Weak Solutions for a System of Nonlinear Klein-Gordon Equations (*).

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Summary. – *We prove the existence and uniqueness of weak solutions of the mixed problem for a class of systems of nonlinear Klein-Gordon equations. Uniqueness is proved when the spatial dimension is either* $n = 1, 2$ *or* 3.

Introduction.

A mathematical model describing the interaction of scalar fields u, v of mass α, β, respectively, with interaction constants g and h, is the following system of nonlinear Klein-Gordon equations:

$$(*) \qquad \Box u + \alpha^2 u + g^2 v^2 u = 0 \,, \quad \Box v + \beta^2 v + h^2 u^2 v = 0 \,,$$

where $\Box$ is the d'Alembertian operator, i.e., $\Box = \partial^2/\partial t^2 - \Delta$. The above system defines the motion of charged mesons in an electromagnetic field and was proposed by I. SEGAL [8]. A number of authors also proposed such systems and among them we can mention K. JÖRGENS [2] and V. G. MAKHANKOV [6]. Recently L. A. MEDEIROS-G. PERLA MENZALA [7] obtained weak solutions of the mixed problem for the system $(*)$ in $\Omega \times]0, T[$, where Ω denotes a bounded domain of $\boldsymbol{R}^n$, $n = 1, 2, 3$, and J. FERREIRA and G. PERLA MENZALA [1] analyzed the decay of the solutions of the system $(*)$. There is no loss of generality if we make $\alpha = 0$, $\beta = 0$, $g = 1$, $h = 1$ in $(*)$.

A significant generalization of $(*)$ is the following system:

$$(**) \qquad \begin{cases} \Box u + |v|^{\varrho+2}|u|^{\varrho} u = f_1 \\ \Box v + |u|^{\varrho+2}|v|^{\varrho} v = f_2 \end{cases}$$

where ϱ is a real number, $\varrho > -1$. In this paper we study the existence and uniqueness of weak solutions of the mixed problem for the system $(**)$. The

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existence is proved in $\Omega\times]0,T[$ where Ω denotes a bounded domain of $\boldsymbol{R}^n$, for any $n\geqslant 1$ and the uniqueness is obtained for $n=1,2,3$ (when $n=3$, $\varrho=0$). Our discussion is based on the method applied by J. L. LIONS [3] to solve the scalar nonlinear wave equation $\Box u+|u|^{\varrho}u=f$, see also LIONS-STRAUSS [5].

1. – Notations and main results.

Let Ω be a regular bounded domain of $\boldsymbol{R}^n$, $T>0$ a real number and Q the cylinder $Q=\Omega\times]0,T[$. By $(\cdot,\cdot)$ and $|\cdot|$ we denote the inner product and norm of $L^2(\Omega)$ and by $a(u,v)$ and $\|\cdot\|$, the inner product and norm of $H_0^1(\Omega)$, respectively. Here $a(u,v)$ denotes the Dirichlet form and $H_0^1(\Omega)$ is the closure in $H^1(\Omega)$ of the space $\mathcal{D}(\Omega)$, where $\mathcal{D}(\Omega)$ denotes the space of infinitely differentiable functions with compact support contained in Ω.

Let X be a Banach space and $1\leqslant p<\infty$ a real number. We shall represent by $L^p(0,T;X)$ the Banach space of vector valued functions $u\colon]0,T[\to X$ which are measurable and $\|u(t)\|_X\in L^p(0,T)$, with the norm:

$$\|u\|_{L^p(0,T;X)}=\left(\int_0^T\|u(t)\|_X^p\,dt\right)^{1/p}.$$

By $L^\infty(0,T;X)$ we shall represent the Banach space of functions $u\colon]0,T[\to X$ which are measurable and essentially bounded in Ω, with the norm

$$\|u\|_{L^\infty(0,T;X)}=\operatorname*{ess\,sup}_{0<t<T}\|u(t)\|_X\,.$$

Furthermore, $\mathcal{D}'(Q)$ and $\mathcal{D}'(0,T)$ will denote the space of distributions on Q and $]0,T[$, respectively. All the scalar functions considered in this paper will be real valued.

