Algebraic Riccati Equations with Non-smoothing Observation Arising in Hyperbolic and Euler-Bernoulli Boundary Control Problems (*).

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Summary. – *This paper considers the optimal quadratic cost problem (regulator problem) for a class of abstract differential equations with unbounded operators which, under the same unified framework, model in particular « concrete » boundary control problems for partial differential equations defined on a bounded open domain of any dimension, including: second order hyperbolic scalar equations with control in the Dirichlet or in the Neumann boundary conditions; first order hyperbolic systems with boundary control; and Euler-Bernoulli (plate) equations with (for instance) control(s) in the Dirichlet and/or Neumann boundary conditions. The observation operator in the quadratic cost functional is assumed to be non-smoothing (in particular, it may be the identity operator), a case which introduces technical difficulties due to the low regularity of the solutions. The paper studies existence and uniqueness of the resulting algebraic (operator) Riccati equation, as well as the relationship between exact controllability and the property that the Riccati operator be an isomorphism, a distinctive feature of the dynamics in question (emphatically not true for, say, parabolic boundary control problems). This isomorphism allows one to introduce a « dual » Riccati equation, corresponding to a « dual » optimal control problem. Properties between the original and the « dual » problem are also investigated.*

1. – Introduction, dynamical model, quadratic cost problems and corresponding Riccati equations.

A main aim of the present paper is to study the infinite horizon quadratic cost problem—culminating with an analysis of the corresponding Algebraic Riccati (operator) Equation (A.R.E.)—for classes of (linear) hyperbolic and Euler-Bernoulli partial differential equations with nonhomogeneous (control) action exercised on the boundary of the bounded open spatial domain. It is meant to encompass, in particular, the following typical situations:

(i) the case of second order hyperbolic scalar equations with Dirichlet or Neumann boundary control;

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(ii) the case of Euler-Bernoulli equations, with controls in the Dirichlet and Neumann B.C.;

(iii) the case of first order hyperbolic systems.

Thus, one feature of our study is that it provides a common unifying operator theoretic framework, which is capable to include, in particular, all three cases (i), (ii), and (iii). This is achieved by means of an abstract operator dynamical model on which we shall impose some conditions, which—in fact—are distinctive properties enjoyed by the hyperbolic dynamics (i), and (iii), as well as by the Euler-Bernoulli dynamics (ii).

1.1. *Abstract dynamical model* (*which covers cases* (i), (ii), *and* (iii) *as illustrated in Appendix* 2).

We shall introduce the relevant abstract dynamical model, which Appendix 2 will then show how to specialize in order to cover all three cases (i), (ii) and (iii) above.

Let U (control space) and Y (state space) be two separable Hilbert spaces with inner products $\langle\,,\rangle$ and $(\,,)$ and corresponding norm $|\;|$ and $\|\;\|$, respectively.

Throughout this paper we are concerned with the following abstract dynamics on Y:

$$(1.1)\qquad \begin{cases} (a)\ \ y(t) = \exp[At]y_0 + (Lu)(t), \qquad y_0 \in Y \\ (b)\ \ (Lu)(t) = A\displaystyle\int_0^t \exp[A(t-\tau)]A^{-1}Bu(\tau)\,d\tau \end{cases}$$

$$(c)\ \ B \in \mathcal{L}(U; [\mathcal{D}(A^*)]')\ \text{ so that }\ A^{-1}B \in \mathcal{L}(U; Y)$$

formally corresponding to the equation

$$\begin{cases} (d)\ \ \dot{y} = Ay + Bu \qquad \text{on } [\mathcal{D}(A^*)]' \\ \quad\ \ y(0) = y_0 \in Y \end{cases}$$

Here, A is the infinitesimal generator of a strongly continuous (s.c.) semigroup on Y denoted for simplicity by $\exp[At]$, $t \geqslant 0$. (Without loss of generality for the problem here considered, we take $0 \in \varrho(A)$, the resolvent set of A, for otherwise we replace (1.1c) with $(A + \lambda_0 I)\int_0^t \exp[A(t-\tau)]R(\lambda_0, A)Bu(\tau)\,d\tau$, $\lambda_0 \in \varrho(A)$). In (1.1c-d), $[\mathcal{D}(A)]'$ and $[\mathcal{D}(A^*)]'$ are the dual spaces of $\mathcal{D}(A)$ and $\mathcal{D}(A^*)$ with respect to the topology of Y.

Throughout this paper, model (1.1) will be studied under the following standing hypothesis (H.1): for any $0 < T < \infty$, there exists $c_T > 0$ such that

(H.1)

(1.2a) $$\int_0^T |B^* \exp[A^* t]x|^2\, dt \leqslant c_T \|x\|^2, \qquad x \in \mathscr{D}(A^*)$$

so that the operator $B^* \exp[A^* t]$ admits a continuous extension—denoted henceforth by the same symbol (1)—satisfying

(1.2b) $$B^* \exp[A^* t]: \text{continuous } Y \to L_2(0, T; U).$$

Here B^*, the dual of B, satisfies $B^* \in \mathcal{L}(\mathscr{D}(A^*), U)$ after identifying $[\mathscr{D}(A^*)]''$ with $\mathscr{D}(A)$. As documented in Appendix 2, assumption (H.1) = (1.2) *always holds true* for second order scalar hyperbolic equations as in (i) above, or for first order hyperbolic systems as in (iii), or Euler-Bernoulli equations as in (ii) and in these cases *represents*, in fact, *a sharp trace theory result* (not obtainable from interior regularity plus use of trace theory), [L.1], [L-L-T.1], [L-T.2], [R.1], [K.1], [C-L.1], [L-T.8]. In the sequel, we shall indicate by L_{0T} the operator L in (1.1b) when viewed as acting from the space $L_2(0, T; U)$ to $L_2(0, T; Y)$. The adjoint L_{0T}^* of L_{0T}

(1.3a) $$(L_{0T}u, v)_{L_2(0,T;Y)} = (u, L_{0T}^* v)_{L_2(0,T;U)}$$

is given by

(1.3b) $$(L_{0T}^* v)(t) = B^* \int_t^T \exp[A^*(\tau - t)]\, v(\tau)\, d\tau .$$

Assumption (H.1) = (1.2) [as remarked, a *trace* regularity result for cases (i)-(iii)] has the following important implications on the regularity of the dynamics of problem (1.1) [*interior* regularity for cases (i)-(ii)]:

i(1.4) $$L_{0T}: \text{continuous } L_2(0, T; U) \to C([0, T]; Y)$$

and

(1.5) $$L_{0T}^*: \text{continuous } L_1(0, T; Y) \to L_2(0, T; U)$$

as is shown in Appendix 1, following [L-T.1-2], [L-T.9].

1.2. *Quadratic cost problems and Riccati equations.*

With model (1.1)-(1.2) we associate a quadratic functional over an infinite horizon

(1.6) $$J_\infty(u, y) = \int_0^\infty (Ry(t), y(t)) + |u(t)|^2\, dt$$

(1) This will not be repeated.

and pose the corresponding optimal control problem (regulator problem): given $y_0 \in Y$,

$$\text{(1.7)} \qquad \text{O.C.P.}(\infty) \begin{cases} \text{Minimize } J_\infty(u, y) \text{ over all } u \in L_2(0, \infty; U), \text{ where } y \text{ is the solu-} \\ \text{tion of (1.1}a\text{) due to } u\,. \end{cases}$$

The main aim of the present paper is to provide a rather complete study of the O.C.P.(∞) which culminates with the issues of existence and uniqueness of the corresponding Algebraic Riccati Equation

$$PA + A^* P + R = PB^* B^* P \tag{1.8}$$

(in a sense to be made precise later), which arises in the pointwise feedback form of the optimal pair $u^0(t, y_0)$, $y^0(t, y_0)$ of O.C.P.(∞) given by

$$u^0(t, y_0) = -\, BB^* P y^0(t, y_0)\,, \qquad \text{a.e. in } 0 \leqslant t \leqslant \infty\,.$$

In addition, we shall study a number of properties of the solution operator P. The entire theory on the O.C.P.(∞) which we shall present will rest on the following *minimal* hypothesis on the « observation » operator R (and nothing more):

$$\text{(1.9)} \qquad \text{(H.2)} \qquad R \in \mathcal{L}(Y)\,, \quad R = R^* \geqslant 0\,.$$

Thus, R may be, in particular, the identity on Y.

In order to study the O.C.P.(∞) and (1.8), we shall find useful to present relevant results for the corresponding quadratic cost problem over a preassigned finite horizon $T < \infty$: given $y_0 \in Y$,

$$\text{(1.10)} \quad \text{O.C.P.}(T) \begin{cases} \text{Minimize } J_T(u, y) \text{ over all } u \in L_2(0, T; U), \text{ where } y \text{ is the solu-} \\ \text{tion of (1.1}a\text{) due to } u \end{cases}$$

where

$$J_T(u, y) = \int_0^T (Ry(t), y(t)) + |u(t)|^2\, dt \tag{1.11}$$

under the same assumptions (H.1) = (1.2) for the dynamics and (H.2) = (1.9) for the observation operator R.

1.3. *Literature and orientation.*

The main difficulties of the problems under study are related to the underlying dynamics—in particular to the low regularity of both open loop and optimal closed loop solutions of Riccati operators; etc. This requires the introduction of new approaches, as it will be documented below. The present article is a successor paper to the following prior work in the area of boundary control problems for hyperbolic and Euler-Bernoulli type dynamics which we find convenient to group into the following three categories.

(i) *Constructive study from an optimal control problem to the corresponding Riccati equation*: paper [L-T.3] for second order scalar hyperbolic partial differential equations with Dirichlet boundary control. both cases $T < \infty$ and $T = \infty$; and the companion paper [C-L.1] for first order hyperbolic systems, case $T < \infty$. Both works use throughout an abstract functional analytic model of the hyperbolic dynamics.

(ii) *Direct study from a Riccati equation to the corresponding optimal control problem*: paper [DaP-L-T.1] for the abstract model (1.1) subject to assumption (H.1) = (1.2) in the case $T < \infty$.

(iii) *Direct study in* [F.2] *in the case where $T < \infty$ and where A is a group generator of the* (*Dual*) *Differential Riccati equation in the unknown $Q_T(t)$*, formally obtained by setting $Q_T(t) = P_T^{-1}(t)$ starting from the Differential Riccati equation in the unknown $P_T(t)$—whose solution however is precisely the unsettled issue—which corresponds to model (1.1) subject to assumption (H.1).

Moreover, the following considerations apply to the foregoing references.

Case $T < \infty$. - An assumption of « ε smoothness » on the observation operator $0 \leqslant R \leqslant R^* \in \mathcal{L}(Y)$ was needed in references [L-T.3], [C-L.1], in order to claim that the correspondingly constructed candidate of the Riccati operator be, in fact, a bona fide solution of the corresponding Differential Riccati Equation (hence of the corresponding so called « first » Riccati Integral Equation, which involves the *original* semi group). Here, the operator $B^*P_T(t)$ is unbounded (an essential feature and difficulty of the problem) but has dense domain. Examples of such « ε smoothness » include, in particular the following cases:

1) $R = \operatorname{diag}[R_1, R_2]$, with $R_1\bar{\Delta}^{-\varepsilon} \in \mathcal{L}(Y)$, $Y = L_2(\Omega)$, $\varepsilon > 0$ arbitrary, and $R_2 = 0$ for the wave equation with Dirichlet boundary control and cost functional which penalizes only the position; here $\bar{\Delta}$ denotes the Laplacian with zero Dirichlet boundary conditions, see [L-T.3];

2) $RA^{-\varepsilon} \in \mathcal{L}(Y)$, $Y = [L_2(\Omega)]^m$, $\varepsilon > 0$ arbitrary, for first order hyperbolic systems, see [C-L.1].

However, no claim of uniqueness of the Riccati solution was made in such generality. On the other hand, in the absence of such « ε smoothness » for R, i.e. for R subject only to assumption (H.2) = (1.9) and in particular for R = identity, references [L-T.3] and [C-L.1] provide the sought after « pointwise feedback synthesis relation » of the optimal pair through an explicitly constructed operator (the candidate of the Riccati operator), which is then shown to satisfy only the so called « second » Riccati Integral Equation, which involves the *evolution operator* of the optimal feedback dynamics. Similar results are then re-proved in [F.2] via a « dual » problem in the sense of (iii) above in the special but important case that A be a generator of a s.c. *group* on Y. In contrast, reference [DaP-L-T.1] does provide, via a direct method, existence and uniqueness for the Differential (or « first » Integral) Riccati Equation, as well as boundedness of the operator $B^*P_T(t)$, provided however that a stronger assumption is made on the smoothness of the observation R in addition to the standing assumption (H.1) = (1.2): namely that

$$R \exp[At]B: \text{ continuous } U \to L_1(0, Y; Y)\,. \tag{1.12}$$

(This assumption is in particular satisfied e.g. when: $R_1 \bar{\Delta}^{\frac{1}{4}+\varepsilon} \in \mathcal{L}(Y)$, $\varepsilon > 0$ and $RA \in \mathcal{L}(Y)$ for the wave equation and first order hyperbolic systems, respectively, mentioned above.)

By contrast, reference [P-S.1] assumes, in place of (1.12), a condition which, in particular, implies the following one:

$$C \exp[At]B: \text{ continuous } U \to L_2(0, T; Y) \tag{1.13}$$

with C bounded output operator, whereby in the notation of the present paper then $R = C^*C$. Condition (1.13) is stronger than (1.12) on two grounds: (i) it requires L_2 rather than L_1; (ii) with C smoothing, the operator $R = C^*C$ which arises from (1.13) is smoothing « twice as much » as the operator R allowed in (1.12).

Hypothesis (1.13) graetly simplifies the analysis of the Riccati equation, as described in [DaP-L-T.1, Remark pp. 44-45]: indeed, direct use of the Schwarz inequality on the Riccati operator formula gives at once that $B^*P_T(t)$ is a bounded operator, and thus a major difficulty of the problem with B unbounded versus B bounded disappears.

Reference [S.1] considers only the problem with $T < \infty$ with output operator possibly unbounded, but no results are given on the (true) Riccati equation in terms of the original semigroup $\exp[At]$. Reference [S.1] gives only (i) the synthesis of the optimal control and (ii) the Riccati Integral equation involving the evolution operator, not the original semigroup, in line with the earlier treatment of [L-T.3] and [C-L.1]. (However, as mentioned before, [L-T.3] [C-L.1] provide also the (true) differential Riccati equation under an additional « ε smoothness » for R.) The case $T = \infty$ is not considered in [S.1].

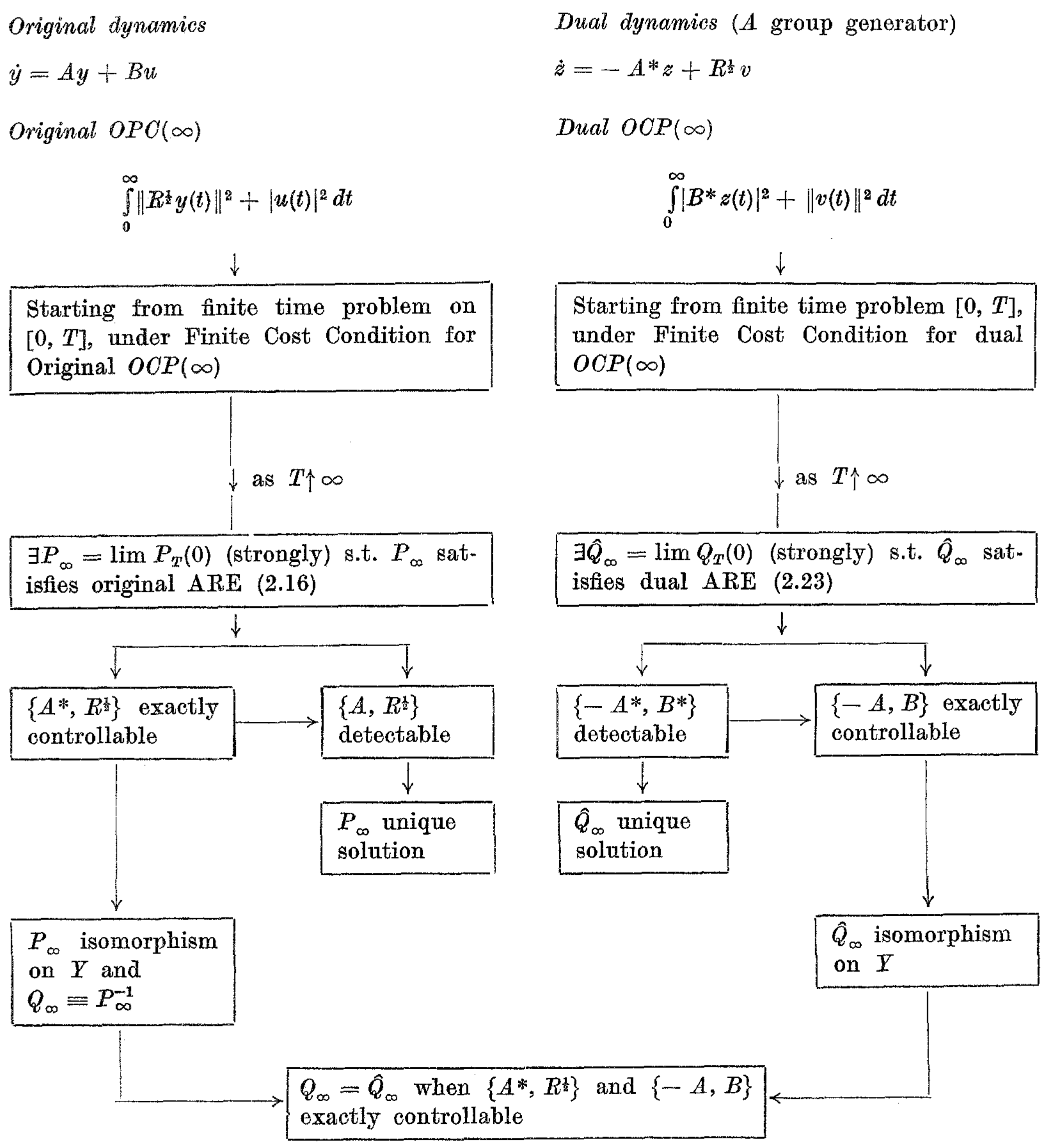
Case $T = \infty$. - Despite the lack of a Differential (or « first » Integral) Riccati Equation for the finite horizon problem $T < \infty$ lamented above in the case where R is only subject to assumption (H.2) = (1.9), reference [L-T.3, Sect. 5] successfully carries out—precisely in this case—a rather complete study of the infinite horizon problem $T = \infty$ as applied to second order scalar hyperbolic equations with Dirichlet boundary control. This study culminates with a statement of existence and uniqueness for the corresponding Algebraic Riccati Equation, as well as a statement of « pointwise feedback synthesis relation » for the optimal pair.

The emphasis in the present article is on the case $T = \infty$: here we provide a rather comprehensive study under the unifying abstract approach of model (1.1) subject to assumption (H.1) = (1.2), with paramount concern that the observation operator R fulfills the sole hypothesis (H.2) = (1.9) that $0 \leqslant R = R^* \in \mathcal{L}(Y)$, and *no* other smoothness. While our study recovers the concrete situation of second order equations with Dirichlet boundary control as in [L-T.2, Sect. 5], it also encompasses other hyperbolic dynamics and Euler-Bernoulli type equations, as documented in Appendix 2. All this despite the absence, as in [L-T.3], [F.2], of a Differential Riccati Equation theory for the finite time problem $T < \infty$. Thus, our approach to the problem $T = \infty$ given in sections 4 and 5 must by necessity differ from the usual or classical one, in that the Algebraic Riccati Equation is *not* recovered as a limit on the Differential Riccati Equation for $[0, T]$, as $T \uparrow \infty$, see e.g. [B.1] (the latter being not available yet, as least for R subject only to (H.2) = (1.9)). Rather, as in [L-T.3, Sect. 5], our approach will be crucially based on « trace regularity » properties of the dynamics, expressed by assumption (H.1) = (1.2).

Conceptually, the present peper may be divided into three parts as follows.

First, sections 2.1 through 2.4 study the original optimal control problem O.C.P.(∞) when $T = \infty$ and culminate with the statements of existence and uniqueness of the Algebraic Riccati Equation, with solution P_∞ given as a strong limit of the corresponding finite time problem as $T \uparrow \infty$. Moreover, under exact controllability assumption of the pair $\{A^*, R^{\frac{1}{2}}\}$, such operator P_∞ turns out to be an isomorphism on Y. (This result is in sharp contrast with, say, the same optimal control problem O.C.P.(∞) for *parabolic* equations with Dirichlet boundary control, where the Riccati Differential and Algebraic operators are, in fact, smooting and compact operators, see e.g. [L-T.5], [L-T.12]). With P_∞ isomorphism, the operator Q_∞ defined by $Q_\infty \equiv P_\infty^{-1} \in \mathcal{L}(Y)$ is a solution of a new (dual) Riccati Algebraic Equation; this, in fact, corresponds to a dual problem, whose dynamics however requires the assumption that A be a generator of a s.c. group, a special but important case. Said duality turns out to be described by the correspondence: $\{A, B, R$ [or $R^{\frac{1}{2}}, R^{*\frac{1}{2}}]\}$ of the original problem to $\{-A^*, R^{\frac{1}{2}}, BB^*$ [or $B^*, B]\}$ of the dual problem, see Tables 2.1-2.2 below in section 2.5. Thus sections 2.1 through 2.4 may be viewed as belonging to the above category (i) and represents the generalization of the treatment of [L-T.3, sect. 5] to the first order abstract model (1.1) subject to hypothesis (H.1) = (1.2).

Second, sections 2.5 and 2.6 study, following an idea of [F.2], the dual Riccati equation (when A is a group generator) by means of the direct method, which reconstructs the corresponding optimal control problem via Dynamic Programming. This may be viewed as belonging to the above categories (ii) and (iii). It then turns out that the dual Algebraic Riccati Equation admits as a solution the operator $\hat{Q}_\infty$ which is obtained as a strong limit of the corresponding finite time dual problem as $T\uparrow\infty$. Under assumption of exact controllability of the pair $\{-A, B\}$ (equivalently, of the pair $\{A, B\}$), such operator $\hat{Q}_\infty$ is the unique solution of the dual Algebraic Riccati Equation and, moreover, $\hat{Q}_\infty$ is an isomorphism on Y. The question arises therefore as to whether or when the analysis of the original problem and the analysis of the dual problem «merge»; more precisely, as to whether or when we have that $Q_\infty = \hat{Q}_\infty$ i.e. $\hat{Q}_\infty = P_\infty^{-1}$. This is the object of section 2.7. In general the answer is in the negative (counter example 2.1 in subsection 2.7). Indeed, the very identification of P_∞ with Q_∞^{-1} requires that P_∞ be an isomorphism on Y. It is most gratifying therefore that the identification $P_\infty = Q_\infty^{-1}$, or $Q_\infty = \hat{Q}_\infty$, holds true



(when A is a group generator) provided that both pairs $\{-A, B\}$ (equivalently, $\{A, B\}$) and $\{A^*, R^{\frac{1}{2}}\}$ are exactly controllable on some $[0, T]$, $T < \infty$; i.e. precisely the conditions under which P_∞ and $\hat{Q}_\infty$ are both isomorphisms on Y. As to the exact controllability problem, we remark that the results needed here have become available very recently for both second order hyperbolic equations (with constant coefficients) and Euler-Bernoulli type equations: see [L-T.4] and, without geometrical conditions on Ω (except for smoothness of $\partial\Omega$), [L.2], [H.1], [T.2] in case of generalized wave equations with Dirichlet boundary control, and [L.2], [L-T.7], [L-T.8] for the Euler-Bernoulli equations considered in Appendix 2; also [L-T.13] and [L-T.11].

We conclude by pointing out that it may be easier to compute (numerically) the solution Q_∞ of the Dual Algebraic Riccati Equation and then invert it (numerically) to obtain $P_\infty = Q_\infty^{-1}$ as desired (under the appropriate assumption mentioned above) rather than to compute (numerically) the solution P_∞ of the original Algebraic Riccati Equation. This may be so since the dual ARE is far simpler to treat than the original ARE.

The accompanying diagram schematically depicts a few main points of the original and dual problem, and their merging at the level of establishing that $Q_\infty = \hat{Q}_\infty$. For a full treatment, we refer to the subsequent sections.

2. – Statement of main results.

To help orient the reader, we shall state in this section the main highlights of the results of the present paper, with the understanding that further properties and claims—which we omit here—will be found in the full technical treatment of the subsequent sections 3-8.

2.1 *Case $T < \infty$. Theorem* 2.1

In section 3 we shall study the O.C.P.(T) and present results which include the following

THEOREM 2.1. – Consider the O.C.P.(T) in (1.10) for the dynamics (1.1) under the standing assumption (H.1) = (1.2) for the dynamics and (H.2) = (1.9) for the observation operator R. Then:

(i) there is a unique solution pair of functions $u_T^0 = u_T^0(t, 0; y_0)$ and $y_T^0 = y_T^0(t, 0; y_0)$, $0 \leq t \leq T$, of the O.C.P.(T), which satisfy

$$u_T^0 \in L_2(0, T; U); \quad y_T^0 \in C([0, T]; Y); \tag{2.4}$$

(ii) u_T^0 and y_T^0 are related by

$$u_T^0(\cdot, 0; y_0) = -L_{0T}^* R\{y_T^0(\cdot, 0; y_0)\} \tag{2.2}$$

and explicitly given by

$$\text{(2.3)}\quad \begin{cases} (a) & -u_T^0(t,0;y_0) = \{L_{0T}^* R[I + L_{0T}L_{0T}^* R]^{-1}[\exp[A\cdot]y_0]\}(t) \\ (b) & y_T^0(t,0;y_0) = \{[I + L_{0T}L_{0T}^* R]^{-1}[\exp[A\cdot]y_0]\}(t) \in C([0,T];Y) \end{cases}$$

where, writing simply L for L_{0T}, we have

$$\text{(2.4)}\qquad [I + LL^*R]^{-1} = I - L[I + L^*RL]^{-1}L^*R \in \mathcal{L}(L_2(0,T;Y));$$

(iii) there exists an operator $P_T(t) \in \mathcal{L}(Y)$, given explicitly by

$$P_T(t)x = \int_t^T \exp[A^*(\tau - t)]R\Phi_T(\tau,t)x\,d\tau$$

where

$$\text{(2.1)}\qquad \Phi_T(t,s)x = y_T^0(t,s;x)$$

which satisfies the following property

$$\text{(2.2)}\qquad P_T(t)\text{: continuous } Y \to C([0,T];Y)\,.$$

Moreover

$$\text{(2.3)}\quad \text{(iv)}\quad u_T^0(t,0;y_0) = -B^*P_T(t)y_T^0(t,0;y_0) \text{ a.e. in } [0,T]$$

$$\text{(2.4)}\quad \text{(v)}\quad (P_T(t)x,z) = \int_t^T (Ry_T^0(\tau,t;x), y_T^0(\tau,t;z)\,d\tau + \int_t^T \langle u_T^0(\tau,t;x), u_T^0(\tau,t;z)\rangle\,d\tau,\quad x,z \in Y$$

$$\text{(2.5)}\quad \text{(vi)}\quad P_T(t) = P_T^*(t) \geqslant 0,\quad 0 \leqslant t \leqslant T.$$

$$\text{(2.6)}\qquad (P_T(0)x,x) = J_T^0 = J_T(u_T^0(\cdot,0;x), y_T^0(\cdot,0;x)) \qquad \square$$

2.2. *The case* $T = +\infty$. *Theorem* 2.2 *Algebraic Riccati Equation*: *existence*.

