Lubin-Hensel Factorization for Laurent Series (*) (**).

Alain Escassut

Summary. – *Let K be a complete ultrametric algebraically closed field. Let D be a bounded closed strongly infraconnected set in K with no T-filter, and let $H(D)$ be the Banach algebra of the analytic elements in D. Let r', r'' be functions from D to $\mathbf{R}$ with bounds a, b such that $0 < a \leqslant r'(x) < r''(x) \leqslant b$. Let $\mathfrak{L}(D, r', r'')$ be the Banach algebra of the Laurent series with coefficients a_s in $H(D)$ such that $\lim_{|s|\to+\infty} \left(\sup_{x\in D} |a_s(x)| \max \left(r'(x)^s, r''(x)^s\right)\right) = 0$, provided with a suitable norm. In $\mathfrak{L}(D, r', r'')$ we give a kind of Hensel Factorization for series whose dominating coefficients at $r'(x)$ and at $r''(x)$ conserve the same rank. We take advantage of this method to correcting a mistake that happened in our previous article on the Hensel Factorization for Taylor series.*

1. – Introduction and theorems.

Let $(K, |\cdot|)$ be a complete ultrametric algebraically closed field.

When A is a ring, we denote by $A[\![Y]\!]$ (resp. $A\langle\!\langle Y\rangle\!\rangle$) the set of the Taylor Series (resp. the Laurent Series) with coefficients in A.

Let D be a bounded closed subset of K. As usual, we will denote by $H(D)$ the Banach algebra of the analytic elements on D [E₁], and $\|\cdot\|_D$ the uniform convergence norm on D defined on $H(D)$.

Let $F(Y) = \sum\limits_{-\infty}^{+\infty} a_s Y^s \in H(D)\langle\!\langle Y\rangle\!\rangle$. For each $x \in D$ we will denote by F_x the series $\sum\limits_{-\infty}^{+\infty} a_s(x) Y^s \in K\langle\!\langle Y\rangle\!\rangle$.

For $a \in K$, $\varrho > 0$ we will denote by $d(a, \varrho)$ (resp. $d^-(a, \varrho)$) the disk $\{x \in K: |x - a| \leqslant \varrho\}$ (resp. $\{x \in K: |x - a| < \varrho\}$).

Also we will denote by $C(a, \varrho)$ the circle $\{x: |x - a| = \varrho\}$.

For every couple $(a, b) \in \mathbf{R}_+ \times \mathbf{R}_+$ with $0 < a < b$, let $L(a, b)$ be the algebra of the Laurent series convergent for $a \leqslant |x| \leqslant b$.

The famous Hensel Lemma gives the classical factorization in the form $P(Y) \cdot G(Y)$ for a Laurent series $F(Y) \in L(a, b)$ with $P(Y)$ a monic polynomial whose

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Indirizzo dell'A.: Université de Bordeaux I, UER de Mathématiques et Informatique, 351 Cours de la Libération, F 33405 Talence, France.

zeroes are those of $F(Y)$ and $G(Y)$ an invertible element of $L(a, b)$ [A, L]. Jonathan Lubin first gave conditions for a Taylor series $F(Y)$ with coefficients in the algebra $H(d(0,1))$ to have a polynomial $P(Y)$ with coefficients in $H(d(0,1))$, such that, for each $x \in d(0,1)$, P_x is the monic polynomial whose zeroes are those of F_x, in a disk $d(0, r(x))$ (1969, unpublished article) and this has been developped by B. DWORK ([D]) who pointed out to my attention this kind of factorization in the algebras $H(D)[\![Y]\!]$. (In 1979 J. LUBIN also gave a kind of factorization for series with coefficients in linearly topologised ring [LU], which does not apply to algebras $H(D)$ in the general case.)

We avoided the technical conditions on Newton Polygon happening in B. DWORK's treatment and tried in [E_4] Theorem 2 to give a Hensel Factorization for Taylor series with coefficients in an algebra $H(D)$ with D open closed, bounded, strongly infraconnected. Unfortunately an error in the use of Euclidean Division in Proposition 2 of [E_4] kindly pointed out to me by D. Bartenwerfer puts that result in doubt.

Here we return to this problem of Hensel factorization in generalizing our study to the Laurent Series with coefficients in algebras $H(D)$. We will particularly use the algebra norms $|\!|\!|\cdot|\!|\!|_{r'}^{r''}$ and $|\!|\!|\cdot|\!|\!|^r$ defined as follows.

Let r, r', r'' be bounded functions defined in the closed bounded set D, with values in $\boldsymbol{R}_+$, with bounds a, b for r', r'' such that $0 < a \leqslant r'(x) \leqslant r''(x) \leqslant b$ whenever $x \in D$.

Let $F(Y) = \sum_{q=0}^{\infty} a_s Y^s \in H(D)[\![Y]\!]$ (resp. $F(Y) = \sum_{-\infty}^{+\infty} a_s Y^s \in H(D)\langle\!\langle Y \rangle\!\rangle$).

We will set $|\!|\!|F|\!|\!|^r = \sup_{x\in D}\left(\sup_{s\in N} |a_s(x)| r(x)^s\right)$ $\left(\text{resp. } |\!|\!|F|\!|\!|_{r'}^{r''} = \sup_{x\in D}\left(\sup_{s\in Z} |a_s(x)| \max\left(r'(x)^s, r''(x)^s\right)\right)\right)$ and we will denote by $\mathfrak{T}(D, r)$ (resp. $\mathfrak{L}(D, r', r'')$) the set of the $F(Y) \in H(D)[\![Y]\!]$ (resp. $F(Y) \in H(D)\langle\!\langle Y \rangle\!\rangle$) such that $\lim_{s\to\infty} |\!|\!|a_s Y^s|\!|\!|^r = 0$ (resp. $\lim_{|s|\to+\infty} |\!|\!|a_s Y^s|\!|\!|_{r'}^{r''} = 0$).

Clearly $\mathfrak{T}(D, r)$ is the Banach algebra completion of $H(D)[Y]$ normed by $|\!|\!|\cdot|\!|\!|^r$. Likewise, let $H(D)\langle Y \rangle$ be the algebra of the Laurent series with a finite number of terms: $\sum_{s=m}^{n} a_s Y^s$, $a_s \in H(D)$, $m, n \in \boldsymbol{Z}$, $m \leqslant n$. Then $\mathfrak{L}(D, r', r'')$ is the Banach algebra completion of $H(D)\langle Y \rangle$ normed by $|\!|\!|\cdot|\!|\!|_{r'}^{r''}$.

Let $f(Y) = \sum_{-\infty}^{+\infty} a_s Y^s \in L(a, b)$. For $\varrho \in [a, b]$ we will also denote by $N^+(f, \varrho)$ (resp. $N^-(f, \varrho)$) the unique integer t (resp. q) such that

$$|a_t|\varrho^t = \sup_{s\in Z} |a_s|\varrho^s \quad \text{and} \quad |a_t|\varrho^t > |a_s|\varrho^s \quad \text{whenever } s > t$$

(resp. $|a_q|\varrho^q = \sup_{s\in Z} |a_s|\varrho^s$ and $|a_q|\varrho^q > |a_s|\varrho^s$ whenever $s < q$).

By classical results [A, L] we know that if f is a Laurent series convergent in the *set* $\{x \in K\colon a \leqslant |x| \leqslant b\}$ f *has exactly* $N^+(f, b) - N^-(f, a)$ *zeroes* (taking account of multiplicities).

If f is a Taylor series convergent for $|x| \leq b$, then f has exactly $N^+(f, b)$ zeroes in $d(0, b)$.

For each $n \in \boldsymbol{N}$ we will denote by $\mathfrak{T}_n(D, r)$ the subset of the $F(Y) \in \mathfrak{T}(D, r)$ such that $N^+(F_x, r(x)) = n$ whenever $x \in D$.

By what precedes we then have the obvious proposition A.

PROPOSITION A. – *Let D be a bounded closed subset of K, let r be a bounded functions from D into $\boldsymbol{R}_+$ with $r(x) > 0$ whenever $x \in D$. Let $n \in \boldsymbol{N}$. Then $\mathfrak{T}_n(D, r)$ is the set of the $F(Y) \in \mathfrak{T}(D, r)$ such that F_x has exactly n zeroes in $d(0, r(x))$ (taking account of multiplicities) for every $x \in D$.*

For every $q, t \in \boldsymbol{Z}$ with $q \leq t$, we will denote by $\mathfrak{L}_{q,t}(D, r', r'')$ the subset of the $F(Y) \in \mathfrak{L}(D, r', r'')$ such that $N^+(F_x, r''(x)) = t$, $N^-(F_x, r'(x)) = q$ whenever $x \in D$.

When no confusion is possible on the set D we will only write $\mathfrak{T}(r)$ instead of $\mathfrak{T}(D, r)$, $\mathfrak{T}_n(r)$ instead of $\mathfrak{T}_n(D, r)$, $\mathfrak{L}(r', r'')$ instead of $\mathfrak{L}(D, r', r'')$, $\mathfrak{L}_{q,t}(r', r'')$ instead of $\mathfrak{L}_{q,t}(D, r', r'')$.

