Compact Sets in the Space $L^p(0, T; B)$ (*).

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Summary. – *A characterization of compact sets in $L^p(0, T; B)$ is given, where $1 \leq p \leq \infty$ and B is a Banach space. For the existence of solutions in nonlinear boundary value problems by the compactness method, the point is to obtain compactness in a space $L^p(0, T; B)$ from estimates with values in some spaces X, Y or B where $X \subset B \subset Y$ with compact imbedding $X \to B$. Using the present characterization for this kind of situations, sufficient conditions for compactness are given with optimal parameters. As an example, it is proved that if $\{f_n\}$ is bounded in $L^q(0, T; B)$ and in $L^1_{loc}(0, T; X)$ and if $\{\partial f_n/\partial t\}$ is bounded in $L^1_{loc}(0, T; Y)$ then $\{f_n\}$ is relatively compact in $L^p(0, T; B)$, $\forall p < q$.*

Introduction.

The question. – Lef f_n be a bounded sequence of functions in $L^p(0, T; B)$ where B is a Banach space and $1 \leq p \leq \infty$. When does there exists a strongly converging sub-sequence, that is to say when is $\{f_n\}$ relatively compact in $L^p(0, T; B)$?

This is a crucial point in the compactness method (1) for existence of solutions in nonlinear boundary value problems; several examples of this method were given by J. L. LIONS [L2].

A first answer which allows to solve a large diversity of problems, as is shown in [L2], was given by J. P. AUBIN [AU]: it suffices that the f_n are bounded in a space $L^p(0, T; X)$ where X is included in B with compact imbedding and the derivatives (2) $\partial f_n/\partial t$ are bounded in a space $L^p(0, T; Y)$ where $B \subset Y$.

Another answer was given by J. L. LIONS [L1] for Hilbert spaces, by replacing the boundedness of derivatives by a fractional hypothesis which is defined by the Fourier transform.

More generally it suffices, as it was proved by the author in [SI1], to replace the boundedness of derivatives by the following uniform (in n) estimates of translations:

$$\int_0^{T-h} \|f_n(t+h) - f_n(t)\|_B^p \, dt \leq \mathcal{O}(h) ,$$

where $\mathcal{O}$ denotes a function such that $\mathcal{O}(h) \to 0$ as $h \to 0$.

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(1) The f_n are « approximated » solutions and the strong convergence is used for passing to the limit in some nonlinear terms, such as $|f|^{\lambda-1}f$ or $f(\partial f/\partial x), \ldots$.

(2) The derivatives are always defined in the sense of distributions on $]0, T[$ in B.

The caracterization of compact sets. – A great question still remained: what are the minimal assumptions for compactness? Question which is sharpened as, in all the results known by the author, some unnecessary restrictions on parameters ($p > 1$, $p < \infty, \ldots$) or on spaces (reflexivity, separability, Hilbert structure ...) are stated and yet in various boundary value problems the excluded cases occur.

An answer is given here (theorem 1): a set F of functions f is relatively compact in $L^p(0, T; B)$, or in $C(0, T; B)$ if $p = \infty$, if and only if

$$(0.1) \qquad \forall t_1 < t_2\,, \quad \int_{t_1}^{t_2} f(t)\,dt \text{ lies in a compact set of } B \text{ independent of } f$$

$$(0.2) \qquad \int_0^{T-h} \|f(t+h) - f(t)\|_B^p\,dt \leqslant \mathcal{O}(h)\,, \quad \text{where } \mathcal{O}(h) \text{ is independent of } f$$

with the usual modifications for $p = \infty$; (0.1) is called the space criterion and (0.2) is the time criterion.

Partial Compactness. – In the boundary value problems, the compactness is often required in $L^p(0, T; B)$ for a set F which is bounded in $L^q(0, T; B)$ with $q > p$ [3]; this is called the partial compactness since $p = q$ is not reached. Then (theorem 2) the time criterion (0.2) in L^p can be replaced by the similar criterion in L^1_{loc}:

$$(0.3) \qquad \forall 0 < t_1 < t_2 < T\,, \quad \int_{t_1}^{t_2} \|f(t+h) - f(t)\|_B\,dt \leqslant \mathcal{O}(h)$$

where $\mathcal{O}(h)$ is independent of f, but can depend on t_1, t_2.

