On the Convergence of the Multigroup, Discrete-Ordinates Solutions for Subcritical Transport Media (*).

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Summary. – *In the multigroup, discrete-ordinates approximations to the linear transport equation, the integration over the directional variable is replaced by a numerical quadrature rule, involving a weighted sum over functional values at selected directions, with the energy dependence discretized by replacing the cross section data by weighted averages over each energy interval. The stability, consistency, and convergence rely fundamentally on the conditions that the maximum fluctuations in the total cross section—and in the expected number of secondary particles arising from each energy level—tend to zero as the energy mesh becomes finer, and as the number of angular nodes becomes infinite. Our analysis is based on using a natural Nyström method of extending the discrete-ordinates, multigroup approximates to all values of the angular and energy variables. Such an extension enables us to employ generalizations of the collectively compact operator approximation theory of P. M. Anselone to deduce stability and convergence of the approximates.*

1. – Introduction.

The steady-state, energy-dependent, linear transport equation is an integro-differential equation, whose dependent variable describes the distribution of particles in a reactor medium with respect to position, direction, and energy. The integral operator describes the generation of particles possessing any of the velocities from a velocity range by means of scattering from other velocities and by production by fission. The differential operator describes the streaming of particles with an arbitrary velocity and the loss of such particles through absorption and scattering into other velocities. Source terms and boundary conditions corresponding to incoming directions may be present, along with an initial distribution of particles, in case time-dependent transport is considered.

Multigroup, discrete-ordinates approximations arise when we replace the exact transport model by an approximate one by first definining average values of the

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problem data over given energy intervals; and secondly, by replacing the integral operator by a numerical (quadrature) operator, thereby obtaining a system of partial differential equations to be solved for the approximate solution at the quadrature points and on each energy interval. This discretization process introduces three fundamental sources of errors: First, the individual energies of the particles within an energy interval or group are uncertain (i.e., a particle is known to be within a group but its energy is unknown); second, the particles are constrained to interact with a single cross-section value (i.e., the group-averaged cross-section); third, much information is potentially lost concerning the streaming of neutral particles in media where absorption is dominant due to the need for many discrete directions to accurately describe a very anisotropic flux. Nevertheless, multigroup, discrete-ordinates approximations are most widely used for approximating the energy distribution of particles in a system modeled by linear transport.

Although there is a vast amount of practical experience in using such approximations, the supporting mathematical theory has been developed within the last six years at least as far as steady state multidimensional transport is concerned. For discrete-ordinates approximations for monoenergetic models, we refer the reader to the Introduction in [8], where contributions to slab transport are also outlined. For multidimensional settings (in the spatial variable), the basic convergence question for monoenergetic discrete-ordinates approximations was settled by NELSON and VICTORY in [8] (two dimensions) and by VICTORY in [14] (three-dimensions). The convergence question for steady state multigroup (with the spatial and angular variables undiscretized) has been investigated by NELSON and VICTORY, and VICTORY, in [9, 15] for slab transport, and by VICTORY in [16] for submultiplying multidimensional transport. For time-dependent transport, the inherent accuracy of the multigroup approximates were obtained by BELLENI-MORANTE and BUSONI [3] for slab media and by YANG MINGZHU and ZHU GUANGHAN [17] for multidimensional media. It is appropriate to remark at this point that a convergence analysis for the fully discretized, monoenergetic slab transport equation was done by J. PITKÄRANTA and R. SCOTT [10] under the assumption of isotropic scattering. Finally, E. ALLEN [1], using finite element techniques, investigated the convergence question for the multigroup approximates, but somewhat restrictive conditions on the behavior of the cross-section data were imposed.

The analysis in this work seeks to demonstrate that the multigroup discrete-ordinates approximations are well-defined and converge to the exact transport solution in any *subcritical* setting. This requirement basically necessitates that our analysis employs techniques different from those used in [3, 9, 16, 17]. We shall, for the most part, focus on transport in two-dimensional Cartesian geometry. A Nyström technique of defining the multigroup discrete-ordinates approximates to all values of the phase space variables enables us to use the collectively compact operator approximation theory of P. M. ANSELONE [2] to study convergence in a functional analytic setting.

In Section 2, we introduce notation and assumptions concerning the transport problem in a two-dimensional medium. The multigroup, discrete-ordinates model is formulated in Section 3, and the convergence proof is given in Section 4. More precisely, consistency and convergence of our approximations are shown under the conditions that the *maximum fluctuations* in the total cross-section, and in the expected number of secondary particles from each energy level, tend to zero as the energy mesh becomes finer.

2. – Regularity properties of transport solutions.

2.1. *The two-dimensional linear transport equation: notation.*

Let Γ be a closed, bounded convex region in $\mathbb{R}^2$, with $\partial\Gamma$ denoting a piecewise C^1-boundary, containing a finite number of (one-dimensional) *exposed faces* [11, pp. 162-3]. The direction cosines with respect to the x_1 and x_2 axes will be denoted by Ω_1 and Ω_2 respectively. The stationary, energy-dependent linear transport equation in two-dimensional (rectangular) geometry for the angular flux Ψ is:

$$\Omega\cdot\nabla_x\Psi(x,\Omega,E)+\sigma(x,\Omega,E)\Psi(x,\Omega,E)=q(x,\Omega,E)+ \\ +\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}}\int_{D^2}\int p(x,\Omega',E',\Omega,E)\sigma(x,\Omega',E')\Psi(x,\Omega',E')(1-|\Omega'|^2)^{-\frac{1}{2}}\,d\Omega'\,dE' \tag{2.1}$$

with

$$\Psi(x,\Omega,E)=h(x,\Omega,E)\,,\qquad x\in\partial\Gamma\,,\qquad \Omega\cdot n_x<0\,,\qquad E\in[E_{\mathfrak{m}},E_{\mathfrak{M}})\,.$$

Here,

(i) x represents a spatial element of Γ;

(ii) Ω represents an element (Ω_1,Ω_2) of $D^2:=\{(\Omega_1,\Omega_2):\Omega_2^2+\Omega_2^2\leqslant 1\}$, with $|\Omega|^2=\Omega_1^2+\Omega_2^2$;

(iii) ∇_x denotes the gradient with respect to the spatial variables;

(iv) n_x is the outward normal at $x\in\partial\Gamma$.

(For values of $x\in\partial\Gamma$ for which n_x is not well-defined, the above boundary condition is taken to apply to those Ω for which $\{(x_1+t\Omega_1,x_2+t\Omega_2):t>0\}\cap\mathring{\Gamma}\neq\emptyset$, where $\mathring{\Gamma}$ indicates the interior of Γ);

(v) E denotes the energy variable, an element of $[E_{\mathfrak{m}},E_{\mathfrak{M}})$, with $E_{\mathfrak{m}}$ and $E_{\mathfrak{M}}$ representing, respectively, the minimum and maximum energy attainable by a particle;

(vi) $\sigma(x, \Omega, E)$ is the total cross-section;

(vii) $p(x, \Omega', E', \Omega, E)$ is the transfer kernel, which describes the expected distribution of particles emerging from scattering events, fissions, etc.;

(viii) $q(x, \Omega, E)$ is the distributed source density;

(ix) $h(x, \Omega, E)$ represents boundary sources.

We assume that the region Γ is subdivided into a finite number of convex subregions, each with a boundary having properties similar to those of $\partial\Gamma$ itself. This requirement means that those subregions, lying wholly within the interior of Γ are polygonal due to the requirement of convexity. In our discussion, we let $\mathfrak{P}$ denote phase space given by $\Gamma \times D^2 \times [E_{\mathfrak{m}}, E_{\mathfrak{M}})$. The transport equation in integral form becomes:

$$\Psi(x, \Omega, E) = \boldsymbol{MQ}\Psi(x, \Omega, E) + \bar{q}(x, \Omega, E) \tag{2.2}$$

where $\bar{q}(x, \Omega, E)$, the uncollided angular flux from internal and boundary sources, is given by

$$\bar{q}(x, \Omega, E) = \boldsymbol{M}q(x, \Omega, E) + u(x, \Omega, E)\,, \tag{2.3}$$

$$u(x, \Omega, E) = \exp\left\{-\int_0^{d(x,\Omega)/|\Omega|} \sigma(x - t\Omega, \Omega, E)\,dt\right\} h(x - d(x, \Omega)|\Omega|^{-1}\Omega, \Omega, E)\,.$$

The quantity $d(x, \Omega)$ measures the distance of x from the exterior of Γ in the direction $-\Omega$, i.e.

$$d(x, \Omega) = \inf\{t > 0 : x - t|\Omega|^{-1}\Omega \notin \mathring{\Gamma}\}\,. \tag{2.4}$$

The operators $\boldsymbol{M}$ and $\boldsymbol{Q}$ are given respectively by

$$\boldsymbol{M}f(x, \Omega, E) = \int_0^{d(x,\Omega)/|\Omega|} f(x - t\Omega, \Omega, E) \exp\left\{-\int_0^t \sigma(x - r\Omega, \Omega, E)\,dr\right\} dt \tag{2.5}$$

and

$$\boldsymbol{Q}f(x, \Omega, E) = \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} \int_{D^2}\!\!\int p(x, \Omega', E', \Omega, E)\sigma(x, \Omega', E') f(x, \Omega', E')(1 - |\Omega'|^2)^{-\frac{1}{2}}\, d\Omega'\, dE'\,. \tag{2.6}$$

The behavior of $d(x, \Omega)/|\Omega|$ as a function of (x, Ω) plays a fundamental role in determining the regularity properties of Ψ. The following lemma describes this behavior and its simple, but tedious proof, uses fundamental properties of convex sets (cf. Theorem 6.1, p. 45, of [11]). The details are omitted.

LEMMA 2.1. – *The quantity* $d(x, \Omega)/|\Omega|(\equiv \infty$ *for* $|\Omega| = 0)$ *is continuous as an extended real-valued function in* $(x, \Omega) \in \bar{\Gamma} \times D^2$ *except at those points* $x \in \partial\Gamma$ *and* $\Omega \in D^2$ *such that either*

(i) $\Omega = 0$;

or

(ii) $\Omega \neq 0$ *and* $\{x - s|\Omega|^{-1}\Omega\}$ *is contained in a one-dimensional exposed face of* Γ *for every* s *such that* $0 \leqslant s \leqslant \alpha$, *for some* $\alpha > 0$.

We are able to conclude that the «singular directions» are precisely the *extreme directions* of Γ and of Γ_i, $i = 1, 2, \ldots, \mathfrak{N}$ where $\mathfrak{N}$ is the number of subregions of Γ (cf. p. 162 of [11]). We define

$$\Pi := \bigcup_i \pi(\Gamma_i) \cup \pi(\Gamma) \cup \{\Omega = 0\} \tag{2.7}$$

where $\pi(\Gamma_i) = \{\Omega \in D^2 \colon |\Omega| > 0$ and $|\Omega|^{-1}\Omega$ is an extreme direction of $\Gamma_i\}$. We observe from the symmetry assumptions in the third spatial variable that Π is the union of a finite number of diameters of D^2 because of the finite number of extreme directions associated with each Γ_i.

2.2. Description of two-dimensional media.

Much of the material in this subsection was motivated by the work of M. BORYSIEWICZ and N. KRUSZYNSKA [4]. In describing the geometry and the intersection of the characteristics of the streaming term with interfaces and their boundaries, we shall for the most part assume that $|\Omega| > 0$ since $\Omega = 0$ is associated with particles which cannot exit from the medium. This assumption is used, in particular, to study the regularity of the optical distance given later in (2.28).

The boundary $\partial\Gamma$ of Γ consists of the segments (the one-dimensional exposed faces), $\partial\Gamma_{0,j}$, $j = 1, 2, \ldots, \mathfrak{N}_0$ along with a curvilinear portion denoted as $\partial\Gamma_{0,c}$. Indeed

$$\partial\Gamma = \bigcup_{j=1}^{\mathfrak{N}_0} \partial\Gamma_{0,j} \cup \partial\Gamma_{0,c}\,. \tag{2.8}$$

For each $\partial\Gamma_i$, $i = 1, 2, \ldots, \mathfrak{N}$, we have the following representation,

$$\partial\Gamma_i = \bigcup_{j \in L_i} \{\partial\Gamma_{i,j} \colon \operatorname{card}(\partial\Gamma_{i,j}) > 1\} \cup \bigcup_{l \in K(i)} \partial\Gamma^i_{0,l} \cup \partial\Gamma^i_{0,c}\,, \tag{2.9}$$

where

(*a*) L_i denotes the set of indices of all neighborhood domains adjacent to Γ_i;

(*b*) $\partial\Gamma_{i,j} := \partial\Gamma_i \cap \Gamma_j$;

(*c*) $K(i)$ is a subset of $1, 2, \ldots, \mathfrak{N}_0$ (the indices of the boundary segments in $\partial\Gamma$) consisting of those indices $j \in \{1, 2, \ldots, \mathfrak{N}_0\}$ for which

$$\operatorname{card}(\partial\Gamma_{0,j} \cap \partial\Gamma_i) > 1;$$

(*d*) $\partial\Gamma^i_{0,l}$ indicates a prototype boundary segment which is common to both $\partial\Gamma_i$ and $\partial\Gamma$;

(*e*) $\partial\Gamma^i_{0,c}$ is the curvilinear portion of the boundary of $\partial\Gamma_i$ which is common also to $\partial\Gamma$.

The nomenclature in the preceding paragraph allows us to represent the boundary of a subregion Γ_i in terms of those segments lying in the interior, and that portion of its boundary lying on $\partial\Gamma$ itself. These two portions of $\partial\Gamma_i$ are denoted as $\partial\Gamma_i^I$ and $\partial\Gamma_i^0$ respectively, and we write

$$\partial\Gamma_i = \partial\Gamma_i^I \cup \partial\Gamma_i^0 \tag{2.10}$$

with

$$\partial\Gamma_i^I = \bigcup_{j\in L_i} \{\partial\Gamma_{i,j} \colon \operatorname{card}(\partial\Gamma_{i,j}) > 1\} \tag{2.11}$$

$$\partial\Gamma_i^0 = \bigcup_{l\in K(i)} \partial\Gamma^i_{0,l} \cup \partial\Gamma^i_{0,c}\,. \tag{2.12}$$

In Appendix A, we discuss an example which illustrates the notation introduced in this subsection. It is obvious that

$$n_{i,j} = -\, n_{j,i}$$

where $n_{i,j}$ is the outward normal vector to the segment $\partial\Gamma_{i,j}$ occurring in (2.9) and (2.11).

The next few paragraphs are devoted to describing precisely the intersection of the ray $\alpha(x, \Omega)$, defined as

$$\alpha(x, \Omega) := \{x' \in R^2 \colon x' = x - s|\Omega|^{-1}\Omega,\ s \in [0, \infty),\ 0 \neq \Omega \in D^2\}\,, \tag{2.13}$$

with the medium Γ. Such a ray will pass through the boundaries of subregions, or vertices thereof, or even coincide with segments of subregion boundaries.

The set of all vertices of the subregion boundaries, $\partial\Gamma_{i,}$, $i = 1, 2, \ldots, \mathfrak{N}$, will be denoted by W. We also define the « honeycomb » of Γ, G_0, to be the set

$$G_0 = \bigcup_{i=1}^{\mathfrak{N}} \partial\Gamma_i \cup \partial\Gamma . \tag{2.14}$$

All the line segments within the set G_0 when extended yield a finite collection of lines in two-dimensional Euclidean space $\mathbb{R}^2$. The direction cosines of these extended lines constitute the singular directions along with $|\Omega| = 0$. We let G_1 be the intersection of these lines with Γ itself. We have $G_0 \subset G_1$:

Our analysis will make crucial use of the intersection of the line $\alpha(x, \Omega)$, with the sets G_0 and G_1: The set of all points from G_0 crossed by $\alpha(x, \Omega)$ is denoted by $W_s(x, \Omega)$:

$$W_s(x, \Omega) = G_0 \cap \alpha(x, \Omega) . \tag{2.15}$$

With any element $w \in W_s(x, \Omega)$, we associate the set of integer pairs,

$$N(w) = \begin{cases} (i, j) , & i = \{1, 2, \ldots, \mathfrak{N}\} ,\ j \in L_i\text{: card } (\partial\Gamma_{i,j}) > 1,\ w \in \partial\Gamma_{i,j} \cap \mathring{\Gamma} \\ (i, j)_0 , & i \in \{1, 2, \ldots, \mathfrak{N}\} ,\ j \in K(i)\text{: } w \in \partial\Gamma^i_{0,j},\ \text{card } (\partial\Gamma^i_{0,j}) > 1 \end{cases} \tag{2.16}$$

where $(\cdot,\cdot)_0$ indicates the parameters of a boundary segment.

The singular directions can be characterized as follows

$$\Pi = \Pi_1 \cup \Pi_2 \tag{2.17}$$

where

$$\Pi_1 := \{\Omega \in D^2\text{: } |\Omega|^{-1}\Omega \cdot n_{i,j} = 0 \text{ for some } i \in \{1, 2, \ldots, \mathfrak{N}\}, j \ni L_i, \\ \text{card } (\partial\Gamma_{i,j}) > 1,\ n_{i,j} \text{ the outward unit normal to } \partial\Gamma_{i,j}\}, \tag{2.18}$$

and

$$\Pi_2 := \{\Omega \in D^2\text{:} |\Omega|^{-1}\Omega \cdot n^i_{0,j} = 0 \text{ for some } (i, j)_0,\ i \in \{1, \ldots, \mathfrak{N}\}, \\ j \in K(i),\ n^i_{0,j} \text{ the outward unit normal to } \partial\Gamma^i_{0,j}\} \cup \{\Omega\text{: } |\Omega| = 0\} . \tag{2.19}$$

For the example in Appendix A, the singular directions can be characterized by

$$\Pi = \{(\mu, 0)\text{: } |\mu| \leqslant 1\} \cup \{(0, \eta)\text{: } |\eta| \leqslant 1\} .$$

Finally, for each $\Omega \in \Pi$ and $x \in G_1$, we define the following set of integer pairs (i, j) to be the set

$$M(x, \Omega) := \{(i, j)\text{: card } (\partial\Gamma_{i,j} \cap \alpha(x, \Omega)) > 1,\ i \geqslant 1,\ |\Omega| = 1\} \cup \\ \cup \{(i, j)_0\text{: card } (\partial\Gamma^i_{0,j} \cap \alpha(x, \Omega)) > 1,\ |\Omega| = 1\} . \tag{2.20}$$

2.3. *Properties of the operators* $\boldsymbol{M}$ *and* $\boldsymbol{Q}$.

The following discussion in the main body of Section 2 provides general results for the operators M and Q defined by (2.5) and (2.6). As R. B. KELLOGG in [6] points out, transport solutions suffer from basically three types of discontinuities: *boundary singularities, shadow singularities,* and *vertex singularities.* In general, boundary singularities refer to the fact that the derivative of the angular flux possesses a logarithmic singularity on the boundary of Γ or of an interface. Vertex singularities account for the discontinuities in the first (spatial) partials in the interior of the medium or subregions. Because of the propagation of discontinuities along characteristics due to the first order hyperbolic nature of the streaming term, shadow singularities are due to the discontinuities in the cross-sections across common boundaries, or linear extensions, thereof. We now proceed to precisely define the topology of the function spaces where transport solutions can be expected to lie.

Our basic conditions on the cross-section data are as follows:

ASSUMPTION A. – *For all* $(x, \Omega) \in \Gamma \times D^2$, *and almost every* $E \in [E_{\mathfrak{m}}, E_{\mathfrak{M}})$, *the total cross-section is positive and bounded away from zero, with lower bound denoted by* $\sigma_{\mathfrak{m}}$. *The restriction of the following mapping to each* $\mathring{\Gamma}_i \times D^2$ ($\mathring{\Gamma}_i$, *the interior of* Γ_i):

$$(x, \Omega) \to \sigma(x, \Omega, \cdot)$$

is a continuous, $L^\infty[E_{\mathfrak{m}}, E_{\mathfrak{M}})$*-valued mapping which has a continuous extension to* $\Gamma_i \times D^2$. *We let* $\sigma_{\mathfrak{M}} := \sup_{(x,\Omega)} \|\sigma(x, \Omega, \cdot)\|_{L^\infty[E_{\mathfrak{m}}, E_{\mathfrak{M}})}$.

ASSUMPTION B. – *The transfer kernel* $p(x, \Omega', E', \Omega, E)$ *is nonnegative and satisfies the following additional assumptions:*

(i) *the mapping*

$$(x, \Omega, \Omega') \to \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} p(x, \Omega', \cdot, \Omega, E)\, dE \tag{2.21}$$

is a $L^\infty[E_{\mathfrak{m}}, E_{\mathfrak{M}})$*-valued mapping, continuous on each* $\Gamma_i \times D^2 \times D^2$, $i = 1, \ldots, \mathfrak{N}$;

(ii) *given* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that whenever* $|x - x_0|^2 + |\Omega - \Omega_0|^2 \leqslant \delta^2$ $x, x_0 \in \Gamma_i$,

$$\left\| \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |p(x, \Omega', \cdot, \Omega, E) - p(x_0, \Omega', \cdot, \Omega_0, E)|\, dE \right\|_{L^\infty[E_{\mathfrak{m}}, E_{\mathfrak{M}})} < \varepsilon\,, \tag{2.22}$$

uniformly for $\Omega' \in D^2$.

(iii) *if we define* $p(x, \Omega', E', E) \equiv 0$ *whenever* E *or* E' *is greater than* $E_{\mathfrak{M}}$, *or whenever* E *or* E' *is less than* $E_{\mathfrak{m}}$, *then, for each* $(x_0, \Omega_0) \in \Gamma_i \times D^2$,

$$\lim_{t\to 0}\left\| \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |p(x_0, \Omega', \cdot, \Omega_0, E+t) - p(x_0, \Omega', \cdot, \Omega_0, E)\, dE \right\|_{L^\infty[E_{\mathfrak{m}} E_{\mathfrak{M}})} = 0 , \tag{2.23}$$

uniformly for $\Omega' \in D^2$.