REMARK. – It seems difficult to obtain a kind of Proposition A for Laurent series. Of course, if a Laurent series $F(Y)$ lies in $\mathfrak{L}_{q,t}(D, r', r'')$ by definition F_x does have exactly $t - q$ zeroes in the annulus $\{\lambda \in K: r'(x) \leq |\lambda| \leq r''(x)\}$ for each $x \in D$. But there is no converse, in the form:

« If $r'(x), r''(x)$ are functions from D to $\boldsymbol{R}_+$ (with bounds a, b such that $0 < a \leq r'(x) \leq r''(x) \leq b$) such that F_x has exactly n zeroes in the annulus $\{\lambda: r'(x) \leq |\lambda| \leq r''(x)\}$ for each $x \in D$, then F lies in some set $\mathfrak{L}_{q,q+n}(D, r', r'')$ ». The following counter-example does show the problem.

Let $\alpha \in K$ with $|\alpha| < 1$, and let $D = \{x \in X: |\alpha| \leq |x| \leq 1/|\alpha|\}$. For $|\alpha| \leq |x| \leq 1$ let $r'(x) = 1$, $r''(x) = 1/|\alpha|$ and for $1 < |x| \leq 1/|\alpha|$ let $r'(x) = 1/|\alpha|^2$, $r''(x) = 1/|\alpha|^3$. Let $F(Y) = 1 + xY + \alpha^3 x^2 Y^2$. Clearly $N^-(F_x, r'(x))$ is not constant in D because when $|x| < 1$, $N^-(F_x, 1) = 0$ while when $|x| = 1/|\alpha|$, $N^-(F_x, 1/|\alpha|^2) = 1$. However we can show that $N^+(F_x, r''(x)) - N^-(F_x, r'(x)) = 1$ whenever $x \in D$. For convenience, for each $x \in D$, let us write $F_x(Y) = a_0 + a_1 Y + a_2 Y^2$.

Suppose first $|x| = |\alpha|$. Then $|a_0| = 1$,

$$\begin{cases} |a_1| r' = |\alpha|, & |a_2| r' = |\alpha|^5 \\ |a_1| r'' = 1, & |a_2| r'' = |\alpha|^3 \end{cases}$$

hence $N^-(F_x, r') = 0$, $N^+(F_x, r'') = 1$.

Suppose now $|\alpha| < |x| < 1$. Then

$$\begin{cases} |a_1| r' = |x| < 1, & |a_2| r'^2 = |\alpha|^3 |x|^2 < 1 \\ |a_1| r'' = \left|\dfrac{x}{\alpha}\right| > 1, & |a_2| r''^2 = |\alpha x^2| < 1 \end{cases}$$

hence $N^-(F_x, r') = 0$, $N^+(F_x, r'') = 1$.

Suppose now $|x| = 1$. Then

$$\begin{cases} |a_1|r' = 1\,, & |a_2|r'^2 = |\alpha|^3 < 1 \\ |a_1|r'' = \dfrac{1}{|\alpha|}, & |a_2|r''^2 = |\alpha| < 1 < \dfrac{1}{|\alpha|} \end{cases}$$

hence $N^-(F_x, r') = 0$, $N^+(F_x, r'') = 1$.

Finally $N^-(F_x, r') = 0$, $N^+(F_x, r'') = 1$ is true for $|\alpha| \leqslant |x| \leqslant 1$. Now, suppose $1 < |x| < 1|\alpha|$. Then

$$\begin{cases} |a_1|r' = \left|\dfrac{x}{\alpha^2}\right| > 1; & |a_2|r'^2 = \left|\dfrac{x^2}{\alpha}\right| < \left|\dfrac{x}{\alpha^2}\right| \\ |a_1|r'' = \left|\dfrac{x}{\alpha^3}\right|; & |a_2|r''^2 = \left|\dfrac{x^2}{\alpha^3}\right| > \left|\dfrac{x}{\alpha^3}\right| \end{cases}$$

hence $N^-(F_x, r') = 1$, $N^+(F_x, r'') = 2$.

At last, suppose $|x| = 1/|\alpha|$. Then

$$\begin{cases} |a_1|r' = \dfrac{1}{|\alpha^2|}; & |a_2|r'^2 = \dfrac{1}{|\alpha|} < \dfrac{1}{|\alpha|^2} \\ |a_1|r'' = \dfrac{1}{|\alpha^3|}; & |a_2|r''^2 = \dfrac{1}{|\alpha|^3} \end{cases}$$

hence $N^-(F_x, r') = 1$, $N^+(F_x, r'') = 2$. Thus this relation is true for $1 < |x| \leqslant 1/|\alpha|$.

A Taylor series $F(Y) \in \mathfrak{T}_n(r)$ will be said *to have Hensel Factorization in* $\mathfrak{T}_n(r)$ (resp. in $\mathfrak{T}_n(r) \cap H(D)[Y]$) if it may be factorised in the form $P(Y) \in G(Y)$ with P a n-degree monic polynomial that lies in $\mathfrak{T}_n(r)$ and $G \in \mathfrak{T}_0(r)$ (resp. $G \in \mathfrak{T}_0(r) \cap H(D)[Y]$).

A Laurent series $F(Y) \in \mathfrak{L}_{q,t}(r', r'')$ will be said *to have Hensel Factorization in* $\mathfrak{L}_{q,t}(r', r'')$ if it may be factorised in the form $P(Y) \in G(Y)$ with P a $(t-q)$-degree monic polynomial that lies $\mathfrak{L}_{0,t-q}(r', r'')$, and $G \in \mathfrak{L}_{q\,q}(r', r'')$.

REMARK. – *When a Taylor series $F(Y)$ (resp. a Laurent series $F(Y)$) has Hensel Factorization in $\mathfrak{T}_n(r)$ (resp. in $L_{q,t}(r', r'')$), Hensel Factorization is unique.* Indeed for each $x \in D$, by classical results [A], we find back the Hensel Lemma for F_x in $T(r(x))$ (resp. in $L(r'(x), r''(x))$).

Recall that set D in K is said to be *infraconnected* if the adherence of the set $\{|x-a| : x \in D\}$ is an interval in $\boldsymbol{R}$ whenever $a \in D$.

Now D is said to be *strongly infraconnected* if for every hole $d^-(a, r)$ of D such that $r \in |K|$, there is a sequence a_n in D such that $|a_n - a| = |a_n - a_m|$ for every $n \neq m$ [E₄].

T-filters are defined in [E₃].

THEOREM 1. – *Let D be an open closed bounded strongly infraconnected subset of K with no T-filter and let r', r'' be functions from D to $\boldsymbol{R}_+$, with bounds a, b such that $0 < a \leqslant r'(x) \leqslant r''(x) \leqslant b$ whenever $x \in D$. Let $t, q \in \boldsymbol{Z}$ and let $F \in \mathfrak{L}_{q\,t}(D, r', r'')$.*

Then F has Hensel Factorization in $\mathfrak{L}_{q,t}(D, r', r'')$.

In the proof of theorem 1 we will particularly use the following proposition B.

PROPOSITION B. – *Let D be an open closed bounded infraconnected set with no T-filter. Let $a, b \in \mathbf{R}$ with $0 < a < b$ and let $q, t \in \mathbf{Z}$ with $q < t$. Let r', r'' be functions from D to $\mathbf{R}_+$ such that $0 < a \leqslant r'(x) \leqslant r''(x) \leqslant b$ whenever $x \in D$. $\mathfrak{L}_{q,t}(D, r', r'')$ is closed in $\mathfrak{L}(D, r', r'')$.*

Let F_m be a convergent sequence in $\mathfrak{L}_{q,t}(D, r', r'')$. Suppose for each $m \in \mathbf{N}$, F_m has Hensel Factorization $P_m \in G_m$ in $\mathfrak{L}_{q,t}(D, r', r'')$ with P_m a $(t-q)$-degree monic polynomial. Then the sequence P_m converges in $\mathfrak{L}_{0,t-q}(D, r', r'')$ to a $(t-q)$-degree monic polynomial P; the sequence G_m converges in $\mathfrak{L}_{q,q}(D, r', r'')$ to a limit G. The limit F of the sequence F_m has Hensel Factorization $F(Y) = P(Y) \in G(Y)$ in $\mathfrak{L}_{q,t}(D, r', r'')$.

When we consider Taylor series instead of Laurent series, we can obtain results with weaker hypothesis.

THEOREM 2. – *Let D be an open closed bounded strongly infraconnected subset of K with no T-filter, let r be a bounded function from D to $\mathbf{R}_+$, let M be a finite subset of D and let $n \in \mathbf{N}$. Let $F(Y) \in \mathfrak{T}(D, r)$ be such that F_x has exactly n zeroes in $d(0, r(x))$ (taking account of multiplicities) whenever $x \in D \setminus M$. Then F has a factorization $P(Y) \in G(Y)$ in $\mathfrak{T}(D, r)$ with P a n-degree monic polynomial in $H(D)[Y]$ and $G(Y) \in \mathfrak{T}(D, r)$ such that $N(G_x, r(x)) = 0$ whenever $x \in D \setminus M$. If $F(Y) \in H(D)[Y]$ then $G(Y) \in H(D)[Y]$.*

COROLLARY. – *Let D be an open closed bounded strongly infraconnected subset of K with no T-filter, let r be a bounded function from D to $\mathbf{R}_+$, let $n \in \mathbf{N}$, let $F \in \mathfrak{T}_n(D, r)$. Then F has Hensel Factorization in $\mathfrak{T}_n(D, r)$.*

If $F \in H(D)[Y]$ then it has Hensel Factorization in $H(D)[Y] \cap \mathfrak{T}_n(D, r)$.