In section 4, we shall begin our study of the O.C.P.(∞). To this end, a necessary assumption to be made at the outset is, as usual [B.1]:

(2.7) (H.3) *Finite Cost Condition*: For each initial condition $y_0 \in Y$, there exists some $\bar{u} \in L_2(0,\infty;U)$ such that if $\bar{y}$ is the corresponding solution of (1.1) due to $\bar{u}$, then $J(\bar{u},\bar{y}) < \infty$.

REMARK 2.1. - It is a highly non-trivial issue to verify assumption (H.3) = (1.3) in the case of hyperbolic dynamics or plate problems. In the case of second order hyperbolic scalar equations with Dirichlet boundary control (case (i)), the answer is fully satisfactory: when the differential elliptic operator has constantcoefficients, these equations are *always exactly controllable* by means of $L_2(0, T; L_2(\Gamma))$-controls, $U = L_2(\Gamma)$, in their *natural state space* $Y = L_2(\Omega_2) \times H^{-1}(\Omega)$, for all $T >$ some universal time $T_0 > 0$ (for which good estimates can be given), *without any geometrical conditions on the spatial domain* Ω (except for minimal smoothness of $\partial\Omega = \Gamma$); see recent results [L.2], [L.3], [T.2], the latter also for non constant coefficients; see also the first result in this space in [L-T.4] as a corollary of the more demanding uniform stabilization problem. As a consequence, the Finite Cost Condition (H.3) is *a-fortiori satisfied* for second order scalar equations with constant coefficients on *arbitrary* Ω, and a rather complete theory for the O.C.P.(∞) is then available under the sole minimal assumption (H.2) = (1.9) on R. Similarly, exact boundary controllability in the natural space of regularity was recently proved in [L-T.6] (under some geometrical conditions on Ω) for multidimensional plate-like equations with boundary control only in the Dirichlet boundary conditions and homogeneous Neumann boundary conditions; or else [L-T.8] with no geometrical conditions when both controls are active. See Appendix 2, C). Here again the Finite Cost Condition (H.3) is satisfied. $\square$

The results of section 4 will show, in particular, the following

Theorem 2.2. – Consider the O.C.P.(∞) in (1.7) for the dynamics (1.1) under the standing assumptions (H.1) = (1.2) for the dynamics (H.2) = (1.9) for the observation operator R, and (H.3) = (2.7) on the Finite Cost Condition. Then:

(i) there exists a unique solution pair of functions $u^0_\infty = u^0_\infty(t, 0; y_0)$ and $y^0_\infty = y^0_\infty(t, 0; y_0)$ of the O.C.P.(∞) which satisfy

$$(2.8) \qquad u^0_\infty \in L_2(0, \infty; U); \qquad R^{\frac{1}{2}} y^0_\infty \in L_2(0, \infty; Y); \qquad y^0_\infty \in C[0, T_0]; Y) \qquad \text{for any } T_0 < \infty;$$

(ii) there exists an operator $P_\infty \in \mathcal{L}(Y)$ given explicitly by

$$(2.9) \qquad P_\infty x = \lim_{T\uparrow\infty} P_T(0)x, \qquad x \in Y$$

satisfying

$$(2.10) \qquad P_\infty = P^*_\infty \geqslant 0, \qquad J^0_T = \lim_{T\uparrow\infty} J(u^0_T(\cdot, 0; x), y^0_T(\cdot, 0; x))$$

$$(2.11) \qquad (Px, x) = J^0_\infty = J_\infty(u^0(\cdot, 0; x), y^0(\cdot, 0; x)) = \lim_{T\uparrow\infty} J^0_T = \lim_{T\uparrow\infty} J(u^0_T(\cdot, 0; x), y^0_T(\cdot, 0; x))$$

and the relation

$$P_\infty x = \int_0^{t_0} \exp[A^* \tau] R \Phi_\infty(\tau) x \, d\tau + \exp[A^* t_0] P_\infty \Phi_\infty(t_0) x, \qquad x \in Y \tag{2.12}$$

where t_0 is an arbitrary point $0 < t_0 < \infty$ and $\Phi_\infty(t)x = y^0_\infty(t, 0; x)$ defines a s.c. semigroup on Y which is uniformly stable:

$$\|\Phi_\infty(t)\|_{\mathcal{L}(Y)} \leqslant C \exp[-\delta t], \qquad \delta > 0, \quad t \geqslant 0 \text{ if } R > 0.$$

Thus for the broad class of problems where $\exp[At]$ is uniformly bounded on $t \geqslant 0$ and $\Phi_\infty(t)x \to 0$ as $t \to +\infty$, then we can take $t_0 = \infty$ in (2.12), thereby obtaining a *defining* formula for P_∞.

(iii) Moreover, for $y_0 \in Y$

$$u^0_\infty(t, 0; y_0) = -B^* P_\infty y^0_\infty(t, 0; y_0) \quad \text{a.e. in } t \in [0, \infty]; \tag{2.13}$$

(2.14) (iv) $$\frac{d\Phi_\infty(t)x}{dt} = [A - BB^* P_\infty] \Phi_\infty(t) x,$$

$x \in \mathcal{D}(A_F)$, $A_F = A - BB^* P_\infty =$ the infinitesimal generator of $\Phi_\infty(\,)$;

(v) P_∞ has the following regularity

$$A^* P_\infty \in \mathcal{L}(\mathcal{D}(A_F); Y); \qquad A_F^* P_\infty \in \mathcal{L}(\mathcal{D}(A); Y) \tag{2.15a}$$

$$B^* P_\infty \in \mathcal{L}(\mathcal{D}(A_F); U) \cap \mathcal{L}(\mathcal{D}(A); U) \tag{2.15b}$$

[so that if $y_0 \in \mathcal{D}(A_F)$, then $y^0(t, 0; y_0) \in C([0, T]; \mathcal{D}(A_F))$ and $u^0_\infty(t, 0; y_0) \in C([0, T]; L^2(U)])$] and moreover satisfies the Algebraic Riccati Equation

$$(P_\infty x, Az) + (P_\infty Ax, z) + (Rx, z) = \langle B^* P_\infty x, B^* P_\infty z \rangle \tag{2.16}$$

for all $x, z \in \mathcal{D}(A)$; or else for all $x, z \in \mathcal{D}(A_F)$. □

2.3. *Case* $T = \infty$. *Theorem* 2.3. *Algebraic Riccati Equation*: *uniqueness.*

For uniqueness, in addition to the preceding hypotheses, we shall need the following hypothesis (which is automatically satisfied if $R > 0$).

Let $K\colon Y \supset \mathscr{D}(K) \to Y$ be a (linear), densely defined operator satisfying the following two conditions:

$$
\text{(H.4)} \quad \begin{cases}
\text{(i) } \|K^* x\|^2 \leqslant C[|B^* x|^2 + \|x\|^2], \text{ for all } x \in \mathscr{D}(B^*) \subset Y \\
\text{(ii) the s.c. semigroup } \exp[A_K t] \text{ on } Y, \text{ with generator} \\
(2.17) \qquad A_K = \overline{A + KR^{\frac{1}{2}}} \\
\text{[as guaranteed by virtue of Lemma 5.1 with } \Pi = KR^{\frac{1}{2}}\text{] is uniformly} \\
\text{stable: there are } M_1,\ k > 0 \text{ such that} \\
\qquad \|\exp[A_K t]\|_{\mathfrak{L}(Y)} \leqslant M_1 \exp[-kt], \quad t \geqslant 0\,.
\end{cases}
$$

REMARK 2.2*a*). – For $R > 0$, we choose $K = -c^2 R^{-\frac{1}{2}}$, with constant c sufficiently large and assumption (H.4) is automatically satisfied.

b) Assumption (i) above in (H.4) implies that $A^{-1}K \in \mathfrak{L}(Y)$ by virtue of assumption (1.1*c*): $A^{-1}B \in \mathfrak{L}(U;\ Y)$. □

THEOREM 2.3. – Consider the O.C.P.(∞) in (1.6) under the standing assumptions (H.1) = (1.2) for the dynamics; (H.2) = (1.9) for the operator R; (H.3) = (2.7) for the Finite Cost Condition; and (H.4) = (2.17) on the existence of the operator K. Then, the Algebraic Riccati Equation (2.16) admits a unique solution $P \in \mathfrak{L}(Y)$ such that $P = P^* \geqslant 0$ and $B^*P \in \mathfrak{L}(\mathscr{D}(A_F)\colon Y)$. This solution is given by the operator P_∞ of Theorem 2.2. □

2.4. *Theorem* 2.4. *Isomorphism of* $P_T(t)$, P_∞ *and exact controllability of* $\{A^*, R^{\frac{1}{2}}\}$. *Dual Algebraic Riccati Equation.*

The dynamical system $\dot{z}(t) = A^* z(t) + R^{\frac{1}{2}} g(t)$, $z(0) = 0$ (in short, the pair $\{A^*, R^{\frac{1}{2}}\}$) is called exactly controllable on Y over $[0, T]$, $0 < T < \infty$ with $g \in L_2(0, T;\ Y)$ in case the totality of all solutions points $z(T)$ fills all of Y as g runs over all of $L_2(0, T;\ Y)$; see Definition 6.1. With this definition we have

THEOREM 2.4. – Consider the O.C.P.(T) in (1.10) and O.C.P.(∞) in (1.7) under the standing assumptions (H.1) = (1.2) on the dynamics, (H.2) = (1.9) on the operator R and, in the case of $T = \infty$, of (H.3) = (2.7) on the Finite Cost Condition. Then:

(i) Case $T < \infty$. The operator $P_T(0)$, [resp. $P_T(t)$] guaranteed by Theorem 2.1, is an isomorphism on Y [at some time $0 \leqslant t < T$] if and only if the pair $\{A^*, R^{\frac{1}{2}}\}$ is exactly controllable on $[0, T]$, [resp. on $[0, T - t]$], whereby $P_s(r)$ is an isomorphism for $s - r \geqslant T$.

(ii) Case $T = \infty$. The operator P_∞ guaranteed by Theorem 2.2 is an isomorphism on Y, provided the pair $\{A^*, R^{\frac{1}{2}}\}$ is exactly controllable on some $[0, T]$, $T < \infty$.

Then setting $Q_\infty = P_\infty^{-1} \in \mathcal{L}(Y)$, we have that Q_∞ satisfies the following Dual Algebraic Riccati Equation

$$(2.18) \quad (\text{DARE}) \begin{cases} (AQ_\infty x, z) + (Q_\infty A^* x, z) + (RQ_\infty x, Q_\infty z) - \langle B^* x, B^* z\rangle = 0 \\ Q_\infty \in \mathcal{L}(\mathcal{D}(A^*); \mathcal{D}(A_F)) \cap \mathcal{L}(\mathcal{D}(A_F^*); \mathcal{D}(A))\,, \\ \qquad\qquad x, z \in \mathcal{D}(A^*) \subset \mathcal{D}(B^*) \subset Y \\ A_F = A - BB^* P_\infty \end{cases}$$

Equation (2.18) will be henceforth referred to as Dual Algebraic Riccati Equation (DARE) with respect to the (original) Algebraic Riccati Equation (2.16). A comparison between (2.18) and (2.16) reveals the following correspondence:

TABLE 2.1. *Correspondence between Original and Dual ARE.*

Original ARE (2.16)	A	R	[or $R^{\frac{1}{2}}$	$R^{*\frac{1}{2}}$]	B	P_∞
Dual ARE (2.18)	$-A^*$	BB^*	B	B^*	$R^{\frac{1}{2}}$	Q_∞

Thus, to the original dynamics (1.1) and to its corresponding (infinite horizon) control problem (1.7), there corresponds the dual dynamics and its corresponding control problem indicated below:

TABLE 2.2. *Original and dual problem.*

Original Problem	*Dual Problem*
dynamics (1.1):	dynamics:
$\dot{y} = Ay + Bu$ on Y	$\dot{z} = -A^* z + R^{\frac{1}{2}} v$ on Y
cost (1.7):	*cost*:
$\int_0^\infty (Ry(t), y(t)) + \lvert u(t)\rvert^2\, dt$	$\int_0^\infty \lvert B^* z(t)\rvert^2 + \lVert v(t)\rVert^2\, dt$

From the correspondence of Table 2.2 we see plainly that the DARE (2.18) is associated to the dynamics $\dot{z} = -A^* z + R^{\frac{1}{2}} v$, whose well-posedness however requires the *additional assumption that* $-A^*$ (equivalently, $-A$) be the generator of a s.c. semigroup on Y; i.e. *that* A^* (*equivalently*, A) *be the generator* of a s.c. *group* on Y. As a consequence of this assumption and of hypothesis (H.1), it will be shown at the beginning of section 7 that $B^* z$ is a well defined element of $L_2(0, T; U)$ for each $T > 0$.

A further analysis and discussion of the dual problem is carried out in the next subsections 2.5 through 2.7, under the standing assumption that A be a s.c. group generator.

2.5. *Case $T < \infty$. Dual Differential Riccati Equation when A is a group generator. Theorem 2.5: existence and uniqueness*

Orientation for subsections 2.5 *through* 2.7. The development of subsections 2.1 through 2.4 originates with the control problems O.C.P.$(T) = (1.10)$ and O.C.P.$(\infty) = (1.7)$ for the dynamics (1.1) and leads to the existence of the operator $P_\infty x = \lim_{T\uparrow\infty} P_T(0)x$, $x \in Y$, (2.9), which is the unique solution of the original ARE (2.16), under the hypotheses (H.1) through (H.4). Moreover, it shows in Theorem 2.4 that, at least when the pair $\{A^*, R^{\frac{1}{2}}\}$ is exactly controllable on some $[0, T]$, $T < \infty$, then the operator P_∞ is an isomorphism on Y and the operator $Q_\infty \equiv P_\infty^{-1}$, wit P_∞ defined by (2.9), is a solution of the DARE (2.18). It should be noted that in subsection 2.1, as well as in [L-T.3], there is no claim however that for R *nonregular* (e.g. $R =$ Identity), the operator $P_T(t)$ satisfies a Differential Riccati Equation (2); indeed, the proofs in sections 3-4 (and in [L-T.3]) show that the ARE for P_∞ is *not* derived as a limit process, as in classical or standard approaches, on Differential Riccati Equations.

In the remaining part of our present development, we shall instead follow in the general *direct* approach on Riccati Equations (in the sense specified e.g. [Da P-L-T.1]) and the idea of [F.2], by which we shall invert the line of argument followed so far and carry out our further investigation through the reversed procedure outlined below.

1) We shall first consider, as a starting point, the Dual Differential Riccati Equation

$$
\left\{
\begin{aligned}
&\frac{d}{dt}(Q_T(t)x, z) = (Q_T(t)x, A^*z) + (A^*x, Q_T(t)z) + \\
&\qquad\qquad\qquad\qquad + (RQ_T(t)x, Q_T(t)z) - \langle B^*x, B^*z\rangle \\
&Q_T(T) = 0 \qquad x, z \in \mathscr{D}(A^*)
\end{aligned}
\right.
\tag{2.19}
$$

(see Tables 2.1-2.2 above) and study directly—through a well established argument [DaP.1]—existence and uniqueness of (2.19). As noted below Table 2.2 at the end of subsection 2.4, this will require, by necessity, the standing assumption that A be the infinitesimal generator of a s.c. *group* $\exp[At]$ on Y. Thus subsections 2.5 through 2.8 will be restricted to apply only to this special but important case. If $Q_T(t)$, $0 \leqslant t \leqslant T$, is the solution of (2.19), then Dynamic Programming will allow us to recover the associated optimal control problem: given $z_0 \in Y$,

(2) Instead, for R e.g. like $A^{-\varepsilon}$, $\varepsilon > 0$ arbitrary, $P_T(t)$ does satisfy a Differential Riccati Equation, see [L-T.3].

Minimize

$$J_T(v, z) = \int_0^T |B^* z(t)|^2 + \|v(t)\|^2 \, dt \tag{2.20}$$

over all $v \in L_2(0, T; Y)$, where z is the solution:

$$z(t) = \exp[-A^* t] z_0 + \int_0^t \exp[-A^*(t-\tau)] R^{\frac{1}{2}} v(\tau) \, d\tau \tag{2.21a}$$

of the dual problem

$$\dot{z} = -A^* z + R^{\frac{1}{2}} v, \tag{2.21b}$$

see Table 2.2. All this is, in essence, Theorem 2.5 below.

2) Next, we shall consider the corresponding infinite horizon dual problem: given $z_0 \in Y$

$$\begin{cases} \text{minimize} \\ J_\infty(v, z) = \int_0^\infty |B^* z(t)|^2 + \|v(t)\|^2 \, dt \\ \text{over all } v \in L_2(0, \infty; Y), \text{ where } z \text{ is the solution (2.21) due to } v. \end{cases} \tag{2.22}$$

Under the finite cost assumption for (2.22), we shall prove the existence of an operator $\hat{Q}_\infty x \equiv \lim_{T \uparrow \infty} Q_T(0) x$, $x \in Y$, solution of the DARE

$$(A\hat{Q}_\infty x, z) + (\hat{Q}_\infty A^* x, z) + (R\hat{Q}_\infty x, \hat{Q}_\infty z) - \langle B^* x, B^* z \rangle = 0 \qquad \forall x, z \in \mathscr{D}(A^*), \tag{2.23}$$

whereby the dual Algebraic Riccati operator $\hat{Q}_\infty$ is obtained as a limit process on the dual Differential Riccati operators $Q_T(\ )$, unlike the original algebraic Riccati operator P_∞ with respect to the original $P_T(\ )$. Dynamic Programming will then again allow us to recover the corresponding optimal control problem (2.22) associated with (2.23). Under the additional assumption that the pair $\{-A, B\}$ is exactly controllable on some $[0, T]$, $T < \infty$ (equivalently, that the pair $\{A, B\}$ is exactly controllable on $[0, T]$ since A is a generator of a s.c. group [3]), we shall

---

[3] Henceforth, we shall freely use that, with A s.c. group generator, then $\{-A, B\}$ is exactly controllable in $[0, T]$ if and only if so is $\{A, B\}$ (i.e. the totality of all solution points $y(T)$ of (1.1) with $y_0 = 0$ fills all of Y as u runs over all of $L_2(0, T; U)$). The proof of this equivalence will be given at the beginning of section 7, in Lemma 7.0 (ii).

further prove that $\hat{Q}_\infty$ is the unique solution of the DARE (2.23) (in a suitably specified class) and that, moreover, $\hat{Q}_\infty$ is an isomorphism on Y. All this is, in essence, the content of Theorem 2.6 below.

3) Finally, it remains to connect the operator $\hat{Q}_\infty^{-1}$ provided by Theorem 2.6 when $\{A, B\}$ is exactly controllable on $[0, T]$ with the operator P_∞ provided by (2.9). More precisely, the question arises as to whether or when we have $P_\infty = \hat{Q}_\infty^{-1}$. In general, this is *not* true, as shown in Example 2.1 below. Indeed, the very identification of P_∞ with $\hat{Q}_\infty^{-1}$ requires that P_∞ be an isomorphism on Y and this—as we have seen in Theorem 2.4 (ii)—holds true in turn provided the pair $\{A^*, R^{\frac{1}{2}}\}$ is exactly controllable on some $[0, T]$, $T < \infty$. It is therefore most gratifying that the identification $P_\infty = \hat{Q}_\infty^{-1}$, (hence $Q_\infty = \hat{Q}_\infty$, with Q_∞ defined in Theorem 2.4 (ii)), holds true when A is a s.c. group generator, provided both pairs $\{-A, B\}$ (equivalently; $\{A, B\}$) and $\{A^*, R^{\frac{1}{2}}\}$ are exactly controllable on some $[0, T]$, $T < \infty$, the conditions under which both P_∞ and $\hat{Q}_\infty$ are isomorphisms. This is Theorem 2.7 below.

In conclusion, in subsections 2.5 through 2.6 we shall proceed from Dual Riccati Equations to the associated Optimal Control Problems, while in subsections 2.1 through 2.4 we proceeded from the original Optimal Control Problems to the associated Original Riccati Equations; then in subsection 2.7 we shall connect these two procedures. □

In section 7 we shall study equation (2.19) and problem (2.20) for the dynamics (2.21). Our main results are given by the following:

THEOREM 2.5 ($T < \infty$). Let A generate a s.c. group on Y and consider eq. (2.19) under the standing assumptions (H.1) = (1.2) on the dynamics, (H.2) = (1.9) on the observation operator. Then on (2.19):

(i) there exists $Q_T(\cdot) \in \mathscr{L}\big(Y;\, C([0, T];\, Y)\big)$ such that $Q_T(t) = Q_T(t)^* \geq 0$, $\forall t \in [0, T]$, $(Q_T(t)x, z)$ is continuously differentiable in t for each x and z in $\mathscr{D}(A^*)$, and $Q_T(\cdot)$ satisfies the Dual Differential Riccati Equation (2.19);

(ii) the Dual Differential Riccati Equation (2.19) admits a unique solution, given by $Q_T(\cdot)$, in the class of operators $Q(\cdot) \in \mathfrak{L}\big(Y;\, C([0, T];\, Y)\big)$ such that $(Q(t)x, z)$ is differentiable in t for each $x, z \in \mathscr{D}(A^*)$; equivalently, $Q_T(\cdot)$ is the unique solution in $\mathscr{L}\big(Y;\, C([0, T];\, Y)\big)$ of the integral Riccati Equation

$$(2.24)\qquad (Q_T(t)x, z) = \int_t^T \langle B^* \exp[-A^*(s-t)]x,\, B^* \exp[-A^*(s-t)]z\rangle\, ds -$$

$$-\int_t^T (RQ_T(s) \exp[-A^*(s-t)]x,\, Q_T(s)\exp[-A^*(s-t)]z\, ds \qquad \forall x, z \in Y;$$

(iii) there exists a unique solution pair of functions $v_T^0 = v_T^0(t, 0; z_0)$ and

$z_T^0 = z_T^0(t, 0; z_0)$, $0 \leqslant t \leqslant T$, of problem (2.20), which satisfy

$$v_T^0 \in C([0, T]; Y), \qquad z_T^0 \in C([0, T]; Y).$$

Moreover, the pair (v_T^0, z_T^0) is characterized by the pointwise feedback formula

$$v_T^0(t) = -R^{\frac{1}{2}} Q_T(t) z_T^0(t), \qquad 0 \leqslant t \leqslant T. \tag{3.35}$$

We finally have

$$(Q_T(0) z_0, z_0) = J_T(v_T^0, z_T^0). \qquad \square \tag{2.26}$$

Further results on (2.19) and (2.20) can be found in section 7.

2.6. *Case $T = \infty$. Dual Algebraic Riccati Equation when A is a group generator. Theorem 2.6: existence and uniqueness.*

In section 8 we shall study the dual infinite horizon problem (2.22) and its corresponding DARE (2.23) for the dynamics (2.21). Here, our main results are collected in the following:

THEOREM 2.6 ($T = \infty$). Let A generate a s.c. group on Y and consider eq. (2.23) under the standing assumptions (H.1) = (1.2) on the dynamics and (H.2) = (1.9) on the observation operator. Assume further the finite Cost Condition on problem (2.22):

$$\begin{cases} \text{for each } z_0 \in Y, \text{ there exists } v \in L_2(0, \infty; Y) \text{ such that } J_\infty(v, z) < \infty, \text{ where} \\ z \text{ is the solution of (2.21) due to } v \end{cases} \tag{2.27}$$

Then on (2.23):

there exists an operator $\hat{Q}_\infty \in \mathcal{L}(Y)$, $\hat{Q}_\infty = \hat{Q}_\infty^* \geqslant 0$, given by

$$\text{(i)} \ \hat{Q}_\infty x = \lim_{T \uparrow \infty} Q_T(0) x, \ x \in Y \tag{2.28}$$

such that

(ii) $\hat{Q}_\infty$ satisfies the dual ARE (2.23);

on (2.22):

(iii) there exists a unique solution pair of functions $v_\infty^0 = v_\infty^0(t, 0, z_0)$ and $z_\infty^0 = z_\infty^0(t, 0, z_0)$ of the problem (2.22), which satisfy

$$v_\infty^0 \in L_2(0, \infty; Y) \cap C([0, T]; Y), \quad \forall T > 0, \qquad B^* z_\infty^0 \in L_2(0, \infty; Y);$$
$$z_\infty \in C([0, T]; Y), \quad \forall T > 0.$$

Moreover, the pair (v_∞^0, z_∞^0) is characterized by the pointwise feedback formula

$$v_\infty^0(t) = - R^{\frac{1}{2}} \hat{Q}_\infty z_\infty^0(t)\,, \qquad t \geqslant 0\,. \tag{2.29}$$

We finally have

$$(\hat{Q}_\infty z_0, z_0) = J_\infty(v_\infty^0, z_\infty^0)\,. \tag{2.30}$$

(iv) If, in addition, the pair $\{-A, B\}$ (equivalently, the pair $\{A, B\}$) is exactly controllable over some interval $[0, T]$ (i.e. the totality of all solution points $y(T)$ of (1.1) with $y_0 = 0$ fills all of Y as u runs over all of $L_2(0, T; U)$), then the DARE (2.23) admits a unique solution, given by $\hat{Q}_\infty$, in the class of all $Q \in \mathcal{L}(Y)$ such that $Q = Q^* \geqslant 0$.

(v) The pair $\{-A, B\}$ (equivalently, the pair $\{A, B\}$) is exactly controllable on some $[0, T]$ if and only if $Q_T(0)$ is an isomorphism on Y, in which case $\hat{Q}_\infty$ is an isomorphism on Y as well. □

For the assumption on exact controllability of $\{A, B\}$ we refer to Remark 2.1.

REMARK 2.3. – In the statement of Theorem 2.6 we have used the symbol $\hat{Q}_\infty$ in place of Q_∞, in order to distinguish between the operator given as the limit of $Q_T(0)$, and the operator Q_∞ given by Theorem 2.4 as $Q_\infty = P_\infty^{-1}$ with P_∞ defined by (2.9). As mentioned in the Orientation in section 2.5, this distinction is not artificial, unless suitable assumptions are imposed. This issue is discussed in subsection 2.7.

