Random Relaxed Dirichlet Problems (*).

MICHELE BALZANO

Summary. – *We investigate sequences of Relaxed Dirichlet Problems of the form:*

$$-\Delta u_h + \mu_h u_h = 0$$

where μ_h are random Borel measures belonging to a suitable class $\mathcal{M}_0$. By means of a variational approach, necessary and sufficient conditions for the convergence in probability of the sequence u_h toward the solution of a deterministic Relaxed Dirichlet Problem are given. Some applications to Dirichlet problems in random perturbated domains and to a Schrödinger equation with random singular potentials are considered.

0. – Introduction.

In this paper we provide a general framework to study both the classical Dirichlet problem in domains with randomly distributed small holes and the stationary Schrödinger equation with rapidly oscillating random potentials.

More precisely, given a bounded open region D of $\mathbf{R}^d$, $d\geqslant 2$, and a function $f\in L^2(D)$, we deal with problems of the form

$$\begin{cases} -\Delta u = f & \text{in } D\setminus F \\ u\in H_0^1(D\setminus F) \end{cases} \tag{0.1}$$

where F is a random subset of D, and of the form

$$\begin{cases} -\Delta u + q(x)u = f & \text{in } D \\ u\in H_0^1(D) \end{cases} \tag{0.2}$$

where q is a random potential.

Problems (0.1) and (0.2) can be considered as particular cases of the so called relaxed Dirichlet problems (see [5], [8], [20], [21], [22]) formally written as

$$\begin{cases} -\Delta u + \mu u = f & \text{in } D \\ \qquad\quad u = 0 & \text{on } \partial D \end{cases} \tag{0.3}$$

(*) Entrata in Redazione il 25 febbraio 1987.

Indirizzo dell'A.: S.I.S.S.A., Strada Costiera 11, 34014 Trieste, Italia.

where μ is a non negative Borel measure on D, which must vanish on sets of (harmonic) capacity zero, but may assume the value $+\infty$ on some subset of positive capacity.

Following [20] we denote by $\mathcal{M}_0$ the class of all Borel measure of this type.

Problem (0.1) can be written in the form (0.3) by taking $\mu = \infty_F$, where ∞_F is the Borel measure on D defined as

$$\infty_F(B) = \begin{cases} 0 & \text{if } \operatorname{cap}(B \cap F) = 0 \\ +\infty & \text{if } \operatorname{cap}(B \cap F) \neq 0 . \end{cases}$$

Problem (0.2) can be written in the form (0.3) by taking

$$\mu(B) = \int_B q(x)\, dx .$$

In this paper we give a variational method for investigating sequences of problems of the form (0.3), where μ are random measures of the class $\mathcal{M}_0$.

The basic tool in our analysis will be the variational μ-capacity defined as

$$C(\mu, B) = \inf\left\{\int_D |Du|^2\, dx + \int_B (u-1)^2\, d\mu;\ u \in H_0^1(D)\right\}$$

for every $\mu \in \mathcal{M}_0$ and for every Borel set $B \subseteq D$.

The probabilistic problem we shall consider can be rigorously stated as follows. Let (Ω, Σ, P) be a probabilistic space. We consider a sequence (M_h) of random measures, i.e. of measurable maps between (Ω, Σ) and $\mathcal{M}_0$, endowed with the minimal σ-algebra $\mathcal{B}(\mathcal{M}_0)$ for which the maps $C(\cdot, K)$ are measurable for every compact subset K of D.

The problem is to analyze the asymptotic behaviour, as $h \to \infty$, of the solutions U_h of the random relaxed Dirichlet problems

$$\begin{cases} -\Delta U_h + M_h U_h = f & \text{in } D \\ U_h = 0 & \text{on } \partial D . \end{cases}$$

We find necessary and sufficient conditions on (M_h) for the convergence in probability of the sequence (U_h) toward the solution of a deterministic relaxed Dirichlet problem of the form

$$\text{(0.4)} \qquad \begin{cases} -\Delta U + \nu U = f & \text{in } D \\ U \in H_0^1(D) \end{cases}$$

where ν is a suitable Radon measure of the class $\mathcal{M}_0$: These conditions are given in terms of the asymptotic behaviour of the expectations of the random variables

$C(M_h, B)$ and of the covariances of the random variables $C(M_h, A)$ and $C(M_h, B)$ for disjoint subsets A and B of D.

When these conditions are satisfied, we obtain also a meaningful characterization of the limit measure ν. In fact, in this case, the expectations of the capacities $C(M_h, B)$ converge weakly (in the sense of [26]) to a countably subadditive increasing set function $\alpha(B)$ (which turns out to be equal to $C(\nu, B)$) and ν is the least measure such that $\nu \geqslant \alpha$. This generalizes a result proved in [6].

As a first application of our results we consider the asymptotic behaviour of a sequence of Dirichlet problems

$$\begin{cases} -\Delta U_h = f & \text{in } D\setminus F_h \\ U_h \in H_0^1(D\setminus F_h) \end{cases} \tag{0.5}$$

in which the random sets F_h have the form

$$F_h = \bigcup_{i=1}^{h} (x_i^h + r_h K) \tag{0.6}$$

where $(x_i^h)_{1\leqslant i\leqslant h}$ is a family of independent identically distributed random variables in D with distribution law β given by

$$\beta(B) = \int_B h(x)\, dx \qquad \big(h \in L^2(D)\big)\,,$$

K is an arbitrary compact subset contained in the unit ball and (r_h) is a sequence of positive real numbers such that

$$\lim_{h\to\infty} h r_h^{d-2} = l < +\infty\,.$$

We prove that in this case the solutions U_h of the random equation (0.5) converge in probability to the solution U of the deterministic equation (0.4) with $\nu = c\beta$, where $c = lC(K, \mathbf{R}^d)$, and

$$C(K, \mathbf{R}^d) = \min\left\{\int_{\mathbf{R}^d} |Du|^2\, dx;\ u \in H^1(\mathbf{R}^d),\ u \geqslant 1 \text{ q.e. on } K\right\}.$$

Problems of this kind have been investigated in [4], [32], [38], [40], by Brownian motion methods and in [36], [37] by Green function methods. Recently the fluctuations around the solution of the limit problem have been investigated in [29].

The corrisponding deterministic case has been studied in [30] by an orthogonal projection method, and in [31], [35] by a capacitary method. Other results on this argument can be found in [34], [13], [14], [15], [16]. Moreover, similar problems on Riemannian manifolds have been studied in [9, Chapter IX], [10], [11].

The second application of our abstract theorem concerns the asymptotic behaviour of a sequence of stationary Schrödinger equations with random potentials of the form

$$\begin{cases} -\Delta U_h + q_h U_h = f & \text{in } D \\ U_h \in H_0^1(D) \end{cases}$$

where q_h is given by

$$q_h(x) = \begin{cases} k_h & \text{if } x \in F_h \\ 0 & \text{otherwise}, \end{cases}$$

F_h are the sets defined in (0.6) with K equal to the closed unit ball, and (k_h) is a sequence of real numbers.

We prove that, in dimension $d = 3$, if $\lim_{h\to\infty} \sqrt{k_h}\, r_h = +\infty$, then the solutions U_h of the random equations converge to the solution of the deterministic equation (0.4), with $\nu = c\beta$, where $c = lC(B_1, \boldsymbol{R}^d)$.

Problems of this kind have been studied in the deterministic case in [2], [3] and [7].

I would like to thank Prof. G. DAL MASO, for suggesting me this research work, with the precious aid of his advice.

1. – Notation and preliminaries.

Troughout the paper we denote by D a fixed bounded open subset of $\boldsymbol{R}^d$ with $d \geqslant 2$. Moreover, we denote by $\mathcal{U}$ the family of all open sets $U \subseteq D$ and by $\mathcal{K}$ the family of all compact sets $K \subseteq D$.

Let us recall some well-known definitions which will be often used in the sequel.

DEFINITION 1.1. – For every compact set $K \in \mathcal{K}$ we define the capacity of K respect to D by

$$C(K, D) = \inf\left\{\int_D |D\varphi|^2,\ \varphi \in C_0^\infty(D),\ \varphi \geqslant 1 \text{ on } K\right\}.$$

The definition is extended to the sets $U \in \mathcal{U}$ by

$$C(U, D) = \sup\{C(K);\ K \subseteq U,\ K \in \mathcal{K}\}$$

and to arbitrary sets $E \subseteq D$ by

$$C(E, D) = \inf\{C(U);\ U \supseteq E,\ U \in \mathcal{U}\}.$$

When no confusion can arise, we will simply write $C(E)$ instead of $C(E, D)$.

Let E be any subset of D. When a property $P(x)$ is satisfied for all $x \in E$ except for a subset $N \subseteq E$ such that $C(N) = 0$, then we say that $P(x)$ holds quasi everywhere on E (q.e. on E).

A set $A \subseteq D$ is said to be *quasi open* (resp. *quasi closed, quasi compact*) in D if for every $\varepsilon > 0$ there exists an open (resp. closed, compact) set $U \subseteq D$ such that $C(A \,\Delta\, U) < \varepsilon$, where Δ denotes the symmetric difference (the topological notions are in the relative topology of D).

We say that a function $f\colon D \to \mathbf{R}$ is *quasi continuous* in D if for every $\varepsilon > 0$ there exists a set $E \subseteq D$ such that $C(D - E) < \varepsilon$ and the restriction of f to E is continuous.

We denote by $H^1(D)$ the Sobolev space of all functions in $L^2(D)$ whose first weak derivatives belong to $L^2(D)$, and by $H^1_0(D)$ the closure of $C^\infty_0(D)$ in $H^1(D)$.

For every $x \in \mathbf{R}^d$ and every $r > 0$ we denote by

$$B_r(x) = \{y \in \mathbf{R}^d\colon |y - x| < r\}$$

the open ball centered at x with radius r.

By the symbol $|B_r(x)|$ we mean the Lebesgue measure of the ball. By B_r we denote the ball of radius r centered at the origin.

Let $u \in H^1(D)$. It is well-known that the limit

$$\lim_{r\to 0} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} u(y)\, dy$$

exists and is finite for quasi every $x \in D$.

In the sequel we always require that for every $x \in D$

$$\liminf_{r\to 0} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} u(y)\, dy \leqslant u(x) \leqslant \limsup_{r\to 0} \frac{1}{|B_r(x)|} \int\limits_{B_r(x)} u(y)\, dy\,.$$

Thus, the pointwise value $u(x)$ is determined quasi everywhere in D, and the function u is quasi continuous in D.

It can be shown that

$$C(E) = \min\left\{\int\limits_D |Du|^2\, dx;\ u \in H^1_0(D),\ u \geqslant 1 \text{ q.e. on } E\right\}$$

for every subset E of D.

For these properties of the capacity and of the function of $H^1(D)$ see [28]. We denote by $\mathcal{B}$ the σ-field of all Borel subsets of D. A nonnegative countable additive set function defined on $\mathcal{B}$ and with value in $[0, +\infty]$ is called a *Borel measure* on D. A Borel measure which assigns finite value to every compact subset of D is called *Radon measure.*

In our paper we deal with a peculiar class of Borel measures, defined as follows:

DEFINITION 1.2. – $\mathcal{M}_0^*$ is the class of all Borel measures μ on D such that:

a) $\mu(B) = 0$ for every $B \in \mathcal{B}$ with $C(B) = 0$;

b) $\mu(B) = \inf\{\mu(A)\colon A$ quasi open, $B \subseteq A\}$ for every $B \in \mathcal{B}$.

An easy example of measure belonging to $\mathcal{M}_0^*$ is the following:

$$\mu(B) = \int_B f\, dx$$

where $f \in L^1_{\mathrm{loc}}(D)$. More generally, every Radon measure μ on D which satisfies *a*) belongs to $\mathcal{M}_0^*$.

We remark that the measures belonging to $\mathcal{M}_0^*$ are not required to be regular nor σ-finite. For istance, the measures introduced in the Definition below belong to the class $\mathcal{M}_0^*$ (see [17], Remark 3.3).

DEFINITION 1.3. – For every quasi closed set F of D we denote by ∞_F the Borel measure defined by

$$\infty_F(B) = \begin{cases} 0 & \text{if } C(F \cap B) = 0 \\ +\infty & \text{if } C(F \cap B) \neq 0 \end{cases}$$

for every $B \in \mathcal{B}$.

Other examples are given in [21].

Now, we give the definition of the variational μ-capacity associated with any measure $\mu \in \mathcal{M}_0^*$. This will be the basic tool in our investigation.

DEFINITION 1.4. – Let $\mu \in \mathcal{M}_0^*$. For every $B \in \mathcal{B}$ we define the μ-capacity of B as:

$$C(\mu, B, D) = \inf\left\{\int_D |Du|^2\, dx + \int_B (u-1)^2\, d\mu;\ u \in H_0^1(D)\right\}.$$

When no confusion can arise, we will simply write $C(\mu, B)$ instead of $C(\mu, B, D)$.

Since the functional is lower semicontinuous in the weak topology of $H_0^1(D)$, the minimum is achieved.

REMARK 1.1. – It is easy to see that if μ is the measure ∞_F of the Definition 1.3 with F quasi closed in D, then $C(\mu, B) = C(B \cap F)$ for every $B \in \mathcal{B}$.

The main properties of the μ-capacity can be summarized in the next Proposition.

PROPOSITION 1.1. – *For every $\mu \in \mathcal{M}_0^*$ the set function $C(\mu, \cdot)$ satisfies the following properties:*

a) $C(\mu, \emptyset) = 0$;

b) *if* $B_1, B_2 \in \mathcal{B}$ *and* $B_1 \subseteq B_2$, *then* $C(\mu, B_1) \leq C(\mu, B_2)$;

c) *if* (B_h) *is an increasing sequence in* $\mathcal{B}$ *and* $\bigcup_{h \in N} B_h = B$, *then*

$$C(\mu, B) = \sup_{h \in N} C(\mu, B_h);$$

d) *if* (B_h) *is a sequence in* $\mathcal{B}$ *and* $B \subseteq \bigcup_{h \in N} B_h$, *then*

$$C(\mu, B) \leq \sum_{h \in N} C(\mu, B_h);$$

e) $C(\mu, B_1 \cup B_2) + C(\mu, B_1 \cap B_2) \leq C(\mu, B_1) + C(\mu, B_2)$ *for every* $B_1, B_2 \in \mathcal{B}$;

f) $C(\mu, B) \leq C(B)$ *for every* $B \in \mathcal{B}$;

g) $C(\mu, B) \leq \mu(B)$ *for every* $B \in \mathcal{B}$;

h) $C(\mu, K) = \inf\{C(\mu, U);\ K \subseteq U,\ U \in \mathcal{U}\}$ *for every* $K \in \mathcal{K}$;

i) $C(\mu, B) = \sup\{C(\mu, K);\ K \subseteq B,\ K \in \mathcal{K}\}$ *for every* $B \in \mathcal{B}$.