REMARK. – The hypothesis « D has no T-filter » could be hardly avoided (unless assuming all the ξ_s are quasi-invertible). Indeed in the proofs of the main results we first divide the series $F(Y) = \sum \xi_s Y^s$ by ξ_n then we use the classical result « if ξ_s/ξ_n is bounded in D then $(\xi_s/\xi_n) \in H(D)$ ». If D has a T-filter this property is sometimes false as it is proved in [S].

Comparison with Theorem 2 in [E_4].

The proof of Theorem 2 in [E_4] is not correct because the proof of Proposition 2 has a mistake. Indeed in Proposition 2 we considered a Euclidean division of a Taylor series $F(Y) = \sum_{s=0}^{\infty} \xi_s Y^s \in H(D)[\![Y]\!]$ with $\xi_n = 1$ by the n-degree monic polynomial $P(Y) = \sum_{s=0}^{n} \xi_s Y^s$. For each fixed $x \in D$; the Euclidean division of F_x by P_x does exist in the Banach algebra of the Taylor series convergent for $|x| \leqslant r(x)$ [A, 4.4.2]. Unfortunately, unless providing $H(D)[\![Y]\!]$ (or a subset S containing F and P) witha suitable topology we cannot deduce a division of F by P in $H(D)[\![Y]\!]$ (or in S). Even if we assume in Proposition 2 the condition

$$\limsup_{s\to\infty} \|(\xi_s^{n-l}\xi_l^{s-n}/\xi_n^{s-l})\|_D < 1$$

that we assumed in Theorem 2, it does not seem possible to prove Proposition 2. However we have no counter-example proving it could be false.

In the present article the Euclidean division of F by P is possible in $\mathfrak{T}(D, r)$. Consider now a Taylor Series $F(Y) = \sum_{s=0}^{\infty} \xi_s Y^s \in \mathfrak{T}_n(D, r)$ satisfying the hypothesis of Theorem 2 in the present article. For $r(x)$ we can simply take the radius $\varrho(x)$ of the smallest disk $d(0, \varrho(x))$ containing exactly n zeroes of F_x (taking account of multiplicities).

Then $\varrho(x) = \max_{l=0,\dots,n-1} \|(\xi^l(x)/\xi_n(x))\|_D^{1/n-l}$ hence the ξ_s satisfy

$$\lim_{s\to\infty} (\sup_{x\in D} \big(|\xi_s||(\xi_l(x)/\xi_n(x))|^{s/n-l}\big)\big) = 0 \qquad \text{whenever } l = 0, \dots, n-1$$

hence

$$\lim_{s\to\infty} \|\xi_s^{n-l}\xi_l^s/\xi_n^s\|_D = 0 \qquad \text{whenever } l = 0, \dots, n-1 \,. \tag{1}$$

Particularly in assuming ξ_n divides all the ξ_s, as we did in Theorem 2 of [E$_4$], we can easily deduce

$$\lim_{s\to\infty} \|\xi_s^{n-l}\xi_l^{s-n}/\xi_n^{s-l}\|_D = 0 \,. \tag{2}$$

Indeed when D is open with no T-filter, every element $f \in H(D)$ is quasi-invertible (i.e. it has a factorization in the form $P(x) \in h(x)$ with P a polynomial and h an invertible element in $H(D)$). Then for any fixed element $u \in H(D)$ there is a constant $c > 0$ such that $\|uf\|_D \geqslant c\|f\|_D$ whenever $f \in H(D)$ because by [E$_1$] $uH(D)$ is a closed ideal, hence $\|uf\|_D$ and $\|\|uf\|\| = \|f\|_D$ define two equivalent topologies by Banach Theorem.

Here, since ξ_n divides ξ_l in $H(D)$, we can factorise $(\xi_s^{n-l}\xi_l^s)/\xi_n^s$ in the form $[(\xi_s^{n-l}\xi_l^{s-n})/\xi_n^{s-l}](\xi_l^n/\xi_n^l)$. Clearly $(\xi_l^n/\xi_n^l) \in H(D)$, hence the hypothesis (1) implies the hypothesis (2).

Thus, our present hypothesis appears to be a little bit stronger than in [E$_4$]. However, in the present article, Theorem 2 *does not require* ξ_n to divide all the ξ_s in $H(D)$. For example the polynomial $F(Y) = x^3 + xY + Y^2$ does satisfy the hypothesis of Theorem 2 with $n = 1$, and $D = d(0, 1)$, $M = \{0\}$.

2. – Basic results.

The following Lemma 1 is immediate

LEMMA 1. – *Let D be a closed bounded subset of K, let r, r', r'' be bounded functions defined in D, with values in $\mathbf{R}_+$, with a, b such that $0 < a \leqslant r'(x) \leqslant r''(x) \leqslant b$ whenever $x \in D$.*

$|||\cdot|||^r$ defines on $\mathfrak{T}(D, r)$ a norm of linear space that makes it a Banach space.

$|||\cdot|||_{r'}^{r''}$ defines on $\mathfrak{L}(D, r', r'')$ a norm of linear space that makes it a Banach space.

Let log be a logarithm function of base $p > 1$ and let v the valuation defined on K by $v(x) = -\log|x|$.

In order to recall easily some classical results and processes used in the rings $L(a, b)$ we define again the valuation function $v(f, \mu)$.

Let $f(Y) = \sum_{-\infty}^{+\infty} a_i Y^i \in L(a, b)$; for $\mu \in [-\operatorname{Log} b, -\operatorname{Log} a]$; let $v(f, \mu) = \inf_{i \in \mathbf{Z}} (v(a_i) + i\mu)$.

LEMMA 2. – *Let D be a bounded closed infraconnected subset of K and let r be a function from D to $\mathbf{R}_+$ with bounds a, b such that $0 < a \leqslant r(x) \leqslant b$ whenever $x \in D$. Let $F(Y) \in \mathfrak{T}(D, r)$ and let $P(Y) \in H(D)[Y] \cap \mathfrak{T}_n(D, r)$ be a n-degree monic polynomial.*

A) There exists $G \in \mathfrak{T}(D, r)$ and $R \in H(D)[Y]$ with $\deg(R) < n$, such that $F = PG + R$. Both G, R are unique.

B) For each $x \in D$ we have

$$v(R_x, -\log r(x)) \geqslant v(F_x, -\log r(x)) \tag{1}$$

$$v(G_x, -\log r(x)) \geqslant v(F_x - \log r(x)) - v(P_x, -\log r(x)) \tag{2}$$

$$N^+(P + R, r(x)) = N^+(P, r(x)) = n\,. \tag{3}$$

C) Let $F(Y) = \sum_{s=0}^{\infty} \xi_s Y^s$ and for each $n \in \mathbf{N}$, let $F_m(Y) = \sum_{s=0}^{m} \xi_s Y^s$, let $F_m = PG_m + R_m$ with G_m, $R_m \in H(D)[Y]$, $\deg R_m < n$. Then the sequence G_m converges to G and R_m converges to 0 in $\mathfrak{T}(D, r)$.

PROOF. – If G and R exist, they are unique. Indeed for each $x \in D$ we have $F_x = P_x G_x + R_x$; this is the Euclidean division of the Taylor series F_x by P_x in the ring of the series convergent in $d(0, r(x))$ and we know they both are unique. Also they verify (1), (2), (3) by classical results ([A], 4.4.2 and [L]).

When F is a polynomial the Euclidean division by the monic polynomial P does exist in $H(D)[Y]$ then we have $G \in H(D)[Y]$ and $R \in H(D)[Y]$ with $\deg R < n$ such that $F = PG + R$.

Now suppose F is a Taylor series $\sum_{s=0}^{\infty} \xi_s Y^s$ $(\xi_s \in H(D))$. For each $m \in \mathbf{N}$ let $F_m(Y) = \sum_{s=0}^{m} \xi_s Y^s$ and let $F_m = PG_m + R_m$ be the Euclidean division of F_m by P. By (1) we can easily show the sequence R_m is a Cauchy sequence in $\mathfrak{T}(D, r)$ for its norm $\|\!|\cdot|\!\|^r$. Indeed the Euclidean Division of $F_{m+1} - F_m$ is $P(G_{m+1} - G_m) + (R_{m+1} - R_m)$ hence $v((R_{m+1} - R_m)_x, -\log r(x)) \geqslant v((F_{m+1} - F_m)_x, -\log r(x))$ there by we have $\|\!|R_{m+1} - R_m|\!\|^r \leqslant \|\!|F_{m+1} - F_m|\!\|^r$ so (R_m) is a Cauchy sequence; it then converges to a polynomial $R \in H(D)[Y]$ of degree $< n$. Similarly since $r(x) \geqslant a > 0$, by (2) we can see the sequence G_m is a Cauchy sequence $\mathfrak{T}(D, r)$; hence it converges to a Taylor series $G(Y) \in \mathfrak{T}(D, R)$ such that $F = PG + R$. Lemma 2 is then proved.