2.7. *The identification of P_∞ with $\hat{Q}_\infty^{-1}$; i.e. of Q_∞ with $\hat{Q}_\infty$, when A is a group generator. Counterexample and Theorem* 2.7.

With reference to Remark 2.3, the following example shows that if P_∞ exists and $\hat{Q}_\infty$ exists and is an isomorphism, we cannot conclude in general that $\hat{Q}_\infty^{-1} = P_\infty$.

EXAMPLE 2.1. – Let $R = 0$, $B \in \mathcal{L}(U, Y)$, $-A^*$ stable, and $\{A, B\}$ exactly controllable over some interval $[0, T]$. Then $P_\infty = 0$, since $P_T(0) = 0$, $\forall T > 0$ (see Theorem 2.1, (iii)). On the other hand, the finite cost condition (2.27) is fulfilled, and

$$(\hat{Q}_\infty x, z) = \int_0^\infty \langle B^* \exp[-A^* t]x, B^* \exp[-A^* t]z\rangle\, dt\,,$$

since

$$(Q_T(0)x, z) = \int_0^T \langle B^* \exp[-A^* t]x, B^* \exp[-A^* t]z\rangle\, dt$$

(from (2.24)).

Finally, $\hat{Q}_\infty$ is an isomorphism, by Theorem 2.6 part (v). Then $\hat{Q}_\infty^{-1} \neq P_\infty$. □

However, as mentioned in the Orientation in subsection 2.5 the important property $P_\infty^{-1} = \hat{Q}_\infty$, (i.e. $Q_\infty = \hat{Q}_\infty$, with Q_∞ defined in Theorem 2.4 (ii) and $\hat{Q}_\infty$ defined in (2.28)) holds true under the assumptions which guarantee that both P_∞ and $\hat{Q}_\infty$ are isomorphisms on Y.

THEOREM 2.7. – Let A generate a s.c. group on Y. If both pairs $\{A, B\}$ and $\{A^*, R^{\frac{1}{2}}\}$ are exactly controllable over some interval $[0, T]$, then

(i) $P_\infty^{-1} = \hat{Q}_\infty$, i.e. $\hat{Q}_\infty$ coincides with the operator Q_∞ defined in Theorem 2.4;

(ii) the optimal solutions pairs (u_∞^0, y_∞^0) and (v_∞^0, z_∞^0) of the original and dual problems, given by Theorem 2.2 and Theorem 2.6 respectively, are related by:

$$u_\infty^0(t, 0; y_0) = -B^* z_\infty^0(t, 0; P_\infty y_0), \qquad v_\infty^0(t, 0; z_0) = -R^{\frac{1}{2}} y_\infty^0(t, 0; Q_\infty z_0).$$

Note that, under these assumptions, P_∞ and $\hat{Q}_\infty$ are well defined (for, in particular, the finite cost conditions (H.3) = (2.7) and (2.27) are satisfied) and are the unique solutions of (2.16) and (2.23), respectively.

3. – The case $T < \infty$. Proof of Theorem 2.1.

3.1. *Proof of parts (i) and (ii) of Theorem 2.1.*

Part (i). – We have already noted in (1.4) the regularity property of the operator L_{0T}. Using this, we see that the functional $J_T(u, y(u))$ is continuous on $L_2(0, T; U)$; since J_T is, moreover, strictly convex, it follows by standard optimization theory that there exists a unique solution pair $u_T^0 \equiv u_T^0(\cdot, 0; y_0)$, $y_T^0 = y_T^0(\cdot, 0; y_0)$ of the optimal control problem O.C.P.(T). Moreover, by (1.1), (1.4) the optimal pair satisfies for $y_0 \in Y$:

$$u_T^0(\cdot, 0; y_0) \in L_2(0, T; U); \qquad R^{\frac{1}{2}} y_T^0(\cdot, 0; y_0) \in C([0, T]; Y) \tag{3.0a}$$

$$y_T^0(t; 0; y_0) = \exp[At] y_0 + \{L_{0T} u_T^0(\cdot, 0; y_0)\}(t) \tag{3.0b}$$

Part (ii). – The Lagrangean of the O.C.P.(T) is

$$\mathscr{L}(u, y, p) \equiv \tfrac{1}{2}\{\|u\|^2_{L_2(0,T;U)} + (Ry, y)_{L_2(0,T;Y)}\} + (p, y - \exp[A^* \cdot] y_0 - L_{0T} u)_{L_2(0,T;Y)}$$

with $p \in L_2(0, T; Y)$. The optimality conditions $\mathscr{L}_y(u_T^0, y_T^0, p_T^0) = \mathscr{L}_u(u_T^0, y_T^0, p_T^0) = 0$ yield, respectively

$$p_T^0 = -R y_T^0; \qquad u_T^0 = L_{0T}^* p_T^0; \qquad \text{hence } u_T^0 = -L_{0T}^* R y_T^0. \tag{3.1}$$

If we eliminate u_T^0 between (1.1*a*) and (3.1), we obtain

$$y_T^0 = [I + L_{0T}L_{0T}^* R]^{-1}[\exp[A\cdot]y_0] \tag{3.2}$$

$$u_T^0 = - L_{0T}^* R[I + L_{0T}L_{0T}^* R]^{-1}[\exp[A\cdot]y_0] \tag{3.2b}$$

as elements of $L_2(0, T; Y)$ and $L_2(0, T; U)$ respectively, where we have to show the existence and boundedness of the inverse operator. In fact, a simple argument as in [L-T.3, below (2.8*e*)] shows that

$$[I + LL^* R]^{-1} = I - L[I + L^* RL]^{-1}L^* R \in \mathcal{L}(L_2(0, T); Y) \tag{3.2c}$$

(we drop for simplicity the subindex « $0T$ »), well defined and bounded in $L_2(0, T; Y)$, since R is self-adjoint nonnegative definite.

3.2. *Proof of part* (iii) *of Theorem* 2.1.

Step 1. – In order to assert the existence of the operator $P_T(t)$, we shall introduce an evolution operator to describe the dynamics of the feedback system. Henceforth, we take s, $0 \leq s < T$, as the new initial time of our optimal control problem with corresponding initial condition $y_s \in T$ at time s; i.e. we consider the optimal control problem over the time interval $[s, T]$ rather than over $[0, T]$. We shall denote the corresponding optimal solution pair by $u_T^0(\cdot, s; y_s)$ and $y_T^0(\cdot, s; y_s)$. The same Lagrange multiplier argument of part (ii), once applied to the new problem, gives then

$$- u_T^0(\cdot, s; y_s) = L_{sT}^* R\{y_T^0(\cdot, s; y_s)\} \tag{3.3a}$$

$$- u_T^0(t, s; y_s) = \big\{L_{sT}^* R[I + L_{sT}L_{sT}^* R]^{-1}[\exp[A(\cdot - s)]y_s]\big\}(t) \in L_2(0, T; U) \tag{3.3b}$$

$$y_T^0(t, s; y_s) = \big\{[I + L_{sT}L_{sT}^* R]^{-1}[\exp[A(\cdot - s)]y_s]\big\}(t) \in C([s, T]; Y) \tag{3.3c}$$

where (compare with (1.1*a*))

$$(L_{sT}u)(t) = A\int_s^t \exp[A(t-\tau)]A^{-1}Bu(\tau)\,d\tau\,, \qquad u \in L_2(s, T; U) \tag{3.4a}$$

:continuous $L_2(s, T; U) \to C([s, T]; Y)$, see (1.4)

$$(L_{sT}^* v)(t) = \begin{cases} (L_{0T}^* v)(t) & s \leq t \leq T \\ 0 & 0 \leq t \leq s \end{cases} \tag{3.5}$$

:continuous $L_1(s, T; Y) \to L_2(s, T; U)$, see (1.5).

We next define an operator $\Phi_T(t, s) \in \mathcal{L}(Y)$, $0 \leqslant s \leqslant t \leqslant T$, by setting

$$\Phi_T(t, s)x = y_T^0(t, s; x) = \big\{[I + L_{sT}L_{sT}^* R]^{-1}[\exp[A(\cdot - s)]x]\big\}(t) \in C([s, T]; Y) \tag{3.6}$$

see (3.3*c*).

Step 2. – The next Lemma collects relevant properties of $\Phi_T(t, s)$ and shows, in particular, that $\Phi_T(t, s)$ is an evolution operator.

LEMMA 3.1. – For the operator $\Phi_T(t, s)$ defined by (3.6) as an operator in $\mathcal{L}(Y)$, the following properties hold true:

a) $\Phi_T(t, t) = I$ (identity on Y), $0 \leqslant t \leqslant T$;

b) $\Phi_T(t, \tau) = \Phi_T(t, s)\Phi_T(s, \tau)$ (transition), $0 \leqslant \tau \leqslant s \leqslant t \leqslant T$;

c) for each fixed s

$$\Phi_T(\cdot, s) \in \mathcal{L}\big(Y; C([s, T]; Y)\big)$$

(strong continuity in the first variable);

d) there is a constant C_T such that

$$\|\Phi_T(t, s)\|_{\mathcal{L}(Y)} \leqslant C_T, \qquad \text{uniformly in } 0 \leqslant s \leqslant t \leqslant T;$$

e) for each fixed t, $0 < t \leqslant T$:

$$\Phi_T(t, \cdot) \in \mathscr{L}\big(Y; C([0, T]; Y)\big)$$

(strong continuity in the second variable).

PROOF OF LEMMA 3.1. – Parts *a*) and *b*) are obvious. Part *c*) was noted explicitly in (3.6). Part *e*) follows in the usual way (e.g. [B.1]) from part *c*) combined with part *d*). To prove part *d*), we first note that

$$\|I_s + L_{sT}^* R L_{sT}\|_{\mathcal{L}(L_2(0,T; U))} \geqslant 1, \qquad \text{hence } \|[I_s + L_{sT}^* R L_{sT}]^{-1}\|_{\mathcal{L}(L_2(0,T; U))} \leqslant 1 \tag{3.7a}$$

uniformly in $s \in [0, T]$. Next, by using these bounds, the version of (3.2*c*) corresponding to the initial time «s» gives

$$\|[I_s + L_{sT}L_{sT}^* R]^{-1}\|_{\mathcal{L}(L_2(0,T; Y))} \leqslant \text{const}_T \tag{3.7b}$$

uniformly in $s \in [0, T]$. Then (3.2*b*) yields by virtue of (3.7*b*)

$$\|u_T^0(\cdot, s; x)\|_{L_2(0,T; U)} \leqslant C_T\|x\| \tag{3.8}$$

uniformly in $s \in [0, T]$. Finally, combining (3.8) and the regularity (3.4) for L_{sT}, yields part *d*) as desired.

Step 3. – We now define an operator $P_T(t) \in \mathscr{L}(Y)$ by setting

$$P_T(t)x = \int_t^T \exp[A^*(\tau - t)] R\Phi_T(\tau, t)x\,d\tau, \qquad 0 \leqslant t \leqslant T. \tag{3.9}$$

By virtue of lemma 3.1 *d*), we plainly obtain $P_T(\cdot) \in \mathfrak{L}(Y; L_\infty(0, T; Y))$; moreover, by adding and subtracting, use of Lemma 3.1, and the Lebesgue dominated theorem [H-P.1, p. 83] we can show that, in fact,

$$P_T(\cdot) \in \mathfrak{L}\big(Y; C([0, T]; Y)\big) \tag{3.10}$$

Part (iii) of Theorem 2.1 is proved. □

3.3. *Proof of parts* (iv), (v), (vi) *of Theorem* 2.1.

Part (iv). – By (3.3*a*), (3.6), (3.5) we obtain

$$\begin{aligned} u_T^0(t, s; x) &= -L_{sT}^* R\Phi_T(\cdot, s)x = \\ &= -B^* \int_t^T \exp[A^*(\tau - t)] R\Phi_T(\tau, s)x\,d\tau \in L_2(s, T; U) \end{aligned} \tag{3.11}$$

where the above expression is well defined for all s, a.e. in $t \in [s, T]$. (See also Lemma 3.1 and property (3.5)). If we now take $s = 0$ in (3.11) for almost every t, we obtain the desired pointwise relation

$$\begin{aligned} u_T^0(t, 0; x) &= -B^* \int_t^T \exp[A^*(\tau - t)] R\Phi_T(\tau, 0)x\,d\tau \\ &= -B^* \int_t^T \exp[A^*(\tau - t)] R\Phi_T(\tau, t)\Phi(t, 0)x\,d\tau = \\ &= -B^* P_T(t)\Phi_T(t, 0)x = -B^* P_T(t) y_T^0(t, 0; x) \end{aligned} \tag{3.12}$$

by Lemma 3.1 *a*), (3.9), and (3.6).

Part (v). – The equation of the optimal dynamics

$$y_T^0(\tau, t; x) = \exp[A(\tau - t)]x + \{L_{tT} u_T^0(\cdot, t; x)\}(\tau) \tag{3.13}$$

can be explicitly re-written by (3.6), (3.4) and (3.12) as

$$(3.14)\qquad \exp[A(\tau-t)]z = \Phi_T(\tau,t)z + A\int_t^\tau \exp[A(\tau-\sigma)]A^{-1}BB^*P_T(\sigma)\Phi_T(\sigma,t)z\,d\sigma$$

Next, from (3.9)

$$(3.15)\qquad (P_T(t)x,z) = \int_t^T \big(R\Phi_T(\tau,t)x, \exp[A(\tau-t)]z\big)\,d\tau$$

Substituting $\exp[A(\tau-t)z]$ from (3.14) into (3.15) yields

$$(3.16)\qquad \begin{cases} (P_T(t)x,z) = \displaystyle\int_t^T \big(R\Phi_T(\tau,t)x, \Phi_T(\tau,t)z\big)\,d\tau + I_T(\tau) \\ I_T(t) = \displaystyle\int_\tau^T \Big(R\Phi_T(\tau,t)x, A\int_t^\tau \exp[A(\tau-\sigma)]A^{-1}BB^*P_T(\sigma)\Phi_T(\sigma,t)z\,d\sigma\Big)\,d\tau \end{cases}$$

(changing the order of integration (4))

$$= \int_t^T \Big(B^*\int_\sigma^T \exp[A^*(\tau-\sigma)]R\Phi_T(\tau,t)x\,d\tau\; B^*P_T(\sigma)\Phi_T(\sigma,t)z\Big)\,d\sigma$$

(using $\Phi_T(\tau,\sigma)\Phi_T(\sigma,t) = \Phi_T(\tau,t)$ and (3.9))

$$(3.18)\qquad = \int_t^T \big(B^*P_T(\sigma)\Phi_T(\sigma,t)x, B^*P_T(\sigma)\Phi_T(\sigma,t)z\big)\,d\sigma$$

Thus, (3.16), (3.17) give

$$(3.18)\qquad (P_T(t)x,z) = \int_t^T \big(R\Phi_T(\tau,t)x, \Phi_T(\tau,t)z\big)\,d\tau + \int_t^T \langle B^*P_T(\tau)\Phi_T(\tau,t)x, B^*P_T(\tau)\Phi_T(\tau,t)z\rangle\,d\tau$$

(4) This step of change in the order of integration can be rigorously justified by using a regularization and approximation argument as in [L-T.3], [C-L.1]. More precisely, $B^*P_T(t)\,\Phi_T(t,0) = \lim_{n\to\infty} B^*P_T^n(t)\,\Phi_T^n(t,0)$, where P^n, Φ^n correspond to (1.11) with $R = R^n$ and where range of $R^n \in \mathscr{D}(A^*)$.

which by virtue of (3.6) and (3.12) produces

$$(3.19)\qquad (P_T(t)x, z) = \int_t^T (Ry_T^0(\tau, t; x), y_T^0(\tau, t; z))\, d\tau + \int_t^T \langle u_T^0(\tau, t; x), u_T^0(\tau, t; z)\rangle\, d\tau$$

Part (vi). – By specializing (3.19) with $x = z$ we obtain

$$(3.20)\qquad P_T^*(t) = P_T(t) \geqslant 0 \qquad t \in [0, T]$$

$$(3.21)\qquad (P_T(0)x, x) = J_T^0 = J_T(u_T^0(\ \ , 0; x), y_T^0(\ \ , 0; x))$$

Theorem 2.1 is fully proved. □

4. – The case $T = \infty$. Proof of Theorem 2.2. Algebraic Riccati equation: existence.

Throughout this section, extension by zero beyond T of the function f_T will be denoted by $\tilde{f}_T$. Thus: $\tilde{f}_T(t) \equiv f_T(t)$, $0 \leqslant t \leqslant T$, while $\tilde{f}_T(t) \equiv 0$, $t > T$.

4.1. *Proof of parts (i) and (ii) of Theorem 2.2.*

Part (i). – By virtue of the finite cost condition—assumption (H.3)—it follows by standard optimization theory that the optimal control problem O.C.P.(∞) admits a unique solution pair $u_\infty^0 = u_\infty^0(\cdot, 0; y_0)$, $y_\infty^0 = y_\infty^0(\cdot, 0; y_0)$. By (1.1), (1.4) the optimal pair satisfies for $y_0 \in Y$

$$(4.0a)\qquad u_\infty^0(\cdot, 0; y_0) \in L_2(0, \infty; U); \qquad R^{\frac{1}{2}} y_\infty^0(\cdot, 0; y_0) \in L_2(0, \infty; Y) \cdot$$

$$\cdot y_\infty^0(\cdot, 0; y_0) \in C([0, T_0]; Y) \qquad \text{for any } 0 < T_0 < \infty.$$

and

$$(4.0b)\qquad y_\infty^0(t, 0; y_0) = \exp[At]y_0 + \{Lu_\infty^0(\cdot, 0; y_0)\}(t) \in C([0, T_0]; Y)$$

Part (ii). – To obtain the operator P_∞ we need a preliminary Lemma, which is an additional property of $\Phi_T(\ ,\ )$ defined in (3.6):

LEMMA 4.1. – For the operator $\Phi_T(\ ,\ )$ defined in (3.6) we have

$$(4.1)\qquad \Phi_{T-t}(\sigma, 0) = \Phi_T(t + \sigma, t) \qquad \text{on } Y,\ 0 \leqslant t \leqslant T,\ 0 \leqslant \sigma \leqslant T - t$$

PROOF. – The equation of the optimal dynamics is

$$(4.2)\qquad \Phi_T(t, s)x = \exp[A(t-s)]x + \{L_{sT}u_T^0(\cdot, s; x)\}(t) =$$

$$= \exp[A(t-s)]x - \{L_{sT}L_{sT}^* R\Phi_T(\cdot, s)x\}(t)$$

obtained via (3.3a) and (3.6). From (4.2) with $s=0$ and $t=\sigma$ and with T replaced by $T-t$ we obtain

$$\exp[A\sigma]x = \Phi_{T-t}(\sigma,0)x + \tag{4.3}$$
$$+ A\int_0^\sigma \exp[A(\sigma-\tau)]A^{-1}B\Big(B^*\int_\tau^{T-t}\exp[A^*(\tau-t)]R\Phi_{T-t}(r,0)\,dr\Big)\,d\tau$$

using (3.4), (3.5), (1.3b). Similarly from (4.2) with s and t replaced by t and $t+\sigma$ respectively, we obtain

$$\exp[A(t+\sigma-t)]x = \Phi_T(t+\sigma,t)x + \tag{4.4}$$
$$+ A\int_t^{t+\sigma}\exp[A(t+\sigma-\tau)]A^{-1}B\Big(B^*\int_\tau^T\exp[A^*(\alpha-\tau)]R\Phi_T(\alpha,t)x\,d\alpha\Big)\,d\tau\,.$$

Setting $\tau-t=\beta$ in the external integral in (4.4) and then $\alpha-t=r$ in the internal integral in (4.4) yields

$$\exp[A\sigma]x = \Phi_T(t+\sigma,t)x + \tag{4.5}$$
$$+ A\int_0^\sigma\exp[A(\sigma-\beta)]A^{-1}B\Big(B^*\int_\beta^{T-t}\exp[A^*(\tau-\beta)]R\Phi_T(t+r,t)x\,dr\Big)\,d\tau\,.$$

Comparison between (4.2) and (4.5) shows that both $\Phi_{T-t}(\sigma,0)x$ and $\Phi_T(t+\sigma,t)x$ satisfy the same equation, say (4.5). But then the difference

$$z(\sigma,t)\equiv\Phi_T(t+\sigma,t)x-\Phi_{T-t}(\sigma,0)x\in C([0,T-t];Y)\qquad(\text{in }\sigma) \tag{4.6}$$

satisfies $[I+L_{0T}L_{0T}^*R]z(\cdot,t)=0$. By (3.2$c$) we deduce that $z(\sigma,t)$ is the zero element in $L_2(0,T-t;Y)$ and by (4.6) in $C([0,T-t];Y)$. $\square$

We can now introduce the operator P_∞ and study some of its preliminary properties.

LEMMA 4.2. - We have

a) the (self-adjoint) operator $P_T(\cdot)\geqslant 0$ converges strongly on Y to a (self-adjoint) operator $P_\infty\geqslant 0$ as $T\uparrow\infty$; i.e.

$$P_\infty x=\lim_{T\uparrow\infty}P_T(0)x=\lim_{T\uparrow\infty}\int_0^T\exp[A^*\tau]R\Phi_T(\tau,0)x\,d\tau \tag{4.7}$$

b) $P_{T-t}(0) = P_T(t) \quad 0 \leqslant t < T$

c) P_∞ in *a*) can likewise be defined by

$$P_\infty x = \lim_{T\uparrow\infty} P_T(t)x = \lim_{T\uparrow\infty} \int_t^T \exp[A^*(\tau - t)]R\Phi_T(\tau, t)x\, d\tau \tag{4.8}$$

independently of t, $0 \leqslant t < T$.

d) For $x \in Y$

$$J^0_\infty \equiv J_\infty(u^0_\infty(\cdot\,; x), y^0_\infty(\cdot\,; x)) = \int_0^\infty |u^0_\infty(t; x)|^2 + (Ry^0_\infty(t; x), y^0_\infty(t; x))\, dt = (P_\infty x, x) \tag{4.9}$$

e) In the notation introduced in the opening paragraph of section 4, we have

$$\tilde{u}^0_T \to u^0_\infty \qquad \text{in } L_2(0, \infty; U) \tag{4.10a}$$

$$R^{\frac12}\tilde{y}_T \to R^{\frac12} y^0_\infty \qquad \text{in } L_2(0, \infty; Y) \tag{4.10b}$$

for a suitable subsequence $T\uparrow\infty$, $\tilde{u}^0_T = \tilde{u}^0_T(\cdot, 0; x)$, $u^0_\infty = u^0_\infty(\cdot\,; 0; x)$ etc.; i.e. the optimai pair on $[0, T]$ for the O.C.P.(T) converges to the optimal pair on $[0, \infty]$ for the O.C.P.(∞), strongly in L_2.

f) For each fixed t, we have

$$y^0_T(t, 0; x) \to y^0_\infty(t; x) \text{ in } Y, \text{ uniformly on bounded } t\text{-intervals as } T\uparrow\infty,\ t < T. \tag{4.11}$$

PROOF. – *Part a*). By optimality of u^0_T, y^0_T and (3.21) we obtain a uniform bound in T for $x \in Y$

$$(P_T(0)x, x) \equiv J_T(u^0_T(\cdot, 0; x), y^0_T(\cdot, 0; x)) < J_\infty(u^0_\infty(\cdot, 0; x), y^0_\infty(\cdot, 0; x)) < \infty\,. \tag{4.12}$$

This, combined with the monotonicity of the self-adjoint non-negative operator $P_T(0)$, implies that the limit in (4.7) exists and defines a self-adjoint non-negative operator $P_\infty \in \mathcal{L}(Y)$.

Part b). – This is a direct consequence of the definition (3.9) of $P_T(t)$ combined with Lemma 4.1.

Part c). – This follows by taking the limit in Part *b*) as $T\uparrow\infty$.

Part d. – First, from

$$\int_0^T |\tilde{u}^0_T(t)|^2 + \|R^{\frac12}\tilde{y}^0_T(t)\|^2\, dt = J_T(\tilde{u}^0_T, \tilde{y}^0_T) = J_T(u^0_T, y^0_T) < J_\infty(u^0_\infty, y^0_\infty) < \infty \tag{4.13}$$

we see that the extended functions $\{\tilde{u}_T^0\}$ and $\{R^{\frac{1}{2}}\tilde{y}_T^0\}$ are contained in a fixed ball of $L_2(0, \infty; U)$ and $L_2(0, \infty; Y)$, respectively. Hence, we can extract subsequences

$$\tilde{u}_T^0 \to \text{some } \tilde{u}, \text{ weakly in } L_2(0, \infty; U) \tag{4.14a}$$

$$R^{\frac{1}{2}}\tilde{y}_T^0 \to \text{some } R^{\frac{1}{2}}\tilde{y}, \text{ weakly in } L_2(0, \infty; Y)\,. \tag{4.14b}$$

Next, we shall prove that the above limits are connected by the underlying dynamics; i.e. for any $0 < T_0 < \infty$

$$R^{\frac{1}{2}}\tilde{y}(t) = R^{\frac{1}{2}}\exp[At]y_0 + R^{\frac{1}{2}}(L\tilde{u})(t) \in C([0, T_0]; Y) \tag{4.15}$$

Indeed, with $T > T_0$, $L_{0T}u_T^0 = L_{0T}\tilde{u}_T^0$ converges weakly to $L\tilde{u}$ in $L_2(0, T; Y)$ by (4.14*a*) and (1.4), while

$$R^{\frac{1}{2}}\tilde{y}_T^0 = R^{\frac{1}{2}}\exp[At]y_0 + R^{\frac{1}{2}}\{L_{0T}\tilde{u}_T^0\}(t)\,, \qquad 0 \leqslant t \leqslant T_0 < T$$

converges weakly to $R^{\frac{1}{2}}\tilde{y}$ in $L_2(0, T; Y)$. By uniqueness of the weak limit, we obtain the identity in (4.15), first in $L_2(0, T_0; Y)$ and then in $C([0, T_0]; Y)$.

Finally, passing to the limit in (4.12) yields

$$(P_\infty x, x) \leqslant J_\infty^0 = J_\infty(u_\infty^0(\cdot, 0; x), y_\infty^0(\cdot, 0; x)) < \infty \tag{4.16}$$

by (4.7), left. On the other hand, the well-known lower semicontinuity of the quadratic cost J_∞ resulting from the weak convergence (4.10), ([E-T.1, p. 11]), completed with (4.15) gives the inequality in

$$(P_T(0)x, x) = J_T(u_T^0, y_T^0) = J_\infty(\tilde{u}_T^0, \tilde{y}_T^0) \geqslant J_\infty(\tilde{u}, \tilde{y})$$

(where $u_T^0 = u_T^0(\cdot, 0; x)$) etc. $\tilde{y} = \tilde{y}(\cdot, 0; x)$, from which taking the limit via (4.7) yields

$$(P_\infty x, x) \geqslant J_\infty(\tilde{u}, \tilde{y}) \geqslant J_\infty(u_\infty^0, y_\infty^0)\,. \tag{4.17}$$

Thus, (4.16), (4.17) give

$$(Px, x) = J_\infty(\tilde{u}, \tilde{y}) = J_\infty(u_\infty^0, y_\infty^0) \tag{4.18}$$

and part *d*) is proved.