Applications. – In the boundary value problems estimates are obtained, see [L2], for the approximated solutions or for their derivatives or integrals in various spaces. In a general way one has [4] $X \subset B \subset Y$ with compact imbedding $X \to B$ and the problem now is: obtain the compactness of F in $L^p(0, T; B)$, from estimates in X, Y or B. Two kind of answer are given:

1° *Partial compactness* (section 9). – If F is bounded in some $L^q(0, T; B)$ where $q > p$, then the compactness hold if:

$$(0.4) \qquad F \text{ is bounded in } L^1_{\text{loc}}(0, T; X) \text{ and } \frac{\partial F}{\partial t} \text{ is bounded in } L^1_{\text{loc}}(0, T; Y)$$

where $\partial F/\partial t = \{\partial f/\partial t : f \in F\}$. The assumption on $\partial F/\partial t$ may be replaced by F is bounded in $W^{s,1}_{\text{loc}}(0, T; Y)$ with $s > 0$, or by an estimate on translations in $L^1_{\text{loc}}(0, T; Y)$.

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(3) If the approximated solutions f_n are bounded in $L^q(0, T; B)$, and if the convergence in $L^p(0, T; B)$ is sufficient for the convergence in nonlinear terms.

(4) $\subset$ denotes an algebraic and topologic inclusion. The imbedding $B \to Y$ is then continuous.

2° *Limit compactness* (section 8). – For compactness up to the largest order for which p is bounded, and particularly for $p = \infty$, stronger assumptions are required such as:

$$\text{(0.5)} \qquad F \text{ is bounded in } L^p(0, T; X) \text{ and } \frac{\partial F}{\partial t} \text{ is bounded in } L^1(0, T; Y) \qquad (p < \infty)\,.$$

This holds for $p = \infty$ if $\partial F/\partial t$ is bounded in $L^r(0, T; Y)$ with $r > 1$. The assumption on $\partial F/\partial t$ can be replaced by F is bounded in $W^{s,r}(0, T; Y)$ with $s > 0$ and $s > 1/r - 1/p$ or even by an estimate on translations in $L^p(0, T; Y)$.

Peculiar cases. – If $Y = B$ (section 7) the spaces criterions may be weakened: the compactness hold for every $p < \infty$ if

$$\text{(0.6)} \qquad F \text{ is bounded in } W^{-m,1}_{\text{loc}}(0, T; X) \text{ and } \frac{\partial F}{\partial t} \text{ is bounded in } L^1(0, T; B)$$

where m is any integer. Various generalizations are given.

If B is a space of class θ with respect to X and Y (section 10) some results are improved, and compactness is obtained for intermediate derivatives (corollary 10).

Optimality. – These assumptions, for example the estimates in X or on $\partial F/\partial t$, are not necessary then one has to ensure that every application is optimal: it is verified that there is no useless restriction on the parameters, and particularly that the compactness order p is as large as possible for the assumptions.

Comparison with former results. – The compactness is proved in [AU] (5) if F is bounded in $L^p(0, T; X)$ and $\partial F/\partial t$ is bounded in $L^r(0, T; Y)$, for $r > 1$, $1 < p < \infty$, X and Y reflexive spaces. See equally [L2] theorem 5.2, p. 60. A proof with the same restrictions excepted the reflexivity is given in [D], see equally [L2] theorem 12, p. 141. A proof for $r = 1$, $p = 2$, X and Y-Hilbert spaces is given in [T1], theorem 2.3, p. 76.

The extension with F bounded in $W^{s,p}(0, T; B)$ instead of the assumption on $\partial F/\partial t$ is proved for $r = p = 2$, X and Y Hilbert spaces in [L1], 1st edition chapter 4, see equally [L2] theorem 5.2, p. 61.

The extension with an estimate on translations instead of the assumption on $\partial F/\partial t$ is proved in [SI1] for $1 < p < \infty$, X and Y reflexive spaces.

In the case $Y = B$, the compactness with an hypothesis on translations is announced in [SI2] remark 3.2, and a partial result is proved in [T2] theorem 13.3 p. 100 (6).

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(5) Theorem 1 with $m = 1$, $j = 0$, $A_0 = B = X$, $\hat{B} = B$ and $Y = A_1$.