Let $n\geqslant 3$ the dimension of $\boldsymbol{R}^n$. Let us consider a real number ϱ satisfying the following condition:

$$-1<\varrho<\frac{4}{n-2}\,. \tag{1}$$

Let θ and γ be the following real numbers:

$$\theta=\frac{2n(\varrho+2)}{(n-2)(\varrho+2)+2n(\varrho+1)}\,,\quad \gamma=\frac{2n(\varrho+2)}{(n+2)(\varrho+2)-2n(\varrho+1)}\,. \tag{2}$$

Clearly,

$$\frac{1}{\theta}+\frac{1}{\gamma}=1$$

and

$$1 < \theta < \frac{\varrho + 2}{\varrho + 1}, \quad \gamma > 1 . \tag{3}$$

THEOREM 1. – Let Ω be a regular bounded domain of $\boldsymbol{R}^n$ and ϱ a real number satisfying the condition (1) if $n \geqslant 3$ or $\varrho > -1$ if $n = 1, 2$. Let

$$f_1,\ f_2 \in L^2(0, T; L^2(\Omega)) \tag{4}$$

$$u_0,\ v_0 \in H_0^1(\Omega) \cap L^p(\Omega) \tag{5}$$

$$u_1,\ v_1 \in L^2(0, T; L^2(\Omega)) \tag{6}$$

where $p = \varrho + 2$. Then there exist functions $u, v\colon]0, T[\to L^2(\Omega)$ such that:

$$u, v \in L^\infty(0, T; H_0^1(\Omega)) \tag{7}$$

$$u', v' \in L^\infty(0, T; L^2(\Omega)) \qquad \left(u' = \frac{du}{dt}\right) \tag{8}$$

$$uv \in L^\infty(0, T; L^{\varrho+2}(\Omega)) \tag{9}$$

satisfying the nonlinear system:

$$u'' - \Delta u + |v|^{\varrho+2}|u|^\varrho u = f_1 \quad \text{in} \quad L^2(0, T; H^{-1}(\Omega) + L^\theta(\Omega)) \tag{10}$$

$$v'' - \Delta v + |u|^{\varrho+2}|v|^\varrho v = f_2 \quad \text{in} \quad L^2(0, T; H^{-1}(\Omega) + L^\theta(\Omega))\,; \tag{11}$$

and the initial conditions:

$$u(0) = u_0\,, \quad v(0) = v_0 \tag{12}$$

$$u'(0) = u_1\,, \quad v'(0) = v_1\,. \tag{13}$$

THEOREM 2. – Let $u, v\colon]0, T[\to L^2(\Omega)$ be functions in the classes (7), (8) and (9) satisfying from (10) to (13). Then, $u = v$ provided that $\varrho \geqslant 0$ in case $n = 1$ or 2; $u = v$ if $\varrho = 0$ in case $n = 3$.

REMARK 1. – We observe that if $n = 1, 2$ the above conditions (5) amount say that u_0 and v_0 are any vectors of $H_0^1(\Omega)$ and condition (9) is a consequence of (7).

REMARK 2. – From (7) and (8), we have that $u(0)$ and $v(0)$ belong to $H_0^1(\Omega)$ (see [4]). Thus the initial conditions (12) do make sense. Let α and β be the real numbers

$$\alpha = \frac{\varrho + 2}{(\varrho + 1)\theta}, \quad \beta = \frac{\varrho + 2}{(\varrho + 2) - (\varrho + 1)\theta}\,. \tag{14}$$

Consequently,

$$\frac{1}{\alpha}+\frac{1}{\beta}=1$$

and by (2) and (3):

$$\alpha>1\,,\quad \beta>1\,,\quad \theta\beta=\frac{2n}{n-2}\qquad (n\geqslant 3)\,. \tag{15}$$