For a proof we refer to ([17], Theorem 2.9 - Theorem 3.5 - Theorem 3.7).

The previous properties allow to show an explicit formula to reconstruct a measure $\mu \in \mathcal{M}_0^*$ from the corresponding μ-capacity (see [17], Theorem 4.5).

THEOREM 1.1. – *Let $\mu \in \mathcal{M}_0^*$. Then for every $B \in \mathcal{B}$ we have*

$$\mu(B) = \lim_{h \to \infty} \sum_{i \in \mathbf{Z}^d} C(\mu, B \cap R_h^i)$$

where R_h^i denotes the cube:

$$R_h^i = \prod_{k=1}^{d-1} \left] \frac{i_k}{2^h}, \frac{i_{k+1}}{2^h} \right]$$

for every $h \in \mathbf{N}$ and for every $i = (i_1, \ldots, i_d) \in \mathbf{Z}^d$.

In our paper we are interested in studing a class of equations formally written as

$$\Delta u + \mu u = f \quad \text{in } D \tag{1.1}$$

$$u = g \quad \text{on } \partial D \tag{1.2}$$

where $g \in H^1(D)$, $f \in L^2(D)$ and $\mu \in \mathcal{M}_0^*$.

Following [20] we shall call the equation (1.1) a relaxed Dirichlet problem in D.

In order to give an appropriate sense to the equation (1.1), we need the following definitions.

DEFINITION 1.5. – A function $u \in H^1_{\mathrm{loc}}(D) \cap L^2_{\mathrm{loc}}(D, \mu)$ is said to be a *local weak solution* of the equation (1.1) if

$$\int_D Du\, Dv\, dx + \int_D uv\, d\mu = \int_D f\, dx$$

for every $v \in H^1(D) \cap L^2(\mu, D)$ with compact support in D.

DEFINITION 1.6. – A local weak solution of (1.1) is said to satisfy the boundary condition (1.2) if, in addition, $u - g \in H^1_0(D)$.

The non trivial relationships between the definitions above and the definitions in the sense of distributions are discussed extensively in [21].

REMARK 1.2. – It can be proven (see [20]) that if $g \in H^1(D)$ is given in such a way that there exists some $\omega \in H^1(D) \cap L^2(D, \mu)$ with $\omega - g \in H^1_0(D)$, then there exists a unique weak solution of problem (1.1)-(1.2), this solution belongs to $H^1(D) \cap \cap L^2(D, \mu)$ and coincides with the unique minimum point of the functional

$$F(v) = \int_D |Dv|^2\, dx + \int_D v^2\, d\mu - 2\int_D fv\, dx$$

on the set $\{v\colon v \in H^1(D),\ v - g \in H^1_0(D)\}$.

In what follows we give two examples of relaxed Dirichlet problems which will be essential in the applications of our main theorems.

EXAMPLE 1.1. – *Dirichlet problems in domains with holes.*

Let $K \in \mathcal{K}$. Let ∞_K be the measure introduced in Definition 1.3. If $\mu = \infty_K$ and $g = 0$ then the problem (1.1)-(1.2) becomes

$$\left\{\begin{aligned} -\Delta u + \infty_K u &= f \quad \text{in } D \\ u &= 0 \quad \text{on } \partial D\,. \end{aligned}\right. \tag{1.3}$$

It can be seen in [21] that a function $u \in H^1_{\mathrm{loc}}(D) \cap L^2_{\mathrm{loc}}(D, \mu)$ is a local weak solution of equation (1.3) if and only if $u|_{D\setminus K}$ is a solution in the usual sense of the boundary value problem:

$$\left\{\begin{aligned} &-\Delta u = f \quad \text{in } D\setminus K \\ &u \in H^1_0(D\setminus K) \end{aligned}\right.$$

and $u|_K = 0$ q.e. on K.

EXAMPLE 1.2. – *Schrödinger equation.*

Let $q \in L^1_{\text{loc}}(D)$ with $q \geqslant 0$. If $\mu(B) = \int_B q(x)\,dx$ then the problem (1.1)-(1.2) becomes

$$\begin{cases} -\Delta u + q(x)u = f & \text{in } D \\ u \in H^1_0(D)\,. \end{cases}$$

We shall also study the following relaxed Dirichlet problem:

$$\begin{cases} -\Delta u + (\mu + \lambda m)u = f & \text{in } D \\ \qquad\qquad\qquad\quad u = 0 & \text{on } \partial D \end{cases} \tag{1.4}$$

where $\mu \in \mathcal{M}_0^*$, $f \in L^2(D)$, m denotes the Lebesgue measure on $\boldsymbol{R}^d$ and $\lambda \geqslant 0$.

In view of Remark 1.2 we can define a family of operators from $L^2(D)$ into $L^2(D)$ which are called resolvent operators.

DEFINITION 1.7. – For every $\lambda \geqslant 0$ and for every $\mu \in \mathcal{M}_0^*$, the *resolvent operator* R_μ^λ is the mapping which associates with every $f \in L^2(D)$ the unique weak solution $u \in H^1_0(D) \cap L^2(D, \mu) \subseteq L^2(D)$ of the problem (1.4).

REMARK 1.3. – R_μ^λ is a linear continuous operator between $L^2(D)$ and $L^2(D)$ (see [5], Definition 2.3).

2. – γ-convergence.

In this section we introduce a variational notion of convergence for sequences (μ_h) in $\mathcal{M}_0^*$ which will be useful to study the perturbations of the relaxed Dirichlet problem (1.2)-(1.3).

With every $\mu \in \mathcal{M}_0^*$ we associate the following functional F_μ defined on $L^2(D)$

$$F_\mu(u) = \begin{cases} \displaystyle\int_D |Du|^2 + \int_D u^2\,d\mu & \text{if } u \in H^1_0(D) \\ +\infty & \text{if } u \in L^2(D),\ u \notin H^1_0(D)\,. \end{cases}$$

Since $\mu(B) = 0$ for every $B \in \mathcal{B}$ with $C(B) = 0$, the functional F_μ is lower semicontinuous in $L^2(D)$.

The following definition of γ-convergence for sequences of measures (μ_h) belonging to $\mathcal{M}_0^*$ is given in terms of the Γ-convergence of the corresponding functionals F_{μ_h}. For the definition of Γ-convergence and its applications to the study of perturbation problems in calculus of variations, we refer to [2], [23], [24], [25].

DEFINITION 2.1. – Let (μ_h) be a sequence in $\mathcal{M}_0^*$ and let $\mu \in \mathcal{M}_0^*$; we say that (μ_h) γ-converges to μ if the following conditions are satisfied:

a) for every $u \in H_0^1(D)$ and for every sequence (u_h) in $H_0^1(D)$ converging to u in $L^2(D)$ we have:

$$F_\mu(u) \leq \liminf_{h\to\infty} F_{\mu_h}(u_h) ;$$

b) for every $u \in H_0^1(D)$, there exists a sequence (u_h) in $H_0^1(D)$ converging to u in $L^2(D)$ such that:

$$F_\mu(u) \geq \limsup_{h\to\infty} F_{\mu_h}(u_h) .$$

REMARK 2.1. – There exists a unique metrizable topology on $\mathcal{M}_0^*$ which induces the γ-convergence, which will be called the *topology of γ-convergence*. All topological notions we shall consider on $\mathcal{M}_0^*$ are relative to this topology, with respect to which $\mathcal{M}_0^*$ is compact ([17], Remark 5.4).

A relevant aspect of Definition 1.7 for our purpose is contained in the following Proposition (see [5], Theorem 2.1).

PROPOSITION 2.1. – *Let (μ_h) be a sequence of measures in $\mathcal{M}_0^*$ and let $\mu \in \mathcal{M}_0^*$. Given $\lambda \geq 0$, let $R_{\mu_h}^\lambda$ be a sequence of resolvent operators associated with the measures μ_h and R_μ^λ the resolvent operator associated with μ. The following statements are equivalent:*

a) (μ_h) γ-converges to μ.

b) $(R_{\mu_h}^\lambda)$ converges to R_μ^λ strongly in $L^2(D)$.

The following Proposition states the relationships between the γ-convergence of a sequence of measures (μ_h) and the behaviour of the corrisponding μ-capacities, (see [17], Theorem 6.3 and Theorem 5.9).

PROPOSITION 2.2. – *Let (μ_h) a sequence in $\mathcal{M}_0^*$ and $\mu \in \mathcal{M}_0^*$. Then (μ_h) γ-converges to μ in $\mathcal{M}_0^*$ if and only if the inequalities*

a) $C(\mu, U) \leq \liminf_{h\to\infty} C(\mu_h, U)$

and

b) $C(\mu, K) \geq \limsup_{h\to\infty} C(\mu_h, K)$

hold for every $K \in \mathcal{K}$ and for every $U \in \mathcal{U}$.

REMARK 2.2. – In view of Proposition 2.2 a sub-base for the topology induced by γ-convergence on $\mathcal{M}_0^*$ is given by the set of the form $\{\mu \in \mathcal{M}_0^*: C(\mu, U) > t\}$ and $\{\mu \in \mathcal{M}_0^*: C(\mu, K) < s\}$ with $t, s \in \mathbf{R}^+$, $U \in \mathcal{U}$ and $K \in \mathcal{K}$.

We denote by $\mathcal{B}(\mathcal{M}_0^*)$ the Borel σ-field of $\mathcal{M}_0^*$ endowed with the topology of γ-convergence.

PROPOSITION 2.3. – *$\mathcal{B}(\mathcal{M}_0^*)$ is the smallest σ-field in $\mathcal{M}_0^*$ for which the functions $C(\cdot, U)$ from $\mathcal{M}_0^*$ into $\boldsymbol{R}$ are measurable for every $U \in \mathcal{U}$ (respectively the functions $C(\cdot, K)$ are measurable for every $K \in \mathcal{K}$).*

PROOF. – Denote by Σ_1 the smallest σ-field in $\mathcal{M}_0^*$ for which all functions $C(\cdot, U)$, $U \in \mathcal{U}$, are measurable, and by Σ_2 the smallest σ-field in $\mathcal{M}_0^*$ for which all functions $C(\cdot, K)$, $K \in \mathcal{K}$, are measurable.

First, let us show that $\Sigma_1 = \Sigma_2$. It is enough to prove that

a) any function $C(\cdot, K)$, $K \in \mathcal{K}$, is Σ_1-measurable;

and

b) any function $C(\cdot, U)$, $U \in \mathcal{U}$, is Σ_2-measurable.

Let us prove *a*). For every $K \in \mathcal{K}$, consider the decreasing sequence of open set:

$$U_h = \{x \in D : d(x, K) < 1/h\} .$$

We remark that $U_h \searrow K$. By (*h*) of Proposition 1.1 we have

$$C(\mu, K) = \inf_{h \in N} C(\mu, U_h)$$

for every $\mu \in \mathcal{M}_0^*$, which proves *a*).

Assertion *b*) can be proved in the same way, by choosing, for every $U \in \mathcal{U}$, an increasing sequence (K_h) in $\mathcal{K}$ such that $K_h \nearrow U$ and by using Proposition 1.1, (i).

The proof of the Proposition is complete if we show that $\mathcal{B}(\mathcal{M}_0^*) = \Sigma_1$. The inclusion $\Sigma_1 \subseteq \mathcal{B}(\mathcal{M}_0^*)$ is trivial because $C(\cdot, U)$, $U \in \mathcal{U}$ is lower semicontinuous on $\mathcal{M}_0^*$ by Proposition 2.2 (*a*). In order to show that $\mathcal{B}(\mathcal{M}_0^*) \subseteq \Sigma_1$, we have only to observe that the sub-base for the topology of the γ-convergence given in Remark 2.2 is contained in Σ_1 (because $\Sigma_1 = \Sigma_2$) and that $\mathcal{M}_0^*$ admits a countable basis for the open sets. ■

The next Corollary follows directly from the previous proposition.

COROLLARY 2.1. – *Let (Ω, Σ, P) be a measure space. Let M be a function from Ω* into $\mathcal{M}_0^*$. *The following statements are equivalent:*

a) *M is $\Sigma - \mathcal{B}(\mathcal{M}_0^*)$ measurable;*

b) *$C(M(\cdot), U)$ is Σ-measurable for every $U \in \mathcal{U}$;*

c) *$C(M(\cdot), K)$ is Σ-measurable for every $K \in \mathcal{K}$.*

We need also some result about the measurability of the function $C(\cdot, B)$ for every $B \in \mathcal{B}$. Let us denote by $\widehat{\mathcal{B}}(\mathcal{M}_0^*)$ the σ-algebra of all subset of $\mathcal{M}_0^*$ which are universally measurable with respect to $\mathcal{B}(\mathcal{M}_0^*)$ (i.e. Q-measurable for every probability measure Q on $(\mathcal{M}_0^*, \mathcal{B}(\mathcal{M}_0^*))$).

PROPOSITION 2.4. – *For every $B \in \mathcal{B}$ the function $C(\cdot, B)$ is $\widehat{\mathcal{B}}(\mathcal{M}_0^*)$-measurable.*

PROOF. – Let Q be a probability measure on $\mathcal{B}(\mathcal{M}_0^*)$. For every $B \in \mathcal{U} \cup \mathcal{K}$ we set

$$\alpha(B) = \int_{\mathcal{M}_0^*} C(\mu, B)\, dQ\,.$$

By properties (*h*), (*i*) and (*e*) of $C(\mu, \cdot)$ in Proposition 1.1 we have that:

$$\alpha(K) = \inf\{\alpha(U);\ U \supseteq K,\ U \in \mathcal{U}\} \tag{2.1}$$

for every $K \in \mathcal{K}$,

$$\alpha(U) = \sup\{\alpha(K);\ K \subseteq U,\ K \in \mathcal{K}\} \tag{2.2}$$

for every $U \in \mathcal{U}$, and

$$\alpha(K_1 \cup K_2) + \alpha(K_1 \cap K_2) \leqslant \alpha(K_1) + \alpha(K_2) \tag{2.3}$$

for every $K_1, K_2 \in \mathcal{K}$.