Part e). – The identity in (4.18) together with the uniqueness of the optimal pair (already noted at the opening paragraph of subsection 4.1) yields

$$\tilde{u} = u_\infty^0 \quad \text{in } L_2(0, \infty; U); \qquad \tilde{y} = y_\infty^0 \quad \text{in } L_2(0, \infty; Y)\,. \tag{4.19}$$

Thus, (4.14) becomes

$$\tag{4.20} \begin{cases} \tilde{u}_T^0 \to u_\infty^0 , & \text{weakly in } L_2(0, \infty; U) \\ R^{\frac{1}{2}}\tilde{y}_T^0 \to R^{\frac{1}{2}}\tilde{y} , & \text{weakly in } L_2(0, \infty; Y) . \end{cases}$$

But the established convergence $J_T^0 \to J_\infty^0$ provides norm convergence

$$\|\tilde{u}_T^0\|^2_{L_2(0,\infty;\,U)} + \|R^{\frac{1}{2}}\tilde{y}_T^0\|^2_{L_2(0,\infty;\,Y)} \to \|u_\infty^0\|^2_{L_2(0,\infty;\,U)} + \|R^{\frac{1}{2}}y_\infty^0\|^2_{L_2(0,\infty;\,Y)} \,.$$

This, combined with weak convergence, yields strong convergence (4.10), as desired.

Part f). – For each t fixed, (4.10) implies $(L_{0T}u_T^0)(t) \to (L_{0T}u_\infty^0)(t)$ in Y by the continuity (1.4) of L_{0T}, uniformly on bounded t-intervals and (4.11) follows then by virtue of the optimal dynamics. Lemma 4.2 is proved. □

We next define the operator $\Phi_\infty(t) \in \mathcal{L}(Y)$ by setting

$$\tag{4.21} \Phi_\infty(t)x = y_\infty^0(t, 0; x) , \qquad x \in Y .$$

We then have

COROLLARY 4.3. – In the notation introduced in the opening paragraph of section 4, we have:

(4.22) *a*) $R^{\frac{1}{2}}\tilde{\Phi}_T(\cdot, 0)x \to R^{\frac{1}{2}}\Phi_\infty(\;)x$ in $L_2(0, \infty; Y)$, $x \in Y$;

b) for each fixed $t > 0$:

(4.23) $\Phi_T(t, 0)x \to \Phi_\infty(t)x$, $x \in Y$, uniformly on bounded t-intervals as $T \uparrow \infty$, with $t < T$.

c) $\Phi_\infty(t)$ is a strongly continuous semigroup on Y; moreover if $R > 0$ then there are constants c, $\delta > 0$ such that

$$\tag{4.24} \|\Phi_\infty(t)\|_{\mathcal{L}(Y)} \leqslant c \exp[-\delta t] , \qquad t \geqslant 0 ;$$

d) the operator P_∞ defined on Y by (4.7) or (4.8) satisfies the relation

$$P_\infty x = \int_0^{t_0} \exp[A^*\tau] R\Phi_\infty(\tau)x\,d\tau + \exp[A^*t_0]P_\infty\Phi_\infty(t_0)x \qquad x \in Y$$

where t_0 is an arbitrary point $0 < t_0 < \infty$.

PROOF. – The convergence properties a), b) are nothing but restatements of properties (4.10), (4.11) of Lemma 4.2.

Part c). – By (4.21) and (4.0b), we see that $\Phi_\infty(t)$ is strongly continuous on Y. The semigroup property of $\Phi_\infty(t)$ follows from the evolution properties of $\Phi_T(\cdot,\cdot)$: Indeed, with $x \in Y$

$$
\begin{aligned}
(4.26) \qquad \Phi_T(t+\tau, 0)x &= \Phi_T(t+\tau, \tau)\Phi_T(\tau, 0)x \qquad \text{(by Lemma 3.1}b\text{)}\\
&= \Phi_{T-\tau}(t, 0)\Phi_T(\tau, 0)x \qquad \text{(by Lemma 4.1)}\\
&= \Phi_{T-\tau}(t, 0)[\Phi_T(\tau, 0)x - \Phi_\infty(\tau)x] + \Phi_{T-\tau}(t, 0)\Phi_\infty(\tau)x\,.
\end{aligned}
$$

Taking the limit in (4.26) we obtain by virtue of (4.23)

$$\Phi_\infty(t+\tau)x = \Phi_\infty(t)\Phi_\infty(\tau)x$$

as desired, since for t fixed we have that $\Phi_{T-\tau}(t, 0)$ is uniformly bounded in T in $\mathcal{L}(Y)$ by the principle of uniform boundedness. Moreover, if $R > 0$, then (4.0a) implies $\Phi_\infty(t)x \in L_2(0, \infty; Y)$ for all $x \in Y$ and a well-known result [D.1] yields (4.24)

Part d. – From (4.7) and Lemma 3.1 b) we compute with t_0 arbitrary, $0 \leqslant t_0 < T$:

$$
\begin{aligned}
(4.27) \qquad P_\infty x &= \lim_{T\uparrow\infty}\Bigg[\int_0^{t_0}\exp[A^*\tau]R\Phi_T(\tau, 0)x\,d\tau \,+\\
&\quad + \exp[A^*t_0]\int_{t_0}^{T}\exp[A^*(T-t_0)]R\Phi_T(\tau, t_0)\Phi_T(t_0, 0)x\,d\tau\Bigg] =\\
&= \int_0^{t_0}\exp[A^*\tau]R\Phi_\infty(\tau)x\,d\tau + \exp[A^*t_0]\lim_{T\uparrow\infty}P_T(t_0)\Phi_T(t_0, 0)x\,.
\end{aligned}
$$

For the first term in (4.27) we have used (4.23) and the Lebesgue dominated theorem (or else (4.22)), while for the second term in (4.27) we have recalled (3.9). On the other hand

$$
\begin{aligned}
(4.28) \qquad &\lim P_T(t_0)\Phi_T(t_0, 0)x =\\
&\qquad = \lim\{P_T(t_0)[\Phi_T(t_0, 0)x - \Phi_\infty(t_0)x] + P_T(t_0)\Phi_\infty(t_0)x\} = P_\infty\Phi_\infty(t_0)x
\end{aligned}
$$

by (4.23), the uniform boundedness of $P_T(t_0)$ for t_0 fixed, and (4.8). Thus (4.27) and (4.28) yield (4.25). Corollary 4.3 is proved. □

4.2. *Proof of part* (iii) *of Theorem* 2.2.

THEOREM 4.4. – With P_∞ and Φ_∞ defined by (4.7) and (4.21) respectively, we have

$$u^0_\infty(t;0;x) = -B^*P_\infty y^0_\infty(t,0;x) = -B^*P_\infty\Phi_\infty(t)x \qquad x\in Y, \text{ a.e. in } 0\leqslant t\leqslant\infty \tag{4.29}$$

where

$$B^*P_\infty\Phi_\infty(t)\colon \text{ continuous } Y\to L_2(0,\infty;Y) \tag{4.30}$$

PROOF. – Recalling (3.12) and (3.9) we have

$$-u^0_T(t,0;x) = B^*P_T(t)\Phi_T(t,0)x = I_{1T}(t) + I_{2T}(t) \tag{4.31}$$

$$I_{1T}(t) = B^*\int_t^{t_0}\exp[A^*(\tau-t)]\Phi_T(\tau,0)x\,d\tau \tag{4.32}$$

$$I_{2T}(t) = B^*\int_{t_0}^{T}\exp[A^*(\tau-t)]\Phi_T(\tau,0)x\,d\tau \tag{4.33}$$

for some $t<t_0<T$. Thus, by (4.32) and (1.3*b*), we can write

$$I_{1T}(\cdot) = L^*_{0t_0}[\Phi_T(\cdot,0)x] \tag{4.34}$$

In view of the regularity (1.5) of $L^*_{0t_0}$ and of the convergence (4.23), we conclude that

$$\lim_{T\uparrow\infty} I_{1T}(\cdot) = L^*_{0t_0}[\Phi_\infty(\cdot)x] = B^*\int_t^{t_0}\exp[A^*(\tau-t)]\Phi_\infty(\tau)x\,d\tau \tag{4.35}$$

the limit being taken in the $L_2(0,t_0;U)$-sense. As for I_{2T} we have from (4.33), Lemma 3.1 *b*), and (3.9)

$$\begin{aligned} I_{2T}(t) &= B^*\exp[A^*(t_0-t)]\int_{t_0}^{T}\exp[A^*(\tau-t_0)]\Phi_T(\tau,t_0)\Phi_T(t_0,0)x\,d\tau \\ &= B^*\exp[A^*(t_0-t)]P_T(t_0)\Phi_T(t_0,0)x\,. \end{aligned} \tag{4.36}$$

Finally, invoking assumption (H.1) = (1.2) and (4.28), we take the limit in the $L_2(t_0,T;U)$-sense in (4.36) to get

$$\lim_{T\uparrow\infty} I_{2T}(t) = B^*\exp[A^*(t_0-t)]P_\infty\Phi_\infty(t_0)x\,. \tag{4.37}$$

Returning to (4.31), we use (4.35), (4.37) on its right, and (4.10a) on its left. The result is

$$-u^0_\infty(t,0;x) = -B^*\Big[\int_t^{t_0}\exp[A^*(\tau-t)]\Phi_\infty(\tau)x\,d\tau + \exp[A^*(t_0-t)]P_\infty\Phi_\infty(t_0)x\Big]$$

$$(\sigma=\tau-t)\qquad = -B^*\Big[\int_t^{t_0}\exp[A^*\sigma]\Phi_\infty(\sigma)x\,d\sigma + \exp[A^*(t_0-t)]P_\infty\Phi_\infty(t_0-t)\Big]\Phi_\infty(t)x$$

$$(\text{by }(4.25))\qquad = -B^*P_\infty\Phi_\infty(t)x \quad \text{a.e. in } t. \quad \square$$

4.3. *Proof of part* (iv), (v) *of Theorem* 2.2.

Part (iv), DEFINITION 4.1. – Henceforth, we let A_F (F defined on Y) infinitesimal generator of the s.c. semigroup asserted by Corollary 4.3 c), i.e.

$$\Phi_\infty(t) = \exp[A_F t]\,.$$

Thus

$$\frac{d\Phi_\infty(t)x}{dt} = A_F\Phi_\infty(t)x = \Phi_\infty(t)A_F x\,,$$

We show in this section that the operator P_∞ satisfies the Algebraic Riccati Equation. A first step in this direction consists in establishing some regularity properties of the operator B^*P_∞.

LEMMA 4.5. – With P_∞ defined by (4.7) we have:

$$Y \supset \mathscr{D}(B^*P_\infty) \supset \mathscr{D}(A_F)\,. \tag{4.41}$$

Thus, $\mathscr{D}(B^*P_\infty)$ is dense in Y (5). More precisely, for $x\in\mathscr{D}(A_F)$ we have

$$B^*P_\infty x = B^*A^{*-1}\Big[\exp[A^*t_0]R\Phi_\infty(t_0)x - Rx - \int_0^{t_0}\exp[A^*\tau]R\Phi_\infty(\tau)A_F x\,d\tau\Big] + B^*\exp[A^*t_0]P_\infty\Phi_\infty(t_0)x\in U\,, \tag{4.42}$$

where, by a standing assumption, $B^*A^{*-1}\in\mathcal{L}(Y;U)$ and where

$$B^*\exp[A^*t]P_\infty\Phi_\infty(t)x\in L_2(0,T;U)\,,\qquad x\in\mathscr{D}(A_F) \tag{4.43}$$

(5) No similar claim on $\mathscr{D}(B^*P_T(t))$ was made in section 3 for a non regular R satisfying only (H.2) = (1.9); an « ε smoothness » was needed in [L-T.3], [C-L.1].

so that t_0 in (4.42) can be chosen (depending on x) so that the last term in (4.42) is well defined in U (the measure of the set of all such t_0's contained in $[0, T]$ is equal to T).

PROOF. – Note first that for *any* t_1 and *any* $x \in \mathscr{D}(A_F)$ we have after integration by parts

$$(4.44)\qquad B^*\int_0^{t_1} \exp[A^*\tau]R\Phi_\infty(\tau)x\,d\tau = B^*A^{*-1}\int_0^{t_1} A^*\exp[A^*\tau]R\Phi_\infty(\tau)x\,d\tau =$$

$$= B^*A^{*-1}\Big[\exp[A^*t_1]R\Phi_\infty(t_1)x - Rx - \int_0^{t_1}\exp[A^*\tau]R\Phi_\infty(\tau)A_Fx\,d\tau\Big] \in U$$

all terms being well defined on U, since by a standing assumption $B^*A^{*-1} \in \mathcal{L}(Y, U)$. Next we show (4.43). Let $x \in \mathscr{D}(A_F)$ and integrate by parts

$$(4.45)\qquad B^*\exp[A^*t]P_\infty\Phi_\infty(t)x =$$

$$= B^*\exp[A^*t]P_\infty\int_0^t \Phi_\infty(\tau)A_Fx\,d\tau + B^*\exp[A^*t]P_\infty \in L_2(0, T; U)\,.$$

Indeed, a fortiori from assumption (H.1) = (1.2) we have

$$(4.46)\qquad B^*\exp[A^*t]P_\infty x \in L_2(0, T; U)$$

while

$$B^*\exp[A^*t]P_\infty\int_0^t \Phi_\infty(\tau)A_Fx\,d\tau \in L_2(0, T; U)$$

as it follows from Lemma 3.1 of [L-T.3] with $F(t) = B^*\exp[A^*t]$ (which is legal by assumption (H.1)). Thus (4.46)-(4.47) prove (4.45).

Finally, recalling (4.25) and using (4.44)-(4.45), we have for $x \in \mathscr{D}(A_F)$:

$$(4.48)\qquad B^*P_\infty x = B^*\int_0^{t_0}\exp[A^*\tau]R\Phi_\infty(\tau)x\,d\tau + B^*\exp[A^*t_0]P_\infty\Phi_\infty(t_0)x \in U$$

provided t_0 is chosen (depending on x) so that the left hand side of (4.45) is well defined as an element of U. $\square$

We next provide information on A_F.

LEMMA 4.6. – For $x \in Y$ and $t \geqslant 0$

$$\frac{d\Phi_\infty(t)x}{dt} = [A - BB^* P_\infty]\Phi_\infty(t)x \in [\mathscr{D}(A^*)]' . \tag{4.49}$$

Thus, by (4.39)-(4.40):

$$[A - BB^* P_\infty]\Phi_\infty(t)x = A_F \Phi_\infty(t)x = \Phi_\infty(t) A_F x \in Y , \qquad x \in \mathscr{D}(A_F) \quad t > 0 \tag{4.50a}$$

$$[A - BB^* P_\infty]x = A_F x , \quad x \in \mathscr{D}(A_F) \tag{4.50b}$$

PROOF. – Recalling the optimal dynamics (4.0*b*) and the optimal control in (4.29), we have

$$\begin{aligned}(\Phi_\infty(t)x, z) &= \\ &= (\exp[At]x, z) - \left(A\int_0^t \exp[A(t-\tau)]A^{-1}BB^* P_\infty \Phi_\infty(\tau)x\, d\tau, z\right), \qquad z \in Y\end{aligned} \tag{4.51}$$

We next differentiate (4.51) in t with $x \in Y$ and $z \in \mathscr{D}(A^*)$

$$\begin{aligned}\left(\frac{d\Phi_\infty(t)x}{dt}, z\right) &= (\exp[At]x, A^* z) - (BB^* P_\infty \Phi_\infty(t)x, z) - \\ &\quad - \left(A\int_0^t \exp[A(t-\tau)]A^{-1}BB^* P_\infty \Phi_\infty(\tau)x\, d\tau, A^* z\right)\end{aligned} \tag{4.52}$$

where the second term on the right of (4.52), being equal to $(A^{-1}BB^* P_\infty \Phi_\infty(t)x, A^* z)$, is well defined a.e. in t by the standing assumption $A^{-1}B \in \mathcal{L}(U; Y)$, coupled with (4.30). We next solve (4.51) for $(\exp[At]x, z)$, replace here z by $A^* z$ and substitute into (4.52) thereby obtaining (4.49). $\square$

Part (v). – Further regularity properties of P_∞ are given next.

LEMMA 4.7. – With P_∞ and A_F defined by (4.7) and (4.39), we have

a) $A^* P_\infty \in \mathcal{L}(\mathscr{D}(A_F); Y)$

b) $A_F^* P_\infty \in \mathcal{L}(\mathscr{D}(A); Y)$.

PROOF. – We shall use once more relation (4.25) for P_∞:

Part a). – For $x \in \mathscr{D}(A_F)$ we apply A^* to both sides of (4.25).

$$A^* P_\infty x = A^* \int_0^{t_0} \exp[A^* \tau] R \Phi_\infty(\tau)x\, d\tau + A^* \exp[A^* t_0] P_\infty \Phi_\infty(t_0)x . \tag{4.53}$$

But, integrating by parts, we obtain for any t_0

$$(4.54)\qquad A^*\int_0^{t_0}\exp[A^*\tau]R\Phi_\infty(\tau)x\,d\tau = \\ = \exp[A^*t_0]R\Phi_\infty(t_0)x - Rx - \int_0^{t_0}\exp[A^*\tau]R\Phi_\infty(\tau)A_F x\,d\tau \in Y\,.$$

As to the second term on the right of (4.53), we have again after integration by parts in t_0:

$$(4.55)\qquad \int_0^T A^*\exp[A^*t_0]P_\infty\Phi_\infty(t_0)x\,dt_0 = \\ = \exp[A^*T]P_\infty\Phi_\infty(T)x - P_\infty x - \int_0^T \exp[A^*t_0]P_\infty\Phi_\infty(t_0)A_F x\,dt_0 \in Y\,.$$

Thus, a fortiori integrating (4.53) in the variable t_0 over the interval $[0, T]$ and using (4.54)-(4.55), leads to

$$(4.56)\qquad T(A^*P_\infty x)\in Y$$

and the desired conclusion of part *a*) follows via the closed graph theorem (or direct estimates based on the identity obtained through the procedure described above).

Part b). – By duality, it suffices to show

$$(4.57)\qquad P_\infty A_F \in \mathcal{L}(Y; [\mathcal{D}(A)]')\,.$$

To this end, we take $x\in Y$, $z\in\mathcal{D}(A)$, and compute via (4.25)

$$(4.58)\qquad (P_\infty A_F x, z) = \left(\int_0^{t_0}\exp[A^*\tau]R\Phi_\infty(\tau)A_F x\,d\tau, z\right) + \left(\exp[A^*t_0]R\Phi_\infty(t_0)A_F x, z\right)\,.$$

As to the first term on the right of (4.58), we obtain after integration by parts via (4.40)

$$(4.59)\qquad \left(\int_0^{t_0}\exp[A^*\tau]R\Phi_\infty(\tau)A_F x\,d\tau, z\right) = \left(\exp[A^*t_0]R\Phi_\infty(t_0)x - Rx, z\right) - \\ - \left(\int_0^{t_0}\exp[A^*\tau]R\Phi_\infty(\tau)x\,d\tau, Az\right) = \text{well-defined}\,.$$

As to the second term on the right of (4.58), again after integration by parts via (4.40)

$$(4.60)\qquad \int_0^T (\exp[A^* t_0] R\Phi_\infty(t_0) A_F x, z)\, dt_0 = (\exp[A^* T] R\Phi_\infty(T) x - Rx, z) - \\ - \int_0^T (\exp[A^* t_0] R\Phi_\infty(t_0) x, Az)\, dt = \text{well-defined.}$$

Thus, a fortiori, integrating (4.58) in the variable t_0 over $[0, T]$ and using (4.59)-(4.60), leads to

$$T(P_\infty A_F x, z) = \text{well-defined}, \qquad x \in Y, \quad z \in \mathscr{D}(A).$$

Thus $P_\infty A_F\colon Y \to [\mathscr{D}(A)]'$ and conclusion *b*) follows. □

A more precise version of Lemma 4.7—which however uses Lemma 4.7—follows next.

LEMMA 4.8. – With P_∞ defined by (4.7) we have

$$(4.61a)\qquad -A^* P_\infty x = Rx + P_\infty A_F x \in Y, \; x \in \mathscr{D}(A_F)$$

$$(4.62b)\qquad -A_F^* P_\infty x = Rx + P_\infty Az \in Y, \; z \in \mathscr{D}(A)$$

PROOF. – From (4.25)

$$(4.63)\qquad (P_\infty x, z) = \Big(\int_0^{t_0} \exp[A^* \tau] R\Phi_\infty(\tau) x\, d\tau, z\Big) + (\exp[A^* t_0] P_\infty \Phi_\infty(t_0) x, z).$$

We differentiate (4.63) in t_0 with $x \in \mathscr{D}(A_F)$ and $z \in \mathscr{D}(A)$:

$$0 = (\exp[A^* t_0] R\Phi_\infty(\tau) x, z) + (\exp[A^* t_0] P_\infty \Phi_\infty(t_0) x, Az) + \\ + (\exp[A^* t_0] P_\infty \Phi_\infty(t_0) A_F x, z).$$

Taking $t_0 = 0$ yields

$$(6.64)\qquad 0 = (Rx, z) + (P_\infty x, Az) + (P_\infty A_F x, z), \qquad x \in \mathscr{D}(A_F), \quad z \in \mathscr{D}(A).$$

Using the a-priori regularity of Lemma 4.7, we can extend the above inner products by continuity to all of $z \in Y$ with $x \in \mathscr{D}(A_F)$; and to all of $x \in Y$ with $z \in \mathscr{D}(A)$. This leads to (4.61) and (4.62), respectively. □

COROLLARY 4.9. – With P_∞ defined by (4.7), we have:

$$\langle B^* P_\infty x, B^* P_\infty z\rangle = \text{well-defined for } x, z \in \mathscr{D}(A), \quad \text{or else for } x, z \in \mathscr{D}(A_F) \tag{4.65}$$

Thus (recall also Lemma 4.5)

$$B^* P_\infty \in \mathfrak{L}(\mathscr{D}(A);\, U) \cap \mathfrak{L}(\mathscr{D}(A_F);\, U) \tag{4.66}$$

PROOF. – For $x \in \mathscr{D}(A_F)$, using (4.50*b*)

$$(P_\infty A_F x, z) = (P_\infty[A - BB^* P_\infty]x, z) = (P_\infty A x, z) - \langle B^* P_\infty x, B^* P_\infty z\rangle$$

i.e.

$$-\langle B^* P_\infty x, B^* P_\infty z\rangle = (P_\infty A_F x, z) - (P_\infty A x, z) \tag{4.67a}$$

$$= (x, A_F^* P_\infty z) - (x, A^* P_\infty z)\,. \tag{4.67b}$$

The first term on the right of (4.67*a*) is well defined for

$$x \in \mathscr{D}(A_F) \quad \text{and } z \in Y\,.$$

or else:

$$x \in Y \quad \text{and } z \in \mathscr{D}(A)\,, \quad \text{by Lemma 4.7 } b) \text{ applied to (4.67}b)\,.$$

The second term on the right of (4.67*a*) is well defined for

$$x \in \mathscr{D}(A) \quad \text{and } z \in Y$$

or else:

$$x \in Y \quad \text{and } z \in \mathscr{D}(A_F)\,, \quad \text{by Lemma 4.7 } a) \text{ applied to (4.67}b)\,.$$

Thus, conclusion (4.65) follows; more precisely

$$|\langle B^* P_\infty x, B^* P_\infty z\rangle| \leq C_T \begin{cases} \|Ax\|\,\|Az\|\,, & x, z \in \mathscr{D}(A) \\ \|A_F x\|\,\|A_F z\|\,, & x, z \in \mathscr{D}(A_F)\,. \end{cases}$$

We finally obtain the ultimate goal of our analysis in this section.

THEOREM 4.10. – The operator P_∞ defined by (4.7) satisfies the following Algebraic Riccati Equation.

$$(P_\infty x, Az) + (P_\infty A x, z) + (Rx, z) = \langle B^* P_\infty x, B^* P_\infty z\rangle \tag{4.68}$$

$$\text{for all } x, z \in \mathscr{D}(A); \text{ or else for all } x, z \in \mathscr{D}(A_F)$$

PROOF. – We combine Lemma 4.8 and Corollary 4.9. □

5. – The case $T = \infty$. Proof of Theorem 2.3. Algebraic Riccati equation: uniqueness.

In order to prove the uniqueness of Theorem 2.3, we need two preliminary Lemmas.

LEMMA 5.1. – Consider the dynamics (1.1) under the standing assumption (H.1) = (1.2). Let $\Pi\colon Y \supset \mathscr{D}(\Pi) \to Y$ be an operator satisfying

$$\|\Pi^* x\|^2 \leqslant C[|B^* x|^2 + \|x\|^2], \quad \forall x \in \mathscr{D}(B^*) \subset Y \tag{5.1}$$

so that $A^{-1}\Pi \in \mathcal{L}(Y)$, as remarked in section 2.3, Remark 2.2 *b*.
Then:

a) the perturbed closed operator

$$A_\Pi^* = \overline{A^* + \Pi^*} \tag{5.2}$$

generates a s.c. semigroup $\exp[A_\Pi^* t]$ on Y, $t \geqslant 0$.

b) Moreover, the operator $B^* \exp[A_\Pi^* t]$ admits a continuous extension—denoted by the same symbol (6)—such that

$$\begin{aligned} &(a)\ B^* \exp[A_\Pi^* t]\colon \text{ continuous } Y \to L_2(0, T; Y),\ T < \infty \\ &(b)\ \int_0^T |B^* \exp[A_\Pi^* t] x|^2\, dt \leqslant C_T \|x\|^2,\ x \in Y,\ C_T > 0. \end{aligned} \tag{5.3}$$

PROOF. – Consider the integral equation

$$w(t) = \exp[A^* t]x + \int_0^t \exp[A^*(t - \tau)]\Pi^* w(\tau)\, d\tau, \qquad x \in Y \tag{5.4a}$$

in the Y-valued unknown $w(t) = w(t, 0; x)$, formally corresponding to the problem

$$\dot{w} = (A^* + \Pi^*) w, \qquad w(0) = x \tag{5.4b}$$

Define the operator $\mathscr{F}$ by setting

$$(\mathscr{F} f)(t) \equiv \exp[A^* t]x + \int_0^t \exp[A^*(t - \tau)]\Pi^* f(\tau)\, d\tau, \qquad x \in Y. \tag{5.5}$$

(6) This will not be repeated.