(6) Assuming F bounded in $L^p(0, T; B) \cap L^1(0, T; X)$ and (0.2) it is proved that F is relatively compact in $L^r(0, T, B)$ $\forall r < p$. As it is proved here the conclusion hold for $r = p$

All the proofs relie on one of the following two ideas. The first one is to use weakly converging subsequences and to prove that such a sequence is strongly converging. It requires the reflexivity of $L^p(0, T; X)$ and $L^p(0, T; Y)$ then the restrictions $1 < p < \infty$, $1 < r < \infty$ and X and Y reflexive spaces (7). The second idea, which is used here, is to approximate uniformly the functions of F by some mean-functions which are continuous and to conclude with the Ascoli theorem.

Local regularity, Sobolev spaces and interpolation. – The compactness criterions (0.1) and (0.2) are in particular satisfied if, in the boundary value problems language, one has uniform regularity estimates respectively in space and time. We search for local criterions since the regularity estimates often don't hold up to the boundary of the interval $[0, T]$ on which the problem is stated.

The compact sets characterization of theorems 1 and 2 are some vector valued variations on the Fréchet-Kolmogorov theorem, see remark 4.3. The applications to Sobolev spaces (all the corollaries) are vector valued extensions of the Rellich-Kondrachov theorem.

The fractionnary Sobolev spaces are sometimes defined by interpolation; here the criterion (0.2) on translation suggest the use of the (equivalent) definition by translations. For the intermediate spaces we use the condition of « class θ » which is easy to use and give more spaces than interpolation does. Then most of the result are obtained by easy means.

This work is endebted to J. L. LIONS works for many basic ideas. A. DAMLAMIAN and L. TARTAR contribute to it by fruitful debates.

The outlines are

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(7) It can be extended to the case where $L^p(0, T; X)$ is a dual space, than $1 < p \leqslant \infty$ and X and Y are dual spaces. It cannot be extended for $p = 1$.

1. – The spaces $L^p(0, T; B)$.

Let $[0, T]$ be a bounded interval of R which is provided with the Lebesgue measure dt, let B be a Banach space and $1 \leqslant p \leqslant \infty$. Denote $C(0, T; B)$ the space of continuous functions from $[0, T]$ into B equipped with the uniform convergence norm, and

$$\|f\|_{L^p(0,T;B)} = \left(\int_0^t \|f(t)\|_B^p\, dt\right)^{1/p} \qquad \left(= \operatorname*{Sup\,ess}_{0<t<T} \|f(t)\|_B \text{ if } p = \infty\right).$$

By definition (8) $L^p(0, T; B)$, $p < \infty$, is the seperated completed space of $C(0, T; B)$ for this norm; for $p = \infty$, $L^\infty(0, T; B)$ is the subset of $L^1(0, T; B)$ on which the L^∞ norm is finite. It is a Banach space for $1 \leqslant p \leqslant \infty$ (9).

Then $L^p(0, T; B)$ is a space of class of almost everywhere equal functions; in an usual way, a function will be identified to the class of the a.e. equal functions. Then $C(0, T; B)$ is by definition dense in $L^p(0, T; B)$ for $p < \infty$.

The integral $\int_0^T f(t)\, dt$ is defined if f is a measurable finite valued function (10) and it depends continuously on f for the L^1 norm. The measurable finite valued functions being dense (11) in $L^1(0, T; B)$, the integral is defined (12) in a unique way by continuous extension for $f \in L^1(0, T; B)$. If ω is a measurable subset of $[0, T]$, $\int_\omega f(t)\, dt = \int_0^T 1_\omega(t) f(t)\, dt$.

At last let $L^p_{\text{loc}}(0, T; B)$ be the set of (class of a.e. equal functions) f such that $1_{[t_1,t_2]} f \in L^p(0, T; B)$, $\forall 0 < t_1 < t_2 < T$, equipped with the semi-norms $\|f\|_{L^p(t_1,t_2;B)}$.

REMARK 1.1. – *Measurability*. It follows that (13) $L^p(0, T; B)$ is the set of (class of a.e. equal) measurable functions f such that $\|f\|_B \in L^p(0, T)$. $L^p(0, T; B)$ is sometimes defined by this property, see [Y]. We won't use the measurability, excepted

(8) [B1] definition 2, p. 129 for $p < \infty$ and, for $p = \infty$, definition 2, p. 206 and corollary p. 215.