LEMMA 3. – *Let D be an open closed bounded infraconnected subset of K and let r be a function from D to $\mathbf{R}_+$, with bounds a, b such that $0 < a \leqslant r(x) \leqslant b$ whenever $x \in D$. Let $M = \{\alpha_1, \ldots, \alpha_q\} \subset D$ and for every $\varrho > 0$, let $D_\varrho = D \setminus \bigcup_{i=1}^{q} d(\alpha_i, \varrho)$ and let r_ϱ be the restriction of r in D_ϱ.*

Let $n \in \mathbf{N}$, let $F(Y) \in \mathfrak{T}_n(D, r)$ and assume for every $\varrho > 0$, F has Hensel. Factorization in $\mathfrak{T}_n(D_\varrho, r_\varrho)$. Then F has Hensel Factorization in $\mathfrak{T}_n(D, r)$.

PROOF. – By hypothesis we can define a n-degree monic polynomial $P(Y)$ and a Taylor series $G(Y)$ such that $P(Y) \in H(D_\varrho)[Y]$ and $G(Y) \in \mathfrak{T}(D_\varrho, r_\varrho)$ whenever $\varrho > 0$ and $F_x(Y) = P_x(Y) \in G_x(Y)$ whenever $x \in D \setminus M$ and the n zeroes of P_x are the zeroes of F_x in $d(0, r(x))$. Since $r(x) \leqslant b$ the coefficients of P_x are clearly upper bounded by $\max(1, b^n)$ in $D \setminus M$.

Since they are in $H(D_\varrho)$ whenever $\varrho > 0$, we know by the Lemma 9 of [E4] they do belong to $H(D)$. By Lemma 3 it is obvious $G(Y) \in \mathfrak{T}(D, r)$. Indeed by Euclidean Division in $\mathfrak{T}(D, r)$ we have $F(Y) = P(Y)$, $V(Y) + R(Y)$, $R(Y) \in H(D)[Y]$.

Since $F_x(Y) = P_x(Y) G_x(Y)$, whenever $x \in D \setminus M$ clearly we have $G_x(Y) = V_x(Y)$, $R_x(Y) = 0$ whenever $x \in D \setminus M$, hence finally $R = 0$, $V = G$.

3. – Sets with no T-filter.

We will often use the following Lemmas

LEMMA 4. – *Let D be a closed bounded subset of K with no T-filter and let $f \in H(D)$ be such that $f(x) \neq 0$ whenever $x \in D$.*

Then f is invertible in $H(D)$.

PROOF. – Suppose f is not quasi-invertible. Then it approaches zero on a pierced filter $\mathcal{F}$ [E2]. Since f has no zero in D, then $\mathcal{F}$ is not a Cauchy pierced filter, hence it is a large pierced filter [E3]. Then f also approaches zero on a T-filter [E3] what is impossible by hypothesis. Hence f is quasi-invertible in $H(D)$. Since f has no zero it is invertible.

Lemma 5 is classical [A, K_3, L]

LEMMA 5. – *Let* $P(Y) = \sum_{i=0}^{q} a_i Y^i \in K[Y]$, *let* $n \in \boldsymbol{N}$ *with* $n \leqslant q$, *and let* $r \in \boldsymbol{R}_+$ *be such that* $N^+(P, r) = n > N^-(P, r) = h$. *Then* $r = |a_h \varrho a_n|^{1/(n-h)}$.

LEMMA 6. – *Let D be a closed bounded subset of K with no T-filter and let r be a function from D into* $\boldsymbol{R}_+$ *with bounds a, b such that* $0 < a \leqslant r(x) \leqslant b$ *whenever* $x \in D$. *Let* $n \in \boldsymbol{N}$, *let* $F(Y) = \sum_{s=0}^{\infty} \xi_s Y^s \in \mathfrak{T}_n(D, r)$ *and assume* $N^+(F_x, r(x)) > N^-(F_x, r(x))$ *whenever* $x \in D$.

Then there exists $\lambda < 1$ *such that* $\lambda|\xi_n(x)|r(x)^n \geqslant |\xi_s(x)|r(x)^s$ *whenever* $x \in D$, *whenever* $s > n$.

PROOF. – By Lemma 4 we know that ξ_n is invertible in $H(D)$, hence we can clearly assume $\xi_n = 1$ without loss of generality.

Let $l \in \boldsymbol{N}$ be such that $|\xi_s(x)|r(x)^s \leqslant a/2$ whenever $x \in D$, for $s \geqslant l$. Then it only remains to prove for each $s = n+1, \ldots, l$, there exists $\varrho_s < 1$ such that $\varrho_s|\lambda_n(x)|r(x)^n \geqslant |\xi_s(x)|r(x)^s$ whenever $x \in D$.

By the hypothesis $N^+(F_x, r(x)) > N^-(F_x, r(x))$, by Lemma 5 we know that for each $x \in D$, $r(x)$ is in the form

$$r(x) = |\xi_{l(x)}|^{1/(n-l(x))} \quad \text{with} \quad r(x) \geqslant |\xi_h(x)|^{1/(n-h)}$$

whenever $h = 0, \ldots, n-1$ hence we have

$$|\xi_s(x)||\xi_h(x)|^{(s-n)/(n-h)} < 1 \tag{1}$$

whenever $x \in D$, whenever $h = 0, \ldots, n-1$.

By Lemma 8 in [E_4] it follows that

$$\|\xi_s^{n-h}\xi_h^{s-n}\|_D < 1\,. \tag{2}$$

Indeed, if (2) is false, by (1) we have $\|\xi_s^{n-h}\xi_h^{s-n}\|_D = 1$.

Since D is strongly infraconnected, by Lemma 8 in [E_4] there exists $\alpha \in D$ such that $|\xi_s(\alpha)^{n-h}\xi_h(\alpha)^{s-n}| = 1$ in contradiction with (2).

Then we can take $\varrho_s = (\|\xi_s^{n-h}\xi_h^{s-n}\|_D)^{1/(n-h)}$ and so, Lemma 6 is proved.

PROPOSITION P. – *Let D be a closed bounded strongly infraconnected subset of K with no T-filter. Let r be a function from D into* $\boldsymbol{R}_+$ *with bounds a, b such that* $0 < a \leqslant r(x) \leqslant b$ *whenever* $x \in D$, *let* $n \in \boldsymbol{N}$ *and let* $F(Y) \in H(D)[Y] \cap \mathfrak{T}_n(D, r)$. *Then* $F(Y)$ *has Hensel Factorization in* $H(D)[Y] \cap \mathfrak{T}_n(D, r)$.

PROOF. – Let $F(Y) = \sum_{s=0}^{l} \xi_s Y^s$. By hypothesis ξ_n does not vanish in D; since D has no T-filter, by Lemma 4, ξ_n is invertible in $H(D)$, so we can clearly assume

$\xi_n = 1$ without loss of generality. Then the polynomial $P_1(Y) = \sum_{s=0}^{n} \xi_s Y_s$ is monic and $P_1 \in \mathfrak{T}_n(D, r)$. So we can make the Euclidean Division of F by P_1: $F(Y) = P_1(Y) G_1(Y) + R_1(Y)$, $G_1(Y) \in \mathfrak{T}(D, r)$, $R_1 \in H(D)[Y]$, $\deg R_1 < n$. Then the polynomial $P_2 = P_1 + R_1$ is monic and it lies in $\mathfrak{T}_n(D, r)$ also. Thus, by an immediate induction we can define sequences $P_m = P_{m-1} + R_{m-1}$, $G_m \in \mathfrak{T}(D, r)$, $R_m \in H(D)[Y]$, $\deg R_m < n$ such that $F = P_m G_m + R_m$ with relations (1), (2), (3) satisfied for each m.

$$v(R_x, -\log r(x)) \geqslant v(F_x, -\log r(x)) \tag{1}$$

$$v(G_x, -\log r(x)) \geqslant v(F_x, -\log r(x)) - v(P_x - \log r(x)) \tag{2}$$

$$N^+(P_x + R_x, r(x)) = N^+(P_x, r(x)) = n\,. \tag{3}$$

Now we can assume $d(0, r(x))$ is the smallest disk of center 0 containing exactly n zeroes of F_x (taking account of multiplicities), whenever $x \in D$. Indeed, if we prove Proposition P when r is so, it is obvious it holds when r is bigger as long as $d(0, r(x))$ only contains n zeroes of F_x whenever $x \in D$. Thus we can assume $N^+(F_x, r(x)) > N^-(F_x, r(x)) = l(x)$.