We can extend the definition of α by

$$\alpha(B) = \inf\{\alpha(U);\ U \supseteq B,\ U \in \mathcal{U}\} \tag{2.4}$$

for every $B \in \mathcal{B}$. We infer from (2.1), (2.2), (2.3), (2.4) that α is a Choquet capacity on B (see [27], Theorem 1.5). Applying the capacitabily Theorem (see [12]) we get

$$\alpha(B) = \sup\{\alpha(K);\ K \subseteq B,\ K \in \mathcal{K}\} \tag{2.5}$$

for every $B \in \mathcal{B}$. Now, fix $B \in \mathcal{B}$. By (2.4) it follows that for every $\varepsilon > 0$ there exists $U \in \mathcal{U}$, $U \supseteq B$ such that

$$\alpha(B) + \varepsilon/2 > \alpha(U)\,. \tag{2.6}$$

Moreover, by (2.5) we also get that for every $\varepsilon > 0$ there exists a $K \in \mathcal{K}$, $K \subseteq B$ such that:

$$\alpha(B) - \varepsilon/2 < \alpha(K)\,. \tag{2.7}$$

By (2.6) and (2.7) we get that for every $\varepsilon > 0$

$$\int_{\mathcal{M}_0^*} [C(\mu, U) - C(\mu, K)]\, dQ < \varepsilon . \tag{2.8}$$

Since $C(\cdot, K) \leqslant C(\cdot, B) \leqslant C(\cdot, U)$, (2.8) gives the measurability of $C(\cdot, B)$ respect to the σ-field of all subsets Q-measurable. Finally, the assertion follows noting that Q is an arbitrary probability measure on $\mathcal{B}(\mathcal{M}_0^*)$. ■

At the end of this Section we recall some probabilistic notions which we use in the sequel.

By $\mathcal{P}(\mathcal{M}_0^*)$ we mean the space of all probability measures defined on $\mathcal{B}(\mathcal{M}_0^*)$, i.e. an element $Q \in \mathcal{P}(\mathcal{M}_0^*)$ is a non negative countably additive set function defined on $\mathcal{B}(\mathcal{M}_0^*)$ with $Q(\mathcal{M}_0^*) = 1$.

We recall the concept of the weak convergence for a sequence (Q_h) of measures belonging to $\mathcal{P}(\mathcal{M}_0^*)$.

DEFINITION 2.2. – We say that a sequence (Q_h) of measures in $\mathcal{P}(\mathcal{M}_0^*)$ converges *weakly* to a measure Q in $\mathcal{P}(\mathcal{M}_0^*)$ if

$$\lim_{h \to \infty} \int_{\mathcal{M}_0^*} f\, dQ_h = \int_{\mathcal{M}_0^*} f\, dQ$$

for every continuous function $f\colon \mathcal{M}_0^* \to \boldsymbol{R}$.

Similar problems of weak convergence of measures on spaces endowed with topology related to Γ-convergence have been studied in [18] and [19].

The two results that we give in the following hold for a generic compact metric space. For the proofs we refer respectively to [1], Theorem 4.5.1 and to [39], Theorem 6.4.

PROPOSITION 2.5. – *Let (Q_h) be a sequence of probability measures in $\mathcal{P}(\mathcal{M}_0^*)$ and let $Q \in \mathcal{P}(\mathcal{M}_0^*)$. The following statement are equivalent:*

a) *(Q_h) converges weakly to Q in $\mathcal{P}(\mathcal{M}_0^*)$.*

b) $\displaystyle\lim_{h \to \infty} \int_{\mathcal{M}_0^*} f\, dQ_h = \int_{\mathcal{M}_0^*} f\, dQ$

for every function $f\colon \mathcal{M}_0^* \to \boldsymbol{R}$ such that

$$Q\{\mu \in \mathcal{M}_0^* \colon f \textit{ is continuous at } \mu\} = 1 .$$

PROPOSITION 2.6. – *For every sequence (Q_h) of measures in $\mathcal{P}(\mathcal{M}_0^*)$ there exists a sub-sequence (Q_{h_k}) weakly convergent in $\mathcal{P}(\mathcal{M}_0^*)$.*

We conclude with some definitions:

DEFINITION 2.3. – For every $\mathcal{B}(\mathcal{M}_0^*)$-measurable function X we denote by $E_Q[X]$ the *expectation* of X in the probability space $(\mathcal{M}_0^*, \mathcal{B}(\mathcal{M}_0^*), Q)$, defined by

$$E_Q[X] = \int_{\mathcal{M}_0^*} X(\mu)\, dQ(\mu)\,.$$

DEFINITION 2.4. – For every $X, Y \in L^2(\mathcal{M}_0^*, \mathcal{B}(\mathcal{M}_0^*), Q)$ we denote by $\mathrm{Cov}_Q[X, Y]$ the *covariance* of X and Y in the probability space $(\mathcal{M}_0^*, \mathcal{B}(\mathcal{M}_0^*), Q)$ defined by

$$\mathrm{Cov}_Q[X, Y] = E_Q[XY] - E_Q[X]E_Q[Y]\,.$$

The variance of X is defined by $\mathrm{Var}_Q[X] = \mathrm{Cov}_Q[X, X]$.

3. – The main result.

In this section we prove the main result of this paper: a necessary and sufficient condition for the convergence of a sequence (Q_h) of measures on $\mathcal{M}_0^*$ of the class $\mathcal{P}(\mathcal{M}_0^*)$ to a measure $\delta_\nu \in \mathcal{P}(\mathcal{M}_0^*)$ of the form

$$\delta_\nu(\mathcal{E}) = \begin{cases} 0 & \text{if } \nu \notin \mathcal{E} \\ 1 & \text{if } \nu \in \mathcal{E} \end{cases} \tag{3.1}$$

for every $\mathcal{E} \in \mathcal{B}(\mathcal{M}_0^*)$, where ν is a finite Borel measure on D of the class $\mathcal{M}_0^*$. This condition is expressed in terms of the asymptotic behaviour, as $h \to \infty$, of the functions $C(\cdot, B)$, $B \in \mathcal{B}$, considered as a random variables on the probability spaces $(\mathcal{M}_0^*, \mathcal{B}(\mathcal{M}_0^*), Q_h)$.

We begin with some definitions. Let (Q_h) be a sequence in $\mathcal{P}(\mathcal{M}_0^*)$. First, for every $U \in \mathcal{U}$, we define:

$$\alpha'(U) = \liminf_{h\to\infty} E_{Q_h}[C(\cdot, U)]$$

and

$$\alpha''(U) = \limsup_{h\to\infty} E_{Q_h}[C(\cdot, U)]$$

where E_{Q_h} denotes the expectation in the probability space $(\mathcal{M}_0^*, \mathcal{B}(\mathcal{M}_0^*), Q_h)$.

Next we consider the inner regularizations α'_- and α''_- of α' and α'' defined for every $U \in \mathcal{U}$ by:

$$\alpha'_-(U) = \sup\{\alpha'(V);\ V \in \mathcal{U},\ \overline{V} \subset U\} \tag{3.2}$$

and

$$\alpha''_-(U) = \sup\{\alpha''(V);\ V \in \mathcal{U},\ \overline{V} \subset U\}. \tag{3.3}$$

Then, we extend the definitions of α'_- and α''_- to the arbitrary Borel sets $B \subseteq D$ by

$$\alpha'_-(B) = \inf\{\alpha'_-(U);\ U \in \mathcal{U},\ U \supseteq B\} \tag{3.4}$$

and

$$\alpha''_-(B) = \inf\{\alpha''_-(U);\ U \in \mathcal{U},\ U \supseteq B\} \tag{3.5}$$

for every $B \in \mathcal{B}$.

Finally, we denote by ν' and ν'' the least superadditive set functions on $\mathcal{B}$ greater than or equal to α'_- and α''_- respectively.

We are now in a position to state our main result.

THEOREM 3.1. – *Let (Q_h) be a sequence of measures on $\mathcal{M}_0^*$ of the class $\mathcal{F}(\mathcal{M}_0^*)$. Assume that*

i) $\nu'(B) = \nu''(B) < +\infty$ *for every* $B \in \mathcal{B}$

and denote by $\nu(B)$ the common value of $\nu'(B)$ and $\nu''(B)$ for every $B \in \mathcal{B}$.

Suppose in addition that

ii) *there exist a constant $\varepsilon > 0$, an increasing continuous function*

$$\xi: \boldsymbol{R}\times\boldsymbol{R} \to \boldsymbol{R}$$

with $\xi(0,0) = 0$ and a Radon measure β on $\mathcal{B}$ such that

$$\limsup_{h\to\infty} |\mathrm{Cov}_{Q_h}[C(\cdot, U), C(\cdot, V)]| \leqslant \xi(\mathrm{diam}\, U, \mathrm{diam}\, V)\beta(U)\beta(V)$$

for every pair $U, V \in \mathcal{U}$ such that $\overline{U} \cap \overline{V} = \emptyset$ with $\mathrm{diam}\, U < \varepsilon$, $\mathrm{diam}\, V < \varepsilon$.

Then

a) *ν is a finite Borel measure on $\mathcal{B}$ of the class $\mathcal{M}_0^*$;*

b) *(Q_h) converges weakly to the probability measure δ_ν defined by*

$$\delta_\nu(\mathcal{E}) = \begin{cases} 0 & \text{if } \nu \notin \mathcal{E} \\ 1 & \text{if } \nu \in \mathcal{E} \end{cases}$$

for every $\mathcal{E} \in \mathcal{B}(\mathcal{M}_0^*)$;

c) $\alpha'_-(B) = \alpha''_-(B) = C(\nu, B)$ *for every* $B \in \mathcal{B}$.

REMARK 3.1. – Let $\alpha_h\colon \mathcal{U} \to \boldsymbol{R}$ be an increasing set function defined by

$$\alpha_h(U) = E_{Q_h}[C(\cdot, U)]$$

and let $\alpha\colon \mathcal{U} \to \boldsymbol{R}$ be an increasing set function defined by

$$\alpha(U) = C(\nu, U) .$$

Then the condition *c)* of Theorem 3.1 is equivalent to say that (α_h) converges weakly to α in the sense of [26] (with respect to the pair $(\mathcal{U}, \mathcal{K})$).

For the proof of Theorem 3.1 we need some preliminary results. We begin with a general probabilistic Lemma.

Let (Ω, Σ, P) be a probability space. The symbols $E[X]$ and $\mathrm{Var}\,[X]$ will denote respectively the expectation and the variance of the random variable X with respect to the measure P.

LEMMA 3.1. – *Consider a sequence (X_h) of non negative random variables on (Ω, Σ, P).*

Suppose that

i) $X_h \in L^2(\Omega, P)$ *for every* $h \in \boldsymbol{N}$.

ii) X_h *converges to* X *for* P*-almost every* $\omega \in \Omega$.

iii) $\lim\limits_{h\to\infty} \mathrm{Var}\,[X_h] = 0$.

Then, there exists a constant X_0 such that $X(\omega) = X_0$ for P-almost every $\omega \in \Omega$.

PROOF. – Choose a non negative sequence ε_h such that

$$\lim_{h\to\infty} \varepsilon_h = 0 \qquad \text{and} \qquad \lim_{h\to\infty} \frac{\mathrm{Var}\,[X_h]}{\varepsilon_h^2} = 0 .$$

Set

$$t_h = \frac{\mathrm{Var}\,[X_h]}{\varepsilon_h^2} .$$

Then there exists a subsequence of t_h, still denoted by t_h, such that $\sum\limits_{h\in N} t_h < +\infty$.

Consider the sets

$$B_h = \{\omega \in \Omega\colon |X_h - E[X_h]| \geqslant \varepsilon_h\} .$$

By Chebychev's inequality we have $P(B_h) < t_h$ for every h and by Borel-Cantelli's Lemma it follows that

$$P(\limsup_{h\to\infty} B_h) = 0 .$$

Consequently, if ω_1, ω_2 are two elements in $\Omega\setminus\limsup_{h\to\infty} B_h$, we obtain

$$|X_h(\omega_1) - X_h(\omega_2)| < 2\varepsilon_h$$

for h large enough. Passing to the limit, as $h\to\infty$, we get the proof of the assertion. ∎

In the next Lemma we prove a result concerning increasing set functions, i.e. functions $\alpha\colon \mathcal{B}\to \boldsymbol{R}$ such that $\alpha(A)\leqslant\alpha(B)$ whenever $A, B\in\mathcal{B}$ and $A\subseteq B$. First we need some elementary definitions.

DEFINITIONS 3.1. – A subset $\mathcal{D}$ of $\mathcal{U}$ is said to be *dense* if for every pair $U, V\in\mathcal{U}$ such that $\overline{U}\subset V$, there exists a set $W\in\mathcal{D}$ such that $\overline{U}\subset W\subset\overline{W}\subset V$.

LEMMA 3.2. – *Let* $\alpha\colon \mathcal{B}\to\boldsymbol{R}$ *be any increasing set function. Then the set*

$$\mathcal{D} = \{W\in\mathcal{U}\colon \overline{W}\subset D,\ \alpha(W) = \alpha(\overline{W})\}$$

is dense in $\mathcal{U}$.

PROOF. – The Lemma is an immediate consequence of Proposition 4.7 of [26]. For the readers convenience we repeat here the proof in our particular case.