REMARKS. – *For item* (i), *we note that*

$$\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} p(x, \Omega', E', \Omega, E)\, dE \tag{2.24}$$

yields the expected number of particles in direction Ω *resulting from a collision of a of a particle at* x *with direction* Ω' *and energy* E'. *The supremum over* E' *yields the maximum expected number of particles with direction* Ω *which result from a collision at* x *of a particle with incipient direction* Ω', *and* (i) *guarantees this be finite. Items* (ii) *and* (iii) *are technical assumptions and are satisfied for any reasonable data*; *in particular, the conditions on the total scattering cross-section data imposed by* Belleni-Morante *and* Busoni *in* [3] *satisfy these hypotheses.*

At this point, we are able to make some rudimentary observations about the mapping properties of $\boldsymbol{M}$ and $\boldsymbol{Q}$. We introduce the function spaces needed in our analysis:

$\mathfrak{C}_0 := \{f(x, \Omega, E) \colon (x, \Omega) \to f(x, \Omega, \cdot)$ is continuous as in $L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})$-valued mapping of $\Gamma \times D^2$, except possibly for x lying along some exposed face of Γ, or the linear extension of an exposed face of some Γ_i and Ω parallel to such a face, such that

$$\left.\sup_{(x,\Omega)} \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |f(x, \Omega, E)|\, dE < \infty\right\};$$

$\mathfrak{C}^P := \{f(x, \Omega, E) \colon (x, \Omega) \to f(x, \Omega, \cdot)$ is continuous from each $\mathring{\Gamma}_i \times D^2$ to $L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})$ and has a continuous extension to $\Gamma_i \times D^2\}$.

Both $\mathfrak{C}_0$ and $\mathfrak{C}^P$ are equipped with the following norm,

$$\|f\| = \sup_{(x,\Omega)} \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |f(x, \Omega, E)|\, dE , \tag{2.25}$$

under which they are clearly Banach spaces. At times, we shall write $\|f\|_{\mathfrak{C}^P}$ or $\|f\|_{\mathfrak{C}_0}$ to indicate which space is being considered.

The assumptions on σ and p, in conjunction with the results of Lemma 2.1, imply that $\boldsymbol{M}$ is a bounded linear operator from $\mathfrak{C}^P$ to $\mathfrak{C}_0$ and that $\boldsymbol{Q}$ is a bounded linear operator from $\mathfrak{C}_0$ to $\mathfrak{C}^P$. The following representation of $\boldsymbol{M}$ shows that $\boldsymbol{M}$ has the asserted properties:

$$(2.26)\qquad \boldsymbol{M}g(x,\Omega,E)=\sum_i\int\limits_{d_i^-(x,\Omega)/|\Omega|}^{d_i^+(x,\Omega)/|\Omega|}g(x-t\Omega,\Omega,E)\exp\left\{-\int\limits_0^t\sigma(x-r\Omega,\Omega,E)\,dr\right\}dt$$

where $d_i^-(d_i^+)$ is the distance from x to the first (last) boundary point of subregion i in direction Ω, with $d_i^-=d_i^+=0$ if the ray from x in direction $-\Omega$ does not encounter subregion i. The $d_i^\pm$ will have continuity properties similar to those of d because each subregion is convex and has a boundary with properties similar to those of $\partial\Gamma$. Furthermore, σ is uniformly continuous in each $\Gamma_i\times D^2$ as a mapping with range in $L^\infty[E_\mathfrak{m},E_\mathfrak{M})$, and g is uniformly continuous in each $\Gamma_i\times\Pi_\delta$, with range in $L^1[E_\mathfrak{m},E_\mathfrak{M})$, where Π_δ is an arbitrary closed subset of D^2, whose distance from Π is δ. As $\Omega\in\Pi$ need not be considered for continuity properties of $\boldsymbol{M}g$, it follows that $\boldsymbol{M}g$ is continuous for $(x,\Omega)\in\Gamma\times\Pi_\delta$ for any $\delta>0$. By exploiting the positivity of σ on $\Gamma\times D^2$, it is easily seen that $\|\boldsymbol{M}\|\leqslant\sigma_\mathfrak{m}^{-1}$, thereby proving that $\boldsymbol{M}$ is a bounded linear operator from $\mathfrak{C}^P$ to $\mathfrak{C}_0$: Assumption B and the integrability of $(1-|\Omega'|^2)^{-\frac{1}{2}}$ over D^2 implies that $\boldsymbol{Q}$ is a bounded linear mapping of $\mathfrak{C}_0$ to $\mathfrak{C}^P$.

Thus the operator $\boldsymbol{MQ}$ is a bounded linear mapping of $\mathfrak{C}_0$ into itself. If we require $h(x,\Omega,E)$ to be continuous for $x\in\partial\Gamma$ and $\Omega\in\bigcup_{x\in\partial\Gamma}\bar{\Xi}_-(x)$, with range in $L^1[E_\mathfrak{m},E_\mathfrak{M})$, where $\Xi_-(x)$ represents the set of ingoing directions at $x\in\partial\Gamma$, and $\bar{\Xi}_-(x)$ its closure, we then see that $\bar{q}\in\mathfrak{C}_0$:

A precise description of the regularity properties of $\boldsymbol{M}f$, $f\in\mathfrak{C}^P$, is governed by the analytic properties of the optical distance between the two points x and x', $\varrho(x,x',\Omega,E)$, given by

$$(2.27)\qquad \varrho(x,x',\Omega,E):=\int\limits_0^{|x-x'|}\sigma(x-s|\Omega|^{-1}\Omega,\Omega,E)\,ds\,.$$

We recall the *optical distance* between a point x and the boundary $\partial\Gamma$ along the direction $-|\Omega|^{-1}\Omega$ is

$$(2.28)\qquad \varrho(x,\Omega,E):=\varrho(x,x-d(x,\Omega)|\Omega|^{-1}\Omega,\Omega,E)\,.$$

As is easily seen $\varrho(x,\Omega,\cdot)$ is a mapping from $\Gamma\times D^2$ to $L^\infty[E_\mathfrak{m},E_\mathfrak{M})$.

For a complete description of the continuity and differentiability properties of $\varrho(x,x',\Omega,E)$, and of $\boldsymbol{M}f$, $f\in\mathfrak{C}^P$, we shall need the following notation. Let x_{Γ_i} be a point on the boundary of Γ_i, and assume for the moment that $x_{\Gamma_i}\in\mathring{\Gamma}$. As we know from the results on the underlying properties of Γ itself, there is a $j\in L_i$ for which $x_{\Gamma_i}\in\partial\Gamma_{i,j}$. Let additionally $f\in\mathfrak{C}^P$ and define $f_{i,j}(x_{\Gamma_i},\Omega,\cdot)$, $i=\{1,2,\ldots,\mathfrak{N}\}$,

$j \in L_i$, to be the limit (in $L^1[E_m, E_{\mathfrak{M}}]$), given by

$$f_{i,j}(x_{\Gamma_i}, \Omega, \cdot) = \lim_{n\to\infty} f(x_n, \Omega, \cdot) \qquad x_n \to x_{\Gamma_i} \in \partial\Gamma_{i,j} \quad x_n \in \Gamma_i\,, \quad i \in \{1, 2, \ldots, \mathfrak{N}\}; \tag{2.29}$$

similarly, for $x_{\Gamma_i} \in \partial\Gamma^i_{0,j}$,

$$f^i_{0,j}(x_{\Gamma_i}, \Omega, \cdot) = \lim_{n\to\infty} f(x_n, \Omega, \cdot) \tag{2.30}$$

where $x_n \in \Gamma_i$, $x_n \to x_{\Gamma_i} \in \partial\Gamma^i_{0,j}$, $j \in K(i)$, $x_n \notin G_0$. The same definition applies, of course, to functions continuous on each $\Gamma_i \times D^2$, $i \in \{1, 2, \ldots, \mathfrak{N}\}$, with continuous extensions to the boundary of each Γ_i and having range in $L^\infty[E_m, E_{\mathfrak{M}}]$. For any positive number η, the set $\Pi(\eta)$ is the set consisting of those $\Omega \in \Pi$ for which $|\Omega| \geqslant \eta$ and D^2_η is the annulus $\{\Omega\colon \eta \leqslant |\Omega| \leqslant 1\}$.

The regularity properties of Mf, $f \in \mathfrak{C}^p$, will be better described by the following Banach space $\mathfrak{C}_1$, a closed subspace of $\mathfrak{C}_0$ under the supremum norm such that the elements of $\mathfrak{C}_1$ satisfy the following seven conditions:

1) f is continuous at all $x \in \mathring{\Gamma}_i$, the interior of the set Γ_i, and at $\Omega \in \Pi_\delta$, $\delta > 0$, $i = 1, 2, \ldots, \mathfrak{N}$;

2) f is continuous at all $x \in \mathring{\Gamma}_i \setminus \mathring{\Gamma}_i \cap G_1$, $\Omega \in D^2_\eta$, $i = 1, 2, \ldots, \mathfrak{N}$;

3) f is continuous at all $x \in \mathring{\Gamma}_i \cap G_1$, $\Omega \in \Pi(\eta)$ such that card $\big(W_s(x, \Omega)\big) < \infty$, $i = 1, 2, \ldots, \mathfrak{N}$.

4) For $x \in \mathring{\Gamma}_i$ and $\Omega \in \Pi(\eta)$ with card $\big(W_s(x, \Omega)\big) = \infty$, f is discontinuous in general, but has one-sided limits in $L^1[E_m, E_{\mathfrak{M}}]$ in the following sense: Let n_Ω be a normal to the ray $\alpha(x, \Omega)$ whose second component in selected positive if nonzero; otherwise, whose first component is selected positive. Let x^+_n, x^-_n, Ω^+_m, and Ω^-_m be sequences tending to x and Ω respectively and characterized by

$$\begin{gathered} x^+_n \to x\,, \quad (x^+_n - x)\cdot n_\Omega \geqslant 0\,, \quad x^+_n \in \mathring{\Gamma}_i\,, \\ x^-_n \to x\,, \quad (x^-_n - x)\cdot n_\Omega \leqslant 0\,, \quad x^-_n \in \mathring{\Gamma}_i\,, \\ \Omega^+_m \to \Omega\,, \quad \Omega^+_m\cdot n_\Omega > 0\,, \\ \Omega^-_m \to \Omega\,, \quad \Omega^-_m\cdot n_\Omega < 0\,. \end{gathered}$$

Then $\lim_{n\to\infty}\lim_{m\to\infty} f(x^+_n, \Omega^+_m, \cdot)$ and $\lim_{n\to\infty}\lim_{m\to\infty} f(x^-_n, \Omega^-_m, \cdot)$ are to exist in $L^1[E_m, E_{\mathfrak{M}}]$ and we define

$$\begin{aligned} f^+(x, \Omega, \cdot) &:= \lim_{n\to\infty}\lim_{m\to\infty} f(x^+_n, \Omega^+_m, \cdot) \\ f^-(x, \Omega, \cdot) &:= \lim_{n\to\infty}\lim_{m\to\infty} f(x^-_n, \Omega^-_m, \cdot)\,. \end{aligned}$$

The following iterative limits are hypothesized to exist in $L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})$ and are equal to f^+ and f^- respectively, i.e.

$$\lim_{m\to\infty}\lim_{n\to\infty} f(x_n^-, \Omega_m^-, \cdot) = \lim_{m\to\infty}\lim_{n\to\infty} f(x_n^+, \Omega_m^-, \cdot) = = \lim_{n\to\infty}\lim_{m\to\infty} f(x_n^+, \Omega_m^-, \cdot) = f^+(x, \Omega, \cdot); \tag{2.31}$$

$$\lim_{m\to\infty}\lim_{n\to\infty} f(x_n^+, \Omega_m^+, \cdot) = \lim_{m\to\infty}\lim_{n\to\infty} f(x_n^-, \Omega_m^+, \cdot) = = \lim_{n\to\infty}\lim_{m\to\infty} f(x_n^-, \Omega_m^+, \cdot) = f^-(x, \Omega, \cdot)\,. \tag{2.32}$$

5) An element f has an extension to the boundary of Γ_i defined in the manner below. Let $x_n \in \mathring{\Gamma}_i$, $\Omega \in \Pi_\delta$, and $x_n \to x_{\Gamma_i}$ Then,: denoting this extension as $f_{i,j}$,

$$f_{i,j}(x_{\Gamma_i}, \Omega, \cdot) := \lim_{n\to\infty} f(x_n, \Omega, \cdot)\,, \qquad \Omega \in \Pi_\delta\,, \quad x_n \in \mathring{\Gamma}_i \quad x_{\Gamma_i} \in \partial\Gamma_{i,j}\,, \tag{2.33}$$

with the limit existing in $L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})$ and f so extended is continuous at such (x_{Γ_i}, Ω). The extended function has the following regularity properties:

(*a*) For $x_{\Gamma_i} \in \partial\Gamma_{i,j}$ and $\Omega \in \Pi(\eta)$ for which card $\big(W_s(x, \Omega)\big) < \infty$,

$$f_{i,j}(x_{\Gamma_i}, \Omega, \cdot) := \lim_{n\to\infty} f(x_n, \Omega, \cdot)\,, \qquad x_n \in \mathring{\Gamma}_i\,, \tag{2.34}$$

with the limit existing in $L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})$ and $f_{i,j}$ is continuous at such points in $\partial\Gamma_{i,j}$, $\Omega \in \Pi(\eta)$.

(*b*) For $x \in \partial\Gamma_{i,j}$, $\Omega \in \Pi(\eta)$ with card $\big(W_s(x, \Omega)\big) = \infty$ and $\alpha(x, \Omega) \cap \mathring{\Gamma}_i \neq \emptyset$, $f_{i,j}$ is discontinuous at such points but possesses one-sided limits in $L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})$ in the sense described in condition (4). The quantities $f_{i,j}^+$ and $f_{i,j}^-$ are defined analogously to f^+ and f^- in (2.31) and (2.32) respectively. The sequence of points x_n^+ and x_n^- themselves can lie on $\partial\Gamma_{i,j}$ and the limits are precisely the limiting values of $f_{i,j}$ at points in (5*a*).

(*c*) Let $\alpha(x_{\Gamma_i}, \Omega) \cap \partial\Gamma_{i,j} \neq \emptyset$, with cardinality exceeding unity; moreover, let $n_{i,j}$ be a normal to $\alpha(x_{\Gamma_i}, \Omega)$ pointing outward from Γ_i. As in (4), let x_n^+, x_n^-, Ω_m^+, and Ω_m^- be sequences converging to x_{Γ_i} and Ω with the following property:

$$\begin{aligned} x_n^- &\to x_{\Gamma_i}\,, & (x_{\Gamma_i} - x_n^-)\cdot n_{i,j} &\geqslant 0\,,\\ x_n^+ &\to x_{\Gamma_i}\,, & (x_n^+ - x_{\Gamma_i})\cdot n_{i,j} &\geqslant 0\,,\\ \Omega_m^+ &\to \Omega\,, & \Omega_m^+\cdot n_{i,j} &> 0\,.\\ \Omega_m^- &\to \Omega\,, & \Omega_m^-\cdot n_{i,j} &< 0\,. \end{aligned}$$

Then $\lim_{m\to\infty} f_{i,j}(x_{\Gamma_i}, \Omega_m^-, \cdot)$ and $\lim_{m\to\infty}\lim_{n\to\infty} f(x_n^+, \Omega_m^+, \cdot)$ are to exist in $L^1[E_\mathfrak{m}, E_\mathfrak{M})$ and

$$f_{i,j}^+(x_{\Gamma_i}, \Omega, \cdot) := \lim_{m\to\infty} f_{i,j}(x_{\Gamma_i}, \Omega_m^-, \cdot)\,,$$

$$f_{i,j}(x_{\Gamma_i}, \Omega, \cdot) := \lim_{m\to\infty}\lim_{n\to\infty} f(x_n^+, \Omega_m^+, \cdot)\,.$$

The following iterative limits are hypothesized to exist in $L^1[E_\mathfrak{m}, E_\mathfrak{M})$ and are equal to $f_{i,j}^+$ and $f_{i,j}$ respectively, namely,

$$\begin{aligned}(2.35)\quad &\lim_{m\to\infty}\lim_{n\to\infty} f(x_n^-, \Omega_m^-, \cdot) = \lim_{m\to\infty}\lim_{n\to\infty} f(x_n^+, \Omega_m^-, \cdot) = \\ &= \lim_{n\to\infty}\lim_{m\to\infty} f(x_n^+, \Omega_m^+, \cdot) = \lim_{n\to\infty}\lim_{m\to\infty} f(x_n^+, \Omega_m^-, \cdot) = f_{i,j}^+(x_{\Gamma_i}, \Omega, \cdot)\,,\end{aligned}$$

$$\begin{aligned}(2.36)\quad &\lim_{m\to\infty}\lim_{n\to\infty} f(x_n^-, \Omega_m^+, \cdot) = \lim_{n\to\infty}\lim_{m\to\infty} f(x_n^-, \Omega_m^-, \cdot) = \\ &= \lim_{n\to\infty}\lim_{m\to\infty} f(x_n^-, \Omega_m^+, \cdot) = f_{i,j}(x_{\Gamma_i}, \Omega, \cdot)\,,\end{aligned}$$

The following compatibility relation is to hold

$$f_{i,j}(x_{\Gamma_i}, \Omega, \cdot) = f_{j,i}^+(x_{\Gamma_i}, \Omega, \cdot)\,.$$

6) Similar hypotheses are imposed for the extension of f up to the boundary of Γ itself. For example, in (5c), we define for $x_{\Gamma_i} \in \partial\Gamma_{0,j}^i$

$$(2.37)\qquad f_{0,j}^{i,+}(x_{\Gamma_i}, \Omega, \cdot) := \lim_{m\to\infty} f_{0,j}^i(x_{\Gamma_i}, \Omega_m^-, \cdot) = \lim_{m\to\infty}\lim_{n\to\infty} f(x_n^-, \Omega_m^-, \cdot)$$

with all limits assumed to exist in $L^1[E_\mathfrak{m}, E_\mathfrak{M})$. Moreover,

$$(2.38)\qquad f_{0,j}^i(x_{\Gamma_i}, \Omega, \cdot) := \lim_{m\to\infty} f_{0,j}^i(x_{\Gamma_i}, \Omega_m^+, \cdot)\,.$$

The following iterative sequences possess limits in $L^1[E_\mathfrak{m}, E_\mathfrak{M})$ and are equal to $f_{0,j}^i(x_{\Gamma_i}, \Omega, \cdot)$, namely

$$\begin{aligned}(2.39)\quad &\lim_{n\to\infty}\lim_{m\to\infty} f(x_n^+, \Omega_m^-, \cdot) = \lim_{m\to\infty}\lim_{n\to\infty} f(x_n^-, \Omega_n^+, \cdot) = \\ &= \lim_{n\to\infty}\lim_{m\to\infty} f(x_n^-, \Omega_m^+, \cdot) = f_{0,j}^i(x_{\Gamma_i}, \Omega, \cdot)\,.\end{aligned}$$

7) $\lim_{t\to 0} f(x, t\Omega, \cdot)$, $|\Omega| = 1$, exists in $L^1[E_\mathfrak{m}, E_\mathfrak{M})$ for both f and extensions to the boundary of each Γ_i.

With the notation used in the definition of $\mathfrak{C}_1$, we are better able to discuss the smoothness of p and $p(x, x', \Omega, \cdot)$ in Appendix B. The mapping properties of $\boldsymbol{M}$

are influenced by the regularity of the optical thickness and are summarized in Lemma 2.2, proved in Appendix C. In particular, conditions (4) and (5*b*, *c*) account for the *shadow* singularities alluded to earlier.

LEMMA 2.2. – *Under Assumptions A and B,* $\boldsymbol{M}$ *is a continuous linear mapping of* $\mathfrak{C}^p$ *to* $\mathfrak{C}_1$.

Our comments concerning the operator $\boldsymbol{Q}$ are summarized in Lemma 2.3:

LEMMA 2.3. – *Under Assumptions A and B,* $\boldsymbol{Q}$ *is a continuous linear mapping of* $\mathfrak{C}_1$ *to* $\mathfrak{C}^p$.

We wish to show that $\boldsymbol{MQ}$ is an eventually compact linear mapping of $\mathfrak{C}_1$ to itself. The compactness properties of $\boldsymbol{MQ}$ will rely crucially on the following abstract version of the Arzela-Ascoli characterization of compact sets in spaces of continuous function [5, p. 137], which we state for convenience:

THEOREM 2.1. – *Let* $\mathfrak{X}$ *be a compact metric space and* $\mathfrak{Y}$ *a Banach space with norm denoted by* $\|-\|_{\mathfrak{Y}}$. *Let* $\mathfrak{C}_{\mathfrak{Y}}(\mathfrak{X})$ *be the Banach space of functions continuous on* $\mathfrak{X}$ *with range in* $\mathfrak{Y}$, *equipped with a norm specified for an element* f *by* $\|f\| := \max_{x\in\mathfrak{X}} \|f(x)\|_{\mathfrak{Y}}$. *In order that a subset* $\mathfrak{H}$ *or* $\mathfrak{C}_{\mathfrak{Y}}(\mathfrak{X})$ *be relatively compact, necessary and sufficient conditions are that* $\mathfrak{H}$ *be equicontinuous and that, for each* $x\in\mathfrak{X}$, *the set* $\{\mathfrak{H}(x)\}$ *of all* $f(x)$, *such that* $f\in\mathfrak{H}$, *be relatively compact in* $\mathfrak{Y}$.