It follows by (15) and Hölder inequality that

$$\int\limits_{\Omega}\big(|v|^{\varrho+2}|u|^{\varrho+1}\big)^{\theta}\,dx\leqslant\Big(\int\limits_{\Omega}|uv|^{(\varrho+1)\theta\alpha}\,dx\Big)^{1/\alpha}\Big(\int\limits_{\Omega}|v|^{\beta\theta}\,dx\Big)^{1/\beta}=\|u\|^{(\varrho+1)\theta}_{L^{\varrho+2}(\Omega)}\|v\|^{\theta}_{L^{\beta\theta}(\Omega)}\,.$$

Now we use (15), Sobolev's embedding theorem and (9) to deduce that:

$$\big\||v(t)|^{\varrho+2}|u(t)|^{\varrho}u(t)\big\|_{L^{\theta}(\Omega)}\leqslant c\|u(t)v(t)\|^{\varrho+1}_{L^{\varrho+2}(\Omega)}\|v(t)\|_{H^1_0(\Omega)}\leqslant C_1\,. \tag{16}$$

Therefore,

$$|v|^{\varrho+2}|u|^{\varrho}u\in L^{\infty}\big(0,T;L^{\theta}(\Omega)\big)\,. \tag{17}$$

The above conclusion and equation (10) imply that:

$$u''\in L^{2}\big(0,T;H^{-1}(\Omega)+L^{\theta}(\Omega)\big)\,, \tag{18}$$

which together with condition (8) imply $u(0)\in L^2(\Omega)$. Analogously $v(0)\in L^2(\Omega)$. Thus, the initial conditions (13) make sense for $n\geqslant 3$. When $n=1,2$ we choose $\theta=2$, $\alpha=(\varrho+2)/(\varrho+1)$, $\beta=\varrho+2$ to obtain

$$|v|^{\varrho+2}|u|^{\varrho}u\,,\qquad |u|^{\varrho+2}|v|^{\varrho}v\in L^{\infty}\big(0,T;L^{2}(\Omega)\big)\,, \tag{18}'$$

hence,

$$u'',\,v''\in L^{2}\big(0,T;H^{-1}(\Omega)\big)\,,$$

therefore, the initial conditions (13) do make sense.

2. – Proof of Theorem 1.

We use Galerkin procedure and compactness method.

i) *Approximate Solutions.* – Let $(w_\nu)_{\nu\in N}$ be a «basis» of $H^1_0(\Omega)\cap L^{p_1}(\Omega)$, $p_1=\max(p,\gamma)$ and V_m the subspace generated by the m first vectors $w_1, w_2, \dots, w_m$. Let

$$u_m(t)=\sum_{j=1}^{m}g_{jm}(t)w_j\,,\qquad v_m(t)=\sum_{j=1}^{m}h_{jm}(t)w_j\,,$$

be the approximate solutions in V_m of the problem (10)-(13). They are defined as solutions of the system:

$$(19)\qquad \big(u_m''(t), w_j\big) + a\big(u_m(t), w_j\big) + \big(|v_m(t)|^{\varrho+2}|u_m(t)|^{\varrho}u_m(t), w_j\big) = \big(f_1(t), w_j\big)$$

$$(20)\qquad \big(v_m''(t), w_j\big) + a\big(v_m(t), w_j\big) + \big(|u_m(t)|^{\varrho+2}|v_m(t)|^{\varrho}v_m(t), w_j\big) = \big(f_2(t), w_j\big)\,, \qquad j = 1, 2, \ldots, m.$$

$$(21)\qquad u_m(0) = u_{0m}\,, \qquad u_{0m} \to u_0 \quad \text{in} \quad H_1^1(\Omega) \cap L^p(\Omega)$$

$$(22)\qquad v_m(0) = v_{0m}\,, \qquad v_{0m} \to v_0 \quad \text{in} \quad H_0^1(\Omega) \cap L^p(\Omega)$$

$$(23)\qquad u_m'(0) = u_{1m}\,, \qquad u_{1m} \to u_1 \quad \text{in} \quad L^2(\Omega)$$

$$(24)\qquad v_m'(0) = v_{1m}\,, \qquad v_{1m} \to v_1 \quad \text{in} \quad L^2(\Omega)$$