Let U, V be in $\mathcal{U}$ such that $\overline{U}\subset V$. By Uryshon's Lemma there exists a function $f\in C_0^0(V)$ such that $0\leqslant f(x)\leqslant 1$ for every $x\in V$ and $f=1$ on U. For every $t\in\,]0,1[\, \equiv T$ we consider the open set:

$$U_t = \{x\in V\colon f(x) > t\}\ .$$

Let $g\colon T\to\boldsymbol{R}$ be the function defined in the following way

$$g(t) = \alpha(U_t)\ .$$

Then g is a decreasing function and for every $t\in T$ we have

$$\inf_{s<t} g(s)\geqslant \alpha(\overline{U}_t)\geqslant\alpha(U_t)\geqslant\sup_{s>t} g(s)\ .$$

Since the function g has at most a countable set of discontinuity points in T, there exists $t\in T$ such that $\alpha(\overline{U}_t) = \alpha(U_t)$ and this proves the Lemma. ∎

In the following we give sufficient conditions in order to have that a probability measure $Q\in\mathcal{P}(\mathcal{M}_0^*)$ be equal to the measure δ_ν defined in (3.1). The conditions are given in terms of the functions $C(\cdot, B)$, $B\in\mathcal{B}$, considered as random variables on $(\mathcal{M}_0^*, \mathcal{B}(\mathcal{M}_0^*), Q)$.

LEMMA 3.3. – *Let Q be a probability measure on $\mathcal{M}_0^*$ of the class $\mathcal{P}(\mathcal{M}_0^*)$. Define $\alpha(U) = E_Q[C(\cdot, U)]$ for every $U \in \mathcal{U}$, and*

$$\alpha(B) = \inf \{\alpha(U);\ U \supseteq B,\ U \in \mathcal{U}\}$$

for every $B \in \mathcal{B}$. Assume that:

(i) *There exists a Radon measure β_1 on $\mathcal{B}$ such that $\beta_1 \geqslant \alpha$ on $\mathcal{B}$;*

(ii) *There exist a constant $\varepsilon > 0$, a Radon measure β_2 on $\mathcal{B}$ and an increasing continuous function ξ: $\boldsymbol{R}\times\boldsymbol{R} \to \boldsymbol{R}$ with $\xi(0, 0)$ such that*

$$|\mathrm{Cov}_Q\,[C(\cdot, U), C(\cdot, V)]| \leqslant \xi(\mathrm{diam}\ U, \mathrm{diam}\ V)\beta_2(U)\beta_2(V) \tag{3.6}$$

for every pair $U, V \in \mathcal{U}$ such that $\overline{U} \cap \overline{V} = \emptyset$, with diam $U < \varepsilon$ *and* diam $V < \varepsilon$.

Let ν be the least superadditive set function on $\mathcal{B}$ such that $\nu \geqslant \alpha$ on $\mathcal{B}$. Then ν is a measure on $\mathcal{B}$ of the clas $\mathcal{M}_0^*$ and

$$Q = \delta_\nu\,.$$

PROOF. – The function α is countably subadditive on $\mathcal{U}$ (hence on $\mathcal{B}$) by the countable subadditivity of $C(\mu, \cdot)$ (Proposition 1.1, (d)). Therefore ν is a measure by Lemma 4.1 of [17]. We observe that the measure ν is in $\mathcal{M}_0^*$ because it is a Radon measure and $\nu(B) = 0$ whenever $C(B) = 0$ by Proposition 1.1, (f). By properties (h) and (i) of Proposition 1.1 we can extend the relation (3.6) to each pair of disjoint sets $A, B \in \mathcal{B}$ and check that

$$\alpha(B) = E_Q[C(\cdot, B)]$$

for every $B \in \mathcal{B}$.

Let us denote by $z(\cdot, B)$ the random variable on the probability space $(\mathcal{M}_0^*, \mathcal{B}(\mathcal{M}_0^*), Q)$ defined by

$$z(\mu, B) = \mu(B)$$

for every $B \in \mathcal{B}$.

By Theorem 1.1 we have that

$$z(\cdot, B) = \lim_{h\to\infty} \sum_{i \in \boldsymbol{Z}^d} C(\cdot, B \cap R_h^i)$$

for every $B \in \mathcal{B}$, where R_h^i denotes the cube defined in Theorem 1.1. We apply now Lemma 3.1 to show that $z(\cdot, B)$ is a constant random variable. Therefore, we have only to prove that:

$$\lim_{h\to\infty} \mathrm{Var}_Q\Big[\sum_{i\in \mathbf{Z}^d} C(\cdot, B\cap R_h^i)\Big] = 0 .$$

Now, let us fix $B\in\mathcal{B}$ with $\bar{B}\subseteq D$. For every $h\in \mathbf{N}$, we have

$$(3.7)\qquad \sum_{i\in\mathbf{Z}^d}\mathrm{Var}_Q[C(\cdot, B\cap R_h^i)] = \sum_{i\in\mathbf{Z}^d}\{E_Q[C(\cdot, B\cap R_h^i)^2]-(E_Q[C(\cdot, B\cap R_h^i)])^2\}\leqslant$$
$$\leqslant \sum_{i\in\mathbf{Z}^d} E_Q[C(\cdot, B\cap R_h^i)^2]\leqslant \sum_{i\in\mathbf{Z}^d} C(B\cap R_h^i)E_Q[C(\cdot, B\cap R_h^i)]\leqslant$$
$$\leqslant \sup_{i\in\mathbf{Z}^d} C(B\cap R_h^i)\sum_{i\in\mathbf{Z}^d}\alpha(B\cap R_h^i)\leqslant s_h\beta_1(B)$$

where we have set

$$s_h = \sup_{i\in\mathbf{Z}^d} C(B\cap R_h^i) .$$

We observe that $s_h\to 0$, as $h\to\infty$, because the dimension d is greater than or equal to 2 and $\bar{B}$ is compact in D. On the other hand, by hypotheses there exists $h_0\in\mathbf{N}$ such that, for every $h\geqslant h_0$,

$$(3.8)\qquad \Big|\sum_{\substack{i,j\in\mathbf{Z}^d\\ i\neq j}} \mathrm{Cov}_Q[C(\cdot, B\cap R_h^i), C(\cdot, B\cap R_h^j)]\Big|\leqslant$$
$$\leqslant \sum_{\substack{i,j\in\mathbf{Z}^d\\ i\neq j}} \xi(\mathrm{diam}\,(B\cap R_h^i), \mathrm{diam}\,(B\cap R_h^j))\,\beta_2(B\cap R_h^i)\beta_2(B\cap R_h^j)\leqslant$$
$$\leqslant \xi(\mathrm{diam}\, R_h^0, \mathrm{diam}\, R_h^0)\sum_{\substack{i,j\in\mathbf{Z}^d\\ i\neq j}}\beta_2(B\cap R_h^i)\beta_2(B\cap R_h^j)\leqslant \xi(\mathrm{diam}\, R_h^0, \mathrm{diam}\, R_h^0)[\beta_2(B)]^2 .$$

By (3.7), (3.8) and by hypothesis we get:

$$\lim_{h\to\infty}\mathrm{Var}_Q\Big[\sum_{i\in\mathbf{Z}^d} C(\cdot, B\cap R_h^i)\Big]\leqslant$$
$$\leqslant\lim_{h\to\infty}\Big\{\sum_{i\in\mathbf{Z}^d}\mathrm{Var}_Q[C(\cdot, B\cap R_h^i)]+\sum_{\substack{i,j\in\mathbf{Z}^d\\ i\neq j}}\mathrm{Cov}_Q[C(\cdot, B\cap R_h^i), C(\cdot, B\cap R_h^j)]\Big\}\leqslant$$
$$\leqslant\lim_{h\to\infty}\{s_h\beta_1(B)+\xi(\mathrm{diam}\, R_h^0, \mathrm{diam}\, R_h^0)[\beta_2(B)]^2\} = 0 .$$

Therefore Lemma 3.2 implies that for every Borel set $z(\cdot, B)$ is a constant random variable. Now, let us compute the expectation of $z(\cdot, B)$. Since the sequence $\Big(\sum_{i\in\mathbf{Z}^d} C(B\cap R_h^i)\Big)_{h\in N}$ is increasing, we get

$$E_Q[z(\cdot, B)] = \lim_{h\to\infty} E_Q\Big[\sum_{i\in\mathbf{Z}^d} C(\cdot, B\cap R_h^i)\Big] = \lim_{h\to\infty}\sum_{i\in\mathbf{Z}^d}\alpha(B\cap R_h^i) = \nu(B)$$

for every $B\in\mathcal{B}$, where the last equality is proved in [17], Lemma 4.2.

Hence for every $B \in \mathcal{B}$ there exists a subset $\mathcal{M}_B$ of $\mathcal{M}_0^*$ with $Q(\mathcal{M}_B = 1)$ such that $z(\mu, B) = \nu(B)$ for every $\mu \in \mathcal{M}_B$. Let $\mathfrak{D}$ be a countable dense set in $\mathcal{U}$ and let us consider

$$\mathcal{M} = \bigcap_{U \in D} \mathcal{M}_U .$$

We obtain that $z(\mu, U) = \nu(U)$ for every $\mu \in \mathcal{M}$ and $Q(\mathcal{M}) = 1$. This implies that $z(\mu, \cdot)$ is a Radon measure on $\mathcal{B}$ for every $\mu \in \mathcal{M}$, and since $z(\mu, \cdot)$ coincides with ν on a dense set $\mathfrak{D}$ in $\mathcal{U}$, we can deduce that $z(\mu, B) = \nu(B)$ for every $B \in \mathcal{B}$ and for every $\mu \in \mathcal{M}$. This concludes the proof of the Lemma. ■

PROOF OF THEOREM 3.1. – The set function α'' is subadditive on $\mathcal{U}$, being the upper limit of a sequence of subadditive set functions on $\mathcal{U}$. Therefore its inner regularization α''_- is countably subadditive on $\mathcal{U}$ by Theorem 5.6 of [26]. It is now easy to see that α''_- is countably subadditive on $\mathcal{B}$, so that ν'' is a measure by Lemma 4.1 of [17]. Moreover, $\nu''(B) = 0$ whenever $C(B) = 0$ by Proposition 1.1 (*f*). This proves assertion (*a*).

Since $\mathcal{P}(\mathcal{M}_0^*)$ is sequentially compact space and ν' and ν'' do not change by passing to a subsequence, in order to prove (*b*) we can assume that (Q_h) converges weakly to a probability measure $Q \in \mathcal{P}(\mathcal{M}_0^*)$ and we have only to prove that $Q = \delta_\nu$.

By Lemma 3.2 the set

$$\mathfrak{D} = \{U \in \mathcal{U}: E_Q[C(\cdot, \overline{U})] = E_Q[C(\cdot, U)]\}$$

is dense in $\mathcal{U}$.

Consequently, for every $U \in \mathfrak{D}$, the equality $C(\mu, U) = C(\mu, \overline{U})$ holds for Q-almost all $\mu \in \mathcal{M}_0^*$. Therefore, by Proposition 2.2,

$$Q\{\mu \in \mathcal{M}_0^*: C(\cdot, U) \text{ is } \gamma\text{-continuous at } \mu\} = 1$$

for every $U \in \mathfrak{D}$. Then, by Proposition 2.5 we have

$$\lim_{h\to\infty} E_{Q_h}[C(\cdot, U)] = E_Q[C(\cdot, U)] = \alpha'(U) = \alpha''(U) \tag{3.9}$$

for every $U \in \mathfrak{D}$, and

$$\lim_{h\to\infty} E_{Q_h}[C(\cdot, U)\, C(\cdot, V)] = E_Q[C(\cdot, U)\, C(\cdot, V)] \tag{3.10}$$

for every $U, V \in \mathfrak{D}$.

By (3.9), (3.10) by hypothesis (ii) and by the properties of the μ-capacity (Proposition 1.1, (*h*) and (*i*)) we get that

$$E_Q[C(\cdot, U)] = \alpha'_-(U) = \alpha''_-(U) \tag{3.11}$$

for every $U \in \mathcal{U}$, and

$$|\mathrm{Cov}_Q[C(\cdot, U), C(\cdot, V)]| \leqslant \xi(\mathrm{diam}\, U, \mathrm{diam}\, V)\beta(U)\beta(V)$$

for every pair $U, V \in \mathcal{U}$ with $\mathrm{diam}\, U < \varepsilon$ and $\mathrm{diam}\, V < \varepsilon$ such that $\overline{U} \cap \overline{V} = \emptyset$.

Assertion (*b*) follows now from Lemma 3.3.

Assertion (*c*) can be obtained from (*b*) and (3.11) by using (3.4), (3.5) and the properties of $C(\mu, \cdot)$ stated in Proposition 1.1, (*h*) and (*i*). ■

REMARK 3.2. – Conditions (i) and (ii) of Theorem 3.1 are also necessary. In fact, if Q_h converges weakly to a probability measure of the form δ_ν (see (3.1)), where ν is a finite Borel measure on $\mathcal{B}$ of the class $\mathcal{M}_0^*$, then (3.9) and (3.10) imply that there exists a family $\mathcal{D}$ dense in $\mathcal{U}$ such that

$$\alpha'(U) = \alpha''(U) = C(\nu, U) \tag{3.12}$$

for every $U \in \mathcal{D}$ and

$$\lim_{h\to\infty} |\mathrm{Cov}_{Q_h}[C(\cdot, U), C(\cdot, V)]| = 0 \tag{3.13}$$

for every $U, V \in \mathcal{U}$ with $\overline{U} \cap \overline{V} = \emptyset$. By the properties of the capacities $C(\mu, \cdot)$ (Proposition 1.1, (*h*), (*i*)), (3.12) implies that

$$\alpha'_-(B) = \alpha''_-(B) = C(\nu, B) \tag{3.14}$$

for every $B \in \mathcal{B}$ and (3.13) implies condition (ii) of Theorem 3.1. The condition (i) follows now from (3.14) and from the characterization of ν as the least superadditive set function greater than or equal to $C(\nu, \cdot)$, (see [17], Theorem 4.3).

4. – Dirichlet problems in domains with random small holes.

In this section we consider an application of our results to a Dirichlet problem in a domain with small holes. In order to simplify the computations we assume $d \geqslant 3$.

Let (Ω, Σ, P) be a probability space. We shall denote by E and by Cov respectively the expectation and the covariance of a random variable, with respect to the measure P.

DEFINITION 4.1. – A measurable function $M: \Omega \to \mathcal{M}_0^*$ will be called *random measure*.