(9) [B1] theorem 2, p. 130 for $p < \infty$ and proposition 2, p. 206 for $p = \infty$.

(10) f is a measurable finite valued function if $f^{-1}(b_i) = \omega_i$ is a measurable set in R for every value b_i; then $\int_0^T f(t)\, dt = \sum_i b_i$ measure (ω_i), see [B1] p. 80 and [B2].

(11) [B1] corollary 1, p. 162.

(12) Then $\int_0^T f(t)\, dt = \lim_{n\to\infty} \int_0^T f_n(t)\, dt$ where f_n is a measurable finite valued function and $f_n \to f$ in $L^1(0, T; B)$; see [B1] definition 1, p. 140.

(13) [B1] theorem 5, p. 184 and definition 2, p. 206; a measurable function is by definition a.e. limit of a sequence of measurable finite valued functions.

for the following

$$\text{(1.1)} \quad \text{if } f \in L^1(0, T; B) \quad \text{and} \quad \|f\|_B \in L^p(0, T), \quad \text{then} \quad f \in L^p(0, T; B) \quad \text{and} \quad \|f\|_{L^p(0,T;B)} = \|\|f\|_B\|_{L^p(0,T)}. \qquad \blacksquare$$

Let us now recall some inequalities

Hölder. If $g \in L^{s_0}(0, T; B)$ and $\varphi \in L^{s_1}(0, T)$, $1 \leqslant s_i \leqslant \infty$, then $g\varphi \in L^s(0, T; B)$ and

$$\text{(1.2)} \qquad \|g\varphi\|_{L^s(0,T;B)} \leqslant \|g\|_{L^{s_0}(0,T;B)} \|\varphi\|_{L^{s_1}(0,T)}, \qquad \frac{1}{s} = \frac{1}{s_0} + \frac{1}{s_1}.$$

Particularly ($\varphi \equiv 1$) if $g \in L^{s_0}(0, T; B)$, $1 \leqslant s_0 \leqslant \infty$ then $g \in L^s(0, T; B)$ and

$$\text{(1.3)} \qquad \|g\|_{L^s(0,T;B)} \leqslant T^{1/s - 1/s_0} \|g\|_{L^{s_0}(0,T;B)}, \qquad 1 \leqslant s \leqslant s_0.$$

Young. If $g \in L^{s_0}(0, T; B)$, $\varphi \in L^{s_1}(0, a)$, $1 \leqslant s_i \leqslant \infty$ and $G(t) = \int_0^a g(t + \lambda)\varphi(\lambda)\, d\lambda$ then $G \in L^s(0, T - a; B)$ and

$$\text{(1.4)} \quad \|G\|_{L^s(0,T-a;B)} \leqslant \|g\|_{L^{s_0}(0,T;B)} \|\varphi\|_{L^{s_1}(0,a)}, \qquad \frac{1}{s} = \frac{1}{s_0} + \frac{1}{s_1} - 1, \qquad \left(\frac{1}{s_0} + \frac{1}{s_1} \geqslant 1\right).$$

Particularly ($\varphi \equiv 1$) if $g \in L^{s_0}(0, T; B)$, $1 \leqslant s_0 \leqslant \infty$, its right-mean function is defined for $a > 0$ by $(M_a g)(t) = 1/a \int_t^{t+a} g(\lambda)\, d\lambda$. Then $M_a g \in C(0, T - a; B)$ and

$$\text{(1.5)} \qquad \|M_a g\|_{L^s(0,T-a;B)} \leqslant \begin{cases} a^{1/s - 1/s_0} \|g\|_{L^{s_0}(0,T)} & \text{if } s_0 \leqslant s \leqslant \infty, \\ T^{1/s - 1/s_0} \|g\|_{L^{s_0}(0,T)} & \text{if } 1 \leqslant s \leqslant s_0. \end{cases}$$

For $s \geqslant s_0$ (1.5) is given by (1.4), and for $s \leqslant s_0$ it follows from the case $s = s_0$ by (1.3). The estimate (1.4) is given by the standard Young convolution inequality if g and φ are extended to R by 0; a proof is given in [SI3], Appendix and note (1) in the proof of lemma 7.