The solutions $u_m(t)$ and $v_m(t)$ are defined in $[0, t_m]$, $t_m > 0$. In order to define them in all $[0, T]$, we need to obtain a priori estimates for $u_m(t)$ and $v_m(t)$.

ii) *A Priori Estimates.* – Multiplying (19) by $g_{jm}'(t)$ and then adding from $j = 0$ to $j = m$, we get:

$$\frac{1}{2}\frac{d}{dt}|u_m'(t)|^2 + \frac{1}{2}\frac{d}{dt}\|u_m(t)\|^2 + \int_\Omega |v_m|^{\varrho+2}|u_m|^{\varrho}u_m u_m'\, dx = (f_1, u_m')\,.$$

We observe that:

$$\int_\Omega |v_m|^{\varrho+2}|u_m|^{\varrho}u_m u_m'\, dx = \frac{1}{\varrho+2}\int_\Omega |v_m|^{\varrho+2}\frac{d}{dt}|u_m|^{\varrho+2}\, dx\,,$$

which implies that:

$$\frac{d}{dt}|u_m'(t)|^2 + \frac{d}{dt}\|u_m(t)\|^2 + \frac{2}{\varrho+2}\int_\Omega |v_m|^{\varrho+2}\frac{d}{dt}|u_m|^{\varrho+2}\, dx = 2(f_1, u_m')\,.$$

Similarly from (20), multiplying by h_{jm}', we obtain:

$$\frac{d}{dt}|v_m'(t)|^2 + \frac{d}{dt}\|v_m(t)\|^2 + \frac{2}{\varrho+2}\int_\Omega |u_m|^{\varrho+2}\frac{d}{dt}|v_m|^{\varrho+2}\, dx = 2(f_2, v_m')\,.$$

Adding the last two identities and then using Schwarz's inequality, we have that:

$$\frac{d}{dt}\left(|u_m'(t)|^2 + |v_m'(t)|^2 + \|u_m(t)\|^2 + \|v_m(t)\|^2 + \frac{2}{\varrho+2}\int_\Omega |u_m|^{\varrho+2}|v_m|^{\varrho+2}\, dx\right) \leqslant$$
$$\leqslant |f_1(t)|^2 + |f_2(t)|^2 + |u_m'(t)|^2 + |v_m'(t)|^2\,.$$

Integrating the above expression from 0 to t, $t \leqslant t_m$, using hypothesis (4) on f_1, f_2 and convergences (21)-(24), we get:

$$(25)\qquad |u_m'(t)|^2 + |v_m'(t)|^2 + \|u_m(t)\|^2 + \|v_m(t)\|^2 + \\ + \frac{2}{\varrho+2}\int_\Omega |u_m(t)v_m(t)|^{\varrho+2}\,dx \leqslant C + \frac{2}{\varrho+2}\int_\Omega |u_{0m}v_{0m}|^{\varrho+2} + \int_0^t \left(|u_m'(\sigma)|^2 + |v_m'(\sigma)|^2\right) d\sigma\,,$$

for m large enough, where C is a constant independent of m and t. We observe, by using Schwarz's inequality and convergences (21) and (22), that:

$$\int_\Omega |u_{0m}v_{0m}|^{\varrho+2}\,dx \leqslant \|u_{0m}\|^{\varrho+2}_{L^{2(\varrho+2)}(\Omega)}\|v_{0m}\|^{\varrho+2}_{L^{2(\varrho+2)}(\Omega)} \leqslant C_1\|u_{0m}\|^{\varrho+2}_{L^{p}(\Omega)}\|v_{0m}\|^{\varrho+2}_{L^{p}(\Omega)} \leqslant C_2\,.$$

Substituting the above estimate in (25) and applying Gronwall's inequality we, finally, obtain the following estimates:

(26) $(u_m)_{m\in N}$, $(v_m)_{m\in N}$ are bounded in $L^\infty(0, T; H_0^1(\Omega))$

(27) $(u_m')_{m\in N}$, $(v_m')_{m\in N}$ are bounded in $L^\infty(0, T; L^2(\Omega))$

(28) $(u_m v_m)_{m\in N}$ is bounded in $L^\infty(0, T; L^{\varrho+2}(\Omega))$.