We recall that necessary and sufficient conditions for the measurability of a function $M\colon \Omega \to \mathcal{M}_0^*$ are given in Corollary 2.1.

Let M be a random measure.

DEFINITION 4.2. – The probability measure in $\mathcal{P}(\mathcal{M}_0^*)$ defined by

$$Q(\mathcal{E}) = P\{M^{-1}(\mathcal{E})\} \qquad \text{for any } \mathcal{E} \in \mathcal{B}(\mathcal{M}_0^*)$$

will be called *the distribution law* of the random measure M.

Let (M_h) be a sequence of random measures and M a random measure. Let (Q_h) be the sequence of the distribution laws of M_h and let Q be the distribution law of M.

DEFINITION 4.3. – We say that (M_h) *converges in law* to the random measure M if and only if the distribution laws Q_h converge weakly in $\mathcal{P}(\mathcal{M}_0^*)$ to the distribution law Q.

Let Q be the distribution of random measure M. It is easy to see that:

$$E_Q[C(\cdot, U)] = E[C(M(\cdot), U)] \qquad \text{for any } U \in \mathcal{U} \tag{4.1}$$

$$\begin{aligned} \operatorname{Cov}_Q[C(\cdot, U)\, C(\cdot, V)] = \\ = E[C(M(\cdot), U)\, C(M(\cdot), V)] - E[C(M(\cdot), U)]E[C(M(\cdot), V)] = \\ = \operatorname{Cov}[C(M(\cdot), U)\, C(M(\cdot), V)] \end{aligned} \tag{4.2}$$

for any pair $U, V \in \mathcal{U}$.

Let (M_h) be a sequence of random measures and let (Q_h) be the corresponding sequence of distribution laws.

Let us define the set functions:

$$\alpha'(U) = \liminf_{h\to\infty} E[C(M_h(\cdot), U)] \tag{4.3}$$

$$\alpha''(U) = \limsup_{h\to\infty} E[C(M_h(\cdot), U)] \tag{4.4}$$

for every $U \in \mathcal{U}$.

In the sequel we will denote by α'_- and α''_- respectively the inner regularization of α' and α'' as defined in (3.2) and (3.3).

The functions ν' and ν'' will be the least superadditive set function on $\mathcal{B}$ greater than or equal to α'_- and α''_-, respectively.

REMARK 4.1. – Equalities (4.1), (4.2), (4.3), (4.4) allow to reformulate the hypotheses of Theorem 3.1 in terms of the expectations and covariances of the random variables $C(M(\cdot), U)$. By definition 4.3 the theses of Theorem 3.1 can be reformulated saying that the sequence (M_h) converges in law to a random measure M such that $M(\omega) = \nu$ for P-almost every $\omega \in \Omega$ (i.e. to the constant random measure $M = \nu$).

REMARK 4.2. – It is well known that, whenever M is a constant random measure, the convergence in law and the convergence in probability toward M of the sequence (M_h) of random measures are equivalent. Thus, by Remark 4.1, we can deduce that, if the assumptions of Theorem 3.1 hold, then the sequence (M_h) converges in probability to the measure ν in $\mathcal{M}_0^*$, that is, for every $\varepsilon > 0$

$$\lim_{h\to\infty} P\{\omega \in \Omega \colon d\gamma(M_h(\omega), \nu) > \varepsilon\} = 0$$

where $d\gamma$ is any metric on $\mathcal{M}_0^*$ which induces γ-convergence (Remark 2.1).

We wish to study the following sequence of random relaxed Dirichlet problems

$$\begin{cases} -\Delta u_h + (M_h + \lambda m) u_h = f & \text{in } D \\ u_h = 0 & \text{on } \partial D \end{cases}$$

where $\lambda \geqslant 0$, $f \in L^2(D)$, m denotes the Lebesgue measure on $\boldsymbol{R}^d$.

Let $\nu \in \mathcal{M}_0^*$ and let R^λ be the resolvent operator associated with ν. The next Theorem states a relationship between the previous results and the convergence of the resolvent operators R_h^λ associated with the random measures M_h.

THEOREM 4.1. – *Let (M_h) be a sequence of random measures. Let α' and α'' be the functions defined in* (4.3) *and* (4.4) *and let ν' and ν'' be the least superadditive set functions on $\mathcal{B}$ greater than or equal to α'_- and α''_- respectively.*

Assume that

(i) $\nu'(B) = \nu''(B) < +\infty$ *for every* $B \in \mathcal{B}$
and denote by $\nu(B)$ the common value of $\nu'(B)$ and $\nu''(B)$ for every $B \in \mathcal{B}$.

Suppose, in addition, that

(ii) *there exist a constant $\varepsilon > 0$, an increasing continuous function*

$$\xi \colon \boldsymbol{R}\times\boldsymbol{R} \to \boldsymbol{R}$$

with $\xi(0, 0) = 0$ and a Radon measure β on $\mathcal{B}$ such that:

$$\limsup_{h\to\infty} |\text{Cov}\,[C(M_h(\cdot), U)\,C(M_h(\cdot), V)]| \leqslant \xi(\text{diam}\, U, \text{diam}\, V)\beta(U)\beta(V)$$

for every pair $U, V \in \mathcal{U}$ such that $\overline{U} \cap \overline{V} = \emptyset$ and with diam $U < \varepsilon$, diam $V < \varepsilon$.
Then, for every $\lambda \geqslant 0$, R_h^λ converges strongly in probability to R^λ, i.e.

$$\lim_{h\to\infty} P\{\omega \in \Omega \colon \|R_h^\lambda(\omega)[f] - R^\lambda[f]\|_{L^2(D)} > \eta\} = 0$$

for every $\eta > 0$, and for any $f \in L^2(D)$.

PROOF. – By Remark 4.2 we have that the sequence (M_h) converges in probability to ν in $\mathcal{M}_0^*$. To get the assertion it is enough to recall that, by Proposition 2.1, for every $\omega \in \Omega$ the sequence of measures (M_h) γ-converges to ν if and only if the resolvent operators $R_h^\lambda(\omega)$ converge to R^λ strongly in $L^2(D)$. ∎

Next, we wish to consider a particular sequence (M_h) of random measures related with Dirichlet problems in domains with random holes.

Let $\mathcal{F}(D)$ be the family of all closed sets contained in D.

DEFINITION 4.4. – A function $F\colon \Omega \to \mathcal{F}(D)$ is called a *random set* if the function $M\colon \Omega \to \mathcal{M}_0^*$ defined by $M(\omega) = \infty_{F(\omega)}$ for each $\omega \in \Omega$ is Σ-measurable, where $\infty_{F(\omega)}$ is the measure in $\mathcal{M}_0^*$ as in Definition 1.3.

REMARK 4.3. – Let $F\colon \Omega \to \mathcal{F}(D)$ be a function. By Corollary 2.1 and by the equality $C(\infty_E, B) = C(E \cap B)$ the following conditions are equivalent:

a) F is a random set.

b) $C(F(\cdot) \cap U)$ is Σ-measurable for every $U \in \mathcal{U}$.

c) $C(F(\cdot) \cap K)$ is Σ-measurable for every $K \in \mathcal{K}$.

Let us take a sequence (F_h) of random sets. Let (M_h) be the sequence of random measures so defined

$$M_h(\omega) = \infty_{F_h(\omega)} \quad \text{for each } \omega \in \Omega\,.$$

Let $f \in L^2(D)$ and $\lambda \geqslant 0$ be a real parameter. We shall consider the weak solutions u_h of the following Dirichlet problems on random domains

$$\begin{cases} -\Delta u_h + \lambda u_h = f & \text{on } D\setminus F_h \\ u_h \in H_0^1(D\setminus F_h)\,. \end{cases} \tag{4.5}$$

In view of the example 1.1, setting $u_h = 0$ on the set F_h, we have that u_h is the local weak solution of the relaxed Dirichlet problem

$$\begin{cases} -\Delta u_h + (\infty_{F_h} + \lambda m) u_h = f & \text{in } D \\ u_h = 0 & \text{on } \partial D \end{cases}$$

where m denotes the Lebesgue measure in $\boldsymbol{R}^d$.

We are interested in the behaviour of the sequence u_h as $h \to \infty$. More specifically, we will study the convergence of the resolvent operators R_h^λ associated with the measures ∞_{F_h}, which are related to the resolvents operators $\hat{R}_h^\lambda$ of the Dirichlet problems (4.5) by

$$R_h^\lambda(f) = \begin{cases} \hat{R}_h^\lambda(f) & \text{on } D \setminus F_h \\ 0 & \text{on } F_h \end{cases}$$

(see example 1.1).

To do that we consider the distribution laws Q_h of the random measures $M_h = \infty_{F_h}$, defined by

$$Q_h(\mathcal{E}) = P\{\infty_{F_h}^{-1}(\mathcal{E})\} \quad \text{for any } \mathcal{E} \in \mathcal{B}(\mathcal{M}_0^*) . \tag{4.6}$$

It is easy to check that

$$E_{Q_h}[C(\cdot, U)] = E[C(F_h(\cdot), U)] \quad \text{for any } U \in \mathcal{U}$$

and

$$\mathrm{Cov}_{Q_h}[C(\cdot, U), C(\cdot, V)] = \mathrm{Cov}\,[C(F_h(\cdot) \cap U), C(F_h(\cdot) \cap V)]$$

for any pair $U, V \in \mathcal{U}$.

In this case the functions α', α'' defined in (4.3) and (4.4), take the following form

$$\alpha'(U) = \liminf_{h\to\infty} E[C(F_h(\cdot) \cap U)] \tag{4.7}$$

$$\alpha''(U) = \limsup_{h\to\infty} E[C(F_h(\cdot) \cap U)] \tag{4.8}$$

for every $U \in \mathcal{U}$.

An interesting case occurs when the probability distribution of the random set is specified. We will assume the following *general hypotheses*:

(i_1) Let β be a probability law on D of the form

$$\beta(B) = \int_B g\, dx$$

for every $B \in \mathcal{B}$, where $g \in L^2(D)$.

(i_2) For every $h \in N$ we set $I_h = \{1, \ldots, h\}$ and we consider h measurable functions $x_i^h\colon \Omega \to D$, $i \in I_h$, such that $(x_i^h)_{i\in I_h}$ is a family of independent identically distributed random variables with probability distribution β.

(i_3) Let r_h be a sequence of strictly positive numbers such that

$$\lim_{h\to\infty} r_h^{d-2} h = l$$

for some constant $l < +\infty$.

Let $x \in \boldsymbol{R}^d$. Let F be a closed set of $\boldsymbol{R}^d$. We define the set $x + F$ by

$$x + F = \{y \in \boldsymbol{R}^d : x - y \in F\}.$$

The next Lemma will be useful to identify a class of random sets.

LEMMA 4.1. – *For every compact set K of $\boldsymbol{R}^d$ the function*

$$(x_1, \ldots, x_h) \to C\Big[\bigcup_{i=1}^{h} (x_i + F) \cap K\Big] \quad \text{from } (\boldsymbol{R}^d)^h \text{ into } \boldsymbol{R}$$

is upper semicontinuous in $\boldsymbol{R}^d$.

PROOF. – For each $n \in \boldsymbol{N}$ we define the set

$$F_n = \left\{x \in \boldsymbol{R}^d : \operatorname{dist}(x, F) < \frac{1}{n}\right\}.$$

Set $\bar{x} = (x_1, \ldots, x_h)$. Let $(\bar{x}_k)_{k\in N}$ be a sequence in $(\boldsymbol{R}^d)^h$ converging to $\bar{x}$ in $(\boldsymbol{R}^d)^h$. Then, for every $n \in \boldsymbol{N}$ there exists $k_0 \in \boldsymbol{N}$ such that

$$(\bar{x}_k)_i + F \subseteq x_i + F_n$$

for every $k \geq k_0$ and for every $i \in \{1, \ldots, h\}$.

Hence, for every $n \in \boldsymbol{N}$ and for every compact set K of $\boldsymbol{R}^d$, we obtain

$$C\Big(\Big(\bigcup_{i=1}^{h} x_i + F_n\Big) \cap K\Big) \geq \limsup_{k\to\infty} C\Big(\Big(\bigcup_{i=1}^{h} (\bar{x}_k)_i + F\Big) \cap K\Big).$$

Since:

$$\bigcap_{n\in N}\Big[\Big(\bigcup_{i=1}^{h} x_i + F_n\Big) \cap K\Big] = \Big(\bigcup_{i=1}^{h} x_i + F\Big) \cap K$$

by property (h) of Proposition 1.1 we get that

$$C\Big(\Big(\bigcup_{i=1}^{h} x_i + F\Big) \cap K\Big) \geq \limsup_{k\to\infty} C\Big(\Big(\bigcup_{i=1}^{h} (\bar{x}_k)_i + F\Big) \cap K\Big)$$

which proves the Lemma. ∎

Let K be a compact set of $\boldsymbol{R}^d$ such that $K \subseteq B_1$: For any $h \in \boldsymbol{N}$, we denote by K^h the following set:

$$K^h = \left\{x \in \boldsymbol{R}^d : \frac{x}{r_h} \in K\right\}$$

and by K_i^h the random sets

$$K_i^h = \left\{ x \in D : \frac{1}{r_h}(x - x_i^h) \in K \right\}$$

we note that $K_i^h \subseteq B_{r_h}(x_i^h)$. Finally, we denote by F_h the random sets:

$$F_h = \bigcup_{i=1}^{h} K_i^h, \qquad h \in \boldsymbol{N}. \tag{4.9}$$

REMARK 4.4. – By Lemma 4.1 and Remark 4.3 the sets F_h, are actually random sets in according to Definition 4.4.