In applying this theorem, we let $\mathfrak{X}=\Gamma\times D^2$ and $\mathfrak{Y}=L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})$. From the results of Lemmas 2.2 and 2.3, we can deduce that $\boldsymbol{QM}$ is a continuous linear mapping of $\mathfrak{C}^p$ into itself. In applying Theorem 2.1 to prove complete continuity of $(\boldsymbol{QM})^2$, we must have criteria for relatively compact sets in L^p-function spaces. Such criteria are provided by the Fréchet-Kolmogorov Theorem [13], stated here for compact subsets of $L^p(\mathbb{R})$, $\mathbb{R}=(-\infty,\infty)$:

THEOREM 2.2 (Fréchet-Kolmogorov). – *Let* $\mathbb{R}$ *denote the real line and* $\mathfrak{R}$, *the* σ*-ring of Baire subsets of* $\mathbb{R}$, *with* $m_{\mathfrak{L}}:(B)=\int_B dt$ *the ordinary Lebesgue measure of* B. *Then a subset* $\mathfrak{K}$ *of* $L^p(\mathbb{R}, \mathfrak{R}:, m_{\mathfrak{L}})$, $1\leqslant p<\infty$, *is relatively compact if and only if it satisfies the conditions*:

$$(a)\ \sup_{f\in\mathfrak{K}}\|f\|=\sup_{f\in\mathfrak{K}}\left\{\int_{\mathbb{R}}|f(t)|^p\,dt\right\}^{1/p}<\infty;$$

$$(b)\ \lim_{t\to 0}\int_{\mathbb{R}}|f(t+s)-f(s)|^p\,ds=0\ \textit{uniformly for}\ f\in\mathfrak{K};$$

$$(c)\ \lim_{\alpha\to\infty}\int_{|s|>\alpha}|f(s)|^p\,ds=0\ \textit{uniformly for}\ f\in\mathfrak{K}.$$

REMARK. – *We shall consider functions in* $L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})$ *to be in* $L^1(\mathbb{R})$ *by defining them to be identically zero outside* $[E_{\mathfrak{m}}, E_{\mathfrak{M}})$. *Therefore, we must only verify the first two cri-*

teria of the Frechet-Kolmogorov theorem in order to show that $\{QMf(x_0, \Omega_0, \cdot)\|f\|_{\mathfrak{C}^P} \leq 1\}$ *is relatively compact whenever* $E_{\mathfrak{M}} < \infty$. *The proof of the following theorem is similar to the proof of* Theorem 4.1 *provided in* Section 4.

THEOREM 2.3. – *The mapping* $(QM)^2$ *is a completely continuous mapping of* $\mathfrak{C}^P$ *into itself.*

REMARK. – *To extend the analysis to include the case* $E_{\mathfrak{M}} = \infty$, *we need only augment part* (iii) *of* Assumption B *by the following condition on* $p(x, \Omega', E', \Omega, E)$: *For any* $(x_0, \Omega_0) \in \Gamma \times D^2$,

$$\lim_{E_0 \to \infty} \sup_{\Omega' \in D^2} \left\| \int_{E_0}^{\infty} p(x_0, \Omega', \cdot, \Omega_0, E)\, dE \right\|_{L^\infty[E_{\mathfrak{m}}, E_{\mathfrak{M}})} = 0 . \tag{2.40}$$

This assumption merely implies that the maximum (over energy) expected number of particles, shunted to energies greater than E_0 having direction Ω_0, becomes smaller with increasing E_0. By defining $p(x, \Omega', E', \Omega, E) \equiv 0$ for E or E' less than $E_{\mathfrak{m}}$, we see that the third requirement of Theorem 2.2 is met for showing compactness of QM when $E_{\mathfrak{M}} = \infty$.

LEMMA 2.4. – *The operator* MQ *is an eventually compact linear operator mapping* $\mathfrak{C}_1$ *into itself.*

PROOF. – We note that $(MQ)^3 = M(QM)^2Q$ and the assertion follows.

Our next assumption is concerned with solving (2.2) in $\mathfrak{C}_1$:

ASSUMPTION C. – *The problem defined by* (2.2) *is subcritical*, i.e.

$$\|MQ\|_{sp} < 1 , \tag{2.41}$$

where $\|MQ\|_{sp}$ *denotes the spectral radius of* MQ *defined to be*

$$\lim_{n \to \infty} \|(MQ)^n\|^{1/n} .$$

From the conditions imposed on the boundary data $h(x, \Omega, E)$ in this section—and from the properties of the optical distance as outlined in Appendix B—we can easily deduce that the function

$$u(x, \Omega, E) := \exp\left\{-\int_0^{d(x,\Omega)} \sigma(x - t|\Omega|^{-1}\Omega, \Omega, E)|\Omega|^{-1} dt\right\} h(x - d(x, \Omega)|\Omega|^{-1}\Omega, \Omega, E) \tag{2.42}$$

is *a-priori* an element of $\mathfrak{C}_0$ which can be extended to possess the continuity features ascribed to $\mathfrak{C}_1$: This extension relies on the properties of $d(x, \Omega)$ described in Lemma 2.1. The details are left to the interested reader. Therefore $\bar{q}$, as defined in (2.3) resides in $\mathfrak{C}_1$ and (2.2) will have a unique nonnegative solution Ψ by virtue of Assumption C.

In order to study the regularity features of $\Omega\cdot\nabla_x\Psi(x,\Omega,\cdot)$, we first observe that for any $g\in\mathfrak{C}^P$, $\boldsymbol{M}g$ will have a derivative with respect to x in direction $|\Omega|^{-1}\Omega$ which is a continuous mapping of each $\Gamma_i\times\Pi_{\delta_0}$ to $L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})$ for any $\delta_0>0$. Indeed, for $s\neq 0$, the difference quotients

$$s^{-1}[\boldsymbol{M}g(x+s|\Omega|^{-1}\Omega,\Omega,E)-\boldsymbol{M}g(x,\Omega,E)]$$

will have this property; the conditions on σ in Assumption A enables us to show:

$$\lim_{s\to 0}\max_{(x,\Omega)\in\Gamma_i\times\Pi_{\delta_0}}\left\|\frac{\boldsymbol{M}g(x+s|\Omega|^{-1}\Omega,\Omega,E)-\boldsymbol{M}g(x,\Omega,E)}{s}-\right.$$
$$\left.-\frac{g(x,\Omega,E)}{|\Omega|}+\frac{\sigma(x,\Omega,E)\boldsymbol{M}g(x,\Omega,E)}{|\Omega|}\right\|_{L^1[E_{\mathfrak{m}},E_{\mathfrak{M}})}=0 \tag{2.43}$$

and, likewise for $u(x,\Omega,E)$

$$\lim_{s\to 0}\max_{(x,\Omega)\in\Gamma_i\times\Pi_{\delta_0}}\left\|\frac{u(x+s|\Omega|^{-1}\Omega,\Omega,E)-u(x,\Omega,E)}{s}+\right.$$
$$\left.+\frac{\sigma(x,\Omega,E)u(x,\Omega,E)}{|\Omega|}\right\|_{L^1[E_{\mathfrak{m}},E_{\mathfrak{M}})}=0 \tag{2.44}$$

(with s such that $x+s|\Omega|^{-1}\Omega\in\mathring{\Gamma}_i$ whenever $x\in\partial\Gamma_i$).

Under the conditions, then, imposed on the total and scattering cross-section data in Assumptions A and B respectively, we can conclude that $\mathfrak{D}(x,\Omega,\cdot):=\Omega\cdot\nabla_x\Psi(x,\Omega,\cdot)$ is a continuous mapping of each $\Gamma_i\times\Pi_{\delta_0}$ to $L^1[E_{\mathfrak{m}},E_{\mathfrak{M}})$ which is precisely equal to

$$\boldsymbol{Q}\Psi(x,\Omega,\cdot)+q(x,\Omega,\cdot)-\sigma(x,\Omega,\cdot)\Psi(x,\Omega,\cdot)\,. \tag{2.45}$$

From (2.45), we note that $\mathfrak{D}(x,\Omega,\cdot)$ can be extended to be a function in $\mathfrak{C}_1$ since the functions in (2.45) lie in $\mathfrak{C}_1$: We summarize:

THEOREM 2.4. – *Suppose the total and scattering cross-section data satisfy Assumptions A, B and C above. Moreover, let* $q\in\mathfrak{C}^P$ *and the boundary data be continuous —with range in* $L^1[E_{\mathfrak{m}},E_{\mathfrak{M}})$*—for* $x\in\partial\Gamma$ *and* $\Omega\in\bigcup_{x\in\partial\Gamma}\bar{\Xi}_-(x)$, *where* $\Xi_-(x)$ *is the set of ingoing directions at* x *and* $\bar{\Xi}_-(x)$ *its closure. Then there exists a function* $\Psi(x,\Omega,\cdot)\in\mathfrak{C}_1$, *with* $\Omega\cdot\nabla_x\Psi(x,\Omega,\cdot)\in\mathfrak{C}_1$, *which satisfies equation* (2.1) *and the accompanying boundary conditions in the sense that*

$$\lim_{x\to x_\Gamma}\max_{\Omega\in\Xi_-(x_\Gamma)\cap D^3_\eta}\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}}|\Psi(x,\Omega,E)-h(x_\Gamma,\Omega,E)|\,dE=0\,. \tag{2.46}$$

3. – The multigroup, discrete-ordinates approximation.

For the discrete-ordinates approximation, we choose a set of directions, Ω_{mi}, $i = 1, 2, \ldots, N_m$, and quadrature weights w_{mi}, such that

$$\lim_{m\to\infty} \sum_{i=1}^{N_m} w_{mi} f(\Omega_{mi}) = \int_{D^2}\int f(\Omega)(1-|\Omega|^2)^{-\frac{1}{2}}\, d\Omega \tag{3.1}$$

for any f continuous on D^2; the points $\Omega \in \Pi$ are not chosen as quadrature points and

$$\sum_i \left\{|w_{mi}|\colon \min_{t\geq 0} |\Omega_{mi} - t\Omega_0| < \varepsilon\right\} \to 0 \tag{3.2}$$

as $(\varepsilon, m) \to (0, \infty)$ uniformly in $\Omega_0 \in D^2$. The principle of uniform boundedness [12, p. 48] enables us to deduce that

$$\sup_m \left\{\sum_{i=1}^{N_m} |w_{mi}|\right\} < \infty . \tag{3.3}$$

The following lemmas are easily shown (cf. [8, p. 358]):

LEMMA 3.1. – *Under the assumption that the quadrature process converges for every continuous function, then to every open subset* $\mathfrak{S} \subset D^2$, *there corresponds a number* n_0 *such that for all* $n \geq n_0$, $\mathfrak{S}$ *contains at least one* Ω_{ni}.

LEMMA 3.2. – *For quadrature formulas with nonnegative weights,* (3.1) *implies* (3.2).

To adequately describe the multigroup approximations to the boundary value problem (2.1), in conjunction with the discrete-ordinates approximations, we partition $[E_\mathfrak{m}, E_\mathfrak{M})$ into G_n subintervals $I_g^n = [E_{g-1}^n, E_g^n)$, $g = 1, 2, \ldots, G_n$, such that $E_\mathfrak{m} = E_0^n < E_1^n < \ldots < E_{G_n}^n = E_\mathfrak{M}$. As is well known, the multigroup equations provide approximations $\Psi_g^n(x, \Omega)$, $g = 1, \ldots, G_n$, to the exact angular flux integrated over each energy interval I_g^n, $g = 1, \ldots, G_n$. The determining equations for the multigroup, discrete-ordinates approximations are:

$$\begin{aligned}\Omega_{mi}\cdot\nabla_x \Psi_{mi,g}^n(x) + \sigma_g^n(x, \Omega_{mi})\Psi_{mi,g}^n(x) &= \\ = q_g(x, \Omega_{mi}) + \sum_{g'=1}^{G_n}\sum_{j=1}^{N_m} w_{mj} p_{gg'}^n(x, \Omega_{mj}, \Omega_{mi})&\sigma_g^n(x, \Omega_{mj})\Psi_{mj,g'}^n(x) , \\ g = 1, \ldots, G_n , \quad i &= 1, \ldots, N_m ,\end{aligned} \tag{3.4}$$

with

$$\Psi_{mi,g}^n(x) = \int_{I_g^n} h(x, \Omega_{mi}, E)\, dE , \qquad x \in \partial\Gamma , \qquad \Omega_{mi} \in \Xi_-(x) , \tag{3.5}$$

and

$$q_g^n(x, \Omega_{mi}) = \int_{I_g^n} q(x, \Omega_{mi}, E)\, dE\,. \tag{3.6}$$

The group-averaged cross-section data $\sigma_g^n(x, \Omega)$ and $p_{gg'}^n(x, \Omega', \Omega)$ are given respectively for each g in terms of a *reference* flux $\Psi_0(x, E)$ by

$$\sigma_g^n(x, \Omega) = \Big[\int_{I_g^n} \Psi_0(x, E)\sigma(x, \Omega, E)\, dE\Big] / \Psi_{0,g}^n(x) \tag{3.7}$$

$$p_{gg'}^n(x, \Omega', \Omega) = \Big[\int_{I_g^n}\int_{I_{g'}^n} p(x, \Omega', E', \Omega, E)\sigma(x, E')\Psi_0(x, E')\, dE'\, dE\Big] / \sigma_{g'}^n \Psi_{0,g'}^n(x) \tag{3.8}$$

with

$$\Psi_{0,g}^n(x) = \int_{I_g^n} \Psi_0(x, E)\, dE \tag{3.9}$$

and $g, g' = 1, \ldots, G_n$.

We refer the reader to [9, Section III] for a detailed discussion of the basic criteria used in selecting a reference flux and of the methods actually used for obtaining it. Also, the fundamental conditions imposed on the reference flux $\Psi_0(x, E)$ necessary for our convergence analysis can be found in the above reference. In addition, we assume that the mapping $x \to \Psi_0(x, \cdot)$ is a continuous $L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}}]$-valued mapping from each Γ_j, $j = 1, \ldots, \mathfrak{N}$. Such an assumption implies that $\sigma_g^n(x, \Omega)$ is continuous on each $\Gamma_j \times D^2$ and bounded above and below by $\sigma_{\mathfrak{M}}$ and $\sigma_{\mathfrak{m}}$ respectively.

At this point, we define $\boldsymbol{P}_n\Psi$, $\Psi \in \mathfrak{C}_1$, to be the G_n-tuple given by,

$$\boldsymbol{P}_n\Psi(x, \Omega) = \Big\{\int_{I_1^n} \Psi(x, \Omega, E)\, dE, \ldots, \int_{I_{G_n}^n} \Psi(x, \Omega, E)\, dE\Big\}\,. \tag{3.10}$$

We note that formally

$$\{\Psi_{mi,g}^n(x),\ g = 1, \ldots, G_n,\ i = 1, \ldots, N_m\} = \{\boldsymbol{P}_n\Psi_m^n(x, \Omega_{mi}),\ i = 1, \ldots, N_m\}$$

where $\Psi_m^n(x, \Omega, E)$, $E \in I_g^n$, solves

$$\begin{aligned} \Omega\cdot\nabla_x\Psi_m^n(x, \Omega, E) + \sigma_g^n(x, \Omega)\Psi_m^n(x, \Omega, E) = \\ = q(x, \Omega, E) + \sum_{g'=1}^{G_n}\sum_{j=1}^{N_m} w_{mj}\Big\{\int_{I_{g'}^n} p(x, \Omega_{mj}, E', \Omega, E)\sigma(x, \Omega_{mj}, E')\frac{\Psi_0(x, E')}{\Psi_{0,g'}^n(x)}\, dE'\Big\}\cdot \\ \cdot\Big(\int_{I_{g'}^n} \Psi_m^n(x, \Omega_{mj}, E'')\, dE''\Big) \end{aligned} \tag{3.11}$$

with boundary condition

$$\Psi_m^n(x, \Omega, E) = h(x, \Omega, E)\,, \quad x \in \partial\Gamma\,, \quad \Omega \in \Xi^-(x)\,, \quad E \in I_g^n\,. \tag{3.12}$$

Equations (3.11) and (3.12) formally allow us to extend the multigroup, discrete ordinates approximations to be functions with common domain $\Gamma \times D^2 \times [E_{\mathfrak{m}}, E_{\mathfrak{M}})$. The task before us now is to ascertain solvability of (3.11) and (3.12) in $\mathfrak{C}_1$. With such a result, we can study the convergence question in a functional-analytic setting and exploit the collectively compact operator approximation theory developed by P. M. Anselone [2].

Solving (3.11) and (3.12) leads to the integral equation for Ψ_m^n:

$$\Psi_m^n(x, \Omega, E) = \boldsymbol{M}_n \boldsymbol{Q}_{mn} \boldsymbol{P}_n \Psi_m^n(x, \Omega, E) + \boldsymbol{M}_n q(x, \Omega, E) + u_n(x, \Omega, E)\,, \tag{3.13}$$

where $\boldsymbol{M}_n$ and $\boldsymbol{Q}_{mn}\boldsymbol{P}_n$ are given respectively by

$$\boldsymbol{M}_n f(x, \Omega, E) = \int_0^{d(x,\Omega)/|\Omega|} f(x - t\Omega, \Omega, E) \exp\left\{-\int_0^t \sigma_g^n(x - r\Omega, \Omega)\, dr\right\} dt\,, \qquad (x, \Omega, E) \in \Gamma \times D^2 \times I_g^n\,, \tag{3.14}$$

and

$$\boldsymbol{Q}_{mn}\boldsymbol{P}_n f(x, \Omega, E) = \sum_{g'=1}^{G_n} \sum_{j=1}^{N_m} w_{mj} \left\{ \int_{I_{g'}^n} p(x, \Omega_{mj}, E', \Omega, E)\, \sigma(x, \Omega_{mj}, E') \frac{\Psi_0(x, E')}{\Psi_{0,g'}^n(x)}\, dE' \right\} (\boldsymbol{P}_n f)_{g'}(x, \Omega_{mj})\,, \tag{3.15}$$

with $(\boldsymbol{P}_n f)_g(x, \Omega_{mj})$ denoting the *gth* component of $\boldsymbol{P}_n f(x, \Omega)$ evaluated at $\Omega = \Omega_{mj}$, $j = 1, \ldots, N_m$: The contribution from boundary sources is given by

$$u_n(x, \Omega, E) = \exp\left\{-\int_0^{d(x,\Omega)/|\Omega|} \sigma_g^n(x - t\Omega, \Omega)\, dt\right\} h(x - d(x, \Omega)|\Omega|^{-1}\Omega, \Omega, E)\,, \qquad (x, \Omega, E) \in \Gamma \times D^2 \times I_g^n\,. \tag{3.16}$$

As a remark, we note that for each n, $\boldsymbol{M}_n$ defined in (3.14) is a continuous linear mapping from $\mathfrak{C}^P$ to $\mathfrak{C}_1$ and the mapping $\boldsymbol{Q}_{mn}\boldsymbol{P}_n$, defined by (3.15) for every m and n, is continuous with domain $\mathfrak{C}_1$ and range in $\mathfrak{C}^P$. The proofs of these results consist of trivial modifications of the proofs of Lemmas 2.2 and 2.3 by using the piecewise continuity of the $L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})$-valued mapping, $x \to \Psi_0(x, E)$. The approximate uncollided flux $\boldsymbol{M}_n q + u_n$ with u_n defined in (3.16), is an element of $\mathfrak{C}_1$, since u_n itself is in $\mathfrak{C}_1$ due to the piecewise continuity of $\{\sigma_g^n(x, \Omega),\ g = 1, 2, \ldots, G_n\}$. The solution $\Psi_m^n(x, \Omega, E)$ to (3.13) has the same regularity properties described for $\Psi(x, \Omega, E)$ in Theorem 2.4.

We now turn to addressing the convergence properties of $\boldsymbol{M}_n$ and $\boldsymbol{Q}_{mn}\boldsymbol{P}_n$. We refer the reader to [8, Section IV] for a discussion of the approximation theory by Anselone [2] needed in our analysis.

4. – The convergence analysis.