It follows, from the above estimates, that we can extract subsequence of $(u_m)_{m\in N}$ and a subsequence of $(v_m)_{m\in N}$ which we still denoted by $(u_m)_{m\in N}$, $(v_m)_{m\in N}$ and determine functions u and v, such that:

(29) $u_m \to u$ and $v_m \to v$ weak star in $L^\infty(0, T; H_0^1(\Omega))$

(30) $u_m' \to u'$ and $v_m' \to v'$ weak star in $L^\infty(0, T; L^2(\Omega))$

(31) $u_m v_m \to \chi$ weak star in $L^\infty(0, T; L^{\varrho+2}(\Omega))$.

By using Lions-Aubin's compactness theorem, [3], we conclude that:

(32) $u_m \to u$ and $v_m \to v$ strong in $L^2(Q)$

(33) $u_m \to u$ and $v_m \to v$ a.e. in Q .

iii) *The nonlinear term.* - From (33), it follows:

$$(34)\qquad |v_m|^{\varrho+2}|u_m|^\varrho u_m \to |v|^{\varrho+2}|u|^\varrho u \quad \text{a.e. in} \quad Q\,.$$

Applying the same argument that we used to obtain (17) or (18), it follows from the estimate (26) and (28) that

(35) $\left(|v_m|^{\varrho+2}|u_m|^\varrho u_m\right)_{m\in N}$ is bounded in $L^\infty(0, T; L^\theta(\Omega))$.

Consequently, from (34), (35) and by LIONS [3], Lemma 1.3, we conclude that:

$$(36)\qquad |v_m|^{\varrho+2}|u_m|^{\varrho}u_m \to |v|^{\varrho+2}|u|^{\varrho}u \quad \text{weak star in} \quad L^\infty(0, T; L^\theta(\Omega)) .$$

Similarly,

$$(37)\qquad |u_m|^{\varrho+2}|v_m|^{\varrho}v_m \to |u|^{\varrho+2}|v|^{\varrho}v \quad \text{weak star in} \quad L^\infty(0, T_2\, L^\theta(\Omega)) .$$

By (31) and (32) we obtain that

$$u_m v_m \to \chi \quad \text{in} \quad \mathfrak{D}'(Q) \quad \text{and} \quad u_m v_m \to uv \quad \text{in} \quad L^1(Q) ,$$

therefore,

$$\chi = uv .$$

iv) *Passage to the Limit.* – Multiplying (19) by $\eta \in \mathfrak{D}(0, T)$ integrating from 0 to T and passing to the limit as $m \to \infty$, $j < m$, we deduce from the convergences (29), (30) and (36) that

$$-\int_0^L (u'(t), \eta'(t)w_j)\, dt + \int_0^T a(u(t), \eta(t)w_j)\, dt + \int_0^T (|v(t)|^{\varrho+2}|u(t)|^{\varrho}u(t), \eta(t)w_j)\, dt = \int_0^T (f_1(t), \eta(t)w_j)\, dt .$$

Observing that $(\eta w_j)_{j\in N}$, $\eta \in \mathfrak{D}(0, T)$ is a total set in $H_1^0(0, T; H_0^1(\Omega) \cap L^{p_1}(\Omega),)$ then,

$$-\int_0^T (u', w)\, dt + \int_0^T a(u, w)\, dt + \int_0^T (|v|^{\varrho+2}|u|^{\varrho}u, w)\, dt = \int_0^T (f, w)\, dt ,$$

for each $w \in H_0^1(0, T; H_0^1(\Omega) \cap L^{p_1}(\Omega))$. Hence taking $w \in \mathfrak{D}'(Q)$ we obtain (10). Analogously, we obtain (11).