We claim that $\mathscr{F}$ is well-defined as an operator $L_2(0, T; \mathscr{D}(B^*)) \to$ itself, where

$$\|z\|^2_{\mathscr{D}(B^*)} \equiv |B^* z|^2 + \|z\|^2, \qquad z \in \mathscr{D}(B^*) \subset Y. \tag{5.6}$$

Indeed, the term $\exp[A^* t]x$ is in $L_2(0, T; \mathscr{D}(B^*))$ by the standing assumption (H.1) = (1.2). As to the integral term in (5.5), setting $v = f_1 - f_2 \in L_2(0, t_0; \mathscr{D}(B^*))$, we compute for $\mathscr{F}f_1 - \mathscr{F}f_2 = \mathscr{F}v$, via (5.5); Schwarz inequality and a change in the order of integration:

$$\int_0^t |B^*(\mathscr{F}v)(t)|^2\, dt = \int_0^{t_0} \left| \int_0^{t_0} B^* \exp[A^*(t-\tau)] \Pi^* v(\tau)\, d\tau \right|^2 dt \leqslant$$

$$\leqslant t_0 \int_0^t \int_0^t |B^* \exp[A^*(t-\tau)] \Pi^* v(\tau)|^2\, d\tau\, dt = t_0 \int_0^{t_0} \int_\tau^t |B^* \exp[A^*(t-\tau)] \Pi^* v(\tau)|^2\, dt\, d\tau =$$

$$= t_0 \int_0^{t_0} \int_0^{t_0-\tau} |B^* \exp[A^* \sigma] \Pi^* v(\tau)|^2\, dt\, d\tau$$

(extending $\int_0^{t_0-\tau}$ to $\int_0^{t_0}$ and using first assumption (H.1) = (1.2) and then assumption (5.1))

$$\leqslant t_0 C_{t_0} \int_0^{t_0} \|\Pi^* v(\tau)\|^2\, d\tau \leqslant t_0 C_{t_0} C \int_0^{t_0} |B^* v(\tau)|^2 + \|v(\tau)\|^2\, d\tau = t_0 C_{t_0} C \|v\|^1_{L_2(0,t_0;\mathscr{D}(B^*))} \tag{5.7}$$

where C_{t_0} and C are the constants in (1.2) and (5.1), respectively. Thus, using (5.7), (5.5) and (5.1)

$$\|\mathscr{F}v\|^2_{L_2(0,t_0;\mathscr{D}(B^*))} = \int_0^{t_0} |B^*(\mathscr{F}v)(t)|^2 + \|(\mathscr{F}v)(t)\|^2\, dt \leqslant t_0 C C_{1t_0} \|v\|^2_{L_2(0,t_0;\mathscr{D}(B^*))} \tag{5.8}$$

where $C_{1t_0} = C_{t_0} + M^2_{t_0}$, $\|\exp[A^* t]\|_{\mathfrak{L}(Y)} \leqslant M_{t_0}$, $0 \leqslant t \leqslant t_0$. Taking t_0 sufficiently small so that $t_0 C C_{1t_0} < 1$, we get that $\mathscr{F}$ is a contraction on $L_2(0, t_0; \mathscr{D}(B^*))$. Hence, the integral equation (5.4*a*) has a unique solution $w(t) = w(t, 0; x)$ such that $B^* w(t, 0; x) \in L_2(0, t_0; U)$, $x \in Y$. Indeed, using this result and the convolution theorem on the integral (5.4*a*) via (5.1), we can improve the regularity of this solution to read $w(t, 0; x) \in C([0, t_0]; Y)$, $x \in Y$. For any preassigned $T < \infty$, we can then repeat the preceding procedure a finite number of time as, say in [Da P-L-T.1] and conclude that problem (5.4) admits a unique global solution

$$w(t, 0; x) \in C([0, T]; Y) \text{ such that } B^* w(t, 0; x) \in L_2(0, T; U)$$

for any $x \in Y$. Moreover, one verifies that $w(t, x)$ satisfies the semi-group property. Hence, we can write $w(t, 0; x) \equiv S(t)x$, with $S(t)$ a s.c. semigroup on Y. Then, finally, $w(t, 0; x) = \exp[A_\Pi^* t]x$ by [ ]. Both parts *a*) and *b*) are proved. $\square$

LEMMA 5.2. – As in Lemma 5.1, consider the dynamics (1.1) under the standing assumption (H.1) = (1.2) and let Π be an operator satisfying assumption (5.1). Thus, Lemma 5.1 guarantees that $\exp[A_\Pi^* t]$ is a s.c. semi-group on Y. Assume in addition that Π is such that the semi-group $\exp[A_\Pi^* t]$ is uniformly stable on Y; i.e. there are M, $\alpha > 0$ such that

$$\|\exp[A_\Pi^* t]\|_{\mathcal{L}(Y)} \leqslant M \exp[-\alpha t], \quad t \geqslant 0. \tag{5.9}$$

Then, conclusion (5.3*b*) of Lemma 5.1 can be strengthened as follows

$$\int_0^\infty |B^* \exp[(A_\Pi^* + \varepsilon I)t]x|^2 dt \leqslant C_\varepsilon \|x\|^2, \qquad \begin{array}{l}\text{for all } x \in Y \\ \text{for all } \varepsilon > 0 \text{ s.t. } -\alpha + \varepsilon < 0\end{array} \tag{5.10}$$

PROOF. – We already know that (5.3*b*) holds true for any $T < \infty$. Thus, we compute using the semi-group property and (5.3*b*).

$$\begin{aligned}\int_T^{2T} |B^* \exp[A_\Pi^* t]x|^2 dt &= \int_T^{2T} |B^* \exp[A_\Pi^*(t - T)] \exp[A_\Pi^* T]x|^2 dt = \\ &= \int_0^T |B^* \exp[A_\Pi^* \sigma] \exp[A_\Pi^* T]x|^2 d\sigma \leqslant C_T \|\exp[A_\Pi^* T]x\|^2.\end{aligned} \tag{5.11}$$

Generally, using (5.9)

$$\int_{(n-1)T}^{nT} |B^* \exp[A_\Pi^* t]x|^2 dt \leqslant C_T \|\exp[A_\Pi^*(n-1)T]x\|^2 \leqslant C_T M \exp[-2\alpha(n-1)T]\|x. \tag{5.12}$$

Thus, choosing T large enough so that $\exp[-2\alpha T] < 1$, we have for $\|x\| \leqslant 1$:

$$\begin{aligned}\int_0^\infty |B^* \exp[A_\Pi^* t]x|^2 dt &= \sum_{n=1}^\infty \int_{(n-1)T}^{nT} |B^* \exp[A_\Pi^* t]x|^2 dt \leqslant \\ &\leqslant C_T M \sum_{n=1}^\infty \exp[-2\alpha(n-1)T] = C_T \frac{M}{1 - \exp[-2\alpha T]} < \infty\end{aligned} \tag{5.13}$$

which shows (5.10) for $\varepsilon = 0$. The proof for $\varepsilon > 0$ is exactly the same since $-\alpha + \varepsilon < 0$. $\square$

The main result of the present section is Theorem 2.3, which we reformulate here as

THEOREM 5.3. – Consider the optimal control problem O.C.P.(∞) (1.6) for the dynamics (1.1) subject to the following standing assumptions:

(H.1) = (1.2) for the dynamics

(H.2) = (1.9) for the observation operator R

(H.3) = (2.7) for the Finite Cost Condition.

Then, section 4 yields that the operator P_∞ defined by (4.7) is a solution of the Algebraic Riccati Equation (4.68) with properties specified in that section, in particular in (4.66) and Theorem 4.10.

Assume, in addition, (H.4) = (2.17); i.e. let $K\colon Y \supset \mathscr{D}(K) \to Y$ be a (linear) operator satisfying the following two conditions:

(5.14) i) $\|K^*x\|^2 \leqslant C[|B^*x|^2\| + \|x\|^2]$, for all $x \in \mathscr{D}(B^*) \subset Y$;

ii) the s.c. semi-group $\exp[A_k t]$ generated by the operator

$$A_K = A + KR^{\frac{1}{2}} \tag{5.15}$$

(as guaranteed by Lemma 5.1 with $\Pi = KR^{\frac{1}{2}}$) is uniformly stable; i.e. there are M_1, $k > 0$ such that

$$\|\exp[A_K t]\|_{\mathcal{L}(Y)} \leqslant M_1 \exp[-kt]\,, \qquad t \geqslant 0\,. \tag{5.16}$$

Then the solution to the A.R.E. (4.68) for $x, z \in \mathscr{D}(A_F)$ is unique within the class of linear self-adjoint operators $P \in \mathcal{L}(Y)$ such that $B^*P \in \mathcal{L}(\mathscr{D}(A_F); Y)$, a condition satisfied by the solution operator P_∞ in (4.7), in view of Lemma 4.7.

PROOF. – Let $P_1 \in \mathcal{L}(Y)$ be another self-adjoint solution with $B^*P_1 \in \mathcal{L}(\mathscr{D}(A_F); Y)$. Following standard arguments given e.g. in [B.1, pp. 272-273], in order to show that in fact $P_1 = P_\infty$, it suffices that, under the present assumptions, the semigroup $\Phi_\infty(t)$ of Corollary 4.3c) is uniformly stable:

$$\|\Phi_\infty(t)\|_{\mathcal{L}(Y)} \leqslant C \exp[-\delta t]\,, \qquad t \geqslant 0\,, \qquad \delta > 0;$$

equivalently [D.1] that

$$\int_0^\infty \|\Phi_\infty(t)x\|^2\, dt < \infty \qquad \text{for all } x \in Y\,. \tag{5.17}$$

It then suffices to prove (5.17). To this end, recalling Lemma 4.6, (4.49), we have for $x \in Y$

$$\frac{d\Phi_\infty(t)x}{dt} = (A + KR^{\frac{1}{2}})\Phi_\infty(t)x - DR^{\frac{1}{2}}\Phi_\infty(t)x - BB^*P_\infty\Phi_\infty(t)x \tag{5.18a}$$

on $[\mathscr{D}(A^*)]'$. Using Lemma 5.1 *a*), we write the integral version of (5.18*a*)

$$\Phi_\infty(t)x = \exp[A_K t]x - \int_0^t \exp[A_K(t-\tau)]KR^{\frac{1}{2}}\Phi_\infty(\tau)x\,d\tau - \\ - \int_0^t \exp[A_K(t-\tau)]BB^*P_\infty\Phi_\infty(\tau)x\,d\tau$$

By assumption (5.16), we have

$$\int_0^\infty \|\exp[A_K t]x\|^2\,dt \leqslant C\|x\|^2, \quad x \in Y \tag{5.19}$$

To estimate the last two terms on the right of (5.18*b*), we employ Lemmas 5.1 and 5.2 with $\Pi = KR^{\frac{1}{2}}$, $\Pi^* = R^{\frac{1}{2}}K^*$, whose legitimacy is guaranteed by assumption (5.14) on K^*. The conclusion (5.10) of Lemma 5.2 specialized to the present case is

$$\int_0^\infty |B^*\exp[(A_K^* + \varepsilon I)t]x|^2\,dt \leqslant C_\varepsilon\|x\|^2, \quad x \in Y \qquad \text{for all } \varepsilon > 0 \text{ s.y. } -k + \varepsilon < 0\,. \tag{5.20}$$

This, combined with assumptions (5.14) and (5.16) yields likewise

$$\int_0^\infty \|K^*\exp[(A_K^* + \varepsilon I)t]x\|^2\,dt \leqslant C_\varepsilon\|x\|^2, \quad x \in Y \qquad \text{for all } \varepsilon > 0 \text{ s.t. } -k + \varepsilon < 0 \tag{5.21}$$

To complete the proof of Theorem 5.3 we must show that

$$\left.\begin{aligned} (L_{KB}g)(t) &= \int_0^t \exp[A_K(t-\tau)]Bg(\tau)\,d\tau \\ (L_{KK}f)(t) &= \int_0^t \exp[A_K(t-\tau)]Kf(\tau)\,d\tau \end{aligned}\right\} : \text{continuous} \quad \left.\begin{aligned} &L_2(0, \infty; U) \\ &L_2(0, \infty; Y) \end{aligned}\right\} \to L_2(0, \infty; Y) \tag{5.22}$$

or, equivalently, that

$$
\left.\begin{aligned}
(L_{KB}^{*}v)(t) &= \int_t^{\infty} B^{*}\exp\,[A_K^{*}(\tau-t)]\,v(\tau)\,d\tau \\
(L_{KK}^{*}\varphi)(t) &= \int_t^{\infty} K^{*}\exp\,[A_K^{*}(\tau-t)]\varphi(\tau)\,d\tau
\end{aligned}\right\}:\ \text{continuous } L_2(0,\infty;Y)\to \tag{5.23}
$$

$$
\to\left\{\begin{aligned} &L_2(0,\infty;U) \\ &L_2(0,\infty;Y)\,. \end{aligned}\right.
$$

Here, the expression for L_{KB}^{*} is obtained from

$$
\begin{aligned}
(L_{KB}\,g, v)_{L_2(0,\infty;Y)} &= \int_0^{\infty}\big((L_{KB}\,g)(t), v(t)\big)\,dt = \int_0^{\infty}\int_0^{t}\big(\exp\,[A_K(t-\tau)]Bg(\tau), v(t)\big)\,d\tau\,dt = \\
&= \int_0^{\infty}\int_0^{t}\langle g(\tau), B^{*}\exp\,[A_K^{*}(t-\tau)]v(t)\rangle\,d\tau\,dt = \int_0^{\infty}\int_\tau^{\infty}\langle g(\tau), B^{*}\exp\,[A_K^{*}(t-\tau)]v(t)\rangle\,dt\,d\tau \\
&= \int_0^{\infty}\langle g(\tau), \int_\tau^{\infty} B^{*}\exp\,[A_K^{*}(t-\tau)]v(t)\,dt\rangle\,d\tau = \int_0^{\infty}\langle g(\tau), (L_{KB}^{*}v)(\tau) > d\tau
\end{aligned}
$$

and similarly for L_{KK}^{*}.

To show (5.23*a*) we compute with $\varepsilon > 0$:

$$
\begin{aligned}
\int_0^{\infty}\|(L_{KB}^{*}v)(t)\|^2\,dt &= \int_0^{\infty}\left\|\int_t^{\infty} B^{*}\exp\,[A_K^{*}(\tau-t)]v(\tau)\,d\tau\right\|^2 dt = \\
&= \int_0^{\infty}\left\|\int_t^{\infty}\|\exp\,[-\varepsilon(\tau-t)]B^{*}\exp\,[(A_K^{*}+\varepsilon I)(\tau-t)]v(\tau)\,d\tau\right\|^2 dt \\
\text{(by Schwarz ineq.)}\quad &\leqslant \int_0^{\infty}\left(\int_t^{\infty}\exp\,[-2\varepsilon(\tau-t)]\,d\tau\right)\cdot \\
&\qquad\cdot\left(\int_t^{\infty}\|B^{*}\exp\,[(A_K^{*}+\varepsilon I)(\tau-t)]v(\tau)\|^2\,d\tau\right)dt \\
&= \frac{1}{2\varepsilon}\int_0^{\infty}\int_t^{\infty}\|B^{*}\exp\,[(A_K^{*}+\varepsilon I)(\tau-t)v(\tau)\|^2\,d\tau\,dt
\end{aligned} \tag{5.25}
$$

(change order of integration) $$= \frac{1}{2\varepsilon} \int_0^\infty \int_0^\tau \|B^* \exp[(A_K^* + \varepsilon I)(\tau - t)] v(\tau)\|^2 \, dt \, d\tau$$

$(\tau - t = \sigma)$ $$= \frac{1}{2\varepsilon} \int_0^\infty \int_0^\tau \|B^* \exp[(A_K^* + \varepsilon I)\sigma] v(\tau)\|^2 \, d\sigma \, d\tau$$

$$\leqslant \frac{1}{2\varepsilon} \int_0^\infty \int_0^\infty \|B^* \exp[(A_K^* + \varepsilon I)\sigma] v(\tau)\|^2 \, d\sigma \, d\tau$$

$$\leqslant \frac{C_\varepsilon}{2\varepsilon} \int_0^\infty \|v(\tau)\|^2 \, d\tau < \infty \tag{5.26}$$

where in going from (5.25) to (5.26) we have taken $\varepsilon > 0$ sufficiently small and used (5.20). A similar computation gives

$$\int_0^\infty \|(L_{KK}\varphi)(t)\|^2 \, dt \leqslant \frac{C_\varepsilon}{2\varepsilon} \|\varphi\|^2_{L_2(0,\infty;\,Y)} \tag{5.27}$$

using now (5.21). Thus, (5.26)-(5.27) prove (5.23*a*-*b*), as desired. □

6. – Proof of Theorem 2.4. Isomorphism of $P_T(t)$, $0 \leqslant t < T$ and of P_∞, and exact controllability of the pair $\{A^*, R^{\frac{1}{2}}\}$.

DEFINITION 6.1. – The dynamical system of Y

$$\dot{z}(t) = A^* z(t) + R^{\frac{1}{2}} g(t)\,, \qquad z(0) = 0 \tag{6.0}$$

in short, the pair $\{A^*, R^{\frac{1}{2}}\}$ is exactly controllable on the space Y over the time interval $[0, T]$ with controls $g \in L_2(0, T; Y)$ in case the totality of all solutions points $z(T)$ fills all of Y when g runs over $L_2(0, T; Y)$. Equivalently, in case

$$S_T\colon L_2(0, T; Y) \xrightarrow{\text{onto}} Y \tag{6.1}$$

where S_T is the (obviously bounded) solution operator

$$S_T g = \int_0^T \exp[A^*(T - t)] R^{\frac{1}{2}} g(t) \, dt\,. \quad \square \tag{6.2}$$

Another equivalent formulation is (as is well known): there is $C_T > 0$ such that

$$\|S_T^* y\|_{L_2(0,T;\,Y)} \geq C_T \|y\| \qquad \forall y \in Y. \tag{6.3}$$

But

$$(S^* y)(t) = R^{\frac{1}{2}} \exp [A(T-t)] y. \tag{6.4}$$

Thus, writing (6.3) explicitly, we have that the pair $\{A^*, R^{\frac{1}{2}}\}$ is exactly controllable in the sense of Definition 6.1 in case

$$\int_0^T \|R^{\frac{1}{2}} \exp [At] y\|^2 \, dt \geq C_T^2 \|y\|^2, \qquad \forall y \in Y. \tag{6.5}$$

Having recalled the above well-known facts for problem (6.0), we now unveil a relationship between exact controllability on Y over $[0, T]$ of the pair $\{A^*, R^{\frac{1}{2}}\}$ and the property that $P_T(t)$, $0 \leq t < T$ and/or P_∞ be isomorphisms of $\mathcal{L}(Y)$.

LEMMA 6.1. – Consider the dynamics (1.1) under the standing assumption (H.1) = [1.2), and let the observation operator R satisfy the standing assumption (H.2) = (1.9).

a) Case $T < \infty$. – The pair $\{A^*, R^{\frac{1}{2}}\}$ is exactly controllable on Y over $[0, T-t]$, $t < T$, in the sense of Definition 6.1 if and only if the operator $P_T(t) \in \mathcal{L}(Y)$ defined by (3.9) is an isomorphism on Y, for some t, $0 \leq t < T$.

b) *Case* $T = \infty$. – Assume further the Finite Cost Condition (H.3) so that the operator $P_\infty \in \mathcal{L}(Y)$ can be defined as in (4.7). Then, if the pair $\{A^*, R^{\frac{1}{2}}\}$ is exactly controllable on Y over some interval $[0, T]$, $T < \infty$ in the sense of Definition 6.1, then P_∞ is an isomorphism on Y.

PROOF. *Part a*). – We first show the claimed equivalence for $P_T(0)$.

If: we return to (3.21) with $u_T^0(t) = u_T^0(t, 0; x)$, $y_T^0(t) = y_T^0(t, 0; x)$

$$(P_T(0)x, x) = J_T^0 = J_T(u_T^0(t), y_T^0(t)) \geq \int_0^T (Ry_T^0(t), y_T^0(t)) \, dt =$$
$$= \int_0^T \| \{R^{\frac{1}{2}}[I + L_{0T} L_{0T}^* R]^{-1}[\exp [A \cdot] x]\}(t)\|^2 \, dt$$

where in the last step we have used the explicit representation (3.2*a*). Writing throughout L for L_{0T} we have

$$R^{\frac{1}{2}}[I + LL^*R]^{-1} = [(I + LL^*R)R^{\frac{1}{2}}]^{-1} = [R^{-\frac{1}{2}} + LL^*R^{\frac{1}{2}}]^{-1} = \tag{6.7a}$$
$$= [R^{-\frac{1}{2}}(I + R^{\frac{1}{2}} LL^* R^{\frac{1}{2}})]^{-1} = [I + R^{\frac{1}{2}} LL^* R^{\frac{1}{2}}]^{-1} R^{\frac{1}{2}}$$

where, a fortiori from the regularity property (1.4) of L—a consequence of (H.1) = = (1.2)—we have

$$\|I + R^{\frac{1}{2}}LL^*R^{\frac{1}{2}}\|_{\mathcal{L}(L_2(0,T;Y))} \leqslant C_T . \tag{6.7b}$$

Thus, using (6.7) in (6.6),

$$(P_T(0)x, x) \geqslant C_T \int_0^T \|R^{\frac{1}{2}} \exp[At]x\|^2 \, dt . \tag{6.8}$$

Thus, if the pair $\{A^*, R^{\frac{1}{2}}\}$ is exactly controllable on $[0, T]$, then characterization (6.5) applies and (6.8) yields

$$(P_T(0)x, x) \geqslant \text{const}_T \|x\|^2 , \quad \text{const}_T > 0 , \quad x \in Y \tag{6.9}$$

so that $P_T^{-1}(0) \in \mathcal{L}(Y)$ as desired. A similar argument shows that $P_T^{-1}(t) \in \mathcal{L}(Y)$, $0 \leqslant t < T$, under exact controllability on $[0, T-t]$.

Only if. Let (6.9) be true. Using again (3.21) we have from (6.9)

$$\int_0^T (R \exp[At]x, \exp[At]x) \, dt = J_T(u \equiv 0, y = \exp[At]x) \geqslant J_T(u_T^0, y_T^0) = \\ = (P_T(0)x, x) \geqslant \text{const}_T \|x\|^2 , \quad x \in Y \tag{6.10}$$

and by the characterization (6.5) we conclude that $\{A^*, R^{\frac{1}{2}}\}$ is exactly controllable on Y over $[0, T]$. A similar argument applies if $P_T(t)$, $0 < t < T$, rather than $P_T(0)$ is assumed an isomorphism on Y.

Part b). – Part *b*) follows from

$$J_\infty^0 = (P_\infty x, x) \geqslant (P_T(0)x, x) = J_T^0 \quad x \in Y \tag{6.11}$$

(see (4.9), (4.13)) and the « if » direction of part. *a*).

REMARK 6.1. – One should note that, at the level of (6.7*b*), the « if » argument in Lemma 6.1 *a*) does not use the full strength of assumption (H.1) = (1.2) of the dynamics (1.1)—which guarantees $L_{0T} \in \mathcal{L}(L_2(0, T; U); C[0, T]; Y)$. (See (1.4))—but rather the *weaker* property that (*): $L_{0T} \in \mathcal{L}(L_2(0, T; U); L_2(0, T; Y))$. This latter property (*) is satisfied also by parabolic equations with, say, Dirichlet boundary control in $L_2(0, T; L_2(\Gamma))$, $U = L_2(\Gamma)$, $Y = L_2(\Omega)$, —the parabolic counterpart of case *A*) in the Appendix 2—, which, however, fails to satisfy (1.4). But in the parabolic case, the operator A (equivalently, A^*) generates a s.c., analytic semigroup on $Y = L_2(\Omega)$ and thus, the pair $\{A^*, R^{\frac{1}{2}}\}$ cannot be exactly controllable on $L_2(\Omega)$ of *any* finite interval $[0, T]$, $T < \infty$. We conclude that in the described parabolic case, the corresponding operators $P_T(t)$, $0 \leqslant t < T$ cannot be isomorphisms on $L_2(\Omega)$. Indeed, the Riccati theory for this parabolic case shows that $P_T(t)$ (as well as P_∞) are smoothing, compact operators on $L_2(\Omega)$: see e.g. [L-T.5], [L-T.12].

LEMMA 6.2. – In addition to the standing hypotheses of Lemma 6.1, assume that the pair $\{A^*, R^{\frac{1}{2}}\}$ is exactly controllable on some $[0, T]$ on Y, so that Lemma 6.1 guarantees that $Q_\infty \equiv P_\infty^{-1} \in \mathcal{L}(Y)$. Then Q_∞ satisfies the following Dual Algebraic Riccati equation

$$(AQ_\infty x, z) + (Q_\infty A^* x, z) + (RQ_\infty x, Q_\infty z) = \langle B^* x, B^* z\rangle \tag{6.12}$$

for all $x, z \in \mathscr{D}(A^*) \subset \mathscr{D}(B^*) \subset Y$.

PROOF. – First we show that the DARE (6.12) holds true for all $x, z \in \mathscr{D}$, where $\mathscr{D}$ is the subspace of Y defined by

$$\mathscr{D} \equiv P_\infty \mathscr{D}(A_F) = \{d \in Y : d = P_\infty \bar{d}, \bar{d} \in \mathscr{D}(A_F)\} ,$$

$\mathscr{D}(A_F)$ defined in turn by (4.39) or (4.50*b*), which satisfies $\mathscr{D} \subset \mathscr{D}(A^*)$, and $\mathscr{D}$ is dense in Y. In fact, let $x, y \in \mathscr{D}$ so that $\bar{x} = P_\infty^{-1} x$, $\bar{z} = P_\infty^{-1} z \in \mathscr{D}(A_F)$. Therefore, $\bar{x}, \bar{z}$ satisfy the Algebraic Riccati Equation (4.68), i.e.

$$(P_\infty P_\infty^{-1} x, AP_\infty^{-1} z) + (P_\infty A P_\infty^{-1} x, P_\infty^{-1} z) + (RP_\infty^{-1} x, P_\infty^{-1} z) = \langle B^* x, B^* z\rangle$$

or equivalently (6.12), since $Q_\infty = Q_\infty^* = P_\infty^{-1}$.

That $\mathscr{D}$ is dense in Y is obvious, since $\mathscr{D} = P_\infty \mathscr{D}(A_F)$ and $\mathscr{D}(A_F)$ is dense in Y, with $P_\infty^{-1} \in \mathcal{L}(Y)$.

Finally, if $x \in \mathscr{D}(A_F)$, then Lemma 4.7 *a*) and Lemma 4.5 give $A^* P_\infty x \in Y$ and $B^* P_\infty x \in U$ respectively, from which we deduce that $z = P_\infty x \in \mathscr{D}$ belongs to $\mathscr{D}(A^*)$ as well as to $\mathscr{D}(B^*)$, respectively. Thus, $\mathscr{D} \subset \mathscr{D}(A^*)$ and $\mathscr{D} \subset \mathscr{D}(B^*)$. But $\mathscr{D}(B^*) \supset \mathscr{D}(A^*)$, since $B^* A^{*-1} \in \mathcal{L}(Y; U)$, by our standing assumption on the model (1.1).