Then we can apply Lemma 6 and we have $\theta > 0$ such that $v(\xi_s(x)) - s \log r(x) \geqslant -n \log r(x) + \theta$ whenever $x \in D$, whenever $s > n$, hence

$$v((F - P_1)_x, -\log r(x)) - v(F_x, -\log r(x)) \geqslant \theta$$

whenever $x \in D$.

Then we can follow the classical way like in [A], 4.4.4. First we can prove that

$$v(G_{m+1} - G_m)_x, -\log r(x)) \geqslant m\theta\,.$$

Then the sequence G_m is a Cauchy sequence for the norm $|||\cdot|||^r$ in the space $H(D)[Y]$.

Hence (G_m) converges to a limit $G \in H(D)[Y]$. Likewise, the sequence P_m converges in $H(D)[Y]$ to a n-degree monic polynomial P that also lies in $\mathfrak{T}_n(D, r)$. At last the sequence R_m converges to 0. So we have $F = PG$ in $H(D)[Y]$.

NOTATIONS. - Let $a, b \in \mathbf{R}_+^*$ with $a \leqslant b$. We will denote by $\Delta(a, b)$ the set $\{x \in K: a \leqslant |x| \leqslant b\}$. Let $q, t \in \mathbf{Z}$ with $q \leqslant t$; $L_{q,t}(a, b)$ will denote the set of the $f \in L(a, b)$ with $N^-(f, a) = q$ and $N^+(f, b) = t$. Let $\|\cdot\| = \|\cdot\|_{\Delta(a,b)}$.

LEMMA 7. - *Let $t, q \in \mathbf{Z}$, $t > q$, let $a, b \in \mathbf{R}_+$ $(0 < a < 1,\ a < b)$ and let $F_1, F_2 \in L_{q,t}(a, b)$. Assume $\|F_1\| = \|F_2\| > \|F_1 - F_2\|$. Let $F_2(x) = \sum_{-\infty}^{+\infty} \xi_s Y^s$ and let $A = |\xi_t|$ and assume $A \leqslant 1$. Let $P_i G_i$ be the Hensel factorization of F_i with P_i the monic polynomials whose zeroes are the $t - q$ zeroes of F_i in $\Delta(a, b)$ $(i = 1, 2)$ and let $\varrho > 0$ be such that for each zero α of P_2 $d(\alpha, \varrho) \subset \Delta(a, b)$.*

Then

$$\|G_1 - G_2\| \leqslant \frac{\|F_1 - F_2\|^{1/(t-q)!}}{A^{t-q}\varrho^{2(t-q)}a^q}, \qquad \|P_1 - P_2\| \leqslant \frac{b^{t-q}\|F_1 - F_2\|^{1/(t-q)!}}{A^{t-q+1}\varrho^{2(t-q)}a^q}.$$

PROOF. – Let α_1 be a zero of F_1 in $\Delta(a, b)$. Then we have

$$|F_1(\alpha_1) - F_2(\alpha_1)| = |P_2(\alpha_1)G_2(\alpha_1)|.$$

Let $G_2(Y) = \sum_{-\infty}^{+\infty} g_s Y^s$; then $|G_2(\lambda)| = |g_q||\lambda|^q$ whenever $\lambda \in \Delta(a, b)$. Set $n = t - q$.

On the other hand $\|F_2\| \geqslant |\xi_t|b^t$, $|\xi_t|b^t = b^n|g_q|b^q$ hence $|g_q| = |\xi_t|$, $|G_2(\alpha_1)| = |g_q||\alpha_1|^q = |\xi_t||\alpha_1|^q$.

Then

$$|P_2(\alpha_1)| \leqslant \frac{\|F_1 - F_2\|}{|G_2(\alpha_1)|} = \frac{\|F_1 - F_2\|}{|\xi_t||\alpha_1|^q}$$

hence

$$|P_2(\alpha_1)| \leqslant \frac{\|F_1 - F_2\|}{|\xi_t|a^q}.$$

Then it is easily seen the nearest zero α_2 from α_1 of P_2 is such that

$$|\alpha_2 - \alpha_1|^n \leqslant |P_2(\alpha_1)| \leqslant \frac{\|F_1 - F_2\|}{Aa^q}$$

thereby we have

$$(1) \qquad |\alpha_2 - \alpha_1| \leqslant \frac{\|F_1 - F_2\|^{1/n}}{(Aa^q)^{1/n}}.$$

Let $F_1(x) = (x - \alpha_1)\overline{F}_1$, $F_2(x) = (x - \alpha_2)\overline{F}_2$; let $D = \Delta(a, b)\setminus\big(d(\alpha_1, \varrho) \cup d(\alpha_2, \varrho)\big)$. By classical results [B, E$_1$] we know that if $F \in L(a, b)$ then $\|F\|_D = \|F\|$. Here we have

$$(2) \qquad \|\overline{F}_1 - \overline{F}_2\| = \|\overline{F}_1 - \overline{F}_2\|_D \leqslant \max\left(\left\|\frac{F_1 - F_2}{x - \alpha_1}\right\|_D, \left\|F_2\left(\frac{1}{x - \alpha_1} - \frac{1}{x - \alpha_2}\right)\right\|_D\right).$$

On the first hand we have

$$(3) \qquad \left\|\frac{F_1 - F_2}{x - \alpha_1}\right\|_D \leqslant \|F_1 - F_2\|_D \left\|\frac{1}{x - \alpha_1}\right\|_D = \|F_1 - F_2\|\left(\frac{1}{\varrho}\right).$$

On the second hand

$$(4) \qquad \left\|F_2\left(\frac{1}{x - \alpha_1} - \frac{1}{x - \alpha_2}\right)\right\|_D \leqslant \|F_2\|_D \left\|\frac{\alpha_2 - \alpha_1}{(x - \alpha_1)(x - \alpha_2)}\right\|_D = \frac{|\alpha_2 - \alpha_1|}{a^2} \leqslant \frac{\|F_1 - F_2\|^{1/n}}{A^{1/n}\varrho^2 a^{q/n}},$$

Since $\|F_1 - F_2\| < 1$, we have $\|F_1 - F_2\|^{1/n} > \|F_1 - F_2\|$. Also $A \leqslant 1$, $a \leqslant \varrho \leqslant 1$ hence

$$\frac{1}{a} \leqslant \frac{1}{A^{1/n}\varrho^2 a^{q/n}}.$$

Finally we have

$$\|\bar{F}_1 - \bar{F}_2\| \leqslant \frac{\|F_1 - F_2\|^{1/n}}{A^{1/n}\varrho^2 a^{q/n}}$$

hence with greater reason

$$\|\bar{F}_1 - \bar{F}_2\| \leqslant \frac{\|F_1 - F_2\|^{1/n}}{A\varrho^2 a^{q/n}}. \tag{5}$$

Let $\bar{F}_2(Y) = \sum_{-\infty}^{+\infty} \hat{\xi}_s Y^s$. Then by classical results [A] we know that

$$N^+(F_2, -\log b) = t - 1, \qquad |\hat{\xi}_s| = |\xi_s|$$

hence we are set back to the same problem with $n-1$ instead of n, $\bar{F}_i$ instead of F_i. Let us remind now that $A \leqslant 1$, $\varrho \leqslant 1$, $a \leqslant 1$.

By an immediate decreasing induction after n similar operations we obtain

$$\|G_1 - G_2\| \leqslant \frac{\|F_1 - F_2\|^{1/n!}}{A^n \varrho^{2n} a^q}.$$

Then $\|(P_2 - P_1)G_2\| \leqslant \max\left(\|F_2 - F_1\|, \|P_1(G_1 - G_2)\|\right)$. By hypothesis $|G_2(\lambda)| = |g_q||\lambda|^q = A|\lambda|^q$ hence $|G_2(\lambda)| \geqslant A \min(a^q, b^q)$ thereby

$$\|P_1 - P_2\| \leqslant \frac{\max\left(\|F_1 - F_2\|, \|P_1(G_1 - G_2)\|\right)}{A \min(a^q, b^q)}.$$

Now

$$\|P_1(G_1 - G_2)\| \leqslant \|P_1\|\|G_1 - G_2\| \leqslant \frac{b^n\|F_1 - F_2\|^{1/n!}}{A^n\varrho^{2n}a^q} = \frac{b^n\|F_1 - F_2\|^{1/n!}}{A^n\varrho^{2n}a^q}$$

hence clearly:

$$\max\left(\|F_1 - F_2\|, \|P_1(G_1 - G_{\_})\|\right) \leqslant \frac{b^n\|F_1 - F\ \|^{1/n!}}{A^n\varrho^{2n}a^q}.$$

Finally

$$\|P_1 - P\ \| \leqslant \frac{b^n\|F_1 - F\ \|^{1/n!}}{A^{n+1}\varrho^{2n}a^q}$$

and so lemma is proved.

COROLLARY. – *Let $F_m(Y)$ be a convergence sequence of limit F in $L(a, b)$ for the canonical norm $\|\cdot\|$, and let $P_m G_m$ be the Hensel Factorization of F_m in $L(a, b)$, let PG be the Hensel Factorization of F. Then*

$$P = \lim_{m\to\infty} P_m\,, \qquad G = \lim_{m\to\infty} G_m\,.$$

There are integers $q < t$ such that $F \in L_{q,t}(a, b)$, $F_m \in L_{q\,t}(a, b)$ for m high enough.