Let Ψ_m be the approximations to Ψ which result from discretizing the angular variable only, and let $q_n := \boldsymbol{M}_n q + u_n$, where u_n is given for each n by (3.16). Moreover, let the approximate scattering operator $\boldsymbol{Q}_m$ be defined analogously to $\boldsymbol{Q}$ in (2.6), with the integral over Ω' replaced by quadrature expressions with nodes $\{\Omega_{mi},\ i = 1, 2, \dots, N_m\}$. By a careful manipulation of the integral equations (2.2) and (3.13) for Ψ and Ψ_m^n respectively, we can obtain the following error estimate for $\Psi - \Psi_m^n$:

$$
(4.1) \qquad \Psi - \Psi_m^n = (\Psi - \Psi_m) + (u - u_n) + (\boldsymbol{M} - \boldsymbol{M}_n)(q + \boldsymbol{Q}\Psi) + \\ + (\boldsymbol{M} - \boldsymbol{M}_n)(\boldsymbol{Q}_m \Psi_m - \boldsymbol{Q}\Psi) + \boldsymbol{M}_n(\boldsymbol{Q}_n \Psi_m - \boldsymbol{Q}_{mn}\boldsymbol{P}_n \Psi_m^n)\,,
$$

where

$$
\begin{aligned}
(4.2) \qquad (\boldsymbol{Q}_m \Psi_m - \boldsymbol{Q}_{mn}\boldsymbol{P}_n \Psi_m^n) &= \\
&= [\boldsymbol{I} - (\boldsymbol{Q}_{mn}\boldsymbol{P}_n \boldsymbol{M}_n)^2]^{-1} \cdot \{(\boldsymbol{Q}_m - \boldsymbol{Q}_{mn}\boldsymbol{P}_n)(\boldsymbol{M}\boldsymbol{Q}_m)^2 \Psi_m + \\
&+ \boldsymbol{Q}_{mn}\boldsymbol{P}_n \boldsymbol{M}(\boldsymbol{Q}_m - \boldsymbol{Q}_{mn}\boldsymbol{P}_n)\boldsymbol{M}\boldsymbol{Q}_m \Psi_m + (\boldsymbol{Q}_m - \boldsymbol{Q}_{mn}\boldsymbol{P}_n) q_n + \\
&+ \boldsymbol{Q}_{mn}\boldsymbol{P}_n(\boldsymbol{M} - \boldsymbol{M}_n)(\boldsymbol{Q}_{mn}\boldsymbol{P}_n - \boldsymbol{Q}_m)\boldsymbol{M}_n \boldsymbol{Q}\Psi + (\boldsymbol{Q}_n - \boldsymbol{Q}_{mn}\boldsymbol{P}_n)\boldsymbol{M}_n \boldsymbol{Q}_m q_n + \\
&+ \boldsymbol{Q}_{mn}\boldsymbol{P}_n \boldsymbol{M}_n(\boldsymbol{Q}_m - \boldsymbol{Q}_{mn}\boldsymbol{P}_n) q_n + \boldsymbol{Q}_{mn}\boldsymbol{P}_n(\boldsymbol{M}_n - \boldsymbol{M})\boldsymbol{Q}_m(\boldsymbol{M} - \boldsymbol{M}_n)\boldsymbol{Q}\Psi + \\
&+ \boldsymbol{Q}_{mn}\boldsymbol{P}_n(\boldsymbol{M}\boldsymbol{Q}_{mn}\boldsymbol{P}_n \boldsymbol{M} - \boldsymbol{M}_n \boldsymbol{Q}_{mn}\boldsymbol{P}_n \boldsymbol{M}_n)(\boldsymbol{Q}_n \Psi_m - \boldsymbol{Q}\Psi) + \\
&+ \boldsymbol{Q}_{mn}\boldsymbol{P}_n(\boldsymbol{M} - \boldsymbol{M}_n)(\boldsymbol{Q}_m - \boldsymbol{Q})\boldsymbol{M}\boldsymbol{Q}\Psi + \boldsymbol{Q}_m(\boldsymbol{M} - \boldsymbol{M})_n(\boldsymbol{Q}_m - \boldsymbol{Q})\bar{q} + \\
&+ \boldsymbol{Q}_{mn}\boldsymbol{P}_n(\boldsymbol{M}\boldsymbol{Q}_{mn}\boldsymbol{P}_n)(\boldsymbol{M} - \boldsymbol{M}_n)\boldsymbol{Q}\Psi + \boldsymbol{Q}_{mn}\boldsymbol{P}_n(\boldsymbol{M} - \boldsymbol{M}_n)\boldsymbol{Q}\boldsymbol{M}\boldsymbol{Q}\Psi + \\
&+ \boldsymbol{Q}_m(\bar{q} - q_n) + \boldsymbol{Q}_m \boldsymbol{M}_n \boldsymbol{Q}_m(\bar{q} - q_n) + \boldsymbol{Q}_m(\boldsymbol{M} - \boldsymbol{M}_n)\boldsymbol{Q}\bar{q}\}\,.
\end{aligned}
$$

From these equations, we must show that given $\varepsilon > 0$ there are integers m_0 and n_0 such that $\|\Psi - \Psi_m^n\|_{\mathfrak{C}_1} < \varepsilon$ for all $m \geqslant m_0$ and $n \geqslant n_0$. Toward this end, we must establish the convergence of Ψ_m to Ψ in $\mathfrak{C}_1$, a task which entails the pointwise convergence of $\boldsymbol{Q}_m$ to $\boldsymbol{Q}$ and the collective compactness of $\{(\boldsymbol{Q}_m\boldsymbol{M})^2,\ m \geqslant 1\}$ (cf. Section V of [8]). Under the assumption that all operators are uniformly bounded with respect to m and n, we see that the convergence properties of $\boldsymbol{Q}_m$ will establish error estimates for terms 8, 9 and 10 in (4.2). Moreover, from the multigroup approximations to the quantity Ψ_m, we need to investigate the pointwise convergence of $\boldsymbol{M}_n$ to $\boldsymbol{M}$, the convergence of $\boldsymbol{Q}_{mn}\boldsymbol{P}_n$ to $\boldsymbol{Q}_m$ for each m, and the collective compactness of $(\boldsymbol{Q}_{mn}\boldsymbol{P}_n\boldsymbol{M}_n)^2$ in order to show that $\|(\boldsymbol{I} - (\boldsymbol{Q}_{mn}\boldsymbol{P}_n\boldsymbol{M}_n)^2)^{-1}\|$ is uniformly bounded. With such results, we may be able to derive error estimates for the second and third terms in the expression for $\Psi - \Psi_m^n$ (in (4.1)) and in the remainder of terms in (4.2). In all, our analysis requires showing the pointwise convergence of $\boldsymbol{Q}_m$ to $\boldsymbol{Q}$ and of $\boldsymbol{M}_n$

to $\boldsymbol{M}$, along with proving the collective compactness of $\{(\boldsymbol{Q}_m\boldsymbol{M}_n)^2, m\geqslant 1, n\geqslant 1\}$ (the proof of collective compactness of $\{(\boldsymbol{Q}_m\boldsymbol{M})^2\colon m\geqslant 1\}$ is precisely the same). The latter statement in the preceding sentence results from combining Proposition 1.8 in [2, p. 8] with the following result.

LEMMA 4.1. – *Under the condition that:*

$$(4.3)\qquad \Upsilon_n := \max_{g'} \sup_{E'\in I^n_{g'}} \Bigg\{ \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} \Big| p(x, \Omega', E', \Omega, E)\sigma(x, \Omega', E') - \int_{I^n_{g'}} p(x, \Omega', E'', \Omega, E)\sigma(x, \Omega', E'') \frac{\Psi_0(x, E'')}{\Psi^n_{0,g'}(x)}\, dE'' \Big| dE \Bigg\} \to 0$$

as $n\to\infty$ *uniformly for all* $x, \Omega, \Omega' \in \Gamma\times D^2\times D^2$, $\|\boldsymbol{Q}_m - \boldsymbol{Q}_{mn}\boldsymbol{P}_n\| \to 0$ *as* $n\to\infty$ *uniformly for all* m.

PROOF. – Now

$$(4.4)\qquad \|(\boldsymbol{Q}_m - \boldsymbol{Q}_{mn}\boldsymbol{P}_n)f\|_{\mathfrak{C}^P} =$$

$$= \max_{(x,\Omega)} \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} \Bigg| \Bigg[\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} \sum_{i=1}^{N_m} w_{mi} p(x, \Omega_{mi}, E', \Omega, E)\sigma(x, \Omega_{mi}, E') f(x, \Omega_{mi}, E')\, dE' \Bigg] - \Bigg[\sum_{g'=1}^{G_n} \sum_{i=1}^{N_m} w_{mi} \int_{I^n_{g'}} p(x, \Omega_{mi}, E'', \Omega, E)\sigma(x, \Omega_{mi}, E'') \frac{\Psi_0(x, E'')}{\Psi^n_{0,g'}(x)}\, dE'' \cdot \int_{I^n_{g'}} f(x, \Omega_{mi}, E')\, dE' \Bigg] \Bigg| dE =$$

$$= \max_{(x,\Omega)} \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} \Bigg| \sum_{g'=1}^{G_n} \sum_{i=1}^{N_m} \omega_{mi} \Bigg\{ \int_{I^n_{g'}} p(x, \Omega_{mi}, E', \Omega, E)\sigma(x, \Omega_{mi}, E') f(x, \Omega_{mi}, E')\, dE' - \Bigg[\int_{I^n_{n'}} p(x, \Omega_{mi}, E'', \Omega, E)\sigma(x, \Omega_{mi}, E'') \frac{\Psi_0(x, E'')}{\Psi^n_{0,g'}(x)}\, dE'' \int_{I^n_{g'}} f(x, \Omega_{mi}, E')\, dE' \Bigg] \Bigg\} \Bigg| dE =$$

$$= \max_{(x,\Omega)} \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} \Bigg| \sum_{g'=1}^{G_n} \sum_{i=1}^{N_m} w_{mi} \Bigg\{ \int_{I^n_{g'}} f(x, \Omega_{mi}, E') \Bigg[p(x, \Omega_{mi}, E', \Omega, E)\sigma(x, \Omega_{mi}, E') - \int_{I^n_{g'}} p(x, \Omega_{mi}, E'', \Omega, E)\sigma(x, \Omega_{mi}, E'') \frac{\Psi_0(x, E'')}{\Psi^n_{0,g'}(x)}\, dE'' \Bigg] dE' \Bigg\} \Bigg| dE \leqslant$$

$$\leqslant \max_{(x,\Omega)} \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} \Bigg| \sum_{g'=1}^{G_n} \sum_{i=1}^{N_m} |w_{mi}| \Bigg\{ \int_{I^n_{g'}} |f(x, \Omega_{mi}, E')| \Bigg| p(x, \Omega_{mi}, E', \Omega, E)\sigma(x, \Omega_{mi}, E') - \int_{I^n_{g'}} p(x, \Omega_{mi}, E'', \Omega, E)\sigma(x, \Omega_{mi}, E'') \frac{\Psi_0(x, E'')}{\Psi^n_{0,g'}(x)}\, dE'' \Bigg| dE' \Bigg\} dE \leqslant$$

$$\leqslant \max_{(x,\Omega)} \sum_{g'=1}^{G_n} \sum_{i=1}^{N_m} |w_{mi}| \left\{ \int_{I^n_{g'}} |f(x, \Omega_{mi}, E')| \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} \left| p(x, \Omega_{mi}, E', \Omega, E)\sigma(x, \Omega_{mi}, E') - \right.\right.$$
$$\left.\left. - \int_{I^n_{g'}} p(x, \Omega_{mi}, E'', \Omega, E)\sigma(x, \Omega_{mi}, E'')\, dE'' \right| dE\, dE' \right\} \leqslant \left(\sum_{i=1}^{N_m} |w_{mi}|\right) \|f\|_{C_1} \Upsilon_n ,$$

and the assertion of the lemma follows.

To show the strong or pointwise convergence of $\boldsymbol{Q}_m$ as $m \to \infty$, we shall need a result on the strong or pointwise convergence of the quadrature rules (3.1) in $\mathfrak{C}^P$. The proof of the following lemma depends on approximating an arbitrary $f \in \mathfrak{C}^P$ by a sequence of functions whose value at each $(x, \Omega) \in \Gamma \times D^2$ is an element of the dense set of bounded continuous functions on $[E_{\mathfrak{m}}, E_{\mathfrak{M}})$. This is accomplished by first extending the L^1 function $f(x, \Omega, \cdot)$, for each (x, Ω), to be identically zero outside $[E_{\mathfrak{m}}, E_{\mathfrak{M}})$ and convolving the extended function with a mollifier in the energy variable. The details are left to the interested reader.

LEMMA 4.2. – *For $f \in \mathfrak{C}^P$, the quantity*

$$\text{(4.5)} \qquad \sup_{x \in \Gamma} \left\| \sum_{i=1}^{N_m} w_{mi} f(x, \Omega_{mi}, \cdot) - \int_{D^2}\int f(x, \Omega, \cdot)(1 - |\Omega|^2)^{-\frac{1}{2}}\, d\Omega \right\|_{L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})} \to 0 \quad \textit{as } m \to \infty .$$

This lemma enables us to prove the next result.

PROPOSITION 4.1. – *For each m, $\boldsymbol{Q}_m$ is a bounded linear operator from $\mathfrak{C}_1$ to $\mathfrak{C}^P$, and the sequence $\boldsymbol{Q}_m$ converges strongly (i.e. pointwise) in $\mathfrak{C}_1$ to $\boldsymbol{Q}$ as $m \to \infty$.*

PROOF. – That each $\boldsymbol{Q}_m$ is a bounded linear map from $\mathfrak{C}_1$ to $\mathfrak{C}^P$ follows from Assumptions A and B concerning p and σ. Because both $\boldsymbol{Q}$ and $\boldsymbol{Q}_m$ have ranges in $\mathfrak{C}^P$, it suffices to show strong convergence whenever the spatial domain is restricted to a Γ_i, $i = 1, 2, \ldots, \mathfrak{N}$.

Let $f \in \mathrm{C}_1$ be fixed and suppose K is an upper bound for

$$\text{(4.6)} \qquad \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} p(x, \Omega', E', \Omega, E)\sigma(x, \Omega', E')|f(x, \Omega', E')|\, dE'\, dE =$$
$$= \|p(x, \Omega', \cdot, \Omega, \cdot)\sigma(x, \Omega', \cdot) f(x, \Omega', \cdot)\|_{L^1([E_{\mathfrak{m}}, E_{\mathfrak{M}}))^2}$$

(with $([E_{\mathfrak{m}}, E_{\mathfrak{M}}))^2$ denoting the Cartesian product of $[E_{\mathfrak{m}}, E_{\mathfrak{M}})$ with itself). As in [8, p. 360], condition (3.2) implies that for given $\varepsilon > 0$ there exists a positive ε_0 and m_0 such that $m > m_0$,

$$\text{(4.7)} \qquad \sum_i \left\{ |w_{mi}| \colon \min_{t \geqslant 0} (|\Omega_{mi} - t\Omega|) < \varepsilon_0,\ \Omega \in \Pi \right\} < \varepsilon/6K .$$

(Furthermore, we can assume $\varepsilon_0 < \max\{1, \varepsilon/24lK\}$ without loss of generality where l denotes the number of rays generated by the singular directions). Next, let $F(x, \Omega', \Omega, \cdot, \cdot)$ be a function, continuous on $\Gamma_i \times D^2 \times D^2$ with range in $L^1([E_{\mathfrak{m}}, E_{\mathfrak{M}}))^2$, which agrees with $p(x, \Omega', \cdot, \Omega, \cdot)\sigma(x, \Omega', \cdot) f(x, \Omega', \cdot)$ as a function in $L^1([E_{\mathfrak{m}}, E_{\mathfrak{M}}))^2$ for values of Ω' such that $\Omega' \in \Pi_{\varepsilon_0}$ which also has its $L^1([E_{\mathfrak{m}}, E_{\mathfrak{M}}))^2$-norm bounded by K. The existence of such an extension is assured by a result of E. Micheal [7, p. 802 (Theorem 7.1)]. From Assumptions A and B, concerning σ and p respectively, we can see that the following family of functions of Ω', parametrized by $(x, \Omega) \in \Gamma_i \times D^2$,

$$\left\{g(\Omega', \cdot, x, \Omega): g(\Omega', \cdot, x, \Omega) := \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} F(x, \Omega', \Omega, E', \cdot)\, dE', (x, \Omega) \in \Gamma_i \times D^2\right]$$

will satisfy the compactness criteria of the abstract Arzela-Ascoli Theorem (Theorem 2.1). In particular, for fixed Ω', we can apply the Fréchet-Kolmogorov Theorem (Theorem 2.2) to show that $\{g(\Omega', \cdot, x, \Omega)\}$ is relatively compact in $L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})$. Hence the above family is relatively compact in the Banach space of functions, con tinuous on D^2 with range in $L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})$, equipped with norm

$$\sup_{\Omega \in D^2} \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |h(\Omega, E)|\, dE$$

for an arbitrary element h. Proposition 1.7 of [2, p. 7] used in conjunction with Lemma 4.2, shows that the quadrature limit (3.1) is uniform on this family. Therefore, for m sufficiently large, we have

$$(4.8)\qquad \|\boldsymbol{Q}_m f - \boldsymbol{Q} f\|_{\mathfrak{C}^p} \leqslant \sup_{(x,\Omega)} \left[\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} \left| \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} \left\{ \iint_{D^2} F(x, \Omega', \Omega, E', E)(1 - |\Omega'|^2)^{-\frac{1}{2}} d\Omega' - \right.\right.\right.$$

$$\left.\left.\left. - \sum_{i=1}^{N_m} w_{mi} F(x, \Omega_{mi}, \Omega, E', E)\right\} dE' \right| dE\right] +$$

$$+ \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} \left| \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} \int_{\{\Omega' : \min_{t\geqslant 0} |\Omega' - t\Omega_0'| \leqslant \varepsilon_0,\, \Omega_0' \in \Pi\}} \{F(x, \Omega', \Omega, E', E) - \right.$$

$$\left. - p(x, \Omega', E', \Omega, E)\sigma(x, \Omega', E') f(x, \Omega', E')\}(1 - |\Omega'|^2)^{-\frac{1}{2}} d\Omega'\, dE' \right| dE +$$

$$+ \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} \left| \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} \left\{ \sum_{i=1}^{N_m} w_{mi} [F(x, \Omega_{mi}, \Omega, E', E) - p(x, \Omega_{mi}, E', \Omega, E)\sigma(x, \Omega_{mi}, E') f(x, \Omega_{mi}, E')] : \right.\right.$$

$$\left.\left. \min_{t\geqslant 0} |\Omega_{mi} - t\Omega_0| \leqslant \varepsilon_0,\ \Omega_0 \in \Pi \right\} dE' \right| dE \leqslant \varepsilon .$$

This completes the proof.

PROPOSITION 4.2. – *The sequence of operators $\boldsymbol{M}_n$ converges to $\boldsymbol{M}$ pointwise, i.e. $\boldsymbol{M}_n f \to \boldsymbol{M} f$, $f \in \mathfrak{C}^p$, under the condition that $\chi_n \to 0$ as $n \to \infty$, where χ_n is defined by*

$$(4.9)\qquad \chi_n := \max_{1 \leqslant g \leqslant G_n} \left[\sup\{|\sigma_g^n(x, \Omega) - \sigma(x, \Omega, E)| : (x, \Omega, E) \in \Gamma \times D^2 \times I_g^n\}\right].$$

PROOF. – To show the strong convergence, we must prove that given $\varepsilon > 0$ and $f \in \mathfrak{C}^r$, there is an N_0 (possibly depending on ε and f) such that

$$\sup_{(x,\Omega)\in\Gamma\times D^2} \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |\boldsymbol{M}_n f(x, \Omega, E) - \boldsymbol{M} f(x, \Omega, E)|\, dE < \varepsilon \tag{4.10}$$

for all $n \geqslant N_0$: This task entails obtaining estimates of

$$\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |\boldsymbol{M}_n f(x, \Omega, E) - \boldsymbol{M} f(x, \Omega, E)|\, dE$$

whenever (x, Ω) lies in those subsets of $\Gamma \times D^2$ occurring in the description of conditions 1-7 defining $\mathfrak{C}_1$.

We first treat the cases when

(i) $x \in \mathring{\Gamma}_i$, $i = 1, 2, 3, \ldots, \mathfrak{N}$, $\Omega \in \Pi_\eta$, $\eta > 0$;

(ii) $x \in \mathring{\Gamma}_i \setminus \mathring{\Gamma}_i \cap G_1$, $\Omega \in D^2_\eta$ and

(iii) $x \in \mathring{\Gamma}_i \cap G_1$, $\Omega \in \Pi_\eta$, card $(W_s(x, \Omega)) < \infty$, $i = 1, 2, \ldots, \mathfrak{N}$.

We write

$$\begin{aligned} \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |\boldsymbol{M} f(x, \Omega, E) - \boldsymbol{M}_n f(x, \Omega, E)|\, dE &= \\ = \sum_{g=1}^{G_n} \int_{I^n_g} \Bigg| \int_0^{d(x,\Omega)/|\Omega|} f(x - t\Omega, \Omega, E) \Bigg[\exp\Bigg\{-\int_0^t \sigma(x - r\Omega, \Omega, E)\, dr\Bigg\} &- \\ - \exp\Bigg\{-\int_0^t \sigma^n_g(x - r\Omega, \Omega)\, dr\Bigg\}\Bigg] dt \Bigg|\, dE\,. \end{aligned} \tag{4.11}$$

For (x, Ω) in the three subsets of $\Gamma \times D^2$ depicted, we can estimate

$$\begin{aligned} \sup_{(x,\Omega)} \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |\boldsymbol{M} f(x, \Omega, E) - \boldsymbol{M}_n f(x, \Omega, E)|\, dE &\leqslant \\ \leqslant \sum_{g=1}^{G_n} \int_{I^n_g} \int_0^{d(x,\Omega)} |f(x - t|\Omega|^{-1}\Omega, \Omega, E)| \Bigg[\exp\Bigg\{-\int_0^t \sigma(x - r|\Omega|^{-1}\Omega, \Omega, E)|\Omega|^{-1} dr\Bigg\} &- \\ - \exp\Bigg\{-\int_0^t \sigma^n_g(x - r|\Omega|^{-1}\Omega, \Omega)|\Omega|^{-1} dr\Bigg\}\Bigg]\, |\Omega|^{-1}\, dt|\, dE &\leqslant \\ \leqslant \sum_{g=1}^{G_n} \int_{I^n_g} \int_0^{d(x,\Omega)} |f(x - t|\Omega|^{-1}\Omega, \Omega, E)| \exp\Bigg\{-\int_0 \sigma_{\mathfrak{m}}|\Omega|^{-1} dr\Bigg\}\cdot \end{aligned} \tag{4.12}$$

$$\cdot\left|\exp\left\{\eta^{-1}\int_0^{d(x,\Omega)}\chi_n\,dr\right\}-1\right||\Omega|^{-1}\,dt\,dE\leqslant$$

$$\leqslant\left|1-\exp\left\{\eta^{-1}(\operatorname{diam}\Gamma)\chi_n\right\}\right|\int_0^{d(x,\Omega)}|\Omega|^{-1}\exp\left\{-\int_0^t\sigma_{\mathfrak{m}}|\Omega|^{-1}\,dr\right\}\cdot$$

$$\cdot\left(\sum_{g=1}^{G_n}\int_{I_g^n}|f(x-t\Omega,\Omega,E)|\,dE\right)dt\leqslant|\exp\left(\eta^{-1}(\operatorname{diam}\Gamma)\chi_n\right)-1|\sigma_{\mathfrak{m}}^{-1}\|f\|_{\mathfrak{C}^P}\,.$$

Hence, for (x, Ω) residing in the three subsets of $\Gamma\times D^2$ just described,

$$\sup_{(x,\Omega)}\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}}|\boldsymbol{M}_n f(x,\Omega,E)-\boldsymbol{M}f(x,\Omega,E)|\,dE\leqslant\sigma_{\mathfrak{m}}^{-1}\|f\|_{\mathfrak{C}^P}|\exp\left(\eta^{-1}(\operatorname{diam}\Gamma)\chi_n\right)-1|\,. \tag{4.13}$$

In obtaining these estimates, we have made special use of the fact that both $\sigma(x, \Omega, \cdot)$ and $\sigma_g^n(x, \Omega)$ are evaluated at the same point of the segment $\alpha(x, \Omega)$ throughout the integration defining $\boldsymbol{M}$ and $\boldsymbol{M}_n$, and that $\alpha(x, \Omega)$ does not run along interfaces for the values of (x, Ω) considered. Moreover, for those values of (x, Ω) for which $x\in\partial\Gamma_i\subset\mathring{\Gamma}$ and card $(W_s(x,\Omega))<\infty$—and for those (x, Ω) with $x\in\partial\Gamma$ and $\Omega\in\Pi_\eta$ or $\Omega\in\Pi(\eta)$ such that $1\leqslant$ card $(W_s(x,\Omega))<\infty$—the same analysis will produce similar estimates as (4.13).