We will prove the following theorems

THEOREM 4.2. – *Let* (F_h) *be the sequence of random sets defined in* (4.9). *If the general hypotheses* (i_1), (i_2) *and* (i_3) *hold then the sequence* (Q_h) *of distribution laws defined in* (4.6) *converges weakly to the distribution law* δ_ν, *defined by*

$$\delta_\nu(\mathcal{E}) = \begin{cases} 1 & \text{if } \nu \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

for any $\mathcal{E} \in \mathcal{B}(\mathcal{M}_0^*)$, *where* $\nu = c\beta$, $c = lC(K, \boldsymbol{R}^d)$, *and*

$$C(K, \boldsymbol{R}^d) = \inf\left\{ \int_{\boldsymbol{R}^d} |Du|^2;\ u \in H^1(\boldsymbol{R}^d),\ u \geq 1 \text{ q.e. on } K \right\}.$$

THEOREM 4.3. – *Let* (F_h) *be the sequence of random sets defined in* (4.9). *Assume the general hypotheses* (i_1), (i_2) *and* (i_3). *Then, for any* $f \in L^2(D)$ *and for every* $\varepsilon > 0$,

$$\lim_{h \to \infty} P\{\omega \in \Omega : \|R_h^\lambda(\omega)[f] - R^\lambda[f]\|_{L^2(D)} > \varepsilon\} = 0$$

where $R_h^\lambda(\omega)$ *is the sequence of resolvent operators associated with the random measures* ∞_{F_h} *and* R^λ *is the resolvent operator associated with the measure* ν.

Both the theorems will be consequences of the next Proposition 4.1. More specifically, Theorem 4.2 will follow by applying Theorem 3.1 and Proposition 4.1; while the proof of Theorem 4.3 will be obtained by Theorem 4.1 and Proposition 4.1.

PROPOSITION 4.1. – *Let* (F_h) *be the sequence of random sets defined in* (4.9). *Let* α', α'' *be the set functions as defined in* (4.7), (4.8) *respectively. Then, if the general hypotheses* (i_1), (i_2) *and* (i_3) *hold, we have:*

(t_1) $\nu'(B) = \nu''(B) = c\beta(B)$ *for every* $B \in \mathcal{B}$;

(t_2) *there exist a constant* $\varepsilon > 0$, *an increasing continuous function*

$$\xi\colon \boldsymbol{R}\times\boldsymbol{R}\to\boldsymbol{R}$$

with $\xi(0,0)=0$ *and a Radon measure* β_1, *such that*

$$\limsup_{h\to\infty}|\operatorname{Cov}[C(F_h(\cdot)\cap U(, C(F_h(\cdot)\cap V)]|\leqslant\xi(\operatorname{diam} U, \operatorname{diam} V)\beta_1(U)\beta_1(V)$$

for any $U, V\in\mathcal{U}$ *such that* $\overline{U}\cap\overline{V}=\emptyset$ *with* $\operatorname{diam} U<\varepsilon$ *and* $\operatorname{diam} V<\varepsilon$.

For the proof of Proposition 4.1 we need some preliminary results. First, we give a result which allows us to estimate from below the capacity of the union of a family of sets $(E_i)_{i\in I}$ by means of the sum of capacities of the sets E_i.

LEMMA 4.2. – *Let* $(E_i)_{i\in I}$ *be a family of subsets of* D *and let* $E=\bigcup_{i\in I}E_i$. *Assume that there exist a finite family* $(x_i)_{i\in I}$ *of points in* D *and two positive real numbers* r, R *such that*

(i) $0<r<R$;

(ii) $E_i\subseteq B_r(x_i)\subseteq B_R(x_i)\subseteq D$ *for* $i\in I$;

(iii) $B_R(x_i)\cap B_R(x_j)=\emptyset$ *for* $i, j\in I$, $i\neq j$.

Let us set

$$\delta=\delta(E)=4^{d+1}\frac{r^{d-2}}{R^d}\qquad(R\vee\operatorname{diam} E)^2\,.$$

Then, if $\delta<1$ *we have*

$$C(E)\geqslant(1-\delta)^2\sum_{i\in I}C(E_i, B_R(x_i))\,.$$

PROOF. – Let $u\in H_0^1(D)$ be such that

$$C(E)=\int_D|Du|^2\,dx$$

and $u\geqslant 1$ q.e. on E.

It is well known that u is the unique solution of the variational inequality

$$u\in K_E\colon\int_D Du\,D(v-u)\,dx\geqslant 0\qquad\text{for } v\in K_E$$

where

$$K_E=\{v\in H_0^1(D);\ v\geqslant 1\ \text{q.e. on } E\}\,.$$

Assume that

$$u \leq \delta \quad \text{q.e. on } \partial B_R(x_i) \text{ for every } i \in I . \tag{4.10}$$

We prove that the assertion follows. Let us define the function

$$v = \frac{(u-\delta)^+}{1-\delta} .$$

It is easy to see that $v \in H_0^1(D)$, $v \geq 1$ q.e. on E and $v = 0$ q.e. on $\partial B_R(x_i)$ for each $i \in I$. Since (ii) holds, we have

$$C(E_i, B_R(x_i)) \leq \int_{B_R(x_i)} |Dv|^2 \, dx$$

for any $i \in I$. Hence,

$$\int_D |Dv|^2 \, dx \geq \sum_{i \in I} \int_{B_R(x_i)} |Dv|^2 \, dx \geq \sum_{i \in I} C(E_i, B_R(v_i)) . \tag{4.11}$$

On the other hand, by definition of v we also have

$$\int_D |Dv|^2 \, dx = \frac{1}{(1-\delta)^2} \int_D |D(u-\delta)^+|^2 \, dx \leq \frac{1}{(1-\delta)^2} \int_D |Du|^2 \, dx = \frac{1}{(1-\delta)^2} C(E) . \tag{4.12}$$

By (4.11) and (4.12) we obtain the assertion.

Let us verify (4.10). For every $i \in I$ we consider the function u_i defined by

$$u_i(x) = \left[\frac{r^{d-2}}{|x-x_i|^{d-2}} \wedge 1 \right], \quad x \in \mathbf{R}^d .$$

It is not difficult to check that $u_i \in H^1_{\mathrm{loc}}(\mathbf{R}^d)$ and that

$$\begin{cases} -\Delta u_i \geq 0 & \text{in } \mathbf{R}^d \\ u_i = 1 & \text{on } B_r(x_i) \end{cases} \tag{4.13}$$

for any $i \in I$. Let us set

$$z(x) = \sum_{i \in I} u_i(x) , \quad x \in \mathbf{R}^d . \tag{4.14}$$

We see that $z \in H^1_{\mathrm{loc}}(\mathbf{R}^d)$ and it satisfies the following conditions

$$\begin{cases} -\Delta z \geq 0 & \text{in } D \\ z \geq 1 & \text{q.e. on } E \\ z \geq 0 & \text{on } \partial D . \end{cases} \tag{4.15}$$

By a classical comparison Theorem ([33], Chapter II, Theorem 6.4), we can get, by (4.13) and (4.15), that

$$u \leqslant z \quad \text{q.e. on } D. \tag{4.16}$$

Let $y \in \partial B_R(x_i)$ for $i \in I$ fixed. We wish to estimate $z(y)$. By (4.14) we have

$$z(y) \leqslant \sum_{j\in I} \frac{r^{d-2}}{|x_j - y|^{d-2}}. \tag{4.17}$$

To estimate the right-hand side we introduce the following sets

$$C_k(y) = \{x \in \mathbf{R}^d\colon kR \leqslant |x-y| \leqslant (k+1)R\}, \qquad k = 0, 1, \ldots .$$

Moreover, let

$$I_k(y) = \{i \in I\colon x_i \in C_k(y)\}$$

and let $N_k(y)$ be the number of elements of $I_k(y)$. Since $|x_j - y| \geqslant R$ for each $j \in I$, it is easy to see that

$$\sum_{j\in I} \frac{1}{|x_j - y|^{d-2}} \leqslant \sum_{k=1}^{[\operatorname{diam} E/R]+1} \frac{1}{(kR)^{d-2}} N_k(y) \tag{4.18}$$

where $[a.]$ denotes the integer part of a.

Let us estimate $N_k(y)$. Since, for k fixed,

$$\bigcup_{i\in I_k(y)} B_R(x_i) \subseteq \{x \in \mathbf{R}^d\colon (k-1)R \leqslant |x-y| \leqslant (k+2)R\}$$

we have

$$\operatorname{meas}\Big[\bigcup_{i\in I_k(y)} B_R(x_i)\Big] \leqslant \omega_d R^d[(k+2)^d - (k-1)^d]$$

where ω_d is the volume of the unit ball. Then, using (iii), we have

$$N_k(y) \leqslant (k+2)^d - (k-1)^d \leqslant 4^d k^{d-1}. \tag{4.19}$$

By (4.17), (4.18), (4.19), we obtain

$$z(y) \leqslant \frac{r^{d-2}}{R^{d-2}} 4^d \sum_{k=1}^{[\operatorname{diam} E/R]+1} k \leqslant 4^d \frac{r^{d-2}}{R^{d-2}} \left(\left[\frac{\operatorname{diam} E}{R}\right] + 1\right)^2 \leqslant$$

$$\leqslant 4^d \frac{r^{d-2}}{R^{d-2}} \left\{2\frac{(R \vee \operatorname{diam} E)}{R}\right\}^2 = 4^{d+1} \frac{r^{d-2}}{R^d} (R \vee \operatorname{diam} E)^2 .$$

This inequality, toghether with (4.16), shows that assumption (4.10) is always satisfied and this completes the proof of the Lemma. ∎

For each subset $Z \subseteq D$ we define the random set of indices:

$$I_h(Z) = \{i \in I_h : x_i^h \in Z\}$$

and the random variable:

$$N_h(Z) = \text{number of elements of } I_h(Z) . \tag{4.19}$$

For each $h \in \boldsymbol{N}$, let $R_h^s = (s/h)^{1/d}$ where s is a positive real number (we note that by (i_3) $r_h < R_h^s$ for h large enough). For s fixed we also consider

$$I_h^s(Z) = \{i \in I_h(Z) : \exists j \in I_h,\ i \neq j \text{ such that } |x_i^h - x_j^h| < R_h^s\}$$

and

$$N_h^s(Z) = \text{number of elements of } I_h^s(Z) . \tag{4.20}$$

The following estimate is crucial for our result.

LEMMA 4.3. – *Ir* (i_1) *and* (i_2) *hold then*

$$\limsup_{h\to\infty} \frac{E[N_h^s(U)]}{h} \leqslant \omega_d s \int_U g^2\, dx$$

for any $U \in \mathcal{U}$, *where* ω_d *is the volume of the unit ball.*

PROOF. – Fix $U \in \mathcal{U}$. It is easy to check that $i \in I_h^s(U)$ if and only if

$$\sum_{\substack{j=1\\ j\neq i}}^{h} \chi_{B_{R_h^s}(x_j^h)\cap U}(x_i^h) \geqslant 1 .$$

Therefore, we see that

$$N_h^s(U) \leqslant \sum_{i=1}^{h} \sum_{\substack{j=1\\ j\neq i}}^{h} \chi_{B_{R_h^s}(x_j^h)\cap U}(x_i^h) . \tag{4.21}$$

By (4.21) and the assumptions (i_1), (i_2) we obtain

$$\begin{aligned} E[N_h^s(U)] &\leqslant \sum_{i=1}^{h} \sum_{\substack{j=1\\ j\neq i}}^{h} \int_\Omega \chi_{B_{R_h^s}(x_j^h(\omega))\cap U}\big(x_i^h(\omega)\big)\, dP(\omega) = \\ &= \sum_{i=1}^{h} \sum_{\substack{j=1\\ j\neq i}}^{h} \int_D \Big[\int_D \chi_{B_{R_h^s}(y)\cap U}(x)\, d\beta(x)\Big] d\beta(y) = \\ &= \sum_{i=1}^{h} \sum_{\substack{j=1\\ j\neq i}}^{h} \int_D \beta\big(B_{R_h^s}(y)\cap U\big)\, d\beta(y) = h(h-1)\int_D \beta\big(B_{R_h^s}(y)\cap U\big)\, d\beta(y) . \end{aligned} \tag{4.22}$$

Finally, by (4.22) we get

$$\limsup_{h\to\infty}\frac{E[N_h^s(U)]}{h}\leqslant s\limsup_{h\to\infty}\left[\frac{h}{s}\int_D\beta(B_{R_h^s}(y)\cap U)\right]d\beta(y)=$$
$$= s\limsup_{h\to\infty}\int_D\left[\frac{\omega_d}{|B_{R_h^s}(y)|}\int_{B_{R_h^s}(y)\cap U}g(x)\,dx\right]g(y)\,dy = s\omega_d\int_U g^2(y)\,dy$$

by Lebesgue Theorem. ∎

PROOF OF THE PROPOSITION 4.1. – For any $U\in\mathcal{U}$, let

$$U_h' = \{x\in U\colon \operatorname{dist}(x,\partial U) > R_h^s\} \tag{4.23}$$

and

$$U_h'' = \{x\in D\colon \operatorname{dist}(x,U) < R_h^s\}\,. \tag{4.24}$$

We observe that $U_h'\subseteq U\subseteq U_h''$.
Moreover, we note that

$$J_h(U_h') = I_h(U_h')\setminus I_h^s(U) \tag{4.25}$$

is the set of all elements $i\in I_h$ which satisfy the following conditions:

(a_1) $x_i^h\in U$;

(a_2) $B_{R_h^s}(x_i^h)\subseteq U$;

(a_3) $|x_i^h - x_j^h|\geqslant R_h^s$ for any $j\in I_h$ with $i\neq j$.

Denote by F_h' the random set

$$F_h' = \bigcup_{i\in J_h(U_h')} K_h^i\,.$$

We have

(b_1) $K_i^h\subseteq B_{r_h}(x_i^h)\subseteq B_{R_h^s}(x_i^h)$;

(b_2) $B_{R_h^s}(x_i^h)\cap B_{R_h^s}(x_j^h)=\emptyset$ for $i,j\in J_h(U_h')$ with $i\neq j$.

Let us set

$$\delta(U,h) = 4^{d+1}\frac{r_h^{d-2}}{(R_h^s)^d}(\operatorname{diam} U)^2\,. \tag{4.26}$$

Choosing $\varepsilon=\sqrt{s/c_0}$, where $c_0=4^{d+1}l$, by assumption (i_3), we see that $\delta(U,h)$ will be less than 1 for h large enough and diam $U<\varepsilon$.