PROOF. – By the classical Hensel Lemma in $L(a, b)$ there are integers $q < t$ such that $F \in L_{q,t}(a, b)$ [A]. Then P is a $(t - q)$ degree polynomial. Let $F(Y) = \sum_{-\infty}^{+\infty} a_s Y^s$, $F_m(Y) = \sum_{-\infty}^{+\infty} a_{s,m} Y^s$. We know $a_s = \lim_{m\to\infty} a_{s,m}$ whenever $s \in \boldsymbol{Z}$. Since $F \in L_{q,t}(a, b)$ we have

$$|a_t|b^t \geqslant |a_s|b^s \qquad \text{whenever } s \in \boldsymbol{Z}\,,$$

$$|a_t|b^t > |a_s|b^s \qquad \text{whenever } s > t$$

thereby the inequalities hold for $|a_{t,m}|$, $|a_{s,m}|$ when m is high enough. We also have the symetric relations with $|a_q|$, thereby $F_m \in L_{q,t}(a, b)$ for m high enough.

Then Lemma 7 shows easily that

$$P = \lim_{m\to\infty} P_m\,, \qquad G = \lim_{m\to\infty} G_m\,.$$

PROOF OF PROPOSITION B. – First we can shortly prove $\mathfrak{L}_{q,t}(r', r'')$ is closed in $\mathfrak{L}(r', r'')$. Let U_m be a Cauchy sequence in $\mathfrak{L}_{q,t}(r', r'')$ and let U be its limit in $\mathfrak{L}(r', r'')$.

By Corollary $L_{q,t}(a, b)$ is closed in $L(a, b)$ hence

$$N^+\big((F_t)_x, r''(x)\big) = t \qquad \text{whenever } x \in D\,,$$

$$N^-\big((F_q)_x, r'(x)\big) = q \qquad \text{whenever } x \in D\,,$$

so $U \in \mathfrak{L}_{q,t}(r', r'')$.

In the following, in order to apply easily Lemma 7 we will suppose $b < 1$, what we can do (by changing of variable) without loss of generality.

Then we have $r'(x) < 1$. For each $x \in D$, let $\|\cdot\|_x$ be the canonical norm on $L\big(r'(x), r''(x)\big)$ and let $|\!|\!|\cdot|\!|\!|$ be the norm on $\mathfrak{L}(r', r'')$.

Let $F(Y) = \sum_{-\infty}^{+\infty} \xi_s Y^s$ and for each $m \in \boldsymbol{N}$ let $F_m(Y) = \sum_{-\infty}^{+\infty} \xi_{s,m} Y^s$. By what foregoes, $F \in \mathfrak{L}_{q,t}(r', r'')$ hence ξ_t, ξ_q cannot vanish in D. Since D has no T-filter ξ_t, ξ_q are invertible by Lemma 4.

Let $B > 0$ be such that $\min\big(|\xi_t(x)|, |\xi_q(x)|\big) \geqslant B$ whenever $x \in D$.

Let $\theta = \min(b^t, a^q, b^q, a^t)$ and let $n \in \boldsymbol{N}$ be high enough to have $|||F - F_m||| < B\theta$, whenever $m > n$. Then $r''(x)^t \geqslant \theta$, $r'(x)^q \geqslant \theta$ whenever $x \in D$ hence $|\xi_t(x) - \xi_{t,m}(x)| < B$ and $|\xi_q(x) - \xi_{q,m}(x)| < B$, so that

$$|\xi_{t,m}(x)| = |\xi_t(x)| > |\xi_{t,m+1}(x) - \xi_{t\,m}(x)|$$

$$|\xi_{q,m}(x)| = |\xi_q(x)| > |\xi_{q,m+1}(x) - \xi_{q,m}(x)|$$

whenever $m \geqslant u$, whenever $x \in D$.

Then the relations $\|F_{m+1} - F_m\|_x < \|F_m\|_x$ are clearly verified when $m \geqslant u$, (for each $x \in D$). Thus we can apply Lemma 4.

For $m \geqslant u$, we have

$$\|(G_{m+1})_x - (G_m)_x\|_x \leqslant \frac{\|(F_{m+1})_x - (F_m)_x\|_x^{1/(t-q)!}}{|\xi_t(x)|^q r'(x)^{2(t-q)}}$$

hence with greater reason

$$\|(G_{m+1})_x - (G_m)_x\|_x \leqslant \frac{|||F_{m+1} - F_m|||^{1/(t-q)!}}{B^q a^{2(t-q)}}$$

whenever $x \in D$, thereby

$$|||G_{m+1} - G_m||| \leqslant \frac{|||F_{m+1} - F_m|||^{1/(t-q)!}}{B^q a^{2(t-q)}}.$$

Similarly we have

$$|||P_{m+1} - P_m||| \leqslant \frac{b^{t-q}|||F_{m+1} - F_m|||^{1/(t-q)!}}{B^{t-q+1} a^{2t-q}}.$$

Thus the sequences G_m, P_m are clearly convergent in $\mathfrak{L}(r', r'')$. Let $G = \lim_{m\to\infty} G_m$, $P = \lim_{m\to\infty} P_m$. Obviously, $F = PG$ and by classical result $P_x G_x$ is the Hensel Factorization of F_x in $L(r'(x), r''(x))$ hence PG is the Hensel Factorization of F in $\mathfrak{L}(r', r'')$. That concludes the proof of Proposition B.

4. – Proofs of the Theorems.

PROPOSITION Q. – *Let D be an open closed bounded strongly infraconnected subset of K with no T-filter, let r be a function from D to $\boldsymbol{R}_+$ with bounds a, b such that $0 < a \leqslant r(x) \leqslant b$ whenever $x \in D$. Let $n \in \boldsymbol{N}$ and let $F \in \mathfrak{T}_n(D, r)$. Then F has Hensel Factorization in $\mathfrak{T}_n(D, r)$.*

PROOF. – Let $F(Y) = \sum_{s=0}^{\infty} \xi_s Y^s$ and for every $m \in \boldsymbol{N}$, let $F_m(Y) = \sum_{s=0}^{m} \xi_s Y^s$.

Then $F_m \in \mathfrak{T}_n(D, r)$ for every $m \geqslant n$, hence by Proposition P, F_m has Hensel Factorization in $\mathfrak{T}_n(D, r)$ in the form $P_m G_m$ with P_m a n-degree monic polynomial.

Without loss of generality we can assume ξ_0 is not the nul element in $H(D)$. Indeed if $\xi_0 = \xi_h = 0$, $\xi_h \neq 0$, then F is factorized in the form $Y^h \hat{F}(Y)$ with $\hat{F}(Y) = \sum_{s=0}^{\infty} \lambda_s Y^s$ and λ_0, different of the nul element in $H(D)$.

Since D is open with no T-filter, then ξ_0 is quasi-invertible hence the set Z of the zeroes of ξ_0 in D is finite. We obviously have $N^+(F_x, 0) = N^+((F_m)_x, 0)$ whenever $x \in D \setminus Z$, whenever $m \in N$.

Let ϱ be > 0 and let $D_\varrho = D \setminus (\bigcup_{\alpha \in Z} d^-(\alpha, \varrho))$ and let r_ϱ be the restriction of r in D_ϱ. Then ξ_0 is invertible in $H(D_\varrho)$ hence there is $c_\varrho > 0$ such that $|\xi_0(x)| \geqslant c_\varrho$ whenever $x \in D_\varrho$. For each $x \in D_\varrho$, if α is a zero of F_x, by Lemma 5 we know that for every $l = 0, \ldots, n-1$ we have $|\alpha| \geqslant ((|\xi_0|)/(\|\xi_l\|_D))^{1/l}$. In considering the constant function r'_ϱ defined in D_ϱ by

$$r'_\varrho(x) = \frac{1}{2} \min_{0 \leqslant l < u} \left(\frac{c_\varrho}{\|\xi_l\|_D} \right)^{1/l}$$

we can see that each F_m has no zero in $d(0, r'_\varrho(x))$ whenever $x \in D_\varrho$, hence F_m does belong to $\mathfrak{L}_{0,n}(D_\varrho, r'_\varrho, r_\varrho)$. Since r'_ϱ is a strictly positive constant function we can obviously apply Proposition B to the sequence F_m hence F has Hensel Factorization in $\mathfrak{L}_{0,n}(D_\varrho, r'_\varrho, r_\varrho)$ so it also has Hensel Factorization in $\mathfrak{T}_n(D_\varrho, r_\varrho)$. Then by Lemma 3 F has Hensel Factorization in $\mathfrak{T}_n(D, r)$.

THEOREM 1. – *Let D be an open closed bounded strongly infraconnected subset of K without T-filter, and let r', r'' be functions defined on D with bounds a, b such that $0 < a \leqslant r'(x) < r''(x) \leqslant b$ whenever $x \in D$. Let $t, q \in Z$ and let $F \in \mathfrak{L}_{q,t}(r', r'')$. Then F has Hensel Factorization in $\mathfrak{L}_{q,t}(r', r'')$.*

PROOF. – First we will prove Theorem when F is a Taylor series. Then $F(Y) \in \mathfrak{T}_t(r'')$. By Proposition Q, F has the Hensel Factorization $P(Y)$. $G(Y)$ with P a t-degree monic polynomial and $G \in \mathfrak{T}(r'')$.