The latter observations in the preceding paragraphs, in conjunction with the analysis producing (4.13), allow us to derive estimates for

$$\|(\boldsymbol{M}-\boldsymbol{M}_n)f(x,\Omega,\cdot)\|_{L^1[E_{\mathfrak{m}},E_{\mathfrak{M}})}$$

for (x, Ω) depicted in the subsets of 4, 5*b*, 5*c*, and 6 in the definition of $\mathfrak{C}_1$: We take sequences $x_n^+, x_n^-, \Omega_m^+, \Omega_m^-$ tending to (x, Ω), respectively which lie in the subsets of $\Gamma\times D_\eta^2$ just analyzed. For example, for (x, Ω) lying in the subset depicted in (4), we take sequences $x_n^+, x_n^-, \Omega_m^+, \Omega_m^-$ lying in that subset of $\Gamma\times D_\eta^2$ described in condition (3) of $\mathfrak{C}_1$; for (x, Ω) lying in the subsets of 5*b* and 5*c*, we take sequences lying in those subsets of $\Gamma\times D_\eta^2$ described in the preceding paragraph. For all three pairs of sequences described, $W_s(x_n^\pm, \Omega_m^\pm)$ has cardinality finite for each of the pairs (x_n^+, Ω_m^-) taken, since Γ and Γ_i, $i\in\{1, \dots, \mathfrak{N}\}$ are convex and the number of singular directions (of unit length) is finite.

Since $(\boldsymbol{M}-\boldsymbol{M}_n)f\in\mathfrak{C}_1$ for each n, this means that the various iterated limits indicated in conditions 4, 5*b*, and 5*c* exist as functions in $L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})$. By continuity of the L^1-norms, we see that these limits satisfy inequality (4.13). As a consequence, we obtain that $[(\boldsymbol{M}-\boldsymbol{M}_n)f]^{\pm}(x,\Omega)$ (from condition (4)) and $[(\boldsymbol{M}-\boldsymbol{M}_n)f]_{i,j}^{\pm}(x,\Omega)$ (as in 5*b* and 5*c*) satisfy (4.13). A similar resoning suffices to prove (4.13) for those subsets of $\Gamma\times D_\eta^2$ in 5*c* and 6.

For the subsets of $\Gamma \times D_\eta^2$ in 4 or 5b, we estimate the jump in $(\boldsymbol{M}-\boldsymbol{M}_n)f(x, \Omega, \cdot)$ by

$$
(4.14)\quad \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} \Big|\lim_{x_n^+\to x} (\boldsymbol{M}-\boldsymbol{M}_n)f(x_n^+, \Omega, E) - \lim_{x_n^-\to x}(\boldsymbol{M}-\boldsymbol{M}_n)f(x_n^-, \Omega, E)\Big|\,dE =
$$

$$
= \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} \Big| \sum_{\{(i,j)\in M(x,\Omega):\lim_{n\to\infty}(n_{j,i}\cdot(x_n^+-x)/(\|x_n^+-x\|))\geqslant 0\}} \int_{\{s:\, x-s|\Omega|^{-1}\Omega\in\partial\Gamma_{i,j}\}} f_{ij}(x-s|\Omega|^{-1}\Omega, \Omega, E)\cdot
$$

$$
\cdot\Big[\exp\Big\{-\int_0^s \sigma_{i,j}(x-\tau|\Omega|^{-1}\Omega, \Omega, E)|\Omega|^{-1}\,d\tau\Big\} -
$$

$$
-\exp\Big\{-\int_0^s (\sigma_g^n)_{i,j}(x-\tau|\Omega|^{-1}, \Omega, \Omega, E)|\Omega|^{-1}\,d\tau\Big\}\Big]\,|\Omega|^{-1}\,ds -
$$

$$
- \sum_{\{(i,j)\in M(x,\Omega):\lim_{n\to\infty}(n_{i,j}\cdot(x_n^--x)/(\|x_n^--x\|))\geqslant 0\}} \int_{\{s:\, x-s|\Omega|^{-1}\Omega\in\partial\Gamma_{i,j}\}} f_{ij}(x-s|\Omega|^{-1}\Omega, \Omega, E)\cdot
$$

$$
\cdot\Big[\exp\Big\{-\int_0^s \sigma_{i,j}(x-\tau|\Omega|^{-1}\Omega, \Omega, E)|\Omega|^{-1}\,d\tau\Big\} -
$$

$$
-\exp\Big\{-\int_0^s (\sigma_g^n)_{i,j}(x-\tau|\Omega|^{-1}\Omega, \Omega, E)|\Omega|^{-1}\,d\tau\Big\}\Big]\,|\Omega|^{-1}\,ds\Big|\,dE\,,
$$

where $f_{i,j}$, $\sigma_{i,j}$ and $(\sigma_g^n)_{i,j}$ are defined in (2.29). For $x\in\partial\Gamma_{0,j}^i$, and $x_n^-\in\mathring{\Gamma}$, $x_n^-\notin G_1$, with $x_n^-\to x$, we have

$$
(4.15)\quad \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} \Big|\lim_{x_n^-\to x}(\boldsymbol{M}-\boldsymbol{M}_n)f(x_n^-, \Omega, E)\Big|\,dE =
$$

$$
= \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} \Big| \sum_{\{(i,j)_0\in M(x,\Omega):\lim_{n\to\infty}(n_{0,j}^i\cdot(x-x_n^-)/(\|x-x_n^-\|))\geqslant 0\}} \int_{\{s:\, x-s|\Omega|^{-1}\Omega\in\partial\Gamma_{0,j}^i\}} f_{0,j}^i(x-s|\Omega|^{-1}\Omega, \Omega, E)\cdot
$$

$$
\cdot\Big[\exp\Big\{-\int_0^s \sigma_{0,j}^i(x-\tau|\Omega|^{-1}\Omega, \Omega, E)|\Omega|^{-1}\,d\tau\Big\} -
$$

$$
-\exp\Big\{\int_0^s (\sigma_g^n)_{0,j}^i(x-\tau|\Omega|^{-1}\Omega, \Omega)|\Omega|^{-1}\,d\tau\Big\}\Big]\,|\Omega|^{-1}\,ds\Big|\,dE
$$

with $\sigma_{0,j}^i$ and $(\sigma_j^n)_{0,j}^i$ given by (2.30). The expressions (4.14) and (4.15) have the upper bound

$$
2\sigma_{\mathfrak{m}}^{-1}\|f\|_{\mathfrak{C}^p}|\exp(\eta^{-1}(\operatorname{diam}\Gamma)\chi_n)-1|\,.
$$

The strong or pointwise convergence of $\boldsymbol{M}_n$ to $\boldsymbol{M}$ is reflected in the behavior of $\boldsymbol{M}f(x, \Omega, \cdot)$ and of $\boldsymbol{M}_n f(x, \Omega, \cdot)$ for values of Ω near zero. More precisely, we can observe that the analysis for Lemma 2.2 shows that the operators $\boldsymbol{M}$ and $\boldsymbol{M}_n$ per se approach expressions involving Dirac measures as $|\Omega| \to 0$. The task before us now is to show that these expressions become arbitrarily small as $n \to \infty$ when acting on each $f \in \mathfrak{C}^p$.

Toward this end, we consider those values of $(x, \Omega) \in \Gamma \times D^2$ for which $|\Omega| \leqslant \eta$. The parameter δ_0 in the following discussion is defined to be the maximum δ which guarantees for given $\varepsilon > 0$,

$$\max_{\Omega \in D^2} \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |f(x, \Omega, E) - f(x_0, \Omega, E)|\, dE \leqslant \varepsilon \tag{4.16}$$

$$\operatorname{ess\,sup} \{|\sigma(x, \Omega, E) - \sigma(x_0, \Omega, E)| : \Omega \in D^2,\ E \in [E_{\mathfrak{m}}, E_{\mathfrak{M}})\} \leqslant \varepsilon \tag{4.17}$$

for $|x - x_0| \leqslant \delta$, $x, x_0 \in \Gamma_i$, $i = 1, 2, \ldots, \mathfrak{N}$. We first consider those $x \in \Gamma$ whose distance from $\partial\Gamma$ is greater than, or equal to, δ_0. For this case, our estimate will take into account those $x \in \mathring{\Gamma}$ whose distance from points in G_1 is at least δ_0 and those points in $\mathring{\Gamma}$ within δ_0 of G_1. Secondly, we consider that subset of Γ consisting of points whose distance from $\partial\Gamma$ is at most δ_0. For these values of x, we have the following three sets:

(*a*) $\{(x, \Omega) : d(x, \Omega) \leqslant (\delta_0/\eta)|\Omega|,\ |\Omega| \leqslant \eta\}$;

(*b*) $\{(x, \Omega) : (\delta_0/\eta)|\Omega| \leqslant d(x, \Omega) \leqslant \delta_0,\ |\Omega| \leqslant \eta\}$;

(*c*) $\{x, \Omega) : d(x, \Omega) \geqslant \delta_0,\ |\Omega| \leqslant \eta\}$.

To obtain our estimates for the case when $x \in \mathring{\Gamma}$, $\operatorname{dist}(x, G_1) \geqslant \delta_0$, $|\Omega| \leqslant \eta$, we first write

$$\begin{aligned}
(4.18)\quad & |\boldsymbol{M}f(x, \Omega, E) - \boldsymbol{M}_n f(x, \Omega, E)| = |\boldsymbol{M}f(x, \Omega, E) - \boldsymbol{M}f(x, 0, E) + \\
& + \boldsymbol{M}f(x, 0, E) - \boldsymbol{M}_n f(x, 0, E) + \boldsymbol{M}_n f(x, 0, E) - \boldsymbol{M}_n f(x, \Omega, E)| = \\
& = \Bigg| \int_0^{d(x,\Omega)} \sigma(x - t|\Omega|^{-1}\Omega, \Omega, E) \exp\left\{-\int_0^t \sigma(x - r|\Omega|^{-1}\Omega, \Omega, E)|\Omega|^{-1} dr\right\} \cdot \\
& \cdot \left[\frac{f(x - t|\Omega|^{-1}\Omega, \Omega, E)}{\sigma(x - t|\Omega|^{-1}\Omega, \Omega, E)} - \boldsymbol{M}f(x, 0, E)\right] |\Omega|^{-1} dt - \\
& - \boldsymbol{M}f(x, 0, E) \exp\left\{-\int_0^{d(x,\Omega)} \sigma(x - r|\Omega|^{-1}\Omega, \Omega, E)|\Omega|^{-1} dr\right\} + \\
& + \boldsymbol{M}_n f(x, 0, E) \exp\left\{-\int_0^{d(x,\Omega)} \sigma_g^n(x - r|\Omega|^{-1}\Omega, \Omega)|\Omega|^{-1} dr\right\} + \boldsymbol{M}f(x, 0, E) - \boldsymbol{M}_n f(x, 0, E) - \\
& - \int_0^{d(x,\Omega)} \sigma_g^g(x - t|\Omega|^{-1}\Omega, \Omega) \exp\left\{-\int_0^t \sigma_g^n(x - r|\Omega|^{-1}\Omega, \Omega)\, |\Omega|^{-1} dr\right\} \cdot \\
& \qquad \cdot \left[\frac{f(x - t|\Omega|^{-1}\Omega, \Omega, E)}{\sigma_g^n(x - t|\Omega|^{-1}\Omega, \Omega)} - \boldsymbol{M}_n f(x, 0, E)\right] |\Omega|^{-1} dt \,.
\end{aligned}$$

Our estimates become:

$$(4.19)\quad \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |\boldsymbol{M}f(x,\Omega,E)-\boldsymbol{M}_nf(x,\Omega,E)|\,dE\leqslant$$
$$\leqslant 4\sigma_{\mathfrak{M}}\sigma_{\mathfrak{m}}^{-2}\|f\|_{\mathfrak{C}^p}[\exp(-\sigma_{\mathfrak{m}}\delta_0/2\eta)-\exp(-\sigma_{\mathfrak{m}}|\Omega|^{-1}\operatorname{diam}\Gamma)]+$$
$$+2\sigma_{\mathfrak{M}}\sigma_{\mathfrak{m}}^{-1}[1-\exp(-\sigma_{\mathfrak{m}}\delta_0/2|\Omega|)]\cdot$$
$$\cdot\max_{0\leqslant t\leqslant\delta_0/2}\left[\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}}\left|\frac{f(x-t|\Omega|^{-1}\Omega,\Omega,E)}{\sigma(x-t|\Omega|^{-1}\Omega,\Omega,E)}-\frac{f(x,0,E)}{\sigma(x,0,E)}\right|dE\right]+$$
$$+2\sigma_{\mathfrak{m}}\exp(-\sigma_{\mathfrak{m}}\delta_0/\eta)\left(\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}}|f(x,0,E)|\,dE\right)+\sigma_{\mathfrak{m}}^{-2}\chi_n\left(\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}}|f(x,0,E)|\,dE\right)+$$
$$+\sigma_{\mathfrak{M}}\sigma_{\mathfrak{m}}^{-3}[1-\exp(-\sigma_{\mathfrak{m}}\delta_0^{\frac{1}{2}}2|\Omega|)]\cdot$$
$$\cdot\Big\{\max_{0\leqslant t\leqslant\delta_0/2}\max_{|\Omega|\leqslant\eta}\|\sigma(x-t|\Omega|^{-1}\Omega,\Omega,E)-\sigma(x,0,E)\|_{L^\infty[E_{\mathfrak{m}},E_{\mathfrak{M}}]}\|f\|_{\mathfrak{C}^p}+$$
$$+\sigma_{\mathfrak{M}}\max_{0\leqslant t\leqslant\delta_0/2}\max_{|\Omega|\leqslant\eta}\|f(x-t|\Omega|^{-1}\Omega,\Omega,\cdot)-f(x,0,\cdot)\|_{L^1[E_{\mathfrak{m}},E_{\mathfrak{M}}]}\Big\}.$$

The precise same arguments in extending inequality (4.13) to those (x,Ω) in the subsets of conditions 4, 5*b*, 5*c*, and 6 defining $\mathfrak{C}_1$ for $|\Omega|\geqslant\eta$ show that (4.19) holds for all $x\in\Gamma$, $\operatorname{dist}(x,\partial\Gamma)\geqslant\delta_0$, $|\Omega|\leqslant\eta$.

For (x,Ω) in the two sets,

(*a*) $\{(x,\Omega): d(x,\Omega)\leqslant(\delta_0/\eta)|\Omega|,\ |\Omega|\leqslant\eta\}$,

(*b*) $\{(x,\Omega): (\delta_0/\eta)|\Omega|\leqslant d(x,\Omega)\leqslant\delta_0,\ |\Omega|\leqslant\eta\}$,

we express $(\boldsymbol{M}-\boldsymbol{M}_n)f(x,\Omega,E)$ in the form

$$(4.20)\quad (\boldsymbol{M}-\boldsymbol{M}_n)f(x,\Omega,E)=$$
$$=\int_0^{d(x,\Omega)}\sigma(x-t|\Omega|^{-1}\Omega,\Omega,E)\exp\left\{-\int_0^t\sigma(x-r|\Omega|^{-1}\Omega,\Omega,E)|\Omega|^{-1}dr\right\}\cdot$$
$$\cdot\left[\frac{f(x-t|\Omega|^{-1}\Omega,\Omega,E)}{\sigma(x-t|\Omega|^{-1}\Omega,\Omega,E)}-\frac{f(x,0,E)}{\sigma(x,0,E)}\right]|\Omega|^{-1}dt-$$
$$-\int_0^{d(x,\Omega)}\sigma_g^n(x-t|\Omega|^{-1}\Omega,\Omega)\exp\left\{-\int_0^t\sigma_g^g(x-r|\Omega|^{-1}\Omega,\Omega)|\Omega|^{-1}dr\right\}\cdot$$
$$\cdot\left[\frac{f(x-t|\Omega|^{-1}\Omega,\Omega,E)}{\sigma_g^n(x-t|\Omega|^{-1}\Omega,\Omega)}-\frac{f(x,0,E)}{\sigma_g^n(x,0)}\right]|\Omega|^{-1}dt+$$
$$+\left[\frac{1}{\sigma(x,0,E)}-\frac{1}{\sigma_g^n(x,0)}\right]f(x,0,E)+$$

$$+\frac{f(x,0,E)}{\sigma_g^n(x,0)}\exp\Big\{-\int_0^{d(x,\Omega)}\sigma_g^n(x-t|\Omega|^{-1}\Omega,\Omega)|\Omega|^{-1}dt\Big\}-$$

$$-\frac{f(x,0,E)}{\sigma(x,0,E)}\exp\Big\{-\int_0^{d(x,\Omega)}\sigma(x-t|\Omega|^{-1}\Omega,\Omega,E)|\Omega|^{-1}dt\Big\}.$$

The last two terms on the right hand side of (4.20) can be expressed as:

$$\frac{f(x,0,E)}{\sigma_g^n(x,0)}\exp\Big\{-\int_0^{d(x,\Omega)}\sigma_g^n(x-t|\Omega|^{-1}\Omega,\Omega)|\Omega|^{-1}dt\Big\}-$$

$$-\frac{f(x,0,E)}{\sigma(x,0,E)}\exp\Big\{-\int_0^{d(x,\Omega)}\sigma(x-t|\Omega|^{-1}\Omega,\Omega,E)|\Omega|^{-1}dt\Big\}=$$

$$=\frac{f(x,0,E)}{\sigma_g^n(x,0)}\exp\Big\{-\int_0^{d(x,\Omega)}\sigma_g^n(x-t|\Omega|^{-1}\Omega,\Omega)|\Omega|^{-1}dt\Big\}-\frac{f(x,0,E)}{\sigma(x,0,E)}\cdot$$

$$\cdot\exp\Big\{-\int_0^{d(x,\Omega)}\sigma_g^n(x-t|\Omega|^{-1}\Omega,\Omega)|\Omega|^{-1}dt\Big\}\cdot$$

$$\cdot\exp\Big\{-\int_0^{d(x,\Omega)}\big(\sigma(x-t|\Omega|^{-1}\Omega,\Omega,E)-\sigma_g^n(x-t|\Omega|^{-1}\Omega,\Omega)\big)|\Omega|^{-1}dt\Big\},$$

with

$$(4.21)\qquad (x-t|\Omega|^{-1}\Omega,\Omega,\Omega)\in\Big\{[x,x-d(x,\Omega)|\Omega|^{-1}\Omega]_{(s)}\times\{|\Omega|\leqslant\eta\}\times$$

$$\times I_g^n\colon\sigma(x-t|\Omega|^{-1}\Omega,\Omega,E)-\sigma_g^n(x-t|\Omega|^{-1}\Omega,\Omega)\geqslant 0\Big\};$$

$$-\frac{f(x,0,E)}{\sigma(x,0,E)}\exp\Big\{-\int_0^{d(x,\Omega)}\sigma(x-t|\Omega|^{-1}\Omega,\Omega,E)|\Omega|^{-1}dt\Big\}+\frac{f(x,0,E)}{\sigma_g^n(x,0)}\cdot$$

$$\cdot\exp\Big\{-\int_0^{d(x,\Omega)}\sigma(x-t|\Omega|^{-1}\Omega,\Omega,E)|\Omega|^{-1}dt\Big\}\cdot$$

$$\cdot\exp\Big\{-\int_0^{d(x,\Omega)}\big(\sigma_g^n(x-t|\Omega|^{-1}\Omega,\Omega)-\sigma(x-t|\Omega|^{-1}\Omega,\Omega,E)\big)|\Omega|^{-1}dt\Big\},$$

with

$$(x-t|\Omega|^{-1}\Omega,\Omega,E)\in\Big\{[x,x-d(x,\Omega)|\Omega|^{-1}\Omega]_{(s)}\times$$

$$\times\{|\Omega|\leqslant\eta\}\times I_g^n\colon\sigma_g^n(x-t|\Omega|^{-1}\Omega,\Omega)-\sigma(x-t|\Omega|^{-1}\Omega,\Omega,E)\geqslant 0\Big\},$$

where $[x,x']_{(s)}$ for two points x and x' in Γ denotes the directed segment with initial point x and terminal point x'. For those points in (a), we obtain from (4.20) and (4.21)

$$\begin{aligned}(4.22)\quad &\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |(\boldsymbol{M}-\boldsymbol{M}_n)f(x,\Omega,E)|\,dE \leqslant \\ &\leqslant \max_{0\leqslant t\leqslant\delta_0}\max_{|\Omega|\leqslant\eta}\left[\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}}\left|\frac{f(x-t\Omega,\Omega,E)}{\sigma(x-t\Omega,\Omega,E)}-\frac{f(x,0,E)}{\sigma(x,0,E)}\right|dE\right]+ \\ &+\sigma_{\mathfrak{m}}^{-2}\|f\|_{\mathfrak{C}^p}\max_{0\leqslant t\leqslant\delta_0}\max_{|\Omega|\leqslant\eta}\|\sigma(x-t\Omega,\Omega,E)-\sigma(x,0,E)\|_{L^\infty[E_{\mathfrak{m}},E_{\mathfrak{M}}]}+ \\ &+\sigma_{\mathfrak{M}}\sigma_{\mathfrak{m}}^{-2}\max_{0\leqslant t\leqslant\delta_0}\max_{|\Omega|\leqslant\eta}\|f(x-t\Omega,\Omega,E)-f(x,0,E)\|_{L^1[E_{\mathfrak{m}},E_{\mathfrak{M}}]}+ \\ &+2\sigma_{\mathfrak{m}}^{-2}\|f\|_{\mathfrak{C}^p}\chi_n+\sigma_{\mathfrak{m}}^{-2}\|f\|_{\mathfrak{C}^p}(1-\exp(-\chi_n\delta_0/\eta));\end{aligned}$$

For those points in (b), we obtain

$$\begin{aligned}(4.23)\quad &\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |(\boldsymbol{M}-\boldsymbol{M}_n)f(x,\Omega,E)|\,dE \leqslant \\ &\leqslant \max_{0\leqslant t\leqslant\delta_0}\max_{|\Omega|\leqslant\eta}\left[\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}}\frac{f(x-t|\Omega|^{-1}\Omega,\Omega,E)}{\sigma(x-t|\Omega|^{-1}\Omega,\Omega,E)}-\frac{f(x,0,E)}{\sigma(x,0,E)}|dE\right]+ \\ &+\sigma_{\mathfrak{m}}^{-2}\|f\|_{\mathfrak{C}^p}\max_{0\leqslant t\leqslant\delta_0}\max_{|\Omega|\leqslant\eta}\|\sigma(x-t|\Omega|^{-1}\Omega,\Omega,E)-\sigma(x,0,E)\|_{L^\infty[E_{\mathfrak{m}},E_{\mathfrak{M}}]}+ \\ &+\sigma_{\mathfrak{M}}\sigma_{\mathfrak{m}}^{-2}\max_{0\leqslant t\leqslant\delta_0}\max_{|\Omega|\leqslant\eta}\|f(x-t|\Omega|^{-1}\Omega,\Omega,\Omega)-f(x,0,E)\|_{L^1[E_{\mathfrak{m}},E_{\mathfrak{M}}]}+ \\ &+\sigma_{\mathfrak{m}}^{-1}\|f\|_{\mathfrak{C}^p}\exp\{-\sigma_{\mathfrak{m}}\delta_0/\eta\}\,.\end{aligned}$$

From the inequalities (4.13), (4.19), (4.22) and (4.23), we see that, with δ_0, η at our disposal, we can make

$$\|(\boldsymbol{M}-\boldsymbol{M}_n)f\|_{\mathfrak{C}_1}$$

smaller than any preassigned positive number for sufficiently large values of n. We conclude that $\lim_{n\to\infty}\sup\|\boldsymbol{M}f-\boldsymbol{M}_nf\|_{\mathfrak{C}_1}=0$ and the proof of Proposition 4.2 is complete.