To prove that Q_∞ satisfies the DARE (6.12) for all $x, z \in \mathscr{D}(A^*)$ we shall next show that $\mathscr{D}$ is dense in $\mathscr{D}(A^*)$ in the $\mathscr{D}(A^*)$-topology induced by $\|z\|_{\mathscr{D}(A^*)} = \|A^* z\|$ (Recall from the paragraph below (1.1*d*) that without loss of generality we are taking $0 \in \varrho(A)$, the resolvent set of A, throughout the entire paper). Thus let

(6.15) $(d, a)_{\mathscr{D}(A^*)} = 0$ for all $d \in \mathscr{D}$ and for $a \in \mathscr{D}(A^*)$ fixed and show that $a = 0$.

In fact by (6.13) we have for all $x \in \mathscr{D}(A_F)$

$$\begin{aligned} 0 = (A^* P_\infty x, A^* a)_Y &= (A^* P_\infty A_F^{-1} A_F x, A^* a)_Y = \\ &= \left(A_F x, [A^* P_\infty A_F^{-1}]^* A^* a\right)_Y \end{aligned} \tag{6.16}$$

since A_F is boundedly invertible on Y, by Theorem 2.2 (ii)-(iv). Moreover $A^* P_\infty A_F^{-1}$: $\mathcal{L}(Y)$ by (2.15a) on the left, and so $[A^* P_\infty A_F^{-1}]^* A^* a \in Y$. Then (6.16) implies $[A^* P_\infty A_F^{-1}]^* A^* a = A_F^{*-1} P_\infty a = 0$ and since A_F^* and P_∞ are boundedly invertible, we conclude that $a = 0$, as desired in (6.15). Finally let $x, z \in \mathscr{D}(A^*)$. By the density of $\mathscr{D}$ in $\mathscr{D}(A^*)$, there are $x_n, z_n \in \mathscr{D}$ such that $A^* x_n \to A^* x$, $A^* z_n \to A^* z$ in the Y-norm. Then (6.14) can be re-written for x_n, z_n as

$$
\begin{aligned}
(6.17) \qquad & (A^* x_n, Q_\infty A^{*-1} A^* z_n) + (Q_\infty A^{*-1} A^* x_n, A^* z_n) + \\
& + (RQ_\infty A^{*-1} A^* x_n, Q_\infty A^{*-1} A^* z_n) = \langle B^* A^{*-1} A^* x_n, B^* A^{*-1} A^* z_n \rangle \quad x_n, z_n \in \mathscr{D}
\end{aligned}
$$

with $B^* A^{-1} \in \mathcal{L}(Y)$ by our standing assumption on the model (1.1). Taking the limit in (6.17) yields (6.12) for all $x, y \in \mathscr{D}(A^*)$ as desired. $\square$

7. – Case $T < \infty$. Proof of Theorem 2.5. Dual differential Riccati equation when A is a group generator.

7.0 PRELIMINARIES. – *$B^* z \in L_2(0, T; U)$ and equivalence of exact controllability of $\{A, B\}$ and of $\{-A, B\}$, when A is a group generator.*

For later purposes, it is convenient to study the following optimal control problem, which includes (2.20) when $G = 0$: given $z_0 \in Y$, minimize

$$
(7.1a) \qquad J_{T,G}(v, z) = \int_0^T |B^* z(t)|^2 + \|v(t)\|^2 \, dt + (Gz(T), z(T))
$$

over all $v \in L_2(0, T; Y)$, where z is the solution of (2.21) due to v.

Here we assume:

$$
(7.1b) \qquad G \in \mathcal{L}(Y), \quad G = G^* \geqslant 0 .
$$

The differential Riccati Equation corresponding to (7.1) is

$$
(7.2) \qquad \left\{
\begin{aligned}
& \frac{d}{dt}(Q(t)x, z) = (Q(t)x, A^* z) + (A^* x, Q(t) z) - \langle B^* x, B^* z \rangle + \\
& \qquad\qquad\qquad\qquad + (RQ(t)x, Q(t)z), \qquad x, z \in \mathscr{D}(A^*) \\
& Q(T) = G
\end{aligned}
\right.
$$

We formally re-write (7.2) in integral form (the so-called first integral Riccati

Equation) as

$$(7.3)\qquad (Q(t)x, z) = (G \exp[-A^*(T-t)]x, \exp[-A^*(T-t)]z) + \\ + \int_t^T \langle B^* \exp[-A^*(s-t)]x, B^* \exp[-A^*(s-t)]z\rangle\, ds - \\ - \int_t^T (RQ(s)\exp[-A^*(s-t)]x, Q(s)\exp[-A^*(s-t)]z)\, ds\,, \qquad x, z \in Y\,.$$

The rigorous relation between (7.2) and (7.3) is discussed in Lemma 7.3 below. Before providing the proof of Theorem 2.5, we show the following preliminary Lemma.

LEMMA 7.0. – Assume the standing hypothesis (H.1) = (1.2) and, moreover that A is a s.c. group generator on Y.

i) For each $v \in L_2(0, T; Y)$, the corresponding solution z of (2.21) satisfies $B^* z \in L^2(0, T; U)$

ii) The pair $\{-A, B\}$ is exactly controllable on $[0, T]$ by means of $L_2(0, T; U)$—controls if and only if so is the pair $\{A, B\}$.

PROOF. – (i) With reference to the solution formula (2.21a) for z, from (H.1) = (1.2) and the group property of $\exp[tA^*]$ we have

$$(7.5)\qquad \int_0^T |B^* \exp[-A^* t]x|^2\, dt = \int_0^T |B^* \exp[A^*(T-t)]\exp[-A^* T]x|^2\, dt \leqslant \\ \leqslant c_T \|\exp[-A^* T]x\|^2 \leqslant c'_T \|x\|^2 \qquad \forall x \in Y\,,$$

for some constant $c'_T > 0$. Moreover, if $g \in L_1(0, T; Y)$ and $w \in L^2(0, T; Y)$, then

$$(7.6)\qquad \int_0^T \langle B^* \int_0^t \exp[-A^*(t-s)]g(s)\, ds, w(t)\rangle\, dt =$$

(from Fubini's theorem) $\quad = \displaystyle\int_0^T \int_s^T \langle B^* \exp[-A^*(t-s)]g(s), w(t)\rangle\, dt\, ds \leqslant$

(from Hölder inequality and (7.5)) $\quad \leqslant \displaystyle\int_0^T (c'_T)^{\frac12} \|g(s)\| |w|_{L^2(s;T;U)}\, ds \leqslant$

$$\leqslant (c'_T)^{\frac12} \|g\|_{L_1(0,T;Y)} |w|_{L^2(0,T;U)}\,. \qquad T < \infty$$

It follows that the operator $\tilde{L}$ defined as

$$(\tilde{L}g)(t) = B^* \int_0^t \exp[-A^*(t-s)]\, g(s)\, ds$$

is continuous from $L_1(0, T; Y)$ to $L_2(0, T; U)$. (Notice that this argument is essentially the same as the one in Appendix 1 for L_{0T}^*).

Then (7.4) is a consequence of (7.5), (7.6) and (2.21*a*).

ii) This is essentially due to time-reversibility in the group case. Let $\{A, B\}$ be exactly controllable on $[0, T]$, $T < \infty$, by means of $L_2(0, T; U)$-controls, i.e. let the operator $W_T u = A \int_0^T \exp(A(T-t)) A^{-1} B u(\tau)\, d\tau$ be: $L_2(0, T; U)$ *onto* Y. Then, (as in Definition 6.1) equivalently, its adjoint W_T^*,

$$(W_T u, y)_Y = (u, W_T^* y)_{L_2(0,T;U)},$$

given by $W_T^* y = B^* \exp[A^*(T-t)]y$, $0 \leqslant t \leqslant T$ has continuous inverse [ ]: there is $C_T > 0$ such that

$$\|W_T^* y\|^2_{L_2(0,T;U)} = \int_0^T |B^* \exp[A^* \tau] y|^2\, d\tau \geqslant C_T \|y\|^2 . \tag{7.7}$$

Then, since $\exp[A^* t]$ is a group and using (7.7)

$$\int_0^T |B^* \exp[-A^* \tau] y|^2\, d\tau = \int_0^T |B^* \exp[A^*(T-\tau)] \exp[-A^* T] y|^2\, d\tau \geqslant \\ \geqslant C_T \|\exp[-A^* T] y\|^2 \geqslant C_T' \|y\|^2 , \quad \forall y \in Y \tag{7.8}$$

and $\{-A, B\}$ is exactly controllable on $[0, T]$, by means of $L_2(0, T; U)$-controls. The above argument can be easily reversed from $\{-A, B\}$ to $\{A, B\}$. $\square$

7.1. *Proof of parts* (i) *and* (ii) *of Theorem* 2.5.

We study the integral Riccati Equation (7.3) by means of a method introduced by Da Prato (see for instance [Da P.1]), based on a contraction principle and « a priori » estimates. Therefore, only a sketch of the argument will be provided. In order to study (7.3) in the space $\mathcal{L}(Y, C([0, T]; Y))$ we need the following

LEMMA 7.1. – Let $M(t)$ be the linear operator on Y defined as

$$(M(t)x, z) = \int_t^T \langle B^* \exp[-A^*(s-t)]x, B^* \exp[-A^*(s-t)]z \rangle\, ds , \quad x, z \in Y \tag{7.9a}$$

$$= \int_0^{T-t} \langle B^* \exp[-A^* r]x, B^* \exp[-A^* r]z \rangle\, dr \qquad (r = s-t) . \tag{7.9b}$$

Then $M(\,)\in\mathscr{L}\big(Y, C([0,T];Y)\big)$.

PROOF. – Bt Schwarz inequality applied in (7.9*b*) followed by (7.5), we see that $M(t)\in\mathcal{L}(Y)$ and indeed

$$(M(t)x, z)\leqslant C'_T\|x\|\,\|z\| \qquad \forall x, z\in Y\,,\quad 0\leqslant t\leqslant T \tag{7.10}$$

or $M(t)\in\mathcal{L}\big(Y; L_\infty(0,T;Y)\big)$. In fact, $M(t)$ is strongly continuous on Y: from (7.9*b*), Schwarz inequality and (7.5) we obtain more precisely

$$|(M(t)x - M(s)x, z| = \left|\int_{T-t}^{T-s}\langle B^*\exp[-A^*r]x, B^*\exp[-A^*r]z\rangle\,dr\right|\leqslant$$
$$\leqslant\left|\int_{T-t}^{T-s}|B^*\exp[-A^*r]x|^2\,dr\right|^{\frac12}(C'_T)^{\frac12}\|z\|\to 0 \qquad \text{as } s\to t$$

and this, combined with (7.10) yields the conclusion. $\square$

Next, the right hand side (R.H.S.) of (7.3) defines an operator $\Gamma(Q)$ by means of the Right Hand Side of (7.3):

$$\text{R.H.S. of (7.3)} = (\Gamma(Q)x, z) \qquad x, z\in Y \tag{7.11}$$

and Lemma 7.1 yields then that $\Gamma(Q)\in\mathcal{L}\big(Y, C([0,T];Y)\big)$. Equation (7.3) can then be rewritten as $Q=\Gamma(Q)$. Our first goal is to prove

LEMMA 7.2. – There exists a unique solution $Q(\cdot)\in\mathcal{L}\big(Y; C([0,T];Y)\big)$ of (7.3), i.e. of $Q=\Gamma(Q)$ such that $Q(t)=Q^*(t)$.

PROOF. *Step* 1. – (Local existence) With $0\leqslant T_0< T$, denote by $C([T_0,T];\mathcal{L}(Y))$ the space $\mathcal{L}\big(Y, C([T_0,T];Y)\big)$ endowed with the norm

$$|||Q(\cdot)|||_{T_0} = \sup_{T_0\leqslant t\leqslant T}\|Q(t)\|_{\mathcal{L}(Y)}\,. \tag{7.12}$$

It is easy to see that $C([T_0,T];\mathcal{L}(Y))$ is a Banach space. It is also easy to check from the R.H.S. of (7.3) and Lemma 7.0 that the following bounds are attained

$$|||\Gamma(Q(\cdot))|||_{T_0}\leqslant\|G\|C_{1T}+C_{2T}+\|R\|^2c_{1T}^2|||Q(\cdot)|||_{T_0}^2(T-T_0) \tag{7.13a}$$

$$C_{1T}=|||\exp[-A^*\cdot]|||_0;\qquad C_{2T}=|||M(\cdot)|||_0\leqslant C'_T \tag{7.13b}$$

$$|||\Gamma(Q_1(\cdot))-\Gamma(Q_2(\cdot))|||_{T_0}\leqslant \tag{7.14}$$
$$\leqslant[2\|R\|C_{1T}^2(T-T_0)\max\{|||Q_1(\cdot)|||_{T_0}, |||Q_2(\cdot)|||_{T_0}\}]|||Q_1(\cdot)-Q_2(\cdot)|||_{T_0}$$

for each $Q, Q_1, Q_2 \in C([T_0, T]; \mathcal{L}(Y))$ and C'_T as in (7.10). Let now $B_\eta(T_0)$ denote the closed ball in $C([T_0, T]; \mathcal{L}(Y))$ with radius η centered at the origin. Fix

$$\eta > \|G\| C_{1T} + C_{2T} . \tag{7.14}$$

Then, from (7.13)-(7.14) we see that there exists $T_0 \in [0, T)$ sufficiently close to T such that

(7.15) Γ maps $B_\eta(T_0)$ into itself

(7.16) Γ is a contraction on $B_\eta(T_0)$ with contraction constant less, say, than $\frac{1}{2}$.

The contraction principle then yields a unique solution of (7.3), i.e. of $Q = \Gamma(Q)$, on $C([T_0, T], \mathcal{L}(Y))$.

Step 2. - ($Q(t)$ non-negative definite). From (7.3) $Q^*(t)$ is a solution of (7.3) itself, on $[T_0, T]$; therefore, by uniqueness, $Q(t) = Q^*(t)$. Let us now prove that $Q(t)$ is non-negative definite. Let $t_0 \in [T_0, T]$ and $z_0 \in Y$ be fixed. Given $v \in L_2(t_0, T; Y)$, let $z(t)$ be defined by

$$z(t) = \exp[-A^*(t - t_0)] z_0 + \int_{t_0}^{t} \exp[-A^*(t - s)] R^{\frac{1}{2}} v(s)\, ds \tag{7.17}$$

(see (2.21a)). If $z_0 \in \mathscr{D}(A^*)$ and $v \in H^1(t_0, T; Y)$, then $z(t) \in C([t_0, T]; \mathscr{D}(A^*))$, $\dot{z}(t) \in C([t_0, T]; Y)$, and $\dot{z} = -A^* z + R^{\frac{1}{2}} v$. Since $(Q(t)x, z)$ is continuously differentiable in time for each $x, z \in \mathscr{D}(A^*)$, and (7.2) is satisfied (by direct differentiation in time of (7.3)), it follows that the map $t \to (Q(t) z(t), z(t))$ is differentiable, and

$$\frac{d}{dt}(Q(t)z(t), z(t)) = (\dot{Q}(t) z(t), z(t)) + (Q(t)\dot{z}(t), z(t)) + (Q(t) z(t), \dot{z}(t))$$

[using (7.2), and the equation $\dot{z} = -A^* z + R^{\frac{1}{2}} v$ after cancellation with $Q(t) = Q^*(t)$]

$$= (RQ(t) z(t), Q(t) z(t)) - \langle B^* z(t), B^* z(t)\rangle + 2\,\mathrm{Re}\,(R^{\frac{1}{2}} v(t), Q(t) z(t))$$

$$= -\|v(t)\|^2 - |B^* z(t)|^2 + \|v(t) + R^{\frac{1}{2}} Q(t) z(t)\|^2 .$$

Hence integrating over $[t_0, T]$ and using $Q(T) = G$ (see (7.1b)) yields

$$(Q(t_0) z_0, z_0) = \int_{t_0}^{T} |B^* z(t)|^2 + \|v(t)\|^2\, dt + (Gz(T), z(T)) - \int_{t_0}^{T} \|v(t) + R^{\frac{1}{2}} Q(t) z(t)\|^2\, dt . \tag{7.18}$$

Next, the above equality can be extended by density to all $z_0 \in Y$ and all $v \in L_2(t_0, T; Y)$, via Lemma 7.1. Consider now the closed loop equation

$$z(t) = \exp[-A^*(t-t_0)]z_0 - \int_{t_0}^{t} \exp[-A^*(t-s)]RQ(s)z(s)\,ds\,.$$

It has a unique solution, denoted henceforth by $z_T^0(t, t_0; z_0)$, in $C([t_0, T]; Y)$, because $RQ(\cdot)$ is a strongly continuous perturbation of the infinitesimal generator $-A^*$ (see for instance, [B1, section 4.13]). Let $v_T^0(t, t_0; z_0)$ be defined by the feedback formula $v_T^0(t) = -R^{\frac{1}{2}}Q(t)z_T^0(t)$. Then $v_T^0 \in C([t_0, T]; Y)$, and z_T^0 is the solution of (7.17) due to v_T^0. From (7.18) we have

$$\big(Q(t_0)z_0, z_0\big) = \int_{t_0}^{T} |B^*z(t)|^2 + \|v(t)\|^2\,dt + \big(Gz(T), z(T)\big)$$

whence $\big(Q(t_0)z_0, z_0\big) \geqslant 0$, as desired.

Step 3. – (A priori bounds). Since $Q(t)$ is non-negative definite, from (7.3) with, say, $\|x\| \leqslant 1$ we compute

$$\begin{aligned} |(Q(t)x, x)| &\leqslant \|G\|C_{1T}^2 + \int_t^T |B^* \exp[-A^*r]x|^2\,dr \\ (\text{by } (7.5)) &\leqslant \|G\|C_{1T}^2 + C_T' \end{aligned}$$

since $R = R^* \geqslant 0$, where C_{1T} is defined in (7.11*b*).

Step 4. – It is now standard to extend the local solution in step 1 to a global solution $Q(\cdot) \in C([0, T]; \mathcal{L}(Y))$ of (7.3) by means of the a priori estimate in step 3, in finitely many steps. The proof of Lemma 7.1 is complete. □

The following Lemma relates the Differential Riccati Equation (7.2) to the Integral Riccati Equation (7.3).

LEMMA 7.3. – The following statements are equivalent:

(*a*) $Q \in \mathcal{L}\big(Y, C([0, T]; Y)\big)$ satisfies (7.3);

(*b*) $Q \in \mathcal{L}\big(Y, C([0, T]; Y)\big)$ is such that $\langle Q(t)x, z\rangle$ is continuously differentiable in t for each x and z in $\mathscr{D}(A^*)$, and satisfies (7.2).

PROOF. – (*a*) $\to$ (*b*). This follows by direct differentiation in time of (7.3), which is justified for all $x, z \in \mathscr{D}(A^*)$.

$(b) \to (a)$. From the identity

$$\frac{1}{h}\,[(Q(s+h)\exp[-A^*(s+h-t)]x, \exp[-A^*(s+h-t)]z) -$$
$$- (Q(s)\exp[-A^*(s-t)]x, \exp[-A^*(s-t)]z)] =$$
$$= \frac{1}{h}\,([Q(s+h)-Q(s)]\exp[-A^*(s-t)]x, \exp[-A^*(s-t)z) +$$
$$+ \left(Q(s+h)\frac{1}{h}\,[\exp[-A^*(s+h-t)]x - \exp[-A^*(s-t)]x], \exp[-A^*(s-t)]\,z\right) +$$
$$+ \left(Q(s+h)\exp[-A^*(s+h-t)]x, \frac{1}{h}\,[\exp[-A^*(s+h-t)]z - \exp[-A^*(s-t)]z]\right),$$

along with assumption (b), we see that

$(Q(s)\exp[-A^*(s-t)]x, \exp[-A^*(s-t)]z)$ is differentiable in s, $\forall x$, $z \in \mathscr{D}(A^*)$, and

$$\frac{\partial}{\partial s}\,(Q(s)\exp[-A^*(s-t)]x, \exp[-A^*(s-t)]z) =$$
$$= \frac{\partial}{\partial r}\,(Q(r)\exp[-A^*(s-t)]x, \exp[-A^*(s-t)]z)|_{r=s}$$
$$- (Q(s)A^*\exp[-A^*(s-t)]x, \exp[-A^*(s-t)]z)$$
$$\text{(from (7.2))} \qquad - Q(s)\exp[-A^*(s-t)]x, A^*\exp[-A^*(s-t)]z) =$$
$$= (RQ(s)\exp[-A^*(s-t)]x, Q(s)\exp[-A^*(s-t)]z) -$$
$$- \langle B^*\exp[-A^*(s-t)]x, B^*\exp[-A^*(s-t)]z\rangle\,.$$

After integration on $[t, T]$ of this identity, we finally obtain (7.3). $\square$

COROLLARY 7.4. – Parts i) and ii) of Theorem 2.5 hold true.

PROOF. – Part i) follows readily from Lemmas 7.2 and 7.3, with $G = 0$. As to part ii), the uniqueness of the solution to (7.3) in $\mathcal{L}\big(Y, C([0, T]; Y)\big)$ is claimed by Lemma 7.2 and this in turn yields the uniqueness for the Differential Riccati Equation (7.2) via the equivalence result of Lemma 7.3. $\square$

7.2. *Proof of part* (ii) *of Theorem* 2.5.

We prove the statement of part (iii) of Theorem 2.5 for the more general optimal control problem (7.1) via Dinamic Programming.

Let $z_0 \in Y$, $v \in L_2(0, T; Y)$, and let $z(t)$ be the corresponding solution of (2.21a).

Repeating the argument used in step 2 of the proof of Lemma 7.2, we find

$$(Q(0)z_0, z_0) = J_{T,G}(v, z) - \int_0^T \|v(t) + R^{\frac{1}{2}}Q(t)z(t)\|^2\, dt \tag{7.19}$$

(which corresponds to (7.18)); moreover, if z_T^0 denotes the unique solution in $C([0, T]; Y)$ of the closed loop equation

$$z(t) = \exp[-A^* t]z_0 - \int_0^t \exp[-A^*(t-s)]RQ(s)z(s)\, ds \tag{7.20}$$

and v_T^0 is defined via the feedback formula (2.25), then from (7.19) we have

$$(Q(0)z_0, z_0) = J_{T,G}(v_T^0, z_T^0) = J_{T,G}(v_T^0)\,. \tag{7.21}$$

But (7.19) also yields

$$(Q(0)z_0, z_0) \leq J_{T,G}(v, z) \qquad \forall v \in L_2(0, T; Y)\,. \tag{7.22}$$

Then v_T^0 is an optimal control, and (2.26) holds.

Conversely, if $\hat{v}$ is an optimal control, with corresponding optimal solution $\hat{z}$ via (2.21*b*), then by (7.22) $J_{T,G}(\hat{v}, \hat{z}) = J_{T,G}(v_T^0) = (Q(0)z_0, z_0)$; and, from (7.19),

$$\hat{v}(t) = -R^{\frac{1}{2}}Q(t)\hat{z}(t) \qquad \text{for a.e. } t \in [0, T]\,. \tag{7.23}$$

This implies that $\hat{z}$ is a solution of (7.20) in $C([0, T]; Y)$. From the uniqueness result for (7.20) we have $\hat{z} = z_T^0$, whence $\hat{v} = v_T^0$ via (7.23). The proof is complete. □

8. – Case $T = \infty$. Proof of Theorem 2.6 and 2.7. Dual algebraic Riccati equation when A is a group generator.

ORIENTATION. – By section 7, there exist operators $Q_T(t)$, $0 \leq t \leq T$, which satisfy a differential Riccati equation, in fact the dual equation (7.2) [or (7.3)], in sharp contrast with the situation for the operators $P_T(t)$, $0 \leq t \leq T$, of the original problem, for which no claim was made as to whether or not they satisfy a differential (or integral) Riccati equation for R *nonregular*, say $R = I$ (see section 1.3 and Orientation at the beginning of subsection 2.5). Thus, in proceeding from the finite to the infinite horizon problem we are now in a more favorable situation with the operators $Q_T(t)$ and the dual problem than we were in section 4 with the operators $P_T(t)$ and the original problem. Hence, in this section, our line of argument follows along more classical lines than it was possible in section 4. Accordingly, only the major highlights of our proofs will be given.

8.1. *Proofs of parts* (i), (ii) *and* (iii) *of Theorem* 2.6.

Part (i). – [As Lemma 4.2*a*]. It is standard [B-1]. The increasing monotonicity of the optimal cost $J_T(v_T^0, z_T^0)$ with $T\uparrow\infty$ produces, by virtue of (2.26) or (7.21), a monotonically increasing sequence of self-adjoint operators $Q_T(0)$ which—moreover—is upper bounded by virtue of the finite cost condition (2.27) for problem (2.28). Hence, there exists $\hat{Q}_\infty \in \mathcal{L}(Y)$, $\hat{Q}_\infty = \hat{Q}_\infty^* \geqslant 0$ such that

$$\hat{Q}_\infty x = \lim_{T\to\infty} Q_T(0)x\,, \qquad x \in Y \tag{8.1}$$

Part (ii). – We begin to prove some further properties of $Q_T(\ )$ (see Lemma 4.2*b* for the analogous property for $P_T(t)$, proved however in a different way).

LEMMA 8.1.

$$Q_T(t) = Q_{T+\tau}(t+\tau) \qquad \text{for each } T > 0,\ t \in [0, T], \text{ and } \tau > 0\,. \tag{8.2}$$

PROOF. – Rewriting (2.24) with $T+\tau$ in place of T and $t+\tau$ in place of t, and then using the change of variable $\sigma = s - \tau$, we obtain

$$\begin{aligned}(Q_{T+\tau}(t+\tau)x, z) = &\int_t^T \langle B^* \exp[-A^*(\sigma - t)]x, B^* \exp[-A^*(\sigma-t)]z\rangle\, d\sigma - \\ &-\int_t^T (RQ_{T+\tau}(\sigma+\tau)\exp[-A^*(\sigma-t)]x, Q_{T+\tau}(\sigma+\tau)\exp[-A^*(\sigma-t)]z)\, d\sigma\,.\end{aligned} \tag{8.3}$$

This equation has the unique solution $Q_T(t)$, by Theorem 2.5 (ii). Then $Q_{T+\tau}(t+\tau) = Q_T(t)$. $\square$

The counterpart of Lemma 4.2*c* for $P_T(t)$ is now

COROLLARY 8.2. – For each $t_0 > 0$ and $x \in Y$,

$$Q_T(t)x \to \hat{Q}_\infty x \qquad \text{as } T \to +\infty, \text{ uniformly in } t \in [0, t_0] \tag{8.4}$$

where $\hat{Q}_\infty$ was defined in (8.1).