Since $F \in \mathfrak{L}_{q,t}(r'', r')$ for each $x \in D$, F_x has exactly $t - q$ zeroes in the annulus $\Delta(r'(x), r''(x))$ hence it has exactly q zeroes in $d^-(0, r'(x))$.

For each $x \in D$ let $d(0, \varrho(x))$ be the smallest disk of center 0 containing exactly q zeroes of F_x. Then we can clearly apply Theorem to F in $\mathfrak{T}_q(\varrho)$ so that we have a Hensel Factorization in the form $Q(Y)H(Y)$ with Q a q-degree monic polynomial. We can easily verify Q does divide P in $H(D)[Y]$. Indeed, for each $x \in D$ if we write

$$P_x(Y) = \prod_{j=1}^{t} (Y - \alpha_j) \qquad \text{with } |\alpha_j| \leqslant |\alpha_{j+1}|,$$

we obviously have $Q_x = \prod_{j=1}^{q} (Y - \alpha_j)$, hence Q_x divides P_x in $K[Y]$. In considering the Euclidean division of P by Q in $H(D)[Y]$ the remainder R does verify $R_x = 0$ whenever $x \in D$ hence $R = 0$.

Let $P(Y) = T(Y)Q(Y)$ in $H(D)[Y]$. Then T_x has eactly $t-q$ zeroes in $\Delta(r'(x), r''(x))$ whenever $x \in D$, so in writing $F(Y) = T(Y)[Q(Y)G(Y)]$ we have the Hensel factorization for $F(Y)$ in $\mathfrak{L}_{q,t}(r', r'')$.

Immediately we also have the Hensel Factorization in $\mathfrak{L}_{q,t}(r', r'')$ for Laurent series $F(Y)$ in the form $\sum_{s=-h}^{h} \xi_s Y^s$ $(n > 0)$, in considering $\overline{F}(Y) = Y^h F(Y)$.

At last in the general case, let $F(Y) = \sum_{-\infty}^{+\infty} \xi_s Y^s$. By definition we have

$$\lim_{m\to\infty} \|\|F - F_m\|\|_{r'}^{r''} = 0 \qquad \text{with} \qquad F_m = \sum_{s=-m}^{m} \xi_s Y^s .$$

We have just proved the Hensel Factorization exists for each F_m, so it holds for F by Proposition B.

NOTATION. – Let $a \in K$, $\varrho_1, \varrho_2 \in \boldsymbol{R}_+$. We will denote by $\Gamma(a, \varrho_1, \varrho_2)$ the set $\{x \in K: \varrho_1 < |x-a| < \varrho_2\}$.

LEMMA 8. – *Let D be a closed bounded subset of K that contains a disk $(d(0, \delta)$. Let r be a bounded function from D to $\boldsymbol{R}_+$ such that $\lim_{x\to 0} r(x) = 0$.*

For each $\varrho \in]0, \delta[$, let $D_\varrho = D \setminus d^-(0, \varrho)$, let r_ϱ be the restriction of r in D_ϱ, and let a_ϱ be the lower bound of r in $\Gamma(0, \varrho, \delta)$.

We suppose $a_\varrho > 0$ whenever $\varrho > 0$.

Let $G = \sum_{s=0}^{\infty} b_s Y^s$ be a Taylor series that lies in $\mathfrak{T}(D_\varrho, r_\varrho)$ whenever $\varrho > 0$. Let G_m be a sequence of $\mathfrak{T}(D, r)$ that converges to G in $\mathfrak{T}(D_\varrho, r_\varrho)$ whenever $\varrho > 0$. Then $G \in \mathfrak{T}(D, r)$ and G_m converges to G in $\mathfrak{T}(D, r)$.

PROOF. – We first prove that $G(Y) \in H(D)[\![Y]\!]$. Suppose $b_l \notin H(D)$ for some $l \in \boldsymbol{N}$. Either b_l has a pole at 0, or it has an essential singularity at 0. In all cases, by classical results, for every $\varrho > 0$, there exist ϱ_1, ϱ_2 with $0 < \varrho_1 < \varrho_2 < \varrho$ such that $v(b_l, \mu)$ is in the form $A + t\mu$ for $\mu \in [-\log \varrho_2, -\log \varrho_1]$, with $t \in \boldsymbol{Z}$, $t < 0$ and

$$v(b_l(x)) = v(b_l, v(x)) . \tag{1}$$

Let a_ϱ be the lower bound of r in $\Gamma(0, \varrho_1, \varrho_2)$. Let $\eta = p^{-v(b_l, -\log \varrho_2)}$, and let $\varepsilon = \eta(a_\varrho)^l$. There does exist $n \in \boldsymbol{N}$ such that $\|G_m - G\|_{r_{\varrho_1}} > \varepsilon$. Let $G_m = \sum_{s=0}^{\infty} b_{s,m} Y^s$. Then $|b_{l,m}(x) - b_l(x)| a_\varrho^l < \varepsilon$ for all $x \in D_{\varrho_1}$ hence $|b_{l,m}(x) - b_l(x)| < \eta$ for all $x \in \Gamma(0, \varrho_1, \varrho_2)$.

Then (1) shows that $v(b_{l,m}(x) - b_l(x)) > v(b_l(x))$ hence $v(b_{l,m}(x)) = v(b_l(x)) = A + tv(x)$, when $x \in \Gamma(0, \varrho_1, \varrho_2)$, thereby

$$v(b_{l,m}, \mu) = A + t\mu \qquad \text{for } \mu \in [-\log \varrho_2, -\log \varrho_1] .$$

But $b_{l,m} \in H(D)$, hence $b_{l,m} \in H(d(0, \varrho_2))$ hence the function $\mu \to v(b_{l,m}, \mu)$ cannot be $A + t\mu$ with $t < 0$. So we see that $b_s \in H(D)$ whenever $s \in \mathbf{N}$.

In the same way we can prove now that $\lim_{s\to\infty} \|b_s Y^s\|_r = 0$. Let $\varrho_1 \in]0, \delta[$ and set $\lambda = a_{\varrho_1}$, and let $\varrho_0 \in]0, \varrho_1[$ be such that $r(x) < a$ for $|x| \leqslant \varrho_0$. Let $\varepsilon > 0$ and let $N \in \mathbf{N}$ be such that $|b_s(x)| r(x)^s \leqslant \varepsilon$ whenever $x \in D_{\varrho_0}$, whenever $s \geqslant N$. Then $|b_s(x)| \lambda^s \leqslant \varepsilon$ whenever $x \in \Gamma(0, \varrho_1, \delta)$ hence

$$\|b_s\|_{\Gamma(0,\varrho_1,\delta)} \lambda^s \leqslant \varepsilon .$$

On the other hand, since $d(0, \varrho_0) \subset d(0, \delta) \subset D$ and $b_s \in H(D)$ we have $\|b_s\|_{d(0,\varrho_0)} \leqslant \|b_s\|_{d(0,\delta)} = \|b_s\|_{\Gamma(0,\varrho_1,\delta)}$. Thus for every $s \geqslant N$ when $x \in d(0, \varrho_0)$ we have

$$|b_s(x)|(x)^s \leqslant \|b_s\|_{d(0,\varrho_0)} \lambda^s \leqslant \|b_s\|_{\Gamma(0,\varrho_1,\delta)} \lambda^s \leqslant \varepsilon .$$

Of course, by hypothesis when $x \in D_{\varrho_0}$ we also have $|b_s(x)| r(x)^s \leqslant \varepsilon$, hence finally

$$\|b_s Y^s\|^r \leqslant \varepsilon \qquad \text{for } s \geqslant N .$$

That finishes to prove $G(Y) \in \mathfrak{T}(D, r)$.

LEMMA 9. – *Let D be an open bounded closed infraconnected subset of K with no T-filter and let r be a bounded function from D into $\mathbf{R}_+$. Let $n \in \mathbf{N}$ and let $F(Y) = \sum_{s=0}^{\infty} \xi_s Y^s \in \mathfrak{T}_n(D, r)$. Assume there is some $i \leqslant n-1$ such that $\xi_i \neq 0$, and let $\alpha_1, \ldots, \alpha_q$ be the points in D such that $\xi_0(\alpha_j) = \ldots = \xi_{n-1}(\alpha_j) = 0$ whenever $j = 1, \ldots, q$. For each $\varrho > 0$, let $D_\varrho = D \setminus \bigcup_{i=1}^{q} d^-(\alpha_i, \varrho)$; there exists $a_\varrho > 0$ such that $r(x) \geqslant a_\varrho$ whenever $x \in D_\varrho$.*

PROOF. – By hypothesis $N^+(F_x, r(x)) = n$ whenever $x \in D$, hence by Lemma 5

$$r(x) \geqslant \max_{0 \leqslant i \leqslant n-1} \left|\frac{\xi_i(x)}{\xi_n(x)}\right|^{1/(n-i)} .$$

Since D is open with no T-filter, there does exist $h < n$ such that ξ_h is quasi-invertible. Let $\beta_1, \ldots, \beta_t$ be the zeroes of ξ_r in D_ϱ.