2. – The Ascoli theorem.

A set K of a topological space E is compact iff for every family of open sets covering K there exists a finite sub-family covering K. A set is relatively compact iff its closure is compact.

If E is a normed space, K is relatively compact iff

$$\text{(2.1)} \quad \forall \varepsilon > 0, \ \exists \text{ a finite sub-set } \{e_i : 1 \leqslant i \leqslant I\} \text{ of } K \text{ such that } \forall e \in K, \ \exists e_i \text{ such that } \|e - e_i\|_E \leqslant \varepsilon.$$

It is satisfied if K is the uniform limit of relatively compact sets, that is to say if

(2.2) $\forall \varepsilon > 0$, $\exists$ a relatively compact set K_ε such that
$$\forall e \in K,\ \exists e_\varepsilon \in K_\varepsilon \text{ such that } \|e - e_\varepsilon\|_B \leqslant \varepsilon\,.$$

Let us now recall the Ascoli characterization of compact sets in $C(0, T, B)$.

LEMMA 1. – A set F of $C(0, T; B)$ is relatively compact if and only if:

(2.3) $F(t) = \{f(t): f \in F\}$ is relatively compact in B, $\forall 0 < t < T$,

(2.4) F is uniformly equicontinuous: $\forall \varepsilon > 0$, $\exists \eta$ such that
$$\|f(t_2) - f(t_1)\|_B \leqslant \varepsilon,\ \forall f \in F,\ \forall 0 \leqslant t_1 \leqslant t_2 \leqslant T \text{ such that } |t_2 - t_1| \leqslant \eta\,. \quad \blacksquare$$

PROOF. – Let F be relatively compact in $C(0, T; B)$. Then (2.3) is obvious and (2.4) is satisfied since F can be uniformly approximated by finite sets of continuous functions.

Conversely if (2.3) and (2.4) are satisfied let us first notice that (2.3) is satisfied for $0 \leqslant t \leqslant T$; indeed $\|f(0) - f(\eta)\|_B \leqslant \varepsilon$ and $F(\eta)$ is relatively compact in B then $F(0)$ is relatively compact by (2.2); so does $F(T)$.

For N integer denote f_N the function which equals f for every point nT/N, $0 \leqslant n \leqslant N$, and which is linear between these points. Then $F_N = \{f_N: f \in N\}$ is isomorphic to the product of the sets $F(nT/N)$, $0 \leqslant n \leqslant N$, which is relatively compact in B^{N+1}, then F_N is relatively compact in $C(0, T; B)$. On other hand by (2.4), if $N \geqslant T/\eta$ then $\|f - f_N\|_{C(0,T;B)} \leqslant \varepsilon$. Then F is the uniform limit of the relatively compact sets F_N, and it is relatively compact by (2.2). $\blacksquare$

3. – Characterization of the compact sets of $L^p(0, T; B)$.

Denote $(\tau_h f)(t) = f(t + h)$ for $h > 0$. If f is defined on $[0, T]$, then the translated function $\tau_h f$ is defined on $[-h, T - h]$. The main result of this work is the

THEOREM 1. – Let $F \subset L^p(0, T; B)$. F is relatively compact in $L^p(0, T; B)$ for $1 \leqslant p < \infty$, or in $C(0, T, B)$ for $p = \infty$ (B Banach space) if and only if:

(3.1) $$\left\{\int_{t_1}^{t_2} f(t)\, dt : f \in F\right\} \text{ is relatively compact in } B\,, \quad \forall 0 < t_1 < t_2 < T$$

(3.2) $$\|\tau_h f - f\|_{L^p(0, T-h; B)} \to 0 \quad \text{as } h \to 0\,, \quad \text{uniformly for } f \in F. \quad \blacksquare$$

For $p = \infty$ in comparison with Ascoli's theorem the time criterion (3.2) is exactly the uniform equicontinuity, but the space criterion which was stated on $f(t)$ is replaced here by the similar one on integrals (3.1).