We refer the reader to [9, 15, 16] for a rather thorough interpretation of the quantities Υ_n and χ_n defined in (4.3) and (4.9) respectively. Each represents the maximum of the fluctuations or variations of the total and scattering cross-sections σ and p over the energy intervals, $\{I_g^n\colon g=1,2,\dots,G_n\}$. Also, an analysis similar to that for showing the strong convergence of $\boldsymbol{M}_n$ to $\boldsymbol{M}$ will show the convergence of $u_n(x,\Omega,\cdot)$ (defined in (3.16)) to $u(x,\Omega)$ in $\mathfrak{C}_1$. We now turn to proving the collective compactness of $\{(\boldsymbol{Q}_m\boldsymbol{M}_n)^2\colon m\geqslant1, n\geqslant1\}$.

THEOREM 4.1. – *There exist sequences* $\{\boldsymbol{T}_{mn}\}$ *and* $\{\boldsymbol{R}_{mn}\}$ *of bounded linear operators on* $\mathfrak{C}^p$ *such that* $(\boldsymbol{Q}_m\boldsymbol{M}_n)^2=\boldsymbol{T}_{mn}+\boldsymbol{R}_{mn}$ *where* $\{\boldsymbol{T}_{mn}\colon m\geqslant1, n\geqslant1\}$ *is collectively compact and* $\|\boldsymbol{R}_{mn}\|\to0$ *as* $m\to\infty$ *uniformly in* n.

PROOF. - Let $\mathfrak{B}$ denote the unit ball in $\mathfrak{C}^p$, and let $f \in \mathfrak{C}^p$. We express $(\boldsymbol{Q}_m \boldsymbol{M}_n)^2 f(x, \Omega, E)$ as

$$(\boldsymbol{Q}_m \boldsymbol{M}_n)^2 f(x, \Omega, E) = \sum_{j=1}^{N_m} \sum_{i=1}^{N_m} w_{mi} w_{mj} \, \mathfrak{E}_{i,j,n}(x, \Omega, E) \tag{4.24}$$

where

$$\begin{aligned} \mathfrak{E}_{i,j,n}(x, \Omega, E) = \sum_{g=1}^{G_n} \sum_{g'=1}^{G_n} \int\limits_{I_g^n} & p(x, \Omega_{mi}, E', \Omega, E)\, \sigma(x, \Omega_{mi}, E') \cdot \\ & \cdot \int\limits_0^{d(x,\Omega_{mi})/|\Omega_{mi}|} \exp\left\{ \int\limits_0^t \sigma_g^n(x - r\Omega_{mi}, \Omega_{mi})\, dr \right\} \cdot \\ & \cdot \int\limits_{I_{g'}^n} p(x - t\Omega_{mi}, \Omega_{mj}, E'', \Omega_{mi}, E')\, \sigma(x - t\Omega_{mi}, \Omega_{mj}, E'') \cdot \\ & \cdot \int\limits_0^{d(x - t\Omega_{mi}, \Omega_{mj})/|\Omega_{mj}|} \exp\left\{ - \int\limits_0^s \sigma_{g'}^n(x - t\Omega_{mi} - r\Omega_{mj}, \Omega_{mj})\, dr \right\} \cdot \\ & \cdot f(x - t\Omega_{mi} - s\Omega_{mj}, \Omega_{mj}, E'')\, dE''\, ds\, dE'\, dt\,. \end{aligned} \tag{4.25}$$

As in the proof of Theorem 1 in [8], we define the family of operators $\{\boldsymbol{R}_{mn}: m \geqslant 1, n \geqslant 1\}$ acting on $\mathfrak{C}^p$ to correspond to that portion of the sum (4.24) over indices i and j such that the two-dimensional vectors Ω_{mj}, Ω_{mi}, and the origin are collinear, for each fixed m and n, and $\boldsymbol{T}_{mn}$ to correspond to the remaining pairs of indices. From (3.2) and (3.3), we can deduce that $\|\boldsymbol{R}_{mn}\| \to 0$ as $n \to \infty$ *uniformly* in n by using the basic properties of quadrature sets.

In order to show that the family $\{\boldsymbol{T}_{mn}: m \geqslant 1, n \geqslant 1\}$ is collectively compact, we must, according to Theorems 2.1 and 2.2, show that

(i) $\{\boldsymbol{T}_{mn} f, f \in \mathfrak{B}, m, n \geqslant 1\}$ is equicontinuous and

(ii) for fixed $(x_0, \Omega_0) \in \Gamma_i \times D^2$, $i \in \{1, \ldots, \mathcal{N}\}$, $\{T_{mn} f(x_0, \Omega_0, \cdot): f \in \mathfrak{B}, m, n \geqslant 1\}$ is relatively compact in $L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})$.

For proving equicontinuity of the family $\{\boldsymbol{T}_{mn} f: f \in \mathfrak{B}, m, n \geqslant 1\}$ on each $\Gamma_i \times D^2$, $i \in \{1, 2, \ldots, \mathfrak{N}\}$, we select an $\varepsilon > 0$ and find a $\delta > 0$ independent of $f \in \mathfrak{B}$, of m, of n, and of subregion Γ_i so that

$$\|\boldsymbol{T}_{mn} f(x, \Omega, \cdot) - \boldsymbol{T}_{mn} f(x_0, \Omega_0, \cdot)\|_{L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})} < \varepsilon$$

whenever $|x - x_0|^2 + |\Omega - \Omega_0|^2 \leqslant \delta^2$. The proof follows the procedure discussed in Theorem 1 of [8] with straightforward generalizations. We first find an m_0 and ε_0 such that

$$\max_{\Omega \in D^2} \sum_i \{|w_{mi}|: \min_t |\Omega_{mi} - t\Omega| < \varepsilon_0\} < \varepsilon \sigma_m^2 / 4AP^2, \; m \geqslant m_0\,, \tag{4.26}$$

with

$$(4.27) \qquad P := \max_{(x,\Omega,\Omega')} \left\| \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} p(x, \Omega', E', \Omega, E)\sigma(x, \Omega', E')\, dE \right\|_{L^\infty[E_{\mathfrak{m}}, E_{\mathfrak{M}})}$$

and

$$(4.28) \qquad \sup_{m\geqslant 1} \sum_{i=1}^{N_m} |w_{mi}| \leqslant A .$$

Secondly, for $m \geqslant m_i$, let $\bar{S}_m$ be the set of index pairs (i, j) such that Ω_{mi} and Ω_{mj} are not collinear but satisfy:

(*a*) There exists t such that

$$\min\left(|\Omega_{mi} - t\Omega_{mj}|, |\Omega_{mj} - t\Omega_{mi}|\right) \leqslant \varepsilon_0 ;$$

(*b*) There exists t such that

$$\min_{i,j} \left[|\Omega_{mi} - t\Omega|, |\Omega_{mj} - t\Omega|\right] \leqslant \varepsilon_0 , \qquad \Omega \in \Pi .$$

(If $m \leqslant m_0$, assume $\bar{S}_m$ to be empty set without loss of generality). Thirdly, let S_m denote the pairs (i, j) of indices which are not in $\bar{S}_m$ but such that either Ω_{mj} and Ω_{mi} are not collinear. Then we estimate

$$(4.29) \qquad \|\boldsymbol{T}_{mn} f(x, \Omega, E) - \boldsymbol{T}_{mn} f(x_0, \Omega_0, E)\|_{L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})} \leqslant$$
$$\leqslant \sum_{(i,j)\in \bar{S}_m} |w_{mi}||w_{mj}| \left[\|\mathfrak{E}_{i,j,n}(x, \Omega, \cdot)\|_{L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})} + \|\mathfrak{E}_{i,j,n}(x_0, \Omega_0, \cdot)\|_{L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})}\right] +$$
$$+ \frac{\varepsilon}{2} + A^2 \max_{(i,j)\in S_m} \|\mathfrak{E}_{i,j,n}(x, \Omega, \cdot) - \mathfrak{E}_{i,j,n}(x_0, \Omega_0, \cdot)\|_{L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})} .$$

The proof that

$$(4.30) \qquad \max_{(i,j)\in S_m} \|\mathfrak{E}_{i,j,n}(x, \Omega, \cdot) - \mathfrak{E}_{i,j,n}(x_0, \Omega_0, \cdot)\|_{L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})} \to 0$$

as $(x, \Omega) \to (x_0, \Omega_0)$ uniformly in $m, n, f \in \mathfrak{B}$, and subregion Γ_i, $i \in \{1, 2, ..., \mathfrak{N}\}$, follows the same approach in [8]: Use is made of the uniform continuity of

$$(x, \Omega, \Omega') \to \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} p(x, \Omega', \cdot, \Omega, E)\, dE \sigma(x, \Omega', \cdot) \in L^\infty[E_{\mathfrak{m}}, E_{\mathfrak{M}})$$

with respect to $x \in \Gamma_i$, $i \in \{1, 2, ..., \mathfrak{N}\}$, and $\Omega, \Omega' \in D^2$. Moreover the uniform continuity of $(x, \Omega) \to \sigma_g^n(x, \Omega)$, $g = 1, ..., G_n$, $x \in \Gamma_i$, $\Omega \in D^2$, with $\{\sigma_g^n(x, \Omega), g = 1, ..., G_n\}$ considered as a step function in $L^\infty[E_{\mathfrak{m}}, E_{\mathfrak{M}})$, is also used. (This property follows from the uniform continuity of

$$(x, \Omega) \to \sigma(x, \Omega), \cdot) \in L^\infty[E_{\mathfrak{m}}, E_{\mathfrak{M}})$$

on every $\Gamma_i \times D^2$). We also utilize (2.23) and the fact that $\Omega_{mi}, \Omega_{mj} \in \Pi_{\bar{\varepsilon}_0}$ for some $\bar{\varepsilon}_0$ with $(i, j) \in S_m$. From Lemma 2.1, we have that $d(x, \Omega)/|\Omega|$ is uniformly continuous for $(x, \Omega) \in \Gamma \times \Pi_{\bar{\varepsilon}_0}$, and this guarantees the uniform convergence of the limits of integration to zero when the arguments of the integrand lie in adjacent subregions.

We now turn to showing that the family

$$\{T_{mn} f(x_0, \Omega_0, \cdot), f \in \mathfrak{B}, m, n \geqslant 1\}$$

for

$$x_0 \in \Gamma_i, \quad i \in \{1, ..., \mathfrak{N}\}, \quad \Omega \in D^2$$

is relatively compact in $L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})$. From Assumption A and (2.21) of Assumption B, we have that

$$\sup_{f \in \mathfrak{B}} \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |T_{mn} f(x_0, \Omega_0, E)| \, dE \leqslant \sigma_{\mathfrak{m}}^{-2} \sigma_{\mathfrak{M}}^2 A^2 \max_{(x, \Omega, \Omega')} \left\| \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} p(x, \Omega', E', \Omega, E) \, dE \right\|_{L^\infty[E_{\mathfrak{m}}, E_{\mathfrak{M}})}, \tag{4.31}$$

thereby showing the uniform boundedness of the family $\{T_{mn} f(x_0, \Omega_0, \cdot), f \in \mathfrak{B}, m, n \geqslant 1\}$, with $x_0 \in \Gamma_i$, $i \in \{1, ..., \mathfrak{N}\}$, and $\Omega_0 \in D^2$. Assumption A and (2.21) and (2.23) can be further utilized to produce the following estimates

$$\begin{aligned} &\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |T_{mn} f(x_0, \Omega_0, E + \gamma) - T_{mn} f(x_0, \Omega_0, E)| \, dE \leqslant \\ &\leqslant \sigma_{\mathfrak{M}}^2 \sigma_{\mathfrak{m}}^{-2} A^2 \|f\|_{\mathfrak{C}^p} \max_{(x_0, \Omega_0, \Omega')} \left\| \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} p(x_0, \Omega', E', E', \Omega_0, E) \, dE \right\|_{L^\infty[E_{\mathfrak{m}}, E_{\mathfrak{M}})} \cdot \\ &\cdot \max_{\Omega' \in D^2} \left\| \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |p(x_0, \Omega', E', \Omega_0, E + \gamma) - p(x_0, \Omega', E', \Omega_0, E)| \, dE \right\|_{L^\infty[E_{\mathfrak{m}}, E_{\mathfrak{M}})}. \end{aligned} \tag{4.32}$$

Therefore,

$$\lim_{\gamma \to 0} \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |T_{mn} f(x_0, \Omega_0, E + \gamma) - T_{mn} f(x_0, \Omega_0, E)| \, dE = 0 \tag{4.33}$$

uniformly for $f \in \mathfrak{B}$, and in n and m.

Finally, we remark that if $E_{\mathfrak{M}} = \infty$, we can show that

$$\lim_{E_0 \to \infty} \int_{E_0}^{\infty} |T_{mn} f(x_0, \Omega_0, E)| \, dE = 0$$

uniformly in $f \in \mathfrak{B}$ and in m and n. From (2.40), we see that

$$(4.34) \qquad \int_{E_0}^{\infty} |\boldsymbol{T}_{mn} f(x_0, \Omega_0, E)|\, dE \leqslant \leqslant A^2 \sigma_{\mathfrak{m}}^{-2} \sigma_{\mathfrak{M}}^2 \sup_{\Omega' \in D^2} \left\| \int_{E_0}^{\infty} p(x_0, \Omega', E', \Omega_0, E)\, dE \right\|_{L^\infty[E_{\mathfrak{m}}, E_{\mathfrak{M}})} \|f\|_{\mathfrak{C}^P}$$

and the right hand side approaches zero uniformly for $f \in \mathfrak{B}$ and in m and n. The proof of Theorem 4.1 is complete.

The result of Theorem 4.1, in conjunction with Proposition 4.1, enables us to conclude that the family $\{[\boldsymbol{I} - (\boldsymbol{Q}_{mn}\boldsymbol{P}_n\boldsymbol{M}_n)^2]^{-1}, m \geqslant m_0, n \geqslant n_0\}$ is uniformly bounded because of the subcritically assumption C, with m_0 and n_0 sufficiently large. The treatment in [8] by Nelson and Victory can be generalized in a straightforward manner to show that the family $\{(\boldsymbol{Q}_m\boldsymbol{M})^2, m \geqslant 1\}$ is a perturbation of a collectively compact sequence of operators acting on $\mathfrak{C}^P$. As a consequence an analysis similar to that of Theorem 2 of [8] shows that $\boldsymbol{Q}_m \Psi_m \to \boldsymbol{Q}\Psi$ and hence $\Psi_m \to \Psi$ as $m \to \infty$. From (4.2), $\|\boldsymbol{Q}_m \Psi_m - \boldsymbol{Q}_{mn}\boldsymbol{P}_n \Psi_m^n\| \to 0$ as $m, n \to \infty$, and the convergence of Ψ_m^n to Ψ is immediate from (4.1). We summarize:

Theorem 4.2. – *Let the original transport problem* (2.1) *be subcritical, i.e. let* $\|\boldsymbol{MQ}\|_{sp} < 1$. *Then, under* Assumptions A *and* B *concerning the problem data* σ *and* p *respectively, the approximations* Ψ_m^n *converge to* Ψ *as* $m, n \to \infty$ *under the conditions that both* Υ_n *and* χ_n *converge to zero as* $n \to \infty$.

If we consider the multigroup approximations per se—with Ω undiscretized—we see that precisely the same analysis, as utilized in Theorem 4.1, shows that the family $\{(\boldsymbol{QM}_n)^2 : n \geqslant 1\}$ is a perturbation of a collectively compact sequence of operators acting on $\mathfrak{C}^P$. Toward this end, we subdivide the set $D^2 \times D^2$ into subsets S_{ε_0} and $\bar{S}_{\varepsilon_0}$, for some small ε_0, analogous to S_m and $\bar{S}_m$ respectively in the portion of the proof to Theorem 4.1 concerned with showing equicontinuity of $\{(\boldsymbol{Q}_m\boldsymbol{M}_n)^2 : m \geqslant 1, n \geqslant 1\}$. The proof of the collective compactness properties of $\{(\boldsymbol{QM}_n)^2 : n \geqslant 1\}$ can be carried out under conditions weaker than in Assumption B concerning p. In fact (2.21) is replaced by the requirements that

(i)

$$(4.35) \qquad \int_{D^2} \int \sup_{E'} \left[\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} p(x, \Omega', E', \Omega, E)\, dE \right] (1 - |\Omega'|^2)^{-\frac{1}{2}}\, d\Omega'$$

is bounded above for $(x, \Omega) \in \Gamma \times D^2$;

(ii) given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$(4.36) \qquad \int \int_{\{\Omega' : |\Omega'| \leqslant \delta\}} \sup_{E'} \left[\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} p(x, \Omega', E', \Omega, E)\, dE \right] (1 - |\Omega'|^2)^{-\frac{1}{2}}\, d\Omega' < \varepsilon$$

uniformly in $(x, \Omega) \in \Gamma_i \times D^2$, $i \in \{1, \ldots, \mathfrak{N}\}$; condition (2.22) is replaced by:

(iii) given $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - x_0|^2 + |\Omega - \Omega_0|^2 \leqslant \delta^2$, $x, x_0 \in \Gamma_i$, $i \in \{1, \dots, \mathfrak{N}\}$,

$$(4.37)\qquad \int\limits_{D^2}\int \left\{\sup_{E'} \int\limits_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |p(x, \Omega', E', \Omega, E) - p(x_0, \Omega', E', \Omega_0, E)|\, dE\right\} \cdot \{(1 - |\Omega'|^2)^{-\frac{1}{2}}\, d\Omega' < \varepsilon;$$

and, finally, condition (2.23) is replaced by:

(iv) if we define $p(x, \Omega', E', \Omega, E) \equiv 0$, whenever E or E' is greater than $E_{\mathfrak{M}}$, or whenever E or E' is less than $E_{\mathfrak{m}}$, then for fixed $(x_0, \Omega_0) \in \Gamma_i \times D^2$,

$$(4.38)\qquad \lim_{t\to 0} \int\limits_{D^2}\int \left\{\sup_{E'} \int\limits_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |p(x_0, \Omega', E', \Omega_0, E + t) - p(x_0, \Omega', E', \Omega_0, E)|\, dE\right\} (1 - |\Omega'|^2)^{-\frac{1}{2}}\, d\Omega' = 0\,.$$

A detailed interpretation of these conditions can be defined in Section II of [15].