It follows from the convergences (29), (30) and from the initial conditions (21), (22), that the initial conditions (12) are satisfied. Also, by the convergences (30) and the approximate system (19) and (20), it follows the initial conditions (13). This conclude the proof of Theorem 1. Q.E.D.

3. – Proof of the uniqueness' result (Theorem 2).

Let u, v and $\hat{u}$, $\hat{v}$ satisfying the conditions of Theorem 2. Let

$$U = u - \hat{u} \quad \text{and} \quad V = v - \hat{v} .$$

Then,

$$U'' - \Delta U + |v|^{\varrho+2}|u|^{\varrho}u - |\hat{v}|^{\varrho+2}|\hat{u}|^{\varrho}\hat{u} = 0 \tag{38}$$

$$V'' - \Delta V + |u|^{\varrho+2}|v|^{\varrho}v - |\hat{u}|^{\varrho+2}|\hat{v}|^{\varrho}\hat{v} = 0 \tag{39}$$

$$U(0) = 0\,, \quad V(0) = 0 \tag{40}$$

$$U'(0) = 0\,, \quad V'(0) = 0\,. \tag{41}$$

Remark 3. – We note that

$$U''(t) \in H^{-1}(\Omega) + L^{\theta}(\Omega) \quad \text{and} \quad U'(t) \in L^2(\Omega)\,.$$

Therefore, does not make sense to calculate the duality $\langle U''(t), U'(t)\rangle$. Thus, the uniqueness result via the standard energy method does not work in this case. In order to remedy the above difficulty we shall use a method due to Visik and Ladyshenskaya [9] see also Lions-Magenes [4]. Let

$$\varphi(t) = \begin{cases} -\int_t^s U(\sigma)\,d\sigma & \text{for } 0 \leqslant t \leqslant s \\ 0 & \text{for } s < t \leqslant T \end{cases}$$

$$\psi(t) = \begin{cases} -\int_t^s V(\sigma)\,d\sigma & \text{for } 0 \leqslant t \leqslant s \\ 0 & \text{for } s < t \leqslant T\,. \end{cases}$$

Then

$$\varphi, \psi \in L^{\infty}(0, T; H_0^1(\Omega))\,.$$

Let

$$\varphi_1(t) = \int_0^t U(\sigma)\,d\sigma \quad \text{and} \quad \psi_1(t) = \int_0^t V(\sigma)\,d\sigma\,,$$

then

$$\varphi(t) = \varphi_1(t) - \varphi_1(s)\,, \quad \varphi_1(s) = -\varphi(0)$$

and

$$\psi(t) = \psi_1(t) - \psi_1(s)\,, \quad \psi_1(s) = -\psi(0)\,.$$

First of all we analise the case $n \geqslant 3$. We make the restriction $\varrho \leqslant (8-2n)/(2n-4)$. Then, $\varphi(t), \psi(t) \in L^{\gamma}(\Omega)$, see (2), therefore the inner product of the nonlinear part with φ or ψ make sense. By taking the inner product of (38) and (39) with φ and ψ, respectively, we obtain:

$$(42)\qquad \int_0^s \langle U'', \varphi\rangle\, dt + \int_0^s a(U, \varphi)\, dt + \int_0^s \big(|v|^{\varrho+2}|u|^{\varrho}u - |\hat{v}|^{\varrho+2}|\hat{u}|^{\varrho}\hat{u}, \varphi\big)\, dt = 0$$

$$(43)\qquad \int_0^s \langle V'', \psi\rangle\, dt + \int_0^s a(V, \psi)\, dt + \int_0^s \big(|u|^{\varrho+2}|v|^{\varrho}v - |\hat{u}|^{\varrho+2}|\hat{v}|^{\varrho}\hat{v}, \psi\big)\, dt = 0\,.$$