REMARK 3.1. – The criterion (3.2) can be expressed as:

(3.3) $\forall \varepsilon > 0$, $\exists \eta$ such that: $\forall f \in F$, $\forall h < \eta$ one has $\|\tau_h f - f\|_{L^p(0,T-h;B)} \leqslant \varepsilon$. ■

OPTIMALITY. – The restriction $p < \infty$ is necessary since, if f is a discontinuous bounded function, $F = \{f\}$ is compact in $L^\infty(0, T; B)$ but it does not satisfies (3.2). ■

PROOF OF THEOREM 1. – i) Assume first that F is a relatively compact set of $L^p(0, T; B)$, $p < \infty$. The map $f \to \int_{t_2}^{t_1} f(t)\, dt$ is continuous from $L^p(0, T; B)$ into B then (3.1) is satisfied.

For every $\varepsilon > 0$ there exists, see (2.1), a finite number of $f_i \in L^p(0, T; B)$, $1 \leqslant i \leqslant I_\varepsilon$, such that: $\forall f \in F$, $\exists f_i$ such that $\|f - f_i\|_{L^p(0,T;B)} \leqslant \varepsilon/3$.

As $C(0, T; B)$ is dense in $L^p(0, T; B)$ the f_i may be choosen in $C(0, T; B)$. Then there exists h_i such that,

$$\forall h \leqslant h_i, \quad \text{there holds} \quad \|\tau_h f_i - f_i\|_{L^p(0,T-h;B)} \leqslant \frac{\varepsilon}{3}.$$

Set $\eta = \inf_i h_i$. As $\tau_h f - f = \tau_h(f - f_i) - (f - f_i) + (\tau_h f_i - f_i)$, for every $h \leqslant \eta$ there holds $\|\tau_h f - f\|_{L^p(0,T-h;B)} \leqslant \varepsilon$ which proves (3.3), then (3.2).

ii) Conversely assume that F satisfies (3.1) and (3.2). The relative compactness will follow by three steps.

First step. – For $f \in F$ and $a > 0$ let the right mean function be defined by $(M_a f)(t) = 1/a \int_t^{t+a} f(s)\, ds$. Then $M_a f \in C(0, T - a; B)$.

For every $0 \leqslant t_1 \leqslant t_2 \leqslant T - a$ one has

$$\|(M_a f)(t_2) - (M_a f)(t_1)\|_B = \left\| \frac{1}{a} \int_{t_1}^{t_1+a} (\tau_{t_2-t_1} f - f)(s)\, ds \right\|_B \leqslant \frac{1}{a} \|\tau_{t_2-t_1} f - f\|_{L^1(0,T-(t_2-t_1);B)}.$$

Then the hypothesis (3.2) imply that the set $M_a F = \{M_a f : f \in F\}$ is uniformly equicontinuous in $C(0, T - a; B)$.

For every $0 < t < T - a$ the hypothesis (3.1) with $t_1 = t$, $t_2 = t + a$ imply that $(M_a F)(t)$ is relatively compact in B. Then by Ascoli's characterization (lemma 1)

(3.4) $M_a F$ is relatively compact in $C(0, T - a; B)$.

Second step. – There holds

(3.5) $$M_a f - f = \frac{1}{a} \int_0^a (\tau_h f - f)\, dh \quad \text{in } L^p(0, T - a; B) \ (^{14}).$$

(14) It follows from (3.2) that the map $h \to \tau_h f$ is continuous from $[0, a]$ into $L^p(0, T - a; B)$. Then the right hand side integral in (3.5) is defined in $L^p(0, T - a; B)$ and the equality follows from the definition of $M_a f$.

Then

$$\|M_a f - f\|_{L^p(0,T-a;B)} \leqslant \sup_{0\leqslant h\leqslant a} \|\tau_h f - f\|_{L^p(0,T-a;B)} . \tag{3.6}$$

It follows with (3.3) that, $\forall T_1 < T$, F is the uniform limit of $M_a F$ in $L^p(0, T_1; B)$ as $a \to 0$ $(a \leqslant T - T_1)$. By (3.4) $M_a F$ is relatively compact in $L^p(0, T_1; B)$ then, see (2.2), F is relatively compact in $L^p(0, T_1; B)$.