The approximations Ψ_n, defined as solutions of (3.11), (3.12) when Ω is left undiscretized satisfy error estimates as in (4.1) and (4.2) by formally replacing $\boldsymbol{Q}_m$ by $\boldsymbol{Q}$ and $\boldsymbol{Q}_{mn}\boldsymbol{P}_n$ by $\boldsymbol{Q}_n\boldsymbol{P}_n$. The sequence $\{\boldsymbol{Q}_n\boldsymbol{P}_n\}$ converges uniformly to $\boldsymbol{Q}$ under the stipulation that $\Upsilon_n \to 0$ as $n \to \infty$. We can show by the arguments in this section:

THEOREM 4.3. – *Under* Assumptions A *and* C, *and with the modification of* Assumption B *outlined above, the approximations Ψ_n converge to Ψ as $n \to \infty$ under the conditions that both Υ_n and χ_n converge to zero as $n \to \infty$.*

REMARK. – *From the fact that solutions of* (3.4) *and* (3.5) *generate solutions of* (3.11) *and* (3.12), *and vice-versa, we have rigorously shown the existence of the multigroup discrete-ordinates approximations $\{\Psi^n_{mi,g}(x),\ g = 1, \dots, G_n,\ i = 1, \dots, N_n\}$ in subcritical media at least for n and m sufficiently large. Also*

$$(4.39)\qquad \max_{x\in\Gamma}\max_{\Omega_{mi}} \sum_{g=1}^{G_n} \left|\Psi^n_{mi,g}(x) - \int\limits_{I_g^n} \Psi(x, \Omega_{mi}, E)\, dE\right| \leqslant$$

$$\leqslant \max_{(x,\Omega)\in\Gamma\times D^2} \sum_{g=1}^{G_n} \left|\boldsymbol{P}_n\Psi^n_m(x, \Omega) - \int\limits_{I_g^n} \Psi(x, \Omega, E)\, dE\right| =$$

$$= \max_{(x,\Omega)\in\Gamma\times D^2} \sum_{g=1}^{G_n} \left|\int\limits_{I_g^n} [\Psi^n_m(x, \Omega, E) - \Psi(x, \Omega, E)]\, dE\right| \leqslant$$

$$\leqslant \max_{(x,\Omega)\in\Gamma\times D^2} \sum_{g=1}^{G_n} \int\limits_{I_g^n} |\Psi(x, \Omega, E) - \Psi^n_m(x, \Omega, E)|\, dE = \|\Psi - \Psi^n_m\|_{\mathfrak{C}_1}\,.$$

We note that consistency of the multigroup, discrete-ordinates approximations is due to the strong or pointwise convergence of both $\boldsymbol{M}_n$ to $\boldsymbol{M}$ and of $\boldsymbol{Q}_m$ to $\boldsymbol{Q}$, along with the operator convergence of $\boldsymbol{Q}_{mn}\boldsymbol{P}_n$ to $\boldsymbol{Q}_m$ as $n \to \infty$ uniformly in m. This is guaranteed by requiring Υ_n and χ_n to approach zero uniformly as $n \to \infty$. These are the same results obtained by Nelson and Victory [9] and Victory [16] where different analytical techniques are used in a different normed setting. Moreover the uniform boundedness of the sequence $\{((\boldsymbol{I} - \boldsymbol{Q}_{mn}\boldsymbol{P}_n\boldsymbol{M}_n)^2)^{-1}\}$ has been shown to hold for general subcritical media rather than for the submultiplying ones considered in [9, 16].

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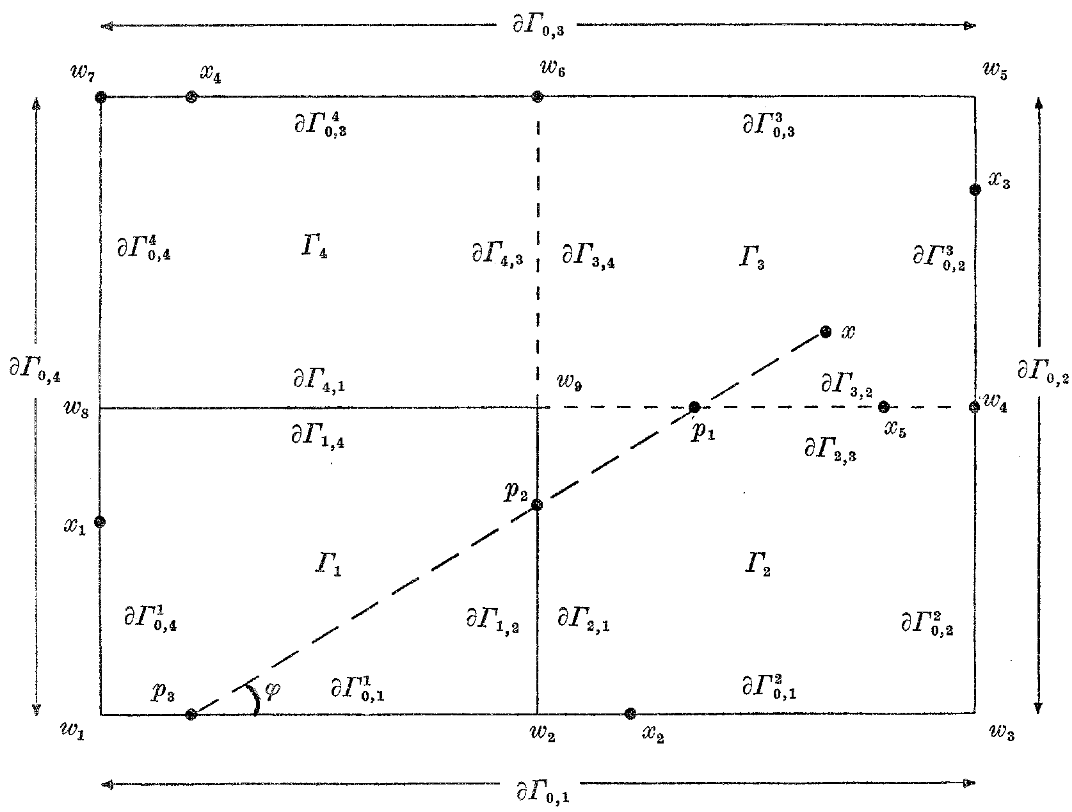
Appendix A



Let Γ be the region in the above figure. We can easily see that $L_1 = \{2, 3, 4\}$, $L_2 = \{1, 3, 4\}$, $L_3 = \{1, 2, 4\}$, and $L_4 = \{1, 2, 3\}$. Moreover:

$$\partial\Gamma_1 = \partial\Gamma^1_{0,1} \cup \partial\Gamma^1_{0,4} \cup \partial\Gamma_{1,2} \cup \partial\Gamma_{1,4}\,,$$
$$\partial\Gamma_2 = \partial\Gamma^2_{0,1} \cup \partial\Gamma^2_{0,2} \cup \partial\Gamma_{2,1} \cup \partial\Gamma_{2,3}\,,$$
$$\partial\Gamma_3 = \partial\Gamma^3_{0,2} \cup \partial\Gamma^3_{0,3} \cup \partial\Gamma_{3,2} \cup \partial\Gamma_{3,4}\,,$$
$$\partial\Gamma_4 = \partial\Gamma^4_{0,3} \cup \partial\Gamma^4_{0,4} \cup \partial\Gamma_{4,3} \cup \partial\Gamma_{4,1}\,.$$

For

$$i = 1\,, \qquad \partial\Gamma^I_1 = \partial\Gamma_{1,2} \cup \partial\Gamma_{1,4}; \quad \partial\Gamma^0_1 = \partial\Gamma^1_{0,1} \cup \partial\Gamma^1_{0,4};$$
$$i = 2\,, \qquad \partial\Gamma^I_2 = \partial\Gamma_{2,1} \cup \partial\Gamma_{2,3}; \quad \partial\Gamma^0_2 = \partial\Gamma^2_{0,2} \cup \partial\Gamma^2_{0,1};$$
$$i = 3\,, \qquad \partial\Gamma^I_3 = \partial\Gamma_{3,2} \cup \partial\Gamma_{3,4}; \quad \partial\Gamma^0_3 = \partial\Gamma^3_{0,2} \cup \partial\Gamma^3_{0,3};$$
$$i = 4\,, \qquad \partial\Gamma^I_4 = \partial\Gamma_{4,3} \cup \partial\Gamma_{4,1}; \quad \partial\Gamma^0_4 = \partial\Gamma^4_{0,3} \cup \partial\Gamma^4_{0,4}\,.$$

We note that

$$(a) \qquad W = \{w_i\colon i = 1, 2, \ldots, 9\}\,,$$

$$(b) \qquad G_0 = \bigcup_{i=1}^{8} [w_i, w_{i+1}] \cup [w_8, w_1] \cup [w_9, w_2] \cup [w_9, w_4] \cup [w_9, w_6]$$

(where here $[w_i, w_{i+1}]$ denotes the line segment joining w_i and w_{i+1}),

$$(c) \qquad G_0 = G_1\,.$$

The points of intersection of the segment $\alpha(x, \Omega)$ with the boundaries of subregions are labeled p_1, p_2, and p_3 (here $\Omega = (\cos\varphi, \sin\varphi)$). For each of these points, we have

$$N(p_1) = \{(3, 2) \cup (2,3)\}\,, \quad N(p_2) = \{(1, 2) \cup (2, 1)\}\,, \quad N(p_3) = \{(1, 1)_0\}\,.$$

Consider the five points labeled x_1, x_2, x_3, x_4, and x_5. Observe that

$$M(x_1, \Omega = (0, -1)) = \{(1, 4)_0 \cup (4, 4)_0\}$$
$$M(x_1, \Omega = (0, 1)) = \{(1, 4)_0\},\ M(x_1, \Omega = (\pm 1, 0)) = \emptyset\,,$$

$$M(x_2, \Omega = (1, 0)) = \{(2, 1)_0 \cup (1, 1)_0\}$$
$$M(x_2, \Omega = (-1, 0)) = \{(2, 1)_0\},\ M(x_2, \Omega = (0, \pm 1)) = \emptyset\,,$$

$$M(x_3, \Omega = (0, 1)) = \{(3, 2)_0 \cup (2, 2)_0\}$$
$$M(x_3, \Omega = (0, -1)) = \{(3, 2)_0\},\ M(x_3, \Omega = (\pm 1, 0)) = \emptyset\,,$$

$$M(x_4, \Omega = (-1, 0)) = \{(4, 3)_0 \cup (3, 3)_0\}$$
$$M(x_4, \Omega = (1, 0)) = \{(4, 3)_0\},\ M(x_4, \Omega = (0, \pm 1)) = \emptyset\,,$$
$$M(x_5, \Omega = (1, 0)) = \{(3, 2) \cup (2, 3) \cup (1, 4) \cup (4, 1)\}$$
$$M(x_5, \Omega = (-1, 0)) = \{(3, 2) \cup (2, 3)\},\ M(x_5, \Omega = (0, \pm 1)) = \emptyset\,.$$

Appendix B

We analyze the regularity of the optical thickness for $\Omega \in D^2_\eta$, as we wish to gauge the influence of boundary, shadow, and vertex singularities on the regularity of angular flux. Without loss of generality, we consider the case for $|\Omega| = 1$; the general case follows by similar arguments. The regularity properties of the optical distance are summarized in:

PROPOSITION B.1. – (i) *The optical distance* $\varrho(x, \Omega, \cdot)$ *is continuous, with range in* $L^\infty[E_m, E_{\mathfrak{M}})$ *in both* x *and* Ω *for all* $x \notin G_1$ *and* $\Omega \in D^2$, $|\Omega| = 1$; (ii) *If* $x \in G_1$, *then* $\varrho(x, \Omega, \cdot)$ *is continuous at* x *for* $\Omega \notin \Pi$, *and for those* $\Omega \in \Pi$ *whenever it happens that*

$$\operatorname{card}\big(W_s(x, \Omega)\big) < \infty;$$

(iii) *Let* $x \in G_1$; *then for those* $\Omega \in \Pi$ *for which*

$$\operatorname{card}\big(W_s(x, \Omega)\big) = \infty$$

the optical distance is discontinuous in general, but has one-sided limits in $L^\infty[E_m, E_{\mathfrak{M}})$. *More precisely, let* n_Ω *be a normal to the ray* $\alpha(x, \Omega)$ *whose second component is selected positive if nonzero; otherwise, whose first component is selected positive. Let* x_n^+ *and* x_n^- *be two sequences tending to* $x \in \Gamma$ *such that*

$$(x_n^+ - x)\cdot n_\Omega \geqslant 0\,, \quad x_n^+ \notin G_1\,,$$
$$(x_n^- - x)\cdot n_\Omega \leqslant 0\,, \quad x_n^- \notin G_1\,.$$

Then

$$(1)\quad \Big|\lim_{x_n^+ \to x} \varrho(x_n^+, \Omega, \cdot) - \lim_{x_n^- \to x} \varrho(x_n^-, \Omega, \cdot)\Big| =$$
$$= \Bigg|\sum_{\{(i,j)\in M(x,\Omega):\, \lim\limits_{n\to\infty}(n_{i,j}\cdot(x_n^+ - x)/(\|x_n^+ - x\|))\geqslant 0\}} \ \int\limits_{\{s:\, x - s\Omega \in \partial\Gamma_{i,j}\}} \sigma_{i,j}(x - s\Omega, \Omega, \cdot)\, ds -$$
$$- \sum_{\{(i,j)\in M(x,\Omega):\, \lim\limits_{n\to\infty}(n_{i,j}\cdot(x_n^- - x)/(\|x_n^- - x\|))\geqslant 0\}} \ \int\limits_{\{s:\, x - s\Omega \in \partial\Gamma_{i,j}\}} \sigma_{ij}(x - s\Omega, \Omega, \cdot)\, ds\Bigg|\,,$$

where $\sigma_{i,j}(x, \Omega, \cdot)$ *is defined by* (2.29). *If* $x \in \partial\Gamma$, *a similar formula is valid, with* $\lim\limits_{x_n^+ \to x} \varrho(x_n^+, \Omega, \cdot) \equiv 0$ (*since* $\sigma \equiv 0$ *outside* Γ). *Hence*

$$(2) \qquad \lim_{n\to\infty} \varrho(x_n^-, \Omega, \cdot) = \sum_{\{(i,j)_0 \in M(x,\Omega) : \lim\limits_{n\to\infty} (n_{0,j}^i \cdot (x - x_n^-)/(\|x - x_n^-\|)) \geq 0\}} \int\limits_{\{s : x - s\Omega \in \partial\Gamma_{0,j}^i\}} \sigma_{0,j}^i (x - s\Omega, \Omega, \cdot)\, ds ,$$

where $x_n \in \mathring{\Gamma}$, $x_n \to x \in \partial\Gamma_{0,j}^i$, $x_n \notin G_1$, *with* $\sigma_{0,j}^i$ *given by* (2.30).

REMARKS. – *The continuity properties of* σ, *as hypothesized in* Assumption A, *along with the definition of* $\varrho(x, \Omega, \cdot)$ *in* (2.28), *enables us to conclude that* ϱ *has the regularity features similar to those described in conditions* 1-6 *defining* $\mathfrak{C}_1$: *Indeed, we can express the optical distance from* x *to the boundary* $\partial\Gamma$ *as*

$$(3) \qquad \varrho(x, \Omega, \cdot) = \sum_{\{i : \alpha(x,\Omega) \cap \Gamma_i \neq \emptyset\}} \int_{d_i^-(x,\Omega)}^{d_i^+(x,\Omega)} \sigma(x - s\Omega, \Omega, \cdot)\, ds$$

where d_i^+ *and* d_i^- *are defined following expression* (2.26). (*For* $x \in \Gamma_i$, *we define* $d_i^-(x, \Omega) = 0$ *and* $d_i^+(x, \Omega)$ *to be the distance to* $\partial\Gamma_i$). *The asserted regularity of* $\varrho(x, \Omega, \cdot)$ *is a straightforward consequence of this formula.*

In order to discuss the spatial derivatives of $\varrho(x, \Omega, \cdot)$, especially along interfaces, we shall need assumptions on the spatial partials of σ similar to those hypothesized for σ in Assumption A. Moreover, in the following discussion, we shall let $W(x, \Omega)$ denote the vertices belonging to $\alpha(x, \Omega)$, i.e.

$$(4) \qquad W(x, \Omega) := W \cup \alpha(x, \Omega) .$$

Our regularity results on the spatial partials of ϱ are summarized in

PROPOSITION B.2. – (i) *Let* $x \notin G_1$, $\Omega \in D^2$, $|\Omega| = 1$. *Suppose, for the moment* $W(x, \Omega) = \emptyset$. *Then we have the following expression for the directional derivative of* ϱ, *with respect to* x *in direction* α:

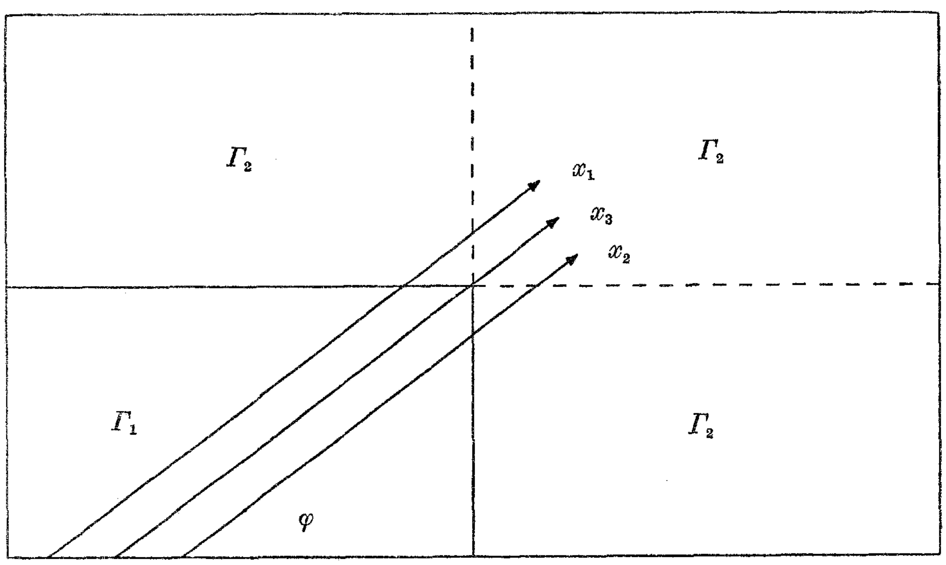
$$(5) \qquad \alpha \cdot \nabla \varrho(x, \Omega, \cdot) = - \sum_{w \in W_s(x,\Omega)} \sum_{\{(i,j) \in N(w)\}} \frac{n_{i,j} \cdot \alpha}{n_{i,j} \cdot \Omega} \cdot \operatorname{sgn}(n_{i,j} \cdot \Omega)\, \sigma_{i,j}(w, \Omega, \cdot) + \varrho_1(x, \Omega, \cdot) ,$$

where $\varrho_1(x, \Omega, \cdot)$ *has the regularity features described for* ϱ *in* Proposition B.1 *and subsequence remarks.* (ii) *Suppose* $x \notin G_1$, *and* $W(x, \Omega) \neq \emptyset$. *Then we cannot define values* (*in* $L^\infty[E_{\mathfrak{m}}, E_{\mathfrak{M}}]$) *of* $\alpha \cdot \nabla_x \varrho(x_n^\pm, \Omega, \cdot)$ *at such points* (x, Ω), *but we do have one sided limits* (*in* $L^\infty[E_{\mathfrak{m}}, E_{\mathfrak{M}}]$). *More precisely, let* x_n^+, x_n^-, *and* n_Ω *be given as in* (iii) *of*

Proposition B.1. *Then the limiting values of* $\alpha\cdot\nabla_x\varrho(x_n^{\pm}, \Omega, \cdot)$ *in* $L^\infty[E_{\mathfrak{m}}, E_{\mathfrak{M}})$ *have a «jump» at* x *given by* (5) *with* $W_s(x, \Omega)$ *replaced by* $W(x, \Omega) - \{x\}$. *Otherwise the derivative is continuous.*

REMARKS. (*A*) *The expression* (5) *for* $\alpha\cdot\nabla_x\varrho(x, Q, \cdot)$ *follows from differentiating* (3) *in conjunction with the hypothesized smoothness of* τ *and the fact that* $\alpha\cdot\nabla_x d_i^{\pm}(x, \Omega) = n_{ij}\cdot\alpha/n_{i,j}\cdot\Omega$ *for appropriate* $(i, j)\in N(\omega)$. (*B*) *For the case when* $x\in G_1$, *and* $W(x, \Omega) = \emptyset$, *then we can define* $\alpha\cdot\nabla_x\varrho(x, \Omega, \cdot)$ *for such* x*'s by taking the limiting values, in* $L^\infty(E_{\mathfrak{m}}, E_{\mathfrak{M}})$, *of the right hand side of* (5) *as* $x_n\to x$, $x_n\notin G_1$. *We can also allow* x *to be a vertex if* $W(x, \Omega) - \{x\} = \emptyset$. (*C*) *Assertions* (i) *and* (ii) *of* Proposition B.2 *implicitly assume that* card $(W_s(x, \Omega)) < \infty$*—or that no interface is a subset of* $\alpha(x, \Omega)$. *If we allow both* x *and* Ω *to vary in assertion* (ii), *the behavior of the derivatives* $\alpha\cdot\nabla_x\varrho(x, \Omega, \cdot)$ *can be even more complicated than previously described. Indeed, if* $x_n^{\pm}$ *and* $\Omega_m^{\pm}$ *are chosen in a manner analogous to that expounded in condition* 4 *describing* $\mathfrak{C}_1$, *the interative limits of* $\alpha\cdot\nabla_x\varrho(x, \Omega, \cdot)$ *will in general be different for any choice of* $x_n^{\pm}$ *with either of* $\Omega_m^{\pm}$. *This is discerned by examining* (5). *The pathologies in the behavior of* $\alpha\cdot\nabla_x\varrho(x, \Omega, \cdot)$ *are primarily responsible for the* vertex singularities *inherent in transport solutions pointed out by* R. B. Kellogg *in* [6]. (*D*) *For values of* x *and* Ω *for which* $\Omega\in\Pi$, $|\Omega| = 1$, $x\in G_1$, $M(x, \Omega)\neq\emptyset$, *then the behavior of* $\alpha\cdot\nabla_x\varrho(x, \Omega, \cdot)$ *is divergent as a result of the divergence of the leading terms in* (5), *since* $n_{i,j}\cdot\Omega_n\to 0$ *whenever* Ω_n *is any sequence tending to such* Ω.