PROOF. – The proof of part i) of Theorem 2.6 showed $Q_T(0) \leqslant \hat{Q}_\infty$ for each $T > 0$. This inquequality, along with the identity $Q_T(t) = Q_{T-t}(0)$ given by Lemma 8.1, implies:

$$\hat{Q}_\infty - Q_T(t) \geqslant 0\,, \qquad \forall T \geqslant t \geqslant 0\,. \tag{8.5}$$

Byt Lemma 8.1 and the monotonicity of $Q_T(0)$ in T yield

$$Q_T(t) = Q_{T-t}(0) \geqslant Q_{T-t_0}(0) \qquad \forall T \geqslant t_0 \geqslant t \geqslant 0\,. \tag{8.6}$$

Then, from (8.5) and (8.6), we have

$$0 \leqslant \hat{Q}_\infty - Q_T(t) \leqslant \hat{Q}_\infty - Q_{T-t_0}(0) , \tag{8.7}$$

or, in other form,

$$\|[\hat{Q}_\infty - Q_T(t)]^{\frac{1}{2}} x\|^2 \leqslant \|[\hat{Q}_\infty - Q_{T-t_0}(0)]^{\frac{1}{2}} x\|^2 \qquad \forall x \in Y . \tag{8.8}$$

Then

$$[\hat{Q}_\infty - Q_T(\cdot)]^{\frac{1}{2}} x \to 0 \quad \text{in } C([0, t_0]; Y) , \qquad \forall x \in Y . \tag{8.9}$$

Since

$$\|[\hat{Q}_\infty - Q_T(t)]x\| = \|[\hat{Q}_\infty - Q_T(t)]^{\frac{1}{2}}[\hat{Q}_\infty - Q_T(t)]^{\frac{1}{2}} x\| \leqslant C\|[\hat{Q}_\infty - Q_T(t)]^{\frac{1}{2}} x\|$$

for some constant $c > 0$ (which exists by (8.9) and the Banach-Steinhaus Theorem), we finally have

$$[\hat{Q}_\infty - Q_T(\cdot)]x \to 0 \quad \text{in } C([0, t_0]; Y) , \qquad x \in Y . \qquad \square$$

We can now prove that $\hat{Q}_\infty$ defined by (8.1) satisfies the DARE (2.23). Indeed, let $T \geqslant t_0 > 0$, and consider equation (2.24) when $0 \leqslant t \leqslant t_0$. Splitting the integrals in (2.24) over $[t, t_0]$ and $[t_0, T]$, we obtain

$$\begin{aligned}(Q_T(t)x, z) = &\int_t^{t_0} \langle B^* \exp[-A^*(s-t)]x, B^* \exp[-A^*(s-t)]z\rangle \, ds - \\ &-\int_t^{t_0} \big(RQ_T(s) \exp[-A^*(s-t)]x, Q_T(s) \exp[-A^*(s-t)]z\big) \, ds + \\ &+ \big(Q_T(t_0) \exp[-A^*(t_0-t)]x, \exp[-A^*(t_0-t)]z\big) .\end{aligned}$$

Now, Corollary 8.2 guarantees that we can take the limit as $T \uparrow \infty$ in the last identity, to obtain

$$\begin{aligned}(Q_\infty x, z) = &\big(Q_\infty \exp[-A^*(t_0-t)]x, \exp[-A^*(t_0-t)]z\big) + \\ &+\int_t^{t_0} \langle B^* \exp[-A^*(s-t)]x, B^* \exp[-A^*(s-t)]z\rangle \, ds - \\ &-\int_t^{t_0} \big(RQ_\infty \exp[-A^*(t_0-t)]x, Q_\infty \exp[-A^*(s-t)]z\big) \, ds .\end{aligned} \tag{8.10}$$

Then, recalling (7.3), we see that Q_∞ is the unique solution of the integral Riccati Equation (7.3) with $G = Q_\infty$. Hence Lemma 7.3 implies that Q_∞ satisfies the Differential Riccati Equation (7.2), which reduces to (2.23) for Q_∞ is independent of t. $\square$

Part (iii). – We first prove the following lemma, of interest in itself, on a « minimality » property of the operator $\hat{Q}_\infty$ defined by (8.1).

LEMMA 8.3. – *If* $\bar{Q}_\infty \in \mathcal{L}(Y)$ satisfies the DARE (2.23) and $\bar{Q}_\infty = \bar{Q}^*_\infty \geqslant 0$, then

(*a*) for each $z_0 \in Y$ the feedback control $\bar{v}(t) = - R^{\frac{1}{2}}\bar{Q}_\infty \bar{z}(t)$ satisfies $J_\infty(\bar{v}) \leqslant (\bar{Q}_\infty z_0, z_0)$ (in particular, $\bar{v} \in L^2(0, \infty; Y)$, and $J_\infty(\bar{v}) < \infty$);

(*b*) $\bar{Q}_\infty \geqslant \hat{Q}_\infty$, where Q_∞ is defined by (8.1).

PROOF. *Part* (*a*). – Consider problem (7.1) with $G = \bar{Q}_\infty$. By assumption, it follows that $\bar{Q}_\infty$ is the corresponding solution of (7.2), or (7.3). Then (7.19) can be rewritten as

$$(\bar{Q}_\infty z_0, z_0) = J_{T,\bar{Q}_\infty}(v) - \int_0^T \|v(t) + R^{\frac{1}{2}}\bar{Q}_\infty z(t)\|^2 \, dt \,, \tag{8.11}$$

for each $v \in L^2(0, T; Y)$, where z is the solution of (2.21) due to v. When $v = \bar{v}$, identity (8.11) reduces to

$$(\bar{Q}_\infty z_0, z_0) = J_{T,\bar{Q}_\infty}(\bar{v}) \,, \qquad \text{whence} \qquad (\bar{Q}_\infty z_0, z_0) \geqslant \int_0^T |B^* \bar{z}(t)|^2 + \|\bar{v}(t)\|^2 \, dt \,, \tag{8.12}$$

by definition of $J_{T,\bar{Q}_\infty}$ in (7.1). Note that $\bar{v}$ does not depend on T. Then the claim of part (*a*) follows from (8.12) as $T \to \infty$.

Part (*b*). – Consider problem (7.1) with $G = 0$, and denote, as in section 2.5, by J_T the corresponding cost functional, and by v^0_T the optimal control. Then $J_T(v^0_T) \leqslant J_T(\bar{v})$. Hence, using also (2.26) and (8.12), we have

$$(Q_T(0) z_0, z_0) = J_T(v^0_T) \leqslant J_T(\bar{v}) \leqslant (\bar{Q}_\infty z_0, z_0) \,.$$

This yields $\hat{Q}_\infty \leqslant \bar{Q}_\infty$ as $T \to \infty$, recalling (8.1) = (2.28). $\square$

Continuing with the proof of Part (iii), when we take $\bar{Q}_\infty = \hat{Q}_\infty$ in Lemma 8.3, we get

$$J_\infty(v^0_\infty) \leqslant (\hat{Q}_\infty z_0, z_0) \,,$$

where v_∞^0 is defined by (2.29). Thus, if we prove that

$$(\hat{Q}_\infty z_0, z_0) \leqslant J_\infty(v) \qquad \forall v \in L^2(0, \infty; Y), \tag{8.13}$$

then v_∞^0 will be optimal control for problem (2.22), whose uniqueness is guaranteed by the strict convexity of J_∞. Then the proof of part (iii) of Theorem 2.6 is complete if we prove (8.13). Let $v \in L^2(0, \infty; Y)$; by definition of J_T and J_∞ we have $J_T(v) \leqslant J_\infty(v)$.

Moreover $(Q_T(0) z_0, z_0) \leqslant J_T(v)$, from Theorem 2.5. The last two inequalities, along with (8.1) = (2.28), yield (8.13) as $T \to \infty$. □

8.2. *Proof of parts* (*iv*) *and* (*v*) *of Theorem* 2.6.

Part (v). – The counterpart of Lemma 6.1 *a*) for the dual problem is

LEMMA 8.4. – Consider the dynamics (1.1) under the standing assumption (H.1) = (1.2) and let A be a s.c. group generator on Y. Then, the pair $\{-A, B\}$ (equivalently, th pair $\{A, B\}$, see Lemma 7.0 *b*)) is exactly controllable over $[0, T]$ by means of $L_2(0, T; U)$-controls if and only if $Q_T(0)$ is an isomorphism on Y.

PROOF. – A proof exactly as in Lemma 6.1 *a*) could be given (for the problem (2.21)-(2.22)). Here, a variation of the same idea will be indicated.

PROOF. *Step* 1. – We prove that: for each $z_0 \in Y$,

$$(Q_T(0) z_0, z_0) \leqslant \int_0^T |B^* \exp[-A^* t] z_0|^2 \, dt \leqslant \nu (Q_T(0) z_0, z_0), \tag{8.14}$$

where ν is a constant independent of z_0. To prove (8.14), denote by $|\cdot|_T$ and $\|\cdot\|_T$ the (canonical) norms in $L_2(0, T; U)$ and $L_2(0, T; Y)$, respectively, and let $\tilde{L}$ be the bounded linear operator from $L_2(0, T; Y)$ (in fact, $L_1(0, T; Y)$) to $L_2(0, T; U)$) defined by (7.6). Moreover, denote by η the function in $L_2(0, T; U)$ defined as $\eta(t) = B^* \exp[-A^* t] z_0$, see Lemma 7.0. The cost J_T in (2.20) can be rewritten as $J_T(v) = |L\tilde{v} + \eta|_T^2 + \|v\|_T^2$. Since $I + \tilde{L}^* \tilde{L}$ is an isomorphism in $L_2(0, T; Y)$ ($I =$ identity in $L_2(0, T; Y)$), we can define an element $\bar{v} \in L_2(0, T; Y)$ as

$$\bar{v} = -(I + \tilde{L}^* \tilde{L})^{-1} \tilde{L}^* \eta .$$

After some manipulations we obtain

$$J_T(v) - J_T(\bar{v}) = |\tilde{L}(v - \bar{v})|_T^2 + \|v - \bar{v}\|_T^2 .$$

This impiies that $\bar{v} = v_T^0$, and that

(8.15) $$J_T(0) = J_T(v_T^0) + |\tilde{L}v_T^0|_T^2 + \|v_T^0\|_T^2 .$$

Since, by definition, $J_T(v_T^0) \geqslant \|v_T^0\|_T^2$, from (8.15) we have

(8.16) $$J_T(0) \leqslant (c+2) J_T(v_T^0) ,$$

where c is the norm of $\tilde{L}$ between $L_2(0, T; Y)$ and $L_2(0, T; U)$. Since v_T^0 is optimal, a converse of (8.16) is also true:

(8.17) $$J_T(v_T^0) \leqslant J_T(0) .$$

If we rewrite (8.16) and (8.17) using the definition of $J_T(0)$ and (2.26), we obtain (8.14).

Step 2. – The assumption and the characterization

$$\int_0^T |B^* \exp[-A^* t] z_0|^2 \, dt \geqslant C_T \|z_0\|^2$$

for exact controllability of $\{-A, B\}$ by means of $L_2(0, T; Y)$-controls, (see Lemma 7.0 ii) easily imply the desired conclusions via (8.14). □

The counterpart of Lemma 6.1 *b*) is now

COROLLARY 8.5. – In addition to the hypothesis of Lemma 8.4 assume further the finite cost condition (2.27) for problem (2.22) so that $\hat{Q}_\infty$ is well defined by (8.1). If the pair $\{A, B\}$ is exactly controllable over some interval $[0, T]$, then $\hat{Q}_\infty$ is an isomorphism.

PROOF. – It is sufficient to recall that $\hat{Q}_\infty \geqslant Q_T(0)$, and to use Lemma 8.4. □

Thus the claim of part (v) of Theorem 2.6 is proved.

Part (iv). – It follows from the following more general uniqueness result, along with Corollary 8.5.

THEOREM 8.6. – Under the assumptions of Theorem 2.6 we have:

(*a*) a solution $\bar{Q}_\infty \in \mathcal{L}(Y)$ of the A.R.E. (2.23), such that $\bar{Q}_\infty = \bar{Q}_\infty^* \geqslant 0$, is equal to $\hat{Q}_\infty$ defined by (8.1) = (2.28) if and only if,

(8.19) $$\text{for each } z_0 \in Y, \quad \big(\bar{Q}_\infty z_\infty^0(t), z_\infty^0(t)\big) \to 0 \quad \text{as } t \uparrow \infty;$$

where $z^0_\infty(t) = z^0_\infty(t, 0; z_0)$ is defined by Theorem 2.6 (iii);

(*b*) if $\hat{Q}_\infty$ is an isomorphism, then, for each $z_0 \in Y$,

$$z^0_\infty(t) \to 0 \quad \text{as } t \uparrow \infty;$$

(*c*) if $\hat{Q}_\infty$ is an isomorphism (this is true, in particular, under the controllability assumption of $\{-A, B\}$ as in Corollary 8.5), then $\hat{Q}_\infty$ is the unique solution of DARE (2.23) in the class of all $Q \in \mathcal{L}(Y)$ such that $Q = Q^* \geqslant 0$.

PROOF. *Part* (*a*). – Recall that (8.11) holds true for each self-adjoint non-negative solution $\bar{Q}_\infty = \hat{Q}_\infty$ of (2.23), and for each $v \in L_2(0, T; Y)$.

Only if. When $\bar{Q}_\infty = \hat{Q}_\infty$ and $v = v^0_\infty$ are used in (8.11), with v^0_∞ given by (2.29) we obtain via the definition of $J_{G,T} = \hat{Q}_\infty$:

$$(\hat{Q}_\infty z_0, z_0) = \int_0^T |B^* z^0_\infty(t)|^2 + \|v^0_\infty(t)\|^2 \, dt + (\hat{Q}_\infty z^0_\infty(T), z^0_\infty(T)) \, .$$

As $T \to +\infty$, recalling (2.30) ,we obtain (8.19), with $\bar{Q}_\infty = \hat{Q}_\infty$.

If. Conversely, assume (8.19), and let $\bar{Q}_\infty$ be a solution of (2.23). From (8.11) with $v = v^0_\infty$ we have

$$(\bar{Q}_\infty z_0, z_0) \leqslant \int_0^T |B^* z^0_\infty(t)|^2 + \|v^0_\infty(t)\|^2 \, dt + (\bar{Q}_\infty z^0_\infty(T), z^0_\infty(T)) \, .$$

As $T \uparrow \infty$, using (8.19) and (2.30), we have $\bar{Q}_\infty \leqslant \hat{Q}_\infty$; then from Lemma 8.3 *b*) we obtain $\bar{Q}_\infty = \hat{Q}_\infty$:

Part b). – From Part *a*), i.e. from $(\hat{Q}_\infty z^0_\infty(t), z^0_\infty(t)) \to 0$.

Part c). – From Part *a*) and *b*). □

REMARK 8.1. – By arguing as in the proof of Theorem 2.3 one can prove that: the uniqueness result in Theorem 8.6 *c*) for the solution of the DARE (2.23) holds true if one replaces the exact controllability assumption of the pair $\{-A, B\}$ (equivalently, of the pair $\{A, B\}$, Lemma 7.0 ii) given there with the following somewhat weaker « detectability » type of condition:

There exists a densely defined (not necessarily bounded) operator $K: U \supset \supset \mathcal{D}(K) \to Y$ such that the following three conditions are fulfilled:

(i) the operator $A_K = A^* + KB^*$ is the generator of a s.c. semigroup $\exp[A_K t]$ on Y;

(ii) the semigroup $\exp[A_K t]$ is uniformly stable: there exist $c_1, \alpha_1 > 0$ s.t.

$$\|\exp[A_K t]\|_{\mathcal{L}(Y)} \leqslant C_1 \exp[-\alpha_1 t], \quad t \geqslant 0;$$

(iii) with $A_K^* = A + BK^*$ we have

$$\int_0^\infty |K^* \exp[A_K^* t]x|_U^2 \leqslant C\|x\|_Y^2 \quad \forall x \in Y \text{ [recall (5.21)]}. \tag{8.20}$$

As to the existence of such K, we have:

PROPOSITION 8.7. – With A the generator of a s.c. group on Y, and B satisfying (H.2), let the pair $\{A, B\}$ be exactly controllable by means of $L_2(0, T; U)$-controls. Then, there exists an operator K satisfying the three conditions of Remark 8.1. Moreover, K is given by

$$K = -P_\infty B \tag{8.21}$$

where P_∞ is the Riccati Algebraic operator for the dynamics (1.1) with respect to the cost (1.6) with $R = I$.

PROOF. – Exact controllability of $\{A, B\}$ guarantees a fortiori the finite cost condition for the optimal control problem (1.1), (1.6) with $R = I$. Then, Theorems 2.2 and 2.3 guarantee the existence (and uniqueness) of the Riccati operator P_∞ such that $A - BB^*P_\infty$ is the generator of a s.c. semigroup $\exp[(A - BB^*P_\infty)t]$ on Y, which moreover is uniformly stable here.

Also P_∞ is the unique solution of the ARE (2.16) with $R = I$

$$(P_\infty x, Az) + (P_\infty Ax, z) + (x, z) = \langle B^*P_\infty x, B^*P_\infty z\rangle, \quad x, z \in \mathscr{D}(A - BB^*P_\infty). \tag{8.22}$$

Thus the choice (8.21) for K guarantees conditions (i)-(ii) of Remark 8.1. To show that such K satisfies also (8.20) we consider the corresponding dynamics

$$w_t = (A - BB^*P_\infty)w, \quad w(0) = x \in Y. \tag{8.23}$$

Taking the Y-inner product of (8.23) with $P_\infty w$ yields

$$\operatorname{Re}(w_t(t), P_\infty w(t)) = \operatorname{Re}(P_\infty Aw(t), w(t)) - |B^*P_\infty w(t)|^2. \tag{8.24}$$

Using now (8.22) with $x = z = w$ for $x \in \mathscr{D}(A - BB^*P_\infty)$ gives

$$2\operatorname{Re}(P_\infty Aw(t), w(t)) + \|w(t)\|^2 = |B^*P_\infty w(t)|^2$$

which inserted in (8.24) yields then the right hand side of

$$\frac{1}{2}\frac{d}{dt}\|P_\infty^{\frac{1}{2}}w(t)\|^2 = \mathrm{Re}\,(w_t(t), P_\infty w(t)) = -\frac{1}{2}\|w(t)\|^2 - \frac{1}{2}|B^*P_\infty w(t)|^2$$

Integrating in t and using the uniform decay of $w(t)$ gives

$$\int_0^\infty |B^*P_\infty w(t)|^2\,dt \leqslant \|x\|^2 + (P_\infty x, x) \leqslant (1 + \|P_\infty^{\frac{1}{2}}\|)\|x\|^2 \qquad x \in \mathscr{D}(A - BB^*P_\infty)\,. \tag{8.24}$$

The inequality in (8.24) is now extended by continuity to all of $x \in Y$ and this proves (8.20) with $K^* = -B^*P_\infty$ as in (8.21). □

8.3. *Proof of Theorem* 2.7.

It follows by combining Theorem 2.4 and Theorem 8.6 *c*) via uniqueness of the solution of the DARE (2.23). In fact, if $\{A^*, R^{\frac{1}{2}}\}$ is exactly controllable on $[0, T]$ by $L_2(0, T; Y)$—controls, then the finite cost condition (2.27) holds a fortiori true for problem (2.21)—(2.22). Moreover, Theorem 2.4 yields that Q_∞ (defined there as P_∞^{-1}) is a solution of the DARE (2.23) for all $x, z \in \mathscr{D}(A^*)$, see (2.18). Similarly, if $\{-A, B\}$ (equivalently, $\{A, B\}$) is exactly controllable on $[0, T]$ by $L_2(0, T; U)$-controls, then the finite cost condition (H.3) = (2.7) holds a fortiori true for the problem (1.1), (1.6). Moreover, Theorem 8.6 *c*) yields that $\hat{Q}_\infty$ is the *unique* solution of the DARE (2.23) for all $x, z \in \mathscr{D}(A^*)$. Thus $Q_\infty = \hat{Q}_\infty$. □

Part (ii). *Step* 1. – Let $S_\infty(t)$ be the C_0-semigroup on Y generated by $-A^* - RQ$. Let us show that

$$S_\infty(t) = P_\infty \Phi_\infty(t) Q_\infty \tag{8.25}$$

where $\Phi_\infty(t)$ is the semigroup generated by A_F, defined in Theorem 2.2 (ii).

Let $T(t) = P_\infty \Phi_\infty(t) Q_\infty$ It can be readily checked that: *a*) $T(t)$ is a C_0-semigroup on Y; *b*) the space $W = \{x \in Y;\, Qx \in \mathscr{D}(A_F)\}$ is invariant for $T(t)$, i.e. $T(t)(W) \subset W$; *c*) for every $x \in W$, $T(t)x$ is differentiable for $t \geqslant 0$, and

$$\frac{d}{dt}T(t)x = P_\infty A Q_\infty T(t)x\,. \tag{8.26}$$

From Theorem 2.2 we have, for every $x \in W$ and $y \in \mathscr{D}(A_F)$,

$$\begin{aligned}(P_\infty A_F Q_\infty T(t)x, y) &= (A_F \Phi_\infty(t) Q_\infty x, P_\infty y) = \\ &= (\Phi_\infty(t)Q_\infty x, A^*P_\infty y) - \langle B^*P_\infty \Phi_\infty(t)Q_\infty x, P_\infty y\rangle =\end{aligned}$$

[from (2.16)]

$$= -(\Phi_\infty(t)Q_\infty x, P_\infty Ay) - (R\Phi_\infty(t)Q_\infty x, y) =$$
$$= -(T(x)x, Ay) - (RQ_\infty T(t)x, y).$$

Thus $T(t)x \in \mathscr{D}(A^*)$, and $P_\infty A_F Q_\infty T(t)x = (-A^* - RQ_\infty)T(t)x$, whence

$$\frac{d}{dt} T(t)x = (-A^* - RQ_\infty)T(t)x, \qquad \text{for every } x \in W$$

(from (8.26)). The uniqueness of the solution of the equation $dw/dt = (-A^* - RQ_\infty)w$, with $w(0) = x$, yields $S_\infty(t)x = T(t)x$ for every $t \geq 0$ and $x \in W$. But W is dense, so that $S_\infty(t) = T(t)$, and (8.25) is proved.

Step 2. – From Theorem 2.6 (iii) we have $S_\infty(t)z_0 = z^0_\infty(t, 0; z_0)$ for every $z_0 \in Y$. Recall also that $\Phi_\infty(t)y_0 = y^0_\infty(t, 0, y_0)$ for every $y_0 \in Y$ (Theorem 2.2(ii)). Thus (2.9) and (8.25) yield

$$v^0_\infty(t, 0, z_0) = -R^{\frac{1}{2}}y^0_\infty(t, 0; Q_\infty z_0).$$

Similarly, the relation

$$u^0_\infty(t, 0; y_0) = -B^* z^0_\infty(t, 0; P_\infty y_0)$$

follows from (2.13). The proof of Theorem 2.7 is complete. □

Appendix 1: Regularity of L_{0T} and L^*_{0T}; proof of (1.4) and (1.5).

We shall prove the regularity of L_{0T} described by (1.4) and the regularity of L^*_{0T} described by (1.5), as a consequence of the standing assumption (H.1) = (1.2*a*), see [L-T.1-2], [L-T.9].

Step 1. – Let $\nu \in L_1(0, T; Y)$ and $u \in L_2(0, T; U)$. Then

$$\int_0^T ((L_{0T}u)(t), \nu(t))\,dt = \int_0^T \Big[\int_0^t \exp[A(t-\tau)]Bu(\tau), \nu(t)\Big]\,d\tau\,dt =$$
$$= \int_0^T \int_0^t \langle u(\tau), B^* \exp[A^*(t-\tau)]\nu(t)\rangle\,d\tau\,dt \leq$$
$$\leq \int_0^T \Big\{\int_0^t |u(\tau)|^2\,d\tau\Big\}^{\frac{1}{2}} \Big\{\int_0^t |B^* \exp[A^*(t-\tau)]\nu(t)|^2\,d\tau\Big\}^{\frac{1}{2}}$$

(replacing t with T and using (H.1) = (1.2a)

$$\leqslant \int_0^T C_T \|\nu(t)\|\, dt \|u\|_{L_2(0,T;\,U)} = C_T \|\nu\|_{L_1(0,T;\,Y)} \|u\|_{L_2(0,T;\,U)}\,.$$

Thus by the closed graph theorem

$$(*) \qquad L_{0T}\colon \text{continuous } L_2(0, T;\, U) \to L_\infty(0, T;\, Y)$$

Step 2. – By taking now u_n smooth, say $u_n \in C^1([0, T];\, U)$ with $u_n \to u$ in $L_2(0, T;\, U)$, and integrating $(L_{0T}u_n)(t)$ by parts, we plainly improve the continuity in $(*)$ to the continuity in (1.4). Then (1.5) follows by duality.

Appendix 2: Illustration of abstract model to (i) **second order scalar hyperbolic equations;** (ii) **plate like equations and** (iii) **first order hyperbolic systems.**

Throughout this Appendix, Ω is an open bounded domain in R^n with sufficiently smooth boundary Γ.

A) Second order scalar hyperbolic equations with Dirichlet boundary control.