For each j there is $l(j)$ such that $\xi_{l(j)}(\beta_j) \neq 0$ hence there exists $\varrho' \in]0, \varrho]$ such that $|\xi_{l(j)}(x)|$ has a lower bound $\theta_j > 0$ in $d^-(\beta_j, \varrho')$.

Let $D' = D_\varrho \setminus \bigcup_{i=1}^{t} d\_(\beta_j, \varrho')$. Then ξ_h is invertible in $H(D')$; hence there exists $\lambda > 0$ such that $|\xi_h(x)| \geqslant \lambda$ whenever $x \in D$. Thus in D' we have

$$r(x) \geqslant \left(\left|\frac{\xi_h(x)}{\xi_n(x)}\right|\right)^{1/(n-h)} \geqslant \left(\frac{\lambda}{b}\right)^{1/(n-h)}$$

then in each $d^-(\beta_j, \varrho')$ we have

$$r(x) \geqslant \left|\frac{\xi_{l(j)}(x)}{\xi_n(x)}\right|^{1/(n-l(j))} \geqslant \left(\frac{\theta_j}{b}\right)^{1/(n-l(j))}.$$

Let $\theta = \min_{1 \leqslant j \wedge t} (\theta_j/b)^{1/(n-l(j))}$, then we can take $a_\varrho = \min\left((\lambda/b)^{1/(n-h)}, \theta\right($.

LEMMA 10. – *Let D be a closed bounded infraconnected subset of K, and let $\alpha_1, \ldots, \alpha_q \in D$. Let $\varrho \in \mathbf{R}_+$ and let $D_\varrho = D \setminus \left(\bigcup_{i=1}^{q} d^-(\alpha_i, \varrho)\right)$.*

Let r be a bounded function from D to $\mathbf{R}_+$. For each $i = 1, \ldots, q$ let $D_{\varrho,i} = D_\varrho \bigcup d^-(\alpha_i, \varrho)$ and let r_i be the restriction of r in $D_{\varrho,i}$.

Let $F(Y) \in \mathfrak{T}(D_\varrho, r)$. If for each $i = 1, \ldots, q$, $F \in \mathfrak{T}(D_{\varrho,i}, r_i)$ then $F \in \mathfrak{T}(D, r)$.

PROOF. – Let $F(Y) = \sum_{s=0}^{\infty} \xi_s Y^s$. For each $s \in \mathbf{N}$, ξ_s lies in $H(D_{\varrho,i})$ whenever $i = 1, \ldots, q$, hence by the Mittag-Lefflerian Decomposition Theorem on the infraconnected set D_ϱ, it is easily seen that $\xi_s \in H(D)$.

On the other hand we have $\lim_{s\to\infty} \|\xi_s Y^s\|_{r_i} = 0$ whenever $i = 1, \ldots, q$. Let $\varepsilon > 0$; there exists $N(\varepsilon)$ such that $|\xi_s(x)| r(x)^s \leqslant \varepsilon$ whenever $x \in D_{\varrho,i}$ whenever $s \geqslant N(\varepsilon)$, whenever $i = 1, \ldots, q$. Hence, we clearly have $|\xi_s(x)| r(x)^s \leqslant \varepsilon$ whenever $x \in D$, whenever $s \geqslant N(\varepsilon)$ and so, Lemma 10 is proved.

THEOREM 2. – *Let D be an open closed bounded strongly infraconnected set with no T-filter. Let M be a finite subset of D. Let r be a bounded function from D to $\mathbf{R}_+$. Let $n \in \mathbf{N}$ and let $F(Y) \in \mathfrak{T}(D, r)$ be such that $N^+(F_x, r(x)) = n$ whenever $x \in D \setminus M$. Then F has Hensel Factorization $P(Y)G(Y)$ in $\mathfrak{T}(D, r)$ with P a n-degree monic polynomial and $G(Y) \in \mathfrak{T}(D, r)$ such that $N^+(G_x, r(x)) = 0$ whenever $x \in D \setminus M$.*

PROOF. – Let $F(Y) = \sum_{s=0}^{\infty} \xi_s(x) Y^s$. Without loss of generality we can assume ξ_0 is not nul in $H(D)$. Indeed if $\xi_0 = \ldots = \xi_h = 0$, $\xi_{h+1} \neq 0$ with $h < n$, we can factorize F in the form $Y^h \hat{F}(Y)$ with $\hat{F}(Y) = \sum_{s=0}^{\infty} \lambda_s Y^s$ and $\lambda_s \neq 0$.

Let Z be the set of the points α in D such that $\xi_0(\alpha) = 0$ and let $M' = M \cup Z$.

For each $\varrho > 0$, let $D_\varrho = D \setminus \left(\bigcup_{\alpha \in M'} d^-(\alpha, \varrho)\right)$ and let r_ϱ be the restriction of r in D_ϱ.

By Lemma 9 there exists $a_\varrho > 0$ such that $r(x) \geqslant a_\varrho$ whenever $x \in D_\varrho$. On the other hand $F(Y)$ belongs to $\mathfrak{T}_n(D_\varrho, r_\varrho)$ hence by Proposition Q, F has Hensel Factorization $P(Y)G(Y)$ in $\mathfrak{T}_n(D_\varrho, r_\varrho)$ (with P a n-degree monic polynomial).

If we can prove that for each $\alpha \in M'$ we have $P(Y) \in H(D)[Y]$ and $G(Y) \in \mathfrak{T}(D_\varrho \cup d_-(\alpha, \varrho), r_{\varrho,\alpha})$ (with $r_{\varrho,\alpha}$ the restriction of r in $D_\varrho \cup d^-(\alpha, \varrho)$) then conclusion will follow by Lemma 10.

Thus without loss of generality we can assume M' has only one point; then we can also assume this point is 0.

First suppose $r(0) \neq 0$. By the hypothesis on D, all the ξ_s are nul or quasi-invertible in $H(D)$; by the hypothesis

(1) $F_x \in T_n(r(x))$ for $x \in D \setminus M'$, we have $\xi_n(x) \neq 0$ for every $x \neq 0$, hence ξ_n has a factorization $x^m \theta_n(x)$ with θ_n invertible in $H(D)$.

Also by (1) we have $|\xi_s(x)| r(x)^{s-n} < |\xi_n(x)|$ hence

$$|\xi_s(x)| a_\varrho^{s-n} < |x|^m |\theta_n(x)| \quad \text{for } |x| \leqslant \varepsilon . \tag{2}$$

Then it clearly appears that x^m does divide ξ_s.

On the other hand, since $r(x)$ is bounded, we can easily verify x^m also divides $\xi_0, \dots, \xi_{n-1}$ because of the inequalities

$$|\xi_i(x)| r(x)^i \leqslant |\xi_n(x)| r(x)^n .$$

Finally F has a factorization in the form $x^m \overline{F}(Y)$ with $\overline{F}(Y) = \sum_{s=0}^{\infty} \theta_s(x) Y^h$, then θ_n invertible in $H(D)$. So $\overline{F} \in \mathfrak{T}_n(D, r)$ and we are sent back to the case $M' = \emptyset$.

Thus we can assume now $r(0) = 0$.

For every $\varrho > 0$, let $D_\varrho = D \setminus d^-(0, \varrho)$ and let r_ϱ be the restriction of r in D_ϱ. Then F, considered as an element of $\mathfrak{T}(D_\varrho, r_\varrho)$, has Hensel Factorization in $\mathfrak{T}_n(D_\varrho, r_\varrho)$ in the form $P(Y)G(Y)$ with $P(Y) \in H(D_\varrho)[Y]$, $G(Y) \in \mathfrak{T}(D_\varrho, r_\varrho)$, and this is true for every $\varrho > 0$.

We can easily show that $P \in H(D)[Y]$. Indeed, since r is bounded by some A, each zero of P is so, hence $\|P_x\|^{r(x)} \leqslant A^n$ whenever $x \in D$, hence each coefficient a_i of P is a bounded function in $D \setminus Z$ that lies in $H(D_\varrho)$ whenever $\varrho > 0$. By Lemma $a_i \in H(D)$, hence $P \in H(D)[Y]$.

Now let us prove that $F(Y) = P(Y)G(Y)$ with $G(Y) \in \mathfrak{T}(D, r)$.

For each $m \in \boldsymbol{N}$ let $F_m = \sum_{s=0}^{m} \xi_s Y^s$, and let $F_m = PG_m + R_m$ be the Euclidean Division of F_m by P in $H(D)[Y]$. For each $\varrho > 0$, by Lemma 9 r_ϱ has a lower bound $a_\varrho > 0$, hence we apply Lemma 2 that shows the sequence R_m converges to 0 and the sequence G_m converges to G in $\mathfrak{T}(D_\varrho, r_\varrho)$. Then Lemma 8 shows G_m converges to G in $\mathfrak{T}(D, r)$, and that ends the proof.

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