ for some $k \leqslant t$. On U_k we have

$$\varrho_k(T_k(\xi, L)) = \sum_{j=1}^{m-r} L_{jk} \int_0^1 \frac{\partial g_{s+k}}{\partial x_{r+j}}(\xi', t\xi'')\, dt = 1\,.$$

Thus $c_g(I(W_1)_{(0,l)}, I(B)_0)$ contains the unit element in $C^{\infty}_{(0,l)}(W)$, and so by our convention is regular of codimension $q - s$ at $(0, l)$. But we have also $\hat{g}(0, l) \in \boldsymbol{E}_1 - Z_1 = \boldsymbol{E}_1 - Z$ (p. 7 statement (i)).

Suppose on the other hand $l_1 = ... = l_t = 0$. Then $c_g(I(W_1)_{(0,l)}, I(B)_0)$ is regular of codimension $q - s$ if and only if $\varrho \circ T_k \circ i'_k$ is a submersion at $\alpha'_k(0, l)$. But the last condition is equivalent to $\hat{g} \pitchfork Z$ at $(0, l)$; this follows by an argument analogous to that for the case $\hat{g} \pitchfork Z_1$ on page 7.

For the second part of the theorem: Suppose again $l_k \neq 0$ for some $k \leqslant t$. Then $c_g(I(W_1)_{(0,l)}, I(B)_0) + I(W_1)_{(0,l)} = C^\infty_{(0,l)}(W)$ and so is regular of codimension $q - s + 1$, and $\hat{g} \pitchfork Z_1$ at $(0, l)$ since $\hat{g}(0, l) \notin Z_1$.

Suppose on the other hand $l_1 = ... = l_t = 0$. Then $c_g(I(W_1)_{(0,l)}, I(B)_0) + I(W_1)_{(0,l)}$ is regular of codimension $q - s + 1$ if and only if $\lambda_k \circ T_k \circ i'_k$ is a submersion at $a'_k(0, l)$. But this is equivalent to $\hat{g} \pitchfork Z_1$ at $(0, l)$ (p. 7).

It follows that g is strongly σ-subtransverse to B at all points of A if and only if $\hat{g} \pitchfork Z_1$ on W_1. This completes the proof of proposition 2.1.

REMARKS. – 1) It follows from corollary 2.3 that the smooth mappings $f: N \to P$ such that f^Δ is transverse to Z_1 form an open dense subset of $C^\infty(N, P)$. For the condition $f^\Delta \pitchfork Z_1$ is equivalent to $\tau f \pitchfork O_P$ outside O_N, and the latter condition is satisfied for an open dense set of mappings f by a standard transversality argument.

One can also prove a general transversality result: *Let $M \subset E$ be a smooth submanifold of E. The smooth mappings $f: N \to P$ such that f^Δ is transverse to M form a dense subset of $C^\infty(N, P)$. If M or N is compact, this subset is also open.*

In general the openness property fails unless there is a compactness condition. The first case holds without compactness because of the special character of the submanifold Z_1.

2) The construction E is tailored to the study of the generic double points of f, as indicated by corollary 2.3. Let $D_f \subset N$ be the locus of genuine double points of f and $S_f \subset N$ the singular locus of f. Thus $x \in D_f$ if $f(x) = f(x')$ for some point $x' \neq x$, and $x \in S_f$ if $\ker \tau_x f \neq \{0\}$. Then for a proper smooth mapping f such that f^Δ is transverse to $Z_1 \bar{D}_f$ equals $D_f \cup S_f$, as is easily seen. In particular $\bar{D}_f = D_f \cup S_f$ is a generic property for proper mappings, satisfied by those mappings $f \in C^\infty_{pr}(N, P)$ such that $\tau f \pitchfork O_P$ outside O_N.

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