EXAMPLE. – *Consider the following medium with constant cross-section data* σ_1 *and* σ_2:



Let $\alpha = (\cos\ \theta_a, \sin\ \theta_a)$. *Note that*

$$\varrho(x, \Omega) = \sigma_2 d_1^+(x, \Omega) + \sigma_1(d(x, \Omega) - d_1^+(x, \Omega)) \tag{6}$$

for any of the positions x_1, x_2, *or* x_3: *From* (5), *we compute*

$$\alpha\cdot\nabla_x\varrho(x_1, \Omega) = \sigma_2 \frac{\sin\theta_a}{\sin\varphi} \tag{7}$$

$$\alpha\cdot\nabla_x\varrho(x_2, \Omega) = (\sigma_2 - \sigma_1)\frac{\cos\theta_a}{\cos\varphi} + \sigma_1\frac{\sin\theta_a}{\sin\varphi} \tag{8}$$

for the points x_1 *and* x_2 *respectively. Formulas* (7) *and* (8) *are also found by direct differentiation of* (6).

In the direction $\alpha = \Omega/|\Omega| = (\cos\ \varphi, \sin\ \varphi)$, *the optical distance tends to increase; for such* α *formulas* (7) *and* (8) *yield* $\alpha\cdot\nabla_x\varrho(x, \Omega) = \sigma_2 > 0$, $x = x_1, x_2$. *Similarly, for* $\alpha = -\Omega/|\Omega|$, *the optical distances at* x_1 *and* x_2 *tend to decrease, i.e.* $\alpha\cdot\nabla_x\varrho(x, \Omega) = -\sigma_2 < 0$, $x = x_1, x_2$. *Finally, the jump in the directional derivatives across the middle ray is precisely*

$$(\sigma_2 - \sigma_1)\left(\frac{\cos\theta_a}{\cos\varphi} - \frac{\sin\theta_a}{\sin\varphi}\right) = 2\ (\sigma_2 - \sigma_1)\left(\frac{\sin(\varphi - \theta_a)}{\sin 2\varphi}\right) \tag{9}$$

unless $\theta_a = \varphi$, *in which case there is no jump.*

Appendix C: Proof of Lemma 2.3.

We have shown that $\boldsymbol{M}f$, $f \in \mathfrak{C}^P$, is uniformly continuous as a mapping from $\Gamma \times \Pi_\delta \subset \Gamma \times D^2$, $\delta > 0$, to $L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})$. In this discussion, we shall focus on showing that $\boldsymbol{M}f$ can be extended so as to have the continuity properties ascribed to $\mathfrak{C}_1$: This will be accomplished by a close perusal of the formula for $\boldsymbol{M}f$, (2,5), $f \in \mathfrak{C}^P$, in the light of the results on the optical distance between x and x' in Appendix B and on the continuity of $d(x, \Omega)$ in Lemma 2.1. We shall also make use of the following expressions for the case when $x_\Gamma \in \partial\Gamma$, $\Omega \in \Xi_-(x_\Gamma)$, $|\Omega| \geq \eta$ and for $\Omega \in \Pi_\delta \setminus \Xi_-(x_\Gamma)$:

$$\boldsymbol{M}f(x_\Gamma, \Omega, E) = \begin{cases} 0, \ \Omega \in \Xi_-(x_\Gamma), \quad |\Omega| \geq \eta \\ \displaystyle\int_0^{d(x_\Gamma,\Omega)} f(x_\Gamma - t|\Omega|^{-1}\Omega, \Omega, E)\,. \\ \exp\left\{-\displaystyle\int_0^t \sigma(x - r|\Omega|^{-1}\Omega, \Omega, E)|\Omega|^{-1} dr\right\}|\Omega|^{-1} dt\,, \\ \Omega \in \Pi_\delta \setminus \Xi_-(x_\Gamma)\,. \end{cases} \tag{1}$$

If $x \in \mathring{\Gamma}_i$, then we define

$$\boldsymbol{M}f(x, 0, E) := f(x, 0, E)/\sigma(x, 0, E) . \tag{2}$$

The regularity results on the optical distance fundamentally determine the smoothness features for $\boldsymbol{M}f$, $f \in \mathfrak{C}^P$, since the integration defining the attenuation term and $\boldsymbol{M}f$ itself takes place over $\alpha(x, \Omega)$. More precisely, in the expression for $\boldsymbol{M}$.

$$\begin{aligned} \text{(3)}\quad & \boldsymbol{M}f(x, \Omega, E) = \\ & = \int\limits_0^{d(x,\Omega)} f(x - t|\Omega|^{-1}\Omega, \Omega, E) \exp\left\{-\int\limits_0^t \sigma(x - r|\Omega|^{-1}\Omega, \Omega, E))|\Omega|^{-1} dr\right\} |\Omega|^{-1} dt , \end{aligned}$$

the continuity properties of the optical distance and of f as an element of $\mathfrak{C}^P$ determine those of $\boldsymbol{M}f$ at least for $\Omega \in D^2_\eta$. We may substitute for $\sigma_{i,j}$ and $\sigma^i_{0,j}$ in equations (1) and (2), respectively, of Appendix B the values $f_{i,j}\Phi_{i,j}$ and $f^i_{0,j}\Phi^i_{0,j}$ where $\Phi_{i,j}$ and $\Phi^i_{i,j}$ are the limiting values of Φ_x given by

$$\Phi_x(x', \Omega, E) = |\Omega|^{-1} \exp\left(-\varrho(x, x', \Omega, E)|\Omega|^{-1}\right) . \tag{4}$$

So, $\boldsymbol{M}f$ can be extended to the boundaries of each Γ_i, $i = 1, 2, ..., \mathfrak{N}$, and of Γ itself, to possess the regularity features ascribed to $\mathfrak{C}_1$ by conditions (1)-(6).

It behooves us, then, to show that for (i) $x, x' \in \mathring{\Gamma}_i$, $i = 1, 2, ..., \mathfrak{N}$,

$$\boldsymbol{M}f(x', \Omega, \cdot) \to f(x, 0, \cdot)/\sigma(x, 0, \cdot)$$

as $x' \to x$, $\Omega \to 0$ with limiting values on the boundary of each Γ_i (as required by condition 7 defining $\mathfrak{C}_1$); (ii) for those Ω residing in $\Pi_\delta \setminus \overline{\Xi_-(x_\Gamma)}$ and in the set $\Pi(\eta)\setminus\overline{\Xi_-(x_\Gamma)}$ for which card $(W_s(x_\Gamma, \Omega)) < \infty$,

$$\boldsymbol{M}f(x, \Omega, \cdot) \to \boldsymbol{M}f(x_\Gamma, \Omega, \cdot)$$

as $x \to x_\Gamma$; (iii) for such Ω in (ii),

$$\boldsymbol{M}f(x_\Gamma, t\Omega, \cdot) \to f(x_\Gamma, 0, \cdot)/\sigma(x_\Gamma, 0, \cdot) ,$$

as $t \to 0$; (iv) for those Ω for which $x_\Gamma - s|\Omega|^{-1}\Omega \cap \mathring{\Gamma} \neq \emptyset$ and card $(W_s(x_\Gamma, \Omega)) = \infty$, $\boldsymbol{M}f(x_\Gamma, \Omega, \cdot)$ possesses one-sided limits in $L^1[E_{\mathfrak{m}}, E_{\mathfrak{M}})$ when sequences $(x_n^\pm, \Omega_m^\pm)$ —as described in conditions (4) and (5b) in the definition of $\mathfrak{C}_1$—are taken; (v) for Ω such that card $(x_\Gamma - s|\Omega|^{-1}\Omega \cap \partial\Gamma) > 1$, we have

$$\boldsymbol{M}f(x_\Gamma, t\Omega, \cdot) \to f(x_\Gamma, 0, ^\infty)/\sigma(x_\Gamma, 0, \cdot);$$

(iv) for Ω for which $\operatorname{card}(x_\Gamma - s|\Omega|^{-1}\Omega \cap \partial\Gamma) = 0$, $s > 0$,

$$M(x_\Gamma, t\Omega, \cdot) \to 0 .$$

To show (i), let us first suppose that the point x lies in that subset of Γ_i whose points are at least a distance ε from $\partial\Gamma_i$ for arbitrary, but fixed, $\varepsilon > 0$. Let x' be a point of the same subset of Γ_i. We proceed to estimate

$$(5)\qquad \int_{E_\mathfrak{m}}^{E_\mathfrak{M}} |Mf(x', \Omega, E) - f(x, 0, E)/\sigma(x, 0, E)|\, dE =$$

$$= \int_{E_\mathfrak{m}}^{E_\mathfrak{M}} \Bigg| \int_0^{d(x',\Omega)} \sigma(x' - t|\Omega|^{-1}\Omega, \Omega, E) \exp\left\{-\int_0^t \sigma(x' - r|\Omega|^{-1}\Omega, \Omega, E)|\Omega|^{-1} dr\right\} \cdot$$

$$\cdot \frac{f(x' - t|\Omega|^{-1}\Omega, \Omega, E)}{\sigma(x' - t|\Omega|^{-1}\Omega, \Omega, E)} |\Omega|^{-1} dt - \frac{f(x, 0, E)}{\sigma(x, 0, E)}\Bigg|\, dE =$$

$$= \int_{E_\mathfrak{m}}^{E_\mathfrak{M}} \Bigg| \int_0^{d(x',\Omega)} \sigma(x' - t|\Omega|^{-1}\Omega, \Omega, E) \exp\left\{-\int_0^t \sigma(x' - r|\Omega|^{-1}\Omega, \Omega, E)|\Omega|^{-1} dr\right\} \cdot$$

$$\cdot \left[\frac{f(x' - t|\Omega|^{-1}\Omega, \Omega, E)}{\sigma(x' - t|\Omega|^{-1}\Omega, \Omega, E)} - \frac{f(x, 0, E)}{\sigma(x, 0, E)} + \frac{f(x, 0, E)}{\sigma(x, 0, E)}\right] |\Omega|^{-1} dt - \frac{f(x, 0, E)}{\sigma(x, 0, E)}\Bigg|\, dE =$$

$$= \int_{E_\mathfrak{m}}^{E_\mathfrak{M}} \Bigg| \int_0^{d(x',\Omega)} \sigma(x' - t|\Omega|^{-1}\Omega, \Omega, E) \exp\left\{-\int_0^t \sigma(x' - r|\Omega|^{-1}\Omega, \Omega, E)|\Omega|^{-1} dr\right\} \cdot$$

$$\cdot \left[\frac{f(x' - t|\Omega|^{-1}\Omega, \Omega, E)}{\sigma(x' - t|\Omega|^{-1}\Omega, \Omega, E)} - \frac{f(x, 0, E)}{\sigma(x, 0, E)}\right] |\Omega|^{-1} dt -$$

$$- \frac{f(x, 0, E)}{\sigma(x, 0, E)} \exp\left\{-\int_0^{d(x',\Omega)} \sigma(x' - t|\Omega|^{-1}\Omega, \Omega, E)|\Omega|^{-1} dt\right\}\Bigg|\, dE \leqslant$$

$$\leqslant \int_{E_\mathfrak{m}}^{E_\mathfrak{M}} \int_0^{\gamma} \sigma(x' - t|\Omega|^{-1}\Omega, \Omega, E) \exp\left\{-\int_0^t \sigma(x' - r|\Omega|^{-1}\Omega, \Omega, E)|\Omega|^{-1} dr\right\} \cdot$$

$$\cdot \left|\frac{f(x' - t|\Omega|^{-1}\Omega, \Omega, E)}{\sigma(x' - t|\Omega|^{-1}\Omega, \Omega, E)} - \frac{f(x, 0, E)}{\sigma(x, 0, E)}\right| |\Omega|^{-1} dt\, dE +$$

$$+ \int_{E_\mathfrak{m}}^{E_\mathfrak{M}} |f(x, 0, E)/\sigma(x, 0, E)| \exp\left\{-\int_0^{d(x',\Omega)} \sigma(x' - t|\Omega|^{-1}\Omega, \Omega, E)|\Omega|^{-1} dt\right\} dE +$$

$$+ \int_{E_\mathfrak{m}}^{E_\mathfrak{M}} \int_{\gamma}^{d(x',\Omega)} \sigma(x' - t|\Omega|^{-1}\Omega, \Omega, E) \exp\left\{-\int_0^t \sigma(x' - r|\Omega|^{-1}\Omega, \Omega, E)|\Omega|^{-1} dr\right\} \cdot$$

$$\cdot \left|\frac{f(x' - t|\Omega|^{-1}\Omega, \Omega, E)}{\sigma(x' - t|\Omega|^{-1}\Omega, \Omega, E)} - \frac{f(x, 0, E)}{\sigma(x, 0, E)}\right| |\Omega|^{-1} dt\, dE \leqslant$$

$$\leqslant 2\sigma_{\mathfrak{M}}\sigma_{\mathfrak{m}}^{-2}\|f\|_{\mathfrak{C}^P}[\exp(-\sigma_{\mathfrak{m}}\gamma/|\Omega|) - \exp(-\sigma_{\mathfrak{m}} d(x,\Omega)/|\Omega|)] +$$

$$+\left(\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |f(x, 0, E)|/\sigma(x, 0, E)\, dE\right)\exp\{-\sigma_{\mathfrak{m}} d(x,\Omega)/|\Omega|\} +$$

$$+\sigma_{\mathfrak{M}}\sigma_{\mathfrak{m}}^{-1}\big(1-\exp(-\sigma_{\mathfrak{m}}\gamma/|\Omega|)\big)\max_{0\leqslant t<\gamma}\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}}\left|\frac{f(x'-t|\Omega|^{-1}\Omega, \Omega, E)}{\sigma(x'-t|\Omega|^{-1}\Omega, \Omega, E)} - \frac{f(x, 0, E)}{\sigma(x, 0, E)}\right| dE\,.$$

These inequalities prove assertion (i), since we can deduce

$$\text{(6)}\qquad \limsup_{x'\to x\; |\Omega|\to 0}\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |\boldsymbol{M}f(x', \Omega, E) - f(x, 0, E)/\sigma(x, 0, E)|\, dE = 0\,.$$

Item (iii) is straightforward to show, by utilizing formula (1), and (iii) is proven by an analysis similar to that for showing (i): Assertion (iv) is easily seen, by using the representation for $\boldsymbol{M}f(x_\Gamma, \Omega, E)$, $\Omega \in \Pi_\delta \diagdown \overline{\Pi_-(x_\Gamma)}$, and for $\boldsymbol{M}f(x, \Omega, E)$, $x \in \mathring{\Gamma}$, in (1) and (2) respectively, when sequences $(x_n^\pm, \Omega_m^\pm)$—as in conditions (4) and (5*b*) in the definition of $\mathfrak{C}_1$—are taken. We can easily deduce (vi) since $M(x_\Gamma, t\Omega, \cdot) \equiv 0$ for all t near zero because of the definition of $d(x_\Gamma, \Omega)$ in the representation of $\boldsymbol{M}f$ in (3) (cf., also, the definition of $\boldsymbol{M}(x_\Gamma, \Omega, E)$ in (1) when $\Omega \in \Xi_-(x_\Gamma)$, $|\Omega| \geqslant \eta$).

It remains to show (v). We recall the extension of $\boldsymbol{M}f$ to $x_\Gamma \in \partial\Omega$ and to such Ω, $|\Omega| = 1$, satisfying card $(x_\Gamma - s\Omega \cap \partial\Gamma) > 1$, as specified by condition (6) in the definition of $\mathfrak{C}_1$. With $d(x_\Gamma, \Omega)$ denoting the length of the intersection of the ray $x_\Gamma - s\Omega$ with $\partial\Gamma$, we have the following estimate similar to (5):

$$\text{(7)}\qquad \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |\boldsymbol{M}f(x_\Gamma, t\Omega, E) - f(x_\Gamma, 0, E)/\sigma(x_\Gamma, 0, E)|\, dE \leqslant$$

$$\leqslant \int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}}\Bigg|\int_0^{d(x_\Gamma,\Omega)} \sigma(x_\Gamma - \tau\Omega, t\Omega, E)\exp\left\{-\int_0^\tau \sigma(x_\Gamma - r\Omega, t\Omega, E)\, t^{-1}\, dr\right\}\cdot$$

$$\cdot\left[\frac{f(x_\Gamma - \tau\Omega, t\Omega, E)}{\sigma(x_\Gamma - t\Omega, t\Omega, E)} - \frac{f(x_\Gamma, 0, E)}{\sigma(x_\Gamma, 0, E)} + \frac{f(x_\Gamma, 0, E)}{\sigma(x_\Gamma, 0, E)}\right] t^{-1}\, d\tau - \frac{f(x_\Gamma, 0, E)}{\sigma(x_\Gamma, 0, E)}\Bigg|\, dE \leqslant$$

$$\leqslant 2\sigma_{\mathfrak{M}}(\sigma_{\mathfrak{m}})^{-2}\|f\|_{\mathfrak{C}^P}[\exp(-\sigma_{\mathfrak{m}}\gamma/t) - \exp(-\sigma_{\mathfrak{m}} d(x_\Gamma, \Omega)/t)] +$$

$$+\left(\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |f(x_\Gamma, 0, E)|/\sigma(x_\Gamma, 0, E)\, dE\right)\exp\{-\sigma_{\mathfrak{m}} d(x_\Gamma, \Omega)/t\} +$$

$$+\sigma_{\mathfrak{M}}\sigma_{\mathfrak{m}}^{-1}\left(1-\exp\left(\frac{-\sigma_{\mathfrak{m}}\gamma}{t}\right)\right)\max_{0\leqslant t\leqslant\gamma}\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}}\left|\frac{f(x_\Gamma - \tau\Omega, t\Omega, E)}{\sigma(x_\Gamma - \tau\Omega, t\Omega, E)} - \frac{f(x_\Gamma, 0, E)}{\sigma(x_\Gamma, 0, E)}\right| dE\,.$$

Hence

$$\limsup_{t\to 0}\int_{E_{\mathfrak{m}}}^{E_{\mathfrak{M}}} |\boldsymbol{M}f(x_\Gamma, t\Omega, E) - f(x_\Gamma, 0, E)/\sigma(x_\Gamma, 0, E)|\, dE = 0\,.$$

If x_Γ happens to be a point on an interface also, then the quantity

$$f(x_\Gamma, 0, E)/\sigma(x_\Gamma, 0, E)$$

should be interpreted as the limit of values (in $L^1[E_m, E_{\mathfrak{M}}]$) of $f(x_\Gamma^{(n)}, 0, \cdot)/\sigma(x_\Gamma^{(n)}, 0, \cdot)$ as $x_\Gamma^{(n)}$ approaches x_Γ along the subregion boundary which the ray $\alpha(x_\Gamma, \Omega)$ intersects. The proof of Lemma 2.3 is complete.

REFERENCES

[1] E. J. ALLEN, *A finite element approach for treating the energy variable in the numerical solution of the neutron transport equation*, Trans. Th. Stat. Phys., **15** (1986), pp. 449-478.

[2] P. M. ANSELONE, *Collectivity Compact Operator Approximation Theory*, Prentice-Hall, Englewood Cliffs, N. J., 1971.

[3] A. BELLENI-MORANTE - G. BUSONI, *Multigroup neutron transport*, J. Math. Phys., **13** (1972), pp. 1146-1150.

[4] M. BORYSIEWICZ - N. KRUSZYNSKA, *Smoothness of the solution of the 2-D neutron trnsport equation in the polygon region*, Atomkernenergie-Kerntechnik, **34** (1979), pp. 11-15.

[5] J. DIEUDONNÉ, *Foundations of Modern Analysis*, 1st ed., New York, Academic Press, 1960.

[6] R. B. KELLOGG, *Numerical analysis of the neutron transport ebuations*, in: Numerical Solution of Partial Differential Equations - III, SYNSPADE 1975, B. Hubbard ed., New York, Academic Press, Inc., 1976.

[7] E. MICHAEL, *Some extension theorems for continuous functions*, Pacific J. Math., **3** (1953), pp. 789-806.

[8] P. NELSON - H. D. VICTORY, JR., *Convergence of two-dimensional Nyström discrete-ordinates in solving the linear transport equation*, Numer. Math., **34** (1980), pp. 353-370.

[9] P. NELSON - H. D. VICTORY, JR., *On the convergence of the multigroup approximations for submultiplying slab media*, Math. Methods Appl. Sci., **4** (1982), pp. 206-229.

[10] J. PITKÄRANTA - L. RIDGWAY SCOTT, *Error estimates for the combined spatial and angular approximations of the transport equation for slab geometry*, SIAM J. Numer. Anal., **20** (1983), pp. 922-950.

[11] R. T. ROCKAFELLAR, *Convex Analysis*, 1st ed., Princeton University Press, 1970.

[12] W. RUDIN, *Real and Complex Analysis*, 1st ed., New York, McGraw Hill, 1968.

[13] J. D. TAMARKIN, *On the compactness of the space L_p*, Bull. Amer. Math. Soc., **38** (1932), pp. 79-84.

[14] H. D. VICTORY, JR., *Convergence properties of discrete-ordinates solutions for neutron transport in three-dimensional media*, SIAM J. Numer. Anal., **17** (1980), pp. 71-83.

[15] H. D. VICTORY, JR., *Convergence of the multigroup approximations for subcritical slab media with applications to shielding calculations*, Adv. Appl. Math., **5** (1984), pp. 227-259.

[16] H. D. VICTORY, JR., *On the convergence of the multigroup approximations for multidimensional media*, Ann. Math. Pura ed Applicata, (IV), **140** (1985), pp. 197-207.

[17] YANG MINGZHU - ZHU GUANGTIAN, *Multigroup theory for neutron transport*, Scientia Sinica, **22** (1979), pp. 1114-1127.