Thus, by Lemma 4.2 we obtain that, for each $\omega \in \Omega$,

$$(4.27)\qquad C(F_h(\omega)\cap U)\geqslant C(F'_h(\omega))\geqslant(1-\delta(U,h))^2\sum_{i\in J_h(U'_h)}C(K_i^h, B_{R_h^s}(x_i^h))\geqslant$$
$$\geqslant(1-\delta(U,h))^2[N_h(U'_h)-N_h^s(U)]\,C(K^h, B_{R_h^s})=$$
$$=(1-\delta(U,h))^2\left[\frac{N_h(U'_h)}{h}-\frac{N_h^s(U)}{h}\right]hr_h^{d-2}C(K, B_{R_h^s/r_h})$$

whenever h is sufficiently large and diam $U<\varepsilon$. On the other hand, by using the elementary properties of the capacity, we immediately get that

$$(4.28)\qquad C(F_h\cap U)\leqslant\sum_{i\in J_h(U''_h)}C(K_i^h, B_{R_h^s}(x_i^h))=\frac{N_h(U''_h)}{h}hr_h^{d-2}C(K, B_{R_h^s/r_h})$$

for every $U\in\mathcal{U}$.

Now we are in position to prove (t_1) and (t_2) of the Proposition 4.1.

PROOF OF (t_1). – First, we observe that by the Law of Large Numbers we have

$$(4.29)\qquad \lim_{h\to\infty}\frac{E[N_h(U'_h)]}{h}=\lim_{h\to\infty}\frac{E[N_h(U''_h)]}{h}=\beta(U)$$

for every $U\in\mathcal{U}$ with $\beta(\partial U)=0$.

Moreover, by (i_3) and (4.26) we obtain

$$(4.30)\qquad \lim_{h\to\infty}\delta(U,h)=\delta(U)=\frac{c_0}{s}(\text{diam } U)^2$$

where $c_0=4^{d+1}l$.

Next, we observe that for every compact subset $K\subseteq B_R$

$$(4.31)\qquad \lim_{R\to\infty}C(K, B_R)=C(K,\mathbf{R}^d)\,.$$

By Lemma 4.3, (4.27), (4.28), (4.29), (4.30) and (4.31) we deduce that

$$(4.32)\qquad \alpha''_-(B)\leqslant c\beta(B)$$

for every $B\in\mathcal{B}$, and

$$(4.33)\qquad \alpha'_-(B)\geqslant\left(1-\frac{c_0}{s}(\text{diam } B)^2\right)^2c\left[\beta(B)-\omega_d s\int_B g^2(y)\,dy\right]$$

for every $B\in\mathcal{B}$ with sufficiently small diameter. By (4.32) we have that

$$\nu''(B)\leqslant c\beta(B)$$

for every $B\in\mathcal{B}$.

Therefore, we have only to prove that

$$\nu'(B) \geqslant c\beta(B) \tag{4.34}$$

for every $B \in \mathcal{B}$. Let us fix $B \in \mathcal{B}$. Next, for arbitrary $\eta > 0$ choose a partition $(B_i)_{i\in I}$ of B such that $B_i \in \mathcal{B}$ and diam $B_i < \eta$ for every $i \in I$. Then, by (4.33) applied with $s = \eta$, we get

$$\nu'(B) = \sum_{i\in I} \nu'(B_i) \geqslant \sum_{i\in I} \alpha'_-(B_i) \geqslant (1 - c_0\eta)^2 c\Big[\beta(B) - \omega_d \eta \int_B g^2(y)\, dy\Big]\,. \tag{4.35}$$

Since η is arbitrary, (4.34) follows from (4.35).

PROOF OF (t_2). – Preliminary, we note that by the Strong Law of Large Numbers we have

$$\frac{N_h(U'_h)}{h} \xrightarrow[h\to\infty]{} \beta(U) \qquad \text{a.e. } \omega \in \Omega \tag{4.36}$$

and

$$\frac{N_h(U'_h)}{h} \xrightarrow[h\to\infty]{} \beta(U) \qquad \text{in } L^1(\Omega) \tag{4.37}$$

for any $U \in \mathcal{U}$. Moreover, since $N_h(U'_h)/h$ is an equibounded sequence of random variables we also have

$$\frac{N_h(U'_h)}{h} \xrightarrow[h\to\infty]{} \beta(U) \qquad \text{in } L^2(\Omega) \tag{4.38}$$

for any $U \in \mathcal{U}$. We observe that (4.36), (4.37) and (4.38) hold also with U'_h replaced by U''_h, provided $\beta(\partial U) = 0$.

By (4.27), (4.31), (4.30) we have

$$\begin{aligned} \liminf_{h\to\infty} E[C(F_h(\cdot) \cap U)C(F_h(\cdot) \cap V)] \geqslant (1 - \delta(U))^2(1 - \delta(V))^2 c^2 \times \\ \times \limsup_{h\to\infty} \Big\{E\Big[\frac{N_h(U'_h)}{h}\cdot\frac{N_h(V'_h)}{h}\Big] - E\Big[\frac{N_h(U'_h)}{h}\cdot\frac{N^s_h(V)}{h}\Big] - E\Big[\frac{N_h(V'_h)}{h}\cdot\frac{N^s_h(U)}{h}\Big]\Big\} \end{aligned} \tag{4.39}$$

for any pair $U, V \in \mathcal{U}$ such that $\overline{U} \cap \overline{V} = \emptyset$, diam $U < \varepsilon$, diam $V < \varepsilon$ with $\varepsilon = \sqrt{s/c_0}$.

By (4.38) we have

$$\lim_{h\to\infty} E\Big[\frac{N_h(U'_h)}{h}\cdot\frac{N_h(V'_h)}{h}\Big] = \beta(U)\beta(V)\,. \tag{4.40}$$

Moreover, by Lemma 4.3 and (4.36) it follows

$$\limsup_{h\to\infty} E\Big[\frac{N_h(U'_h)}{h}\cdot\frac{N^s_h(V)}{h}\Big] \leqslant \omega_d \beta(U)s\int_V g^2\, dx \tag{4.41}$$

and

$$\limsup_{h\to\infty} E\left[\frac{N_h(V_h')}{h}\cdot\frac{N_h^s(U)}{h}\right] \leq \omega_d\beta(V)s\int_U g^2\,dx \tag{4.42}$$

for any $U, V \in \mathcal{U}$.

Then, (4.39), (4.40), (4.41) and (4.42) give

$$\liminf_{h\to\infty} E[C(F_h(\cdot)\cap U)C(F_h(\cdot)\cap V)] \geq (1-2\delta(U)-2\delta(V))c^2\times \\ \times\left[\beta(U)\beta(V)-\beta(U)\,\omega_d s\int_V g^2\,dx - \beta(V)\omega_d s\int_U g^2\,dx\right] \tag{4.43}$$

for every $U, V \in \mathcal{U}$, such that $\overline{U}\cap\overline{V}=\emptyset$ with $\operatorname{diam} U<\varepsilon$, $\operatorname{diam} V<\varepsilon$.

By (4.28) and (4.38) (applied with U_h'' instead of U_h') we also deduce

$$\limsup_{h\to\infty} E[C(F_h(\cdot)\cap U)C(F_h(\cdot)\cap V)] \leq c^2\beta(U)\beta(V) \tag{4.44}$$

for any $U, V \in \mathcal{U}$ with $\beta(\partial U)=\beta(\partial V)=0$.

Estimates like (4.43) and (4.44) for the upper and lower limit of the sequence $E[C(F_h(\cdot)\cap U)]\cdot E[C(F_h(\cdot)\cap V)]$ can be obtained in the same way. Therefore, we deduce that

$$\limsup_{h\to\infty} |\operatorname{Cov}[C(F_h(\cdot)\cap U), C(F_h(\cdot)\cap V)]| \leq \\ \leq c^2\beta(U)\beta(V) - [1-2\delta(U)-2\delta(V)]c^2\left[\beta(U)\beta(V)-\beta(U)\omega_d s\int_V g^2\,dx - \beta(V)\omega_d s\int_U g^2\,dx\right] \leq \\ \leq c^2\left\{\beta(U)\omega_d s\int_V g^2\,dx + \beta(V)\omega_d s\int_U g^2\,dx + 2[\delta(U)+\delta(V)]\beta(U)\beta(V)\right\} \tag{4.45}$$

for every $U, V \in \mathcal{U}$ such that $\overline{U}\cap\overline{V}=\emptyset$ with $\operatorname{diam} U<\varepsilon$, $\operatorname{diam} V<\varepsilon$.

Taking $s=\max\{\operatorname{diam} U, \operatorname{diam} V\}$, by (4.30), formula (4.45) becomes

$$\limsup_{h\to\infty} |\operatorname{Cov}[C(F_h(\cdot)\cap U)C(F_h(\cdot)\cap V)]| \leq \\ \leq c^2\left\{\beta(U)\omega_d s\int_V g^2\,dx + \beta(V)\omega_d s\int_U g^2\,dx + 2c_0 s\beta(U)\beta(V)\right\} \leq \\ \leq c_1 s\left\{\beta(U)\int_V g^2\,dx + \beta(V)\int_U g^2\,dx + \beta(U)\beta(V)\right\} \tag{4.46}$$

for every $U, V \in \mathcal{U}$ such that $\overline{U}\cap\overline{V}=\emptyset$, with $\operatorname{diam} U<\varepsilon$ and $\operatorname{diam} V<\varepsilon$.

In the last inequality we have set $c_1=c^2\max\{\omega_d, 2c_0\}$. The assertion (t_2) follows by (4.46) taking $\beta_1(U)=\beta(U)+\int_U g^2\,dx$ for every $U\in\mathcal{U}$ and $\xi(x,y)=\max\{x,y\}$. ∎

5. – Schrödinger equation with random potentials.

In this section we consider another application of our main Theorem. We study a problem concerning the stationary Schrödinger equation in $\boldsymbol{R}^3$ with particular random potentials.

We still denote by (Ω, Σ, P) a probability space. Moreover, for every $h \in \boldsymbol{N}$ we consider a family $(x_i^h)_{i \in I_h}$ of random variables satisfying the *general hypotheses* (i_1), (i_2), (i_3) in the previous section.

Denote by F_h, $h \in \boldsymbol{N}$ the following random sets

$$F_h = \bigcup_{i=1}^{h} B_{r_h}(x_i^h) .$$

Let (k_h) be a sequence of positive real numbers.

For each $h \in \boldsymbol{N}$ we define the random function:

$$q_h(x) = \begin{cases} k_h & \text{if } x \in F_h \\ 0 & \text{otherwise} . \end{cases}$$

We will study the equations:

$$\text{(5.1)} \qquad \begin{cases} -\Delta u_h + q_h(x) u_h + \lambda u_h = f & \text{in } D \\ u_h \in H_0^1(D) \end{cases}$$

where $\lambda \geqslant 0$ is a real number and $f \in L^2(D)$.

To use the theory developed in section 3 we consider the sequence (M_h) of random measures defined by

$$\text{(5.2)} \qquad M_h(B) = \int_B q_h(x)\, dx$$

for any $B \in \mathcal{B}$.

REMARK 5.1. – For every $U \in \mathcal{U}$ the functions $C(M_h(\cdot), U)$ are Σ-measurable, each of them being the infimum of a sequence of measurable functions. To see this, it is enough to use the variational definition of $C(M_h(\cdot), U)$ and the fact that the functions q_h are bounded so that

$$C(M_h(\cdot), U) = \inf_{v \in H} \left\{ \int_D |Dv|^2\, dx + \int_D (v-1)^2 q_h(\cdot)\, dx \right\}$$

where H is a countable dense subset of $H_0^1(D)$. Therefore the maps $M_h \colon \Omega \to \mathcal{M}_0^*$ are actually random measures by Corollary 1.1.

The problems (5.1) are equivalent to the following relaxed Dirichlet problems

$$\begin{cases} -\Delta u_h + (M_h(\omega) + \lambda m)u_h = f & \text{in } D \\ u_h = 0 & \text{on } \partial D . \end{cases}$$

We shall prove the following theorems:

THEOREM 5.1. – *Let (Q_h) be the sequence of distribution laws on $\mathcal{M}_0^*$ associated with the sequence of random measures (M_h) defined in* (5.2). *Assume that the general hypotheses* (i_1), (i_2), (i_3) *hold. Moreover, we suppose also that*

$$(i_4) \qquad \lim_{h\to\infty} \sqrt{k_h}\, r_h = +\infty .$$

Then (Q_h) converges weakly to the distribution law δ_ν defined by

$$\delta_\nu(\mathcal{E}) = \begin{cases} 1 & \textit{if } \nu \in \mathcal{E} \\ 0 & \textit{otherwise} \end{cases}$$

for any $\mathcal{E} \in \mathcal{B}(\mathcal{M}_0^*)$, where $\nu = c\beta$, $c = lC(B_1, \mathbf{R}^3)$, and $C(B_1, \mathbf{R}^3)$ is defined as in Theorem 4.2.

THEOREM 5.2. – *Let (M_h) be the sequence of random measures defined in* (5.2). *Assume that the general hypotheses* (i_1), (i_2), (i_3) *hold. Suppose also that*

$$(i_4) \qquad \lim_{h\to\infty} \sqrt{k_h}\, r_h = +\infty .$$

Then, for any $f \in L^2(D)$ and for every $\varepsilon > 0$

$$\lim_{h\to\infty} P\{\omega \in \Omega : \|R_h^\lambda(\omega)[f] - R^\lambda[f]\|_{L^2(D)} > \varepsilon\} = 0$$

where R_h^λ is the sequence of resolvent operators associated with the random potentials q_h (i.e. with the random measures M_h) and R^λ is the resolvent operator associated with the constant potential cg (i.e. with the measure $c\beta$).

The proofs of these theorems will depend on the next Proposition 5.1. In particular, the proof of Theorem 5.1 will be obtained by applying Theorem 3.1 and Proposition 5.1; the Theorem 5.2 will follow from Theorem 4.1 and Proposition 5.1.