A simple calculation shows that:

$$\int_0^s \langle U'', \varphi\rangle\, dt = -\tfrac{1}{2}|U(s)|^2$$

and

$$\int_0^s a(U, \varphi)\, dt = -\tfrac{1}{2}\|\varphi(0)\|^2 = -\tfrac{1}{2}\|\varphi_1(s)\|^2\,.$$

Substituting these identities in (42), we deduce that

$$(44)\qquad \tfrac{1}{2}|U(s)|^2 + \tfrac{1}{2}\|\varphi_1(s)\|^2 = \int_0^s \big(|v|^{\varrho+2}|u|^{\varrho}u - |\hat{v}|^{\varrho+2}|\hat{u}|^{\varrho}\hat{u}, \varphi\big)\, dt\,.$$

Similarly it follows from (43) that:

$$(45)\qquad \tfrac{1}{2}|V(s)|^2 + \tfrac{1}{2}\|\psi_1(s)\|^2 = \int_0^s \big(|u|^{\varrho+2}|v|^{\varrho}v - |\hat{u}|^{\varrho+2}|\hat{v}|^{\varrho}\hat{v}, \psi\big)\, dt\,.$$

Adding (44) and (45) and denoting by M and N the right hand side of (44) and (45), respectively, we obtain:

$$(46)\qquad \tfrac{1}{2}|U(s)|^2 + \tfrac{1}{2}|V(s)|^2 + \tfrac{1}{2}\|\varphi_1(s)\|^2 + \tfrac{1}{2}\|\psi_1(s)\|^2 = M + N\,.$$

The key idea of the proof will be to show, by using Sobolev's embedding theorem and Gronwall inequality, that there exists a real number s_0 such that $U(t) = V(t) = 0$ for all $0 \leqslant t \leqslant s_0$. Because of the way s_0 is constructed we will be able to repeat the same technique in the interval $[s_0, 2s_0]$. Then, by iterative process we conclude that $U = V = 0$. (See LIONS-MAGENES [4]).

We write the second member of (44) in the following convenient form:

$$(47)\qquad M = \int_0^s \big([|v|^{\varrho+2} - |\hat{v}|^{\varrho+2}]|u|^{\varrho}u, \varphi\big)\, dt + \int_0^s \big([|u|^{\varrho}u - |\hat{u}|^{\varrho}\hat{u}]|\hat{v}|^{\varrho+2}, \varphi\big)\, dt\,.$$

Next we examine each of the above integrals. By the mean value theorem it follows that:

$$\left|\left(\left[|v|^{\varrho+2}-|\hat{v}|^{\varrho+2}\right]|u|^{\varrho}u,\varphi\right)\right|\leqslant(\varrho+2)\int_{\Omega}\left[\max\left(|v|^{\varrho+1},|\hat{v}|^{\varrho+1}\right]|u|^{\varrho+1}|V||\varphi|\,dx\leqslant \leqslant(\varrho+2)\int_{\Omega}\left[|v|^{\varrho+1}+|\hat{v}|^{\varrho+1}\right]|u|^{\varrho+1}|V||\varphi|\,dx\,.$$

By the Sobolev's embedding theorem we have:

$$\int_{\Omega}|\hat{v}|^{\varrho+1}|u|^{\varrho+1}|V||\varphi|\,dx\leqslant \leqslant\left(\int_{\Omega}|\hat{v}|^{2n(\varrho+1)}\,dx\right)^{1/2n}\left(\int_{\Omega}|u|^{2n(\varrho+1)}\,dx\right)^{1/2n}\left(\int_{\Omega}|V|^{2}\,dx\right)^{1/2}\left(\int_{\Omega}|\varphi|^{q}\,dx\right)^{1/q} \tag{48}$$

where

$$\frac{1}{2}+\frac{1}{q}+\frac{1}{2n}+\frac{1}{2n}=1$$

because $H_0^1(\Omega)\subset L^q(\Omega)$ for $1/q+\frac{1}{2}-1/n$. Since $\hat{v}(t)$ and $u(t)$ are only bounded in $H_0^1(\Omega)$, by the Sobolev's embedding theorem the two last integral exist only when $1\leqslant 2n(\varrho+1)\leqslant 2n/(n-2)$. It then follows that ϱ must to satisfy the condition:

$$\frac{1-2n}{2n}\leqslant\varrho\leqslant\frac{3-n}{2n}\,. \tag{49}$$

Also by using the mean value theorem, we obtain the inequality:

$$\left|\left(\left[|u|^{\varrho}u-|\hat{u}|^{\varrho}\hat{u}\right]|\hat{v}|^{\varrho+2},\varphi\right)\right|\leqslant(\varrho+1)\int_{\Omega}\left[|u|^{\varrho}+|\hat{u}|^{\varrho}\right]|U||\hat{v}|^{\varrho+2}|\varphi|\,dx\,, \tag{50}$$

and for this we must have:

$$\varrho\geqslant 0\,. \tag{51}$$

Due to the restrictions (49), (51) on ϱ, we conclude that if $n=3$ then $\varrho=0$ and if $n\geqslant 4$, this method does not allow us to conclude that we have unique solution of the system, in the conditions of the Theorem 1.

Therefore, we shall prove uniqueness in the case $n=3$, $\varrho=0$. Using estimates (48) and (50), we obtain, from (47):

$$|M|\leqslant C\int_0^s\left(|V(t)|+|U(t)|\right)\|\varphi(t)\|_{L^q(\Omega)}\,dt\leqslant \leqslant\frac{C}{2}\int_0^s|V(t)|^2\,dt+\frac{C}{2}\int_0^s|U(t)|^2\,dt+C\int_0^s\|\varphi(t)\|^2\,dt\leqslant \leqslant\frac{C}{2}\int_0^s|V(t)|^2\,dt+\frac{C}{2}\int_0^s|U(t)|^2\,dt+C\int_0^s\|\varphi_1(t)\|^2\,dt+Cs\|\varphi_1(s)\|^2\,, \tag{52}$$

where C denote various constants. Similarly we have:

$$|N| \leqslant \frac{C}{2}\int_0^s |U(t)|^2\,dt + \frac{C}{2}\int_0^s |V(t)|^2\,dt + C\int_0^s \|\psi_1(t)\|^2\,dt + Cs\|\psi_1(s)\|^2 . \tag{53}$$

Combining estimates (52) and (53) in (46), we obtain:

$$|U(s)|^2 + |V(s)|^2 + (1-2Cs)\|\varphi_1(s)\|^2 + (1-2Cs)\|\psi_1(s)\|^2 \leqslant \leqslant 2C\int_0^s \big(|U(t)|^2 + |V(t)|^2 + \|\varphi_1(t)\|^2 + \|\psi_1(t)\|^2\big)\,dt . \tag{54}$$

By choosing $s_0 = 1/4C$ we have $1 - 2Cs \geqslant \frac{1}{2}$ for all $0 \leqslant s \leqslant s_0$. It follows from (54) and Gronwall inequality that $U(t) = V(t) = 0$ for all $0 \leqslant t \leqslant s_0$.

If $s_0 < T$, we use the same argument with initial data zero in s_0 and we consider $s_0 \leqslant s \leqslant T$. As it was done above, the coefficients of $\|\varphi_1(s)\|^2$ and $\|\psi_1(s)\|^2$ are $1 - 2C(s - s_0)$. Let $s_1 = 2s_0$, then $1 - 2C(s - s_0) \geqslant \frac{1}{2}$ for all $s_0 \leqslant s \leqslant 2s_0$ which implies that $U(t) = V(t) = 0$ on $[s_2, 2s_0]$. By continuing this process we conclude that $U = V = 0$ on $[0, T]$, that is $u = \hat{u}$ and $v = \hat{v}$.

If $n = 1, 2$, we always obtain the estimates (52) and (53) for any real $\varrho \geqslant 0$ and the proof in this case is more direct. This concludes the proof of Theorem 2. Q.E.D.

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