Let $\mathscr{A}(\xi, \partial)$ be a second order, elliptic operator on Ω with symmetric coefficients of its principal part (canonically $-\Delta$) and consider the following mixed problem

$$\text{(A.1)} \qquad \begin{cases} w_{tt} = -\mathscr{A}(\xi, \partial) w & \text{in } (0, T]\times\Omega \equiv Q \\ w|_{t=0} = w_0\,, \quad w_t|_{t=0} = w_1 & \text{in } \Omega \\ w|_\Sigma = u \in L_2(0, T;\, L_2(\Gamma)) & \text{in } (0, T]\times\Gamma \equiv \Sigma\,. \end{cases}$$

We now discuss the connection between problem (A.1) and model (1.1) subject to assumption (H.1) = (1.2). To put problem (A.1) into the abstract form (1.1) we choose [L-T.1], [L-T.2], [L-T.3], [L-L-T.1], [T.1] or [Da P-L-T.1]: $Y = L_2(\Omega)\times \times H^{-1}(\Omega)$, $y = [w, w_t]$; $U = L_2(\Gamma)$ and

$$\text{(A.2)} \qquad A = \begin{vmatrix} 0 & I \\ -\mathscr{A} & 0 \end{vmatrix}; \qquad Bu = \begin{vmatrix} 0 \\ \mathscr{A}Du \end{vmatrix} \text{ (formally)}; \qquad A^{-1}Bu = \begin{vmatrix} -Du \\ 0 \end{vmatrix}$$

$\mathscr{A}$ being the realization of the elliptic operator $\mathscr{A}(\xi, \partial)$ with homogeneous Dirichlet B.C.; D is the « Dirichlet » map defined by (for simplicity, and without loss of generality for the problem of the present paper we assume that $\lambda = 0$ is not an eigenvalue of $\mathscr{A}$)

$$Dv = h \leftrightarrow \begin{cases} -\mathscr{A}(\xi, \partial) h = 0 & \text{in } \Omega \\ h = v & \text{in } \Gamma \end{cases} \tag{A.3}$$

D: continuous $L_2(\Gamma) \to \mathscr{D}(\mathscr{A}^{\frac{1}{4}-\varepsilon}) = H^{\frac{1}{2}-2\varepsilon}(\Omega)$, $\varepsilon > 0$

$$\begin{cases} \exp[At] = \begin{vmatrix} \mathscr{C}(t) & \mathscr{S}(t) \\ -\mathscr{A}\mathscr{S}(t) & \mathscr{C}(t) \end{vmatrix} \text{ in } Y; \\ B^* \begin{vmatrix} z_1 \\ z_2 \end{vmatrix} = D^* \mathscr{A}^{*\frac{1}{2}} \mathscr{A}^{-\frac{1}{2}} z_2, \text{ with dense domain in } Y \end{cases} \tag{A.4}$$

where $\mathscr{C}(t)$ is the s.c. cosine operator generated by $-\mathscr{A}$ and $\mathscr{S}(t)y = \int_0^t \mathscr{C}(\tau) y \, d\tau$. Moreover

$$(Lu)(t) = \begin{vmatrix} \mathscr{A} \int_0^t \mathscr{S}(t-\tau) D u(\tau) \, d\tau \\ \mathscr{A} \int_0^t \mathscr{C}(t-\tau) D u(\tau) \, d\tau \end{vmatrix} \tag{A.5}$$

$$B^* \exp[A^* t] \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} = -D^* \mathscr{A}^* \mathscr{S}^*(t) y_1 + D^* \mathscr{C}^*(t) y_2, \qquad y = [y_1, y_2] \in Y \tag{A.6}$$

The Abstract Assumption (H.1) = (1.2). In view of (A.6), then assumption (H.1) for problem (A.1) means

$$\left.\begin{matrix} D^* \mathscr{A}^* \mathscr{S}^*(t) \\ D^* \mathscr{A}^{*\frac{1}{2}} \mathscr{C}^*(t) \end{matrix}\right\} \text{continuous } L_2(\Omega) \to L_2(0, T; L_2(\Gamma)) \tag{A.7}$$

which holds indeed true, as proved in [L-T.2], [L-L-T.1]. In P.D.E.'s terms, the regularity (A.7) means, in turn that for the following hyperbolic problems with, say, the Laplacian $-\mathscr{A}(\xi, \partial) = \Delta$:

$$\begin{cases} \varphi_{tt} = \Delta\varphi & \text{in } Q \\ \varphi|_{t=0} = \varphi_0;\ \varphi_t|_{t=0} = 0 & \text{in } \Omega \\ \varphi = 0 & \text{in } \Sigma \end{cases} \quad \text{and} \quad \begin{cases} \psi_{tt} = \Delta\psi & \text{in } Q \\ \psi|_{t=0} = 0;\ \psi_t|_{t=0} = \psi_1 & \text{in } \Omega \\ \psi = 0 & \text{in } \Sigma \end{cases} \tag{A.8}$$

with $\varphi_0 \in H_0^1(\Omega)$ and $\psi_1 \in L_2(\Omega)$ we have

$$\left.\frac{\partial\varphi}{\partial\nu}\right|_\Sigma \in L_2(\Sigma) \quad \text{and} \quad \left.\frac{\partial\psi}{\partial\nu}\right|_\Sigma \in L_2(\Sigma) \tag{A.9}$$

a sharp *trace theory* result (*not* obtainable from interior regularity, via standard trace theory): see above references. In the general case of $\mathscr{A}(\xi, \partial)$, the conormal derivative $\partial/\partial\nu_{\mathscr{A}}$ replaces the normal derivative of the Laplacian case.

Thus for problem (A.1) the associated cost is

$$\int_0^T (R_1 w(t), w(t))_{L_2(\Omega)} + (R_2 w_t(t), w_t(t))_{H^{-1}(\Omega)} + |u(t)|^2_{L_2(\Gamma)}\, dt \tag{A.10}$$

and assumption (H.2) = (1.9) means $0 \leq R_1 = R_1^* \in \mathcal{L}(L_2(\Omega))$ and $0 \leq R_2 = R_2^* \in \mathcal{L}(H^{-1}(\Omega))$.

Finite Cost Condition (H.3) = (2.7). As explicitly pointed out in Remark 2.1, exact controllability on the space $Y = L_2(\Omega) \times H^{-1}(\Omega)$ holds true for problem (A.1) with constant coefficients, for an arbitrary domain Ω with sufficiently smooth Γ [L-2; L-3], [T.2], the latter reference also in the variable coefficient case. Thus, a fortiori, the finite cost condition (H.3) is fulfilled in these cases.

B) Second order scalar hyperbolic equations with Neumann boundary control

We consider the canonical model

$$\begin{cases} w_{tt} = \Delta w \\ w|_{t=0} = w_0, \qquad w_t|_{t=0} = w_1, \\ \left.\dfrac{\partial w}{\partial \nu}\right|_{\Sigma} = u \in L_2(0, T; L_2(T)). \end{cases} \tag{B.1}$$

Let $\mathscr{A}_0$ be the (negative self-adjoint) realization of Δ on $L_2(\Omega)$ with homogeneous Neumann boundary conditions. Following [T.1], [T.3], [L-T.1], [L-T.2] introduce the Neumann map (of the translated problem) N defined by

$$Nv = h \Leftrightarrow \begin{cases} (\Delta - 1)h = 0 & \text{in } \Omega \\ \dfrac{\partial h}{\partial \nu} = v & \text{on } \Gamma \end{cases} \tag{B.2}$$

$$N\colon \text{continuous } L_2(\Gamma) \to \mathscr{D}(\mathscr{A}^{\frac{3}{4}-\varepsilon}) \equiv H^{\frac{3}{2}-2\varepsilon}(\Omega) \tag{B.3}$$

where

$$-\mathscr{A} = \mathscr{A}_0 - I.$$

To put problem (B.1) into the abstract form (1.1) we choose (according to recently established regularity results [L-T.6])

$$\text{(B.4)} \qquad \begin{aligned} Y &\equiv H^{\alpha}(\Omega)\times H^{\alpha-1}(\Omega)\,, \qquad y=[w,w_t]\,, \qquad U=L_2(\Gamma) \\ &= \mathscr{D}(\mathscr{A}^{\alpha/2})\times[\mathscr{D}(\mathscr{A}^{(1-\alpha)/2})]' \end{aligned}$$

(with equivalent norms, duality with respect to $L_2(\Omega)$), where

$$\text{(B.5)} \qquad \begin{aligned} &\alpha = 1 && \text{for } \dim\Omega = 1 \\ &\alpha = \tfrac{2}{3} && \text{for } \Omega \text{ a sphere, } \dim\Omega\geqslant 2 \\ &\alpha = \tfrac{3}{4}-\varepsilon && \text{for } \Omega \text{ a parallelepiped, } \dim\Omega\geqslant 2,\ \varepsilon>0 \\ &\alpha = \tfrac{3}{5}-\varepsilon && \text{for general (smooth) domains } \Omega,\ \dim\Omega\geqslant 2 \end{aligned}$$

Following [L-T.1], [T.1], [T.3] etc., the operators A and B of model (1.1) are

$$\text{(B.6)} \qquad A = \begin{vmatrix} 0 & I \\ -\mathscr{A} & 0 \end{vmatrix}; \qquad Bu = \begin{vmatrix} 0 \\ \mathscr{A}Nu \end{vmatrix} \text{ (formally)}: \ A^{-1}Bu = \begin{vmatrix} -Nu \\ 0 \end{vmatrix}$$

$$\text{(B.7)} \qquad \exp[At] = \begin{vmatrix} \mathscr{C}(t) & \mathscr{S}(t) \\ -\mathscr{A}\mathscr{C}(t) & \mathscr{C}(t) \end{vmatrix}$$

with $\mathscr{C}(t)$ the cosine operator generated by the negative self-adjoint operator $-\mathscr{A}$ and $\mathscr{S}(t) = \int_0^t \mathscr{C}(\tau)\,d\tau$. Using (B.6) and the topologies of (B.4) we compute B^* for $y=[y_1, y_2]\in Y$:

$$(Bu, y)_Y = (\mathscr{A}Nu, y_2)_{[\mathscr{D}(\mathscr{A}^{(1-\alpha)/2})]'} = (\mathscr{A}^{\alpha}Nu, y_2)_{L_2(\Omega)} = (u, N^*\mathscr{A}^{\alpha}y_2)_{L^2(\Gamma)}$$

and

$$\text{(B.8)} \qquad B^*\begin{vmatrix} y_1 \\ y_2 \end{vmatrix} = N^*\mathscr{A}^{\alpha}y_2\,.$$

Moreover, by Green second theorem [L-T.1], [T.1], [T.3]

$$\text{(B.9)} \qquad N^*\mathscr{A}f = f|_{\Gamma}\,.$$

Thus by (B.8) and (B.7) with $A = -A^*$:

$$\begin{aligned} B^*\exp[A^*t]\begin{vmatrix} y_1 \\ y_2 \end{vmatrix} &= N^*\mathscr{A}^{\alpha}[\mathscr{A}\mathscr{S}(t)y_1 - \mathscr{C}(t)y_2] \\ \text{(B.10)} \qquad &= N^*\mathscr{A}[-\mathscr{C}(t)\mathscr{A}^{\alpha-1}y_2 + \mathscr{S}(t)\mathscr{A}^{\alpha}y_1] \\ \text{(B.11)} \qquad &= N^*\mathscr{A}^{(\alpha+1)/2}[-\mathscr{C}(t)\mathscr{A}^{(\alpha-1)/2}y_2] + N^*\mathscr{A}^{1+(\alpha/2)}\mathscr{S}(t)\mathscr{A}^{\alpha/2}y_1] \\ & \qquad y=[y_1,y_2]\in Y \end{aligned}$$

The Abstract Assumption (H.1) = (1.2). In view of (B.11), assumption (H.1) is equivalent to

$$\text{(B.11)} \qquad \left.\begin{array}{l} N^*\mathscr{A}^{(\alpha+1)/2}\mathscr{C}(t) \\ N^*\mathscr{A}^{1+(\alpha/2)}\mathscr{S}(t) \end{array}\right\} : \text{continuous } L_2(\Omega) \to L_2(0, T; L_2(\Gamma))$$

which indeed holds true, as proved in [L-T.2], [L-T.6]. By (B.9), (B.10), we have that (B.11) is equivalent in P.D.E.'s terms to

$$\int_\Sigma \varphi^2\, d\Sigma \leqslant C_T \|\{\varphi^0, \varphi^1\}\|^2_{\mathscr{D}(\mathscr{A}^{(1-\alpha)/2}) \times [\mathscr{D}(\mathscr{A}^{\alpha/2})]'}$$

where

$$\text{(B.13)} \qquad \begin{cases} \varphi_{tt} = \Delta\varphi \\ \varphi|_{t=0} = \varphi^0 = -\mathscr{A}^{\alpha-1}y_2; \qquad \varphi_t|_{t=0} = \varphi^1 = \mathscr{A}^\alpha y_1 \\ \left.\dfrac{\partial\varphi}{\partial\nu}\right|_\Sigma \equiv 0\,. \end{cases}$$

Finite Cost Condition (H.3) = (2.7). Here, at present, the situation in the Neuman case (B.1) is quite different from the Dirichlet case (A.1). In fact, exact controllability (or uniform stabilization) results with $L_2(\Sigma)$-Neumann controls have been established so far only on the space $H^1(\Omega)\times L_2(\Omega)$ (of finite « energy ») (under some geometrical conditions on Ω, if $\dim \Omega \geqslant 2$) [C-1, C.2], [L.2], [L.7], [L-T.4], [L-T.13], [T.3]; or else in the larger space $L_2(\Omega)\times[H^1(\Omega)]'$ with a larger class of controls, see [L.2], [L-T.13]; and by interpolation in between. Thus, by (B.5) the space of exact controllability and the regularity space Y coincide for $\dim \Omega = 1$, in which case the finite cost condition is a fortiori fulfilled. Thus, the case $\dim \Omega = 1$ for (B.1) is covered by the theory of the present paper. In higher dimensions, however, the question of the finite cost condition, in particular the question of exact controllability on the space Y in (B.5) with $L_2(\Sigma)$-controls, is open at present.

C) Plate like equations.

Consider the canonical situation

$$\text{(C.1)} \qquad \begin{cases} w_{tt} + \Delta^2 w = 0 & \text{in } (0, T]\times\Omega = Q \quad a) \\ w|_{t=0} = w_0\,, \qquad w_t|_{t=0} = w_1 & \text{in } \Omega \quad b) \\ w|_\Sigma = u_1 & \text{in } (0, T]\times\Gamma \equiv \Sigma \quad c) \\ \left.\dfrac{\partial w}{\partial\nu}\right|_\Sigma = u_2 & \text{in } (0, T]\times\Gamma \equiv \Sigma \quad d)\,. \end{cases}$$

To put problem (C.1) into the abstract form (1.1), we shall specialize to two choices of spaces $\{U, Y\}$, in order to satisfy both the (trace regularity) assumption

(H.1) = (1.2) and the Finite Cost Condition (H.3) = (2.7). Deferring the choice of such spaces to the end of our analysis, we begin by setting

$$A = \begin{vmatrix} 0 & I \\ -\mathscr{A} & 0 \end{vmatrix}, \tag{C.2}$$

where the operator $\mathscr{A}$, defined by $\mathscr{A}f = \Delta^2 f$, $\mathscr{D}(\mathscr{A}) = H^4(\Omega) \cap H_0^2(\Omega)$, is positive self-ajoint on, say, $L_2(\Omega)$. Thus $-\mathscr{A}$ generates a s.c. cosine operator (self-adjoint) $\mathscr{C}(t)$ with $\mathscr{S}(t)y = \int_0^t \mathscr{C}(\tau)y\,d\tau$, say in $L_2(\Omega)$, $t \in R$. Then, as in (A.4), we obtain

$$\exp[At] = \begin{vmatrix} \mathscr{C}(t) & \mathscr{S}(t) \\ -\mathscr{A}\mathscr{S}(t) & \mathscr{C}(t) \end{vmatrix}. \tag{C.3}$$

Next, we introduce [L-T.7], [L-T.8] the following operators (Green maps) G_1 and G_2 defined by:

$$G_1 g_1 = h \Leftrightarrow \begin{cases} \Delta^2 h = 0 & \text{in } \Omega \\ h|_\Gamma = g_1 & \text{in } \Gamma \\ \left.\dfrac{\partial h}{\partial \nu}\right|_\Gamma = 0 & \text{in } \Gamma \end{cases} \tag{C.4}$$

$$G_2 g_2 = h \Leftrightarrow \begin{cases} \Delta^2 h = 0 & \text{in } \Omega \\ h|_\Gamma = 0 & \text{in } \Gamma \\ \left.\dfrac{dh}{\partial \nu}\right|_\Gamma = g_2 & \text{in } \Gamma \end{cases} \tag{C.5}$$

$$G_1\colon \text{continuous } L_2(\Gamma) \to L_2(\Omega) \tag{C.6}$$

$$G_2\colon \text{continuous } L_2(\Gamma) \to L_2(\Omega)\,. \tag{C.7}$$

As operator B we take

$$B\begin{vmatrix} u_1 \\ u_2 \end{vmatrix} = \begin{vmatrix} 0 \\ G_1 u_1 + G_2 u_2 \end{vmatrix} \text{ (formally)}; \qquad \mathscr{A}^{-1}B\begin{vmatrix} u_1 \\ u_2 \end{vmatrix} = \begin{vmatrix} -G_2 u_2 \\ \mathscr{A}G_1 u_1 \end{vmatrix}. \tag{C.8}$$

We now specify the choice of spaces U, Y in model (1.1) for problem (C.1). We consider two cases.

Case 1. - $u_1 \equiv 0$ in (C.1 *c*)). Thus we consider only the control action u_2 in the Neuman boundary condition (C.1 *d*)). We then take $U = U_1 \times U_2$, with $U_1 = \{0\}$ and

$$U_2 = L_2(\Gamma); \qquad Y \equiv L_2(\Omega) \times H^{-2}(\Omega) \equiv L_2(\Omega) \times [\mathscr{D}(\mathscr{A}^{\frac12})]' \tag{C.9}$$

With these topologies we compute B^* with

$$\left(B\begin{vmatrix}u_1\\u_2\end{vmatrix},\begin{vmatrix}y_1\\y_2\end{vmatrix}\right)_Y=\left(\begin{vmatrix}u_1\\u_2\end{vmatrix},B^*\begin{vmatrix}y_1\\y_2\end{vmatrix}\right)_U \tag{C.10}$$

and obtain from (C.8), (C.9):

$$B^*\begin{vmatrix}y_1\\y_2\end{vmatrix}=G_2^*y_2;\qquad (G_2u_2,z)_{L_2(\Omega)}=(u_2,G_2^*z)_{L_2(\Gamma)} \tag{C.11}$$

from which we have using $\exp[A^*t]=\exp[-At]$:

$$B^*\exp[A^*t]\begin{vmatrix}y_1\\y_2\end{vmatrix}=G_2^*\mathscr{A}\mathscr{S}(t)y_1-G_2^*\mathscr{C}(t)y_2=G_2^*\mathscr{A}[\mathscr{S}(t)y_1-\mathscr{C}(t)\mathscr{A}^{-1}y_2]\,. \tag{C.12}$$

by Green's second theorem one obtains as in [L-T.7], [L-T.8]

$$G_2^*\mathscr{A}f=\Delta f|_\Gamma\,,\qquad f\in\mathscr{D}(\mathscr{A}) \tag{C.13}$$

and hence by (C.12)-(C.13):

$$B^*\exp[A^*t]\begin{vmatrix}y_1\\y_2\end{vmatrix}=[\Delta\varphi(t,\varphi^0,\varphi^1)]|_\Gamma \tag{C.14}$$

where

$$\begin{cases}\varphi_{tt}+\Delta^2\varphi\equiv0 & \text{in } Q\\ \varphi|_{t=0}=-\mathscr{A}^{-1}y_2;\quad \varphi_t|_{t=0}=y_1 & \text{in } \Omega\\ \varphi|_\Sigma\equiv0 & \text{in } \Sigma\\ \left.\dfrac{\partial\varphi}{\partial\nu}\right|_\Sigma\equiv0 & \text{in } \Sigma\,.\end{cases} \tag{C.15}$$

The Abstract Assumption (H.1) = (1.2). In view of (C.14), then assumption (H.1) for problem (C.1) with $u_1\equiv0$ means

$$\int_\Sigma|\Delta\varphi|^2\,d\Sigma\leqslant C_T\|\{y_1,y_2\}\|^2_{L_2(\Omega)\times[\mathscr{D}(\mathscr{A}^{1/2})]'}=C_T\|\{\varphi_0,\varphi_1\}\|^2_{H_0^2(\Omega)\times L_2(\Omega)}\,. \tag{C.16}$$

Indeed, condition (C.16) always holds true for any $0<T<\infty$ and any Ω with sufficiently smooth Γ, as it follows by transposition applied to recently established regularity results for the problem dual to (C.1) for which we refer to [L.5], [L-T.8].

Finite Cost Condition (H.3) = (2.7). This follows a fortiori, with the present choice of spaces as in (C.9) corresponding to the case $u_1\equiv0$, from recent results on exact controllability established in [L.2].

Explicitly the cost functional J for problem (C.1) with $u_1 \equiv 0$ is then:

$$J(w, u_2) = \int_0^\infty (R_1 w(t), w(t))_{L_2(\Gamma)} + (R_2 w_t(t), w_t(t))_{H^{-2}(\Omega)} + |u_2(t)|^2_{L_2(\Gamma)} dt \tag{C.17}$$

Case 2. – $u_2 \equiv 0$ in (C.1 d)). Now we consider control action u_1 only in the Dirichlet boundary condition (C.1 c)). We then take $U = U_1 \times U_2$, $U_2 = \{0\}$ and

$$U_1 = L_2(\Gamma); \qquad Y = [\mathscr{D}(\mathscr{A}^{\frac14})]' \times [\mathscr{D}(\mathscr{A}^{\frac34})]' = H^{-1}(\Omega) \times V' , \tag{C.18}$$

$$\mathscr{D}(\mathscr{A}^{\frac14}) = H_0^1(\Omega) \tag{C.19}$$

(with equivalent norms)

$$\mathscr{D}(\mathscr{A}^{\frac34}) = V \tag{C.20}$$

$$V = \left\{ f \in H^3(\Omega) : f|_\Gamma = \left.\frac{\partial f}{\partial \nu}\right|_\Gamma = 0 \right\} \tag{C.21}$$

Then in the topologies of (C.18-C.20), we compute B^* as in (C.10) and obtain

$$B^* \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} = G_1^* \mathscr{A}^{-\frac14} y_2 . \tag{C.22}$$

Recalling (C.3) and using $\exp(A^* t) = \exp(-A\, t)$ we obtain by (C.22)

$$\begin{aligned} B^* \exp[A^* t] \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} &= G_1^*[-\mathscr{A}^{\frac14}\mathscr{S}(t) y_1 + \mathscr{A}^{-\frac14}\mathscr{C}(t) y_2] = \\ &= G_1^* \mathscr{A}[-\mathscr{A}^{-\frac34}\mathscr{S}(t) y_1 + \mathscr{A}^{-\frac54}\mathscr{C}(t) y_2] \end{aligned} \tag{C.23}$$

counterpart of (C.12). Now, however, we have

$$G_1^* \mathscr{A} f = \frac{\partial}{\partial \nu} \Delta f|_\Gamma , \qquad f \in \mathscr{D}(\mathscr{A}) \tag{C.24}$$

see [L-T.7], [L-T.8], instead of (C.13). Thus, (C.23)-(C.24) yield

$$B^* \exp[A^* t] \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} = \frac{\partial}{\partial \nu} \Delta \psi(t, \psi^0, \psi^1)|_\Gamma \tag{C.25}$$

where

$$\begin{cases} \psi_{tt} + \Delta^2 \psi = 0 \\ \psi|_{t=0} = \psi^0 = \mathscr{A}^{-\frac54} y_2 , \qquad \psi_t|_{t=0} = \psi^1 = -\mathscr{A}^{-\frac34} y_1 \\ \psi|_\Sigma \equiv 0 \\ \left.\dfrac{\partial \psi}{\partial \nu}\right|_\Sigma \equiv 0 . \end{cases} \tag{C.26}$$

The Abstract Assumption (H.1) = (1.2). In view of (C.25), then assumption (H.1) for problem (C.1) with $u_2 \equiv 0$ means

$$\int_{\Sigma} \left(\frac{\partial(\Delta\psi)}{\partial\nu}\right)^2 d\Sigma \leqslant C_T \| \{y_1, y_2\} \|^2_{[\mathscr{D}(\mathscr{A}^{1/4})]' \times [\mathscr{D}(\mathscr{A}^{3/4})]'} = C_T \| \{\psi^0, \psi^1\} \|^2_{V \times H_0^1(\Omega)} \tag{C.27}$$

Again, condition (C.27) can be shown to always hold true for all $0 < T < \infty$ and any Ω with sufficiently smooth Γ, by transposition in results of [L.5], see [L-T.8].

Finite Cost Condition (H.3) = (2.7). This follows a fortiori, with the present choice of spaces as in (C.18) corresponding to the case $u_2 \equiv 0$, from recent results on exact controllability established in [L-T.7], [L-T.8] at least under mild geometrical conditions on Ω.

Explicitly, the cost functional J for problem (C.1) with $u_2 \equiv 0$ is then

$$J(w, u_1) = \int_0^\infty (R_1 w(t), w(t))_{[\mathscr{D}(\mathscr{A}^{1/4})]'} + (R_2 w_t(t), w_t(t))_{[\mathscr{D}(\mathscr{A}^{3/4})]'} + |u_1(t)|^2_{L_2(\Gamma)} dt$$

D) First order hyperbolic systems.

Consider the following not necessarily symmetric or dissipative first order hyperbolic system in the unknown $y(\xi_1, \xi_2, \ldots, \xi_n) \in R^m$

$$\begin{cases} \partial_t y = \sum_{j=0}^{n} A_j(\xi)\partial_j y & \text{in } (0, T] \times \Omega \\ y|_{t=0} = y_0 \in [L_2(\Omega)]^m & \text{in } \Omega \\ M(\sigma) y(t, \sigma) = u(t, \sigma) \in L_2(0, T; [L_2(\Gamma)]^k) & \text{in } (0, T) \times \Gamma \end{cases} \tag{D.1}$$

where A_j are smooth $k \times k$ matrix valued functions under the assumptions of (a) strict hyperbolicity and of (b) Γ being non-characteristic and (c) rank $M(\sigma) = k \leqslant m$; here k stands for the number of negative eigenvalues of $A_N = \sum_{j=1}^{n} A_j(\,) N_j$, $N = [N_1, \ldots, N_m]$ outward unit normal. Here the regularity properties of the mixed problem (D.1) are already available form [K.1] [R.1] and are put in a semigroup framework in [C-L.1]. To put problem (D.1) in the abstract form (1.1), we choose $Y = [L_2(\Omega)]^m$, and A = first order differential operator F with homogeneous boundary conditions, where

$$Fy = \sum_{j=0}^{n} A_j(\xi)\partial_j y$$

$B = AD_1$ (formally); $A^{-1}B = D_1$ where (up to a translation)

$$\text{(D.2)} \qquad D_1 g = f \text{ means } \begin{cases} Ff = 0 & \text{in } \Omega \\ Mf = g & \text{in } \Gamma \end{cases}$$

$$\text{(D.3)} \qquad D_1\text{: continuous } [L_2(\Gamma)]^k \to [L_2(\Omega)]^m$$

$$\text{(D.4)} \qquad (Lu)(t) = A\int_0^t \exp[A(t-\tau)]D_1 u(\tau)\,d\tau$$

with $\exp[At]$ the s.c. semigroup on $[L_2(\Omega)]^m$ generated by A.

$$\text{(D.5)} \qquad B^* x = A_N^- x^-|_\Gamma\,, \quad x = [x^-, x^+]\,, \quad \dim x^- = k$$

The Abstract Assumption (H.1) = (1.2). By (D.5), assumption (H.1) for the mixed problem means the *sharp trace regularity* result: $y|_\Sigma \in L_2(0, T; [L_2(\Gamma)]^k)$, which indeed holds true and is *not* obtainable from the interior regularity $y \in C([0, T]; [L_2(\Omega)]^m)$, see [K.1] and [R.1].

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