PROPOSITION 5.1. – *Let (M_h) be the sequence of random measures defined in* (5.2). *Let α' and α'' be the set functions as defined respectively in* (4.3) *and* (4.4). *Assume* *the general hypotheses* (i_1), (i_2), (i_3). *In addition, suppose that*

$$\lim_{h\to\infty} \sqrt{k_h}\, r_h = +\infty . \tag{i_4}$$

Then, the following assertions hold:

(t_1') $\nu'(B) = \nu''(B) = c\beta(B)$ *for every* $B \in \mathcal{B}$;

(t_2') *there exist a constant* $\varepsilon > 0$, *an increasing continuous function* $\xi\colon \boldsymbol{R}\times\boldsymbol{R} \to \boldsymbol{R}$ *with* $\xi(0, 0) = 0$ *and a Radon measure* β_1 *such that:*

$$\limsup_{h\to\infty} |\mathrm{Cov}\,[C(M_h(\cdot), U), C(M_h(\cdot), V)]| \leqslant \xi(\mathrm{diam}\, U, \mathrm{diam}\, V)\beta_1(U), \beta_1(V)$$

for any $U, V \in \mathcal{U}$ *such that* $\overline{U} \cap \overline{V} = \emptyset$ *with* $\mathrm{diam}\, U < \varepsilon$, $\mathrm{diam}\, V < \varepsilon$.

The proof will be based on the following two lemmas.

LEMMA 5.1. – *Let* $\mu \in \mathcal{M}_0^*$. *Then, Lemma* 4.2 *holds if we replace* $C(E)$ *by* $C(\mu, E)$.

PROOF. – It is enough to replace the function u used in the proof of Lemma 4.2 with the μ-capacitary potential of E in D, defined as the unique function $w \in H_0^1(D)$ such that

$$C(\mu, E) = \int_D |Dw|^2\, dx + \int_E (w-1)^2\, d\mu$$

and to use the comparison Theorem for relaxed Dirichlet problems ([20], Theorem 2.10) instead of the classical comparison Theorem for variational inequalities. ∎

We now compute the μ-capacitary potential of a ball with respect to a concentric ball, when μ is the Lebesgue measure (multiplied by a constant).

LEMMA 5.2. – *Let* r, R *be two positive real numbers such that* $r < R$. *Moreover, let* μ *be the Borel measure in* $\mathcal{M}_0^*$ *defined by*

$$\mu(B) = k\int_B dx$$

for any $B \in \mathcal{B}$, *where* k *is constant.*

Then, the μ*-capacity potential associated with* $C(\mu, B_r, B_R)$ *is the function*

$$w(|x|) = \begin{cases} 1 - \left(a|x|^{-1} + b\right) & r \leqslant |x| \leqslant R \\[2ex] 1 - \dfrac{c \sinh \sqrt{k}\, x}{|x|} & 0 < |x| \leqslant r \end{cases} \tag{5.3}$$

for $x \in B_R$, *where*

$$a = R - \frac{\sqrt{k}\,R^2 \cosh(\sqrt{k}\,r)}{\sinh(\sqrt{k}\,r) + (\sqrt{k}\,R - \sqrt{k}\,r)\cosh(\sqrt{k}\,r)}$$

$$b = \frac{\sqrt{k}\,R \cosh(\sqrt{k}\,r)}{\sinh(\sqrt{k}\,r) + (\sqrt{k}\,R - \sqrt{k}\,r)\cosh(\sqrt{k}\,r)}$$

$$c = \frac{R}{\sinh(\sqrt{k}\,r) + (\sqrt{k}\,R - \sqrt{k}\,r)\cosh(\sqrt{k}\,r)}\,.$$

Moreover, setting $d = w(r)$ *we have*

$$(1-d)^2\, C(B_r, B_R) \leqslant C(\mu, B_r, B_R) \leqslant C(B_r, B_R)\,. \tag{5.4}$$

PROOF. – The proof of (5.3) is obtained solving explicitly the Euler equation of the functional

$$F(u) = \int_{B_R} |Du|^2\,dx + k\int_{B_r} u^2\,dx$$

with the boundary condition $u - 1 \in H_0^1(B_R)$. In order to proof (5.4) we note that the relation $C(\mu, B_r, B_R) \leqslant C(B_r, B_R)$ follows by the property (*f*) of Proposition 1.1; moreover let us define

$$u = \frac{(w-d)^+}{1-d}\,.$$

It is easy to see that $u \in H_0^1(B_R)$ and $u \geqslant 1$ q.e. on B_r.
Hence,

$$C(B_r, B_R) \leqslant \int_{B_r} \frac{|D(w-d)^+|^2}{(1-d)^2} \leqslant \frac{1}{(1-d)^2}\int_{B_R} |Dw|^2\,dx = \frac{1}{(1-d)^2}\,C(\mu, B_r, B_R)$$

which proves (5.4). ∎

PROOF OF PROPOSITION 5.1. – For each $h \in \boldsymbol{N}$ let us define a sequence μ_h of Borel measures in the following way:

$$\mu_h(B) = k_h \int_B dx$$

for any $B \in \mathcal{B}$.

Let $U \in \mathcal{U}$. Let U_h' and U_h'' be the sets defined in (4.23) and (4.24) respectively. By $J_h(U_h')$ we denote the set of indices defined in (4.25). Furthermore let $\delta(U, h)$ be as defined in (4.26), $N_h(U)$ as in (4.19) and $N_h^s(U)$ as in (4.20). By hypothesis (i$_3$), by Lemma 5.1 and Lemma 5.2 we can get that, for each $\omega \in \Omega$,

$$
\begin{aligned}
(5.5)\qquad C(M_h, U) &\geq (1-\delta(U,h))^2 \sum_{i\in J_h(U_h')} C(\mu_h, B_{r_h}(x_i^h), B_{R_h}(x_i^h)) = \\
&= (1-\delta(U,h))^2 [N_h(U_h') - N_h^s(U)]\, C(\mu_h, B_{r_h}, B_{R_h}) \geq \\
&\geq (1-\delta(U,h))^2 [N_h(U_h') - N_h^s(U)](1-d_h)^2 C(B_{r_h}, B_{R_h}) = \\
&= (1-\delta(U,h))^2 (1-d_h)^2 \left[\frac{N_h(U_h')}{h} - \frac{N_h^s(U)}{h}\right] h r_h C(B_1, B_{R_h/r_h})
\end{aligned}
$$

whenever h is sufficiently large and diam $U < \varepsilon$, with $\varepsilon = \sqrt{s/c_0}$.

By (5.3) we have that for each $h \in N$

$$
d_h = \frac{1}{r_h/R_h + [\sqrt{k_h}\, R_h(1 - r_h/R_h)]\, r_h/R_h \coth(\sqrt{k_h}\, r_h)}\,.
$$

So, by hypothesis (i_4) it follows that $d_h \to 0$ as $h \to +\infty$.

On the other hand we have by the properties of the μ-capacity

$$
\begin{aligned}
(5.6)\qquad C(M_h, U) &\leq \sum_{i\in I_h(U_h'')} C(\mu_h, B_{r_h}(x_i^h), B_{R_h}(x_i^h)) = N_h(U_h'')\, C(\mu_h, B_{r_h}, B_{R_h}) \leq \\
&\leq N_h(U_h'')\, C(B_{r_h}, B_{R_h}) = \frac{N_h(U_h'')}{h}\, h r_h C(B_1, B_{R_h/r_h})\,.
\end{aligned}
$$

By repeating the same steps made in the proof of the assertions (t_1) and (t_2) of Proposition 4.1, we get by (5.5) and (5.6) immediately the equivalent assertion in this case. ■

REFERENCES

[1] R. ASH, *Real analysis and probability*, Academic Press, New York, 1972.
[2] H. ATTOUCH, *Variational convergence for functions and operators*, Pitman, London, 1984.
[3] H. ATTOUCH - F. MURAT, *Potentiels fortement oscillant*, to appear.
[4] J. R. BAXTER - R. V. CHACON - N. C. JAIN, *Weak limits of stopped diffusions*, Trans. Amer. Math. Soc., **293** (1986), pp. 767-792.
[5] J. R. BAXTER - G. DAL MASO - U. MOSCO, *Stopping times and Γ-convergence*, Trans. Amer. Math. Soc., **303** (1987), pp. 1-38.
[6] J. R. BAXTER - N. C. JAIN, *Asymptotic capacities for finely divided bodies and stopped diffusions*, Illinois J. Math., **31** (1987), pp. 469-495.
[7] A. BRILLARD, *Quelques questions de convergence ... and calcul des variations*, These, Université de Paris Sud, Orsay, 1983.
[8] G. BUTTAZZO - G. DAL MASO - U. MOSCO, *A derivation theorem for capacities with respect to a Radon measure*, J. Funct. Anal., **71** (1987), pp. 263-278.
[9] I. CHAVEL, *Eigenvalues in Riemannian Geometry*, Academic Press, New York, 1984.

[10] I. CHAVEL - E. A. FELDMAN, *The Lenz shift and Wiener sausage in Riemannian manifolds*, to appear.

[11] I. CHAVEL - E. A. FELDMAN, *The Wiener sausage, and a theorem of Spitzer*,in Riemannian manifolds, to appear.

[12] G. CHOQUET, *Forme abstraite du théoreme de capacitabilité*, Ann. Inst. Fourier (Grenoble), **9** (1959), pp. 83-89.

[13] D. CIORANESCU - F. MURAT, *Un terme étrange venu d'ailleurs, I*, in « Nonlinear partial differential equations and their applications », Collège de France Seminar, Volume II, pp. 98-138, Res. Notes in Math., **60**, Pitman, London, 1982.

[14] D. CIORANESCU - F. MURAT, *Un terme étrange venu d'ailleurs, II*, in « Nonlinear partial differential equations and their applications », Collège de France Seminar, Volume III, pp. 154-178, Res. Notes in Math., **70**, Pitman, London, 1983.

[15] D. CIORANESCU - J. SAINT JEAN PAULIN, *Homogénéisation dans des ouverts à cavités*, C. R. Acad. Sci. Paris Sér. A, **284** (1977), pp. 857-860.

[16] D. CIORANESCU - J. SAINT JEAN PAULIN, *Homogenization in open sets with holes*, J. Math. Anal. Appl., **71** (1979), pp. 590-607.

[17] G. DAL MASO, *Γ-convergence and μ-capacities*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **14** (1987), pp. 423-464.

[18] G. DAL MASO - E. DE GIORGI - L. MODICA, *Weak convergence of measures on spaces of lower semicontinuous functions*, in « Integral functionals in calculus of variations » (Trieste, 1985), pp. 59-100, Supplemento ai Rend. Circ. Mat. Palermo, **15** (1987).

[19] G. DAL MASO - L. MODICA, *Nonlinear stochastic homogenization*, Ann. Mat. Pura Appl., (4) **144** (1986), pp. 347-389.

[20] G. DAL MASO - U. MOSCO, *Wiener criteria and energy decay for relaxed Dirichlet problems*, Arch. Rational Mech. Anal., **95** (1986), pp. 345-387.

[21] G. DAL MASO - U. MOSCO, *Wiener's criterion and Γ-convergence*, Appl. Math. Optim., **15** (1987), pp. 15-63.

[22] G. DAL MASO - U. MOSCO, *The Wiener modulus of a radial measure*, Houston J. Math., to appear.

[23] E. DE GIORGI, *G-operators and Γ-convergence*, Proceedings of the « International Congress of Mathematicians », pp. 1175-1191, Warszaw, 1983.

[24] E. DE GIORGI - T. FRANZONI, *Su un tipo di convergenza variazionale*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., (8) **58** (1975), pp. 842-850.

[25] E. DE GIORGI - T. FRANZONI, *Su un tipo di convergenza variazionale*, Rend. Sem. Mat. Brescia, **3** (1979), pp. 63-101.

[26] E. DE GIORGI - G. LETTA, *Une notion générale de convergence faible pour des fonctions croissantes d'ensemble*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., (4) **4** (1977), pp. 61-99.

[27] C. DELLACHERIE, *Ensembles analytiques, capacités, mesures de Hausdorff*, Lecture Notes in Math., **295**, Springer-Verlag, Berlin, 1972.

[28] H. FEDERER - W. ZIEMER, *The Lebesgue set of a function whose distribution derivatives are p-th power summable*, Indiana Univ. Math. J., **22** (1972), pp. 139-158.

[29] R. FIGARI - E. ORLANDI - S. TETA, *The Laplacian in regions with many small obstacles: fluctuation around the limit operator*, J. Statist. Phys., **41** (1985), pp. 465-487.

[30] E. Ya. HRUSLOV, *The method of orthogonal projections and the Dirichlet problems in domains with a fine-grained boundary*, Math. U.S.S.R. Sb., **17** (1972), pp. 37-59.

[31] E. Ya. HRUSLOV, *The first boundary value problem in domains with a complicated boundary for higher order equations*, Math. U.S.S.R. Sb., **32** (1977), pp. 535-549.

[32] M. KAC, *Probabilistic methods in some problems of scattering theory*, Rocky Mountain J. Math., **4** (1974), pp. 511-538.

[33] D. KINDERLEHRER - G. STAMPACCHIA, *Introduction to variational inequalities and their applications*, Academic Press, New York, 1980.

[34] A. V. MARCHENKO - E. Ya. HRUSLOV, *Boundary value problems in domains with closed-grained boundaries (Russian)*, Naukova Dumka, Kiev, 1974.

[35] A. V. MARCHENKO - E. Ya. HRUSLOV, *New results in the theory of boundary value problems for regions with closed-grained boundaries*, Uspekhi Mat. Nauk, **33** (1978), pp. 127-137.

[36] S. OZAWA, *On an elaboration of M. Kac's theorem concerning eigenvalues of the Laplacian in a region with randomly distributed small obstacles*, Comm. Math. Phys., **91** (1983), pp. 473-487.

[37] S. OZAWA, *Random media and the eigenvalues of the Laplacian*, Comm. Math. Phys., **94** (1984), pp. 421-437.

[38] G. C. PAPANICOLAOU - S. R. S. VARADHAN, *Diffusion in regions with many small holes*, in « Stochastic differential systems, filtering and control », Proceedings of the IFIP-WG 7/1 Working Conference (Vilnius, Lithuania, 1978), pp. 190-206, Lecture Notes in Control and Information Sci., Springer-Verlag, **25** (1980).

[39] K. R. PARTHASARATHY, *Probability measures on metric spaces*, Academic Press, New York, 1967.

[40] J. RAUCH - M. TAYLOR, *Potential and scattering theory on wildly perturbed domains*, J. Funct. Anal., **18** (1975), pp. 27-59.