Third step. – The hypothesis (3.1) and (3.2) remains if one changes the time direction: if $\tilde{f}(t) = f(T-t)$ the set $\tilde{F} = \{\tilde{f} : f \in F\}$ satisfies (3.1) and (3.2). Then $\tilde{F}$ is relatively compact in $L^p(0, T_1; B)$, thus F is relatively compact in $L^p(T - T_1, T; B)$ [15].

Choosing $T_1 = T/2$ one finally obtains the relative compactness in $L^p(0, T; B)$.

iii) If $p = \infty$ the proofs are identical in $C(0, T; B)$. ■

REMARK 3.2. – Using the theorem 1 for a set composed of a single function, one find again the standard continuity property of $h \to \tau_h f$:

$$\|\tau_h f - f\|_{L^p(0,T-h;B)} \to 0 \quad \text{as } h \to 0 , \quad \forall f \in L^p(0, T; B),\ 1 \leqslant p < \infty. \quad ■$$

REMARK 3.3. – It is not necessary to assume in theorem 1 that $F \subset L^p(0, T; B)$. It is enough to assume (for giving a meaning to (3.1) and (3.2)):

$$f \in L^1_{\text{loc}}(0, T; B) \quad \text{and} \quad \tau_h f - f \in L^p(0, T-h; B) , \quad \forall f \in F,\ \forall h > 0 . \tag{3.7}$$

Indeed, then $\tau_h f = f + (\tau_h f - f) \in L^1(h, T - h; B)$ which implies $f \in L^1(0, T - 2h; B)$. Similarly one has $f \in L^1(2h, T; B)$ and finally $f \in L^1(0, T; B)$.

Then (3.5) is satisfied in $L^1(0, T - a; B)$. By (3.2) the map $h \to \tau_h f - f$ belongs to $C(0, a; L^p(0, T - a; B))$, then the integral in the right hand side of (3.5) converges in $L^p(0, T - a; B)$, whence $f \in L^p(0, T - a; B)$. Similarly one proves $f \in L^p(a, T; B)$ and finally $f \in L^p(0, T; B)$. ■

REMARK 3.4. – The proof of theorem 1 is easy but one has to take care.

For example it fails if the right mean $M_a f$ is replaced by the centered mean $(J_a f)(t) = 1/2a \int_{t-a}^{t+a} f(s)\, ds$. Indeed $J_a F$ don't converge to F up to the boundary, then one obtains the compactness only in $L^p_{\text{loc}}(0, T; B)$.

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[15] This result may equally be obtained by replacing the right mean by the left mean $(\tilde{M}_a f)(t) = (1/a) \int_{t-a}^{t} f(s)\, ds$.

One may equally try to reduce the problem to functions on R by using the extension $\bar{f}$ of f by 0. Then the necessary condition (3.2) on $\bar{f}$ gives again (3.2), on f, but it also gives the extra condition (4.4), see [T2, remark 13.1, p. 100]. Without this extra hypothesis one obtains only partial compactness [T2, thm. 13.3]. ∎

Application to real valued functions. – A set F is relatively compact in $L^p(0, T)$ when $1 \leqslant p < \infty$, or in $C(0, T)$ when $p = \infty$, if and only if:

$$\exists a_1 < a_2 \text{ such that } \int_{a_1}^{a_2} f(t)\, dt \text{ is bounded uniformly for } f \in F\,, \tag{3.8}$$

$$\int_0^{T-h} |f(t+h) - f(t)|^p\, dt \to 0 \quad \text{as} \quad h \to 0 \text{ uniformly for } f \in F\,. \tag{3.9}$$

PROOF. – It follows from (3.8) and (3.9) that, $\forall 0 \leqslant t_1 \leqslant t_2 \leqslant T$, $\int_{t_1}^{t_2} f(s)\, ds$ is bounded then (3.1) is satisfied. ∎

It is a variation on Fréchet-Kolmogorov's theorem which states ([16]): A set F of $L^p(0, T)$, $1 \leqslant p < \infty$ is relatively compact if and only if:

$$\forall a > 0,\ \int_a^{T-a} |f(t+h) - f(t)|^p\, dt \to 0 \text{ as } h \to 0, \text{ uniformly for } f \in F\,,$$

$$\int_0^a |f(t)|^p\, dt + \int_{T-a}^T |f(t)|^p\, dt \to 0 \qquad \text{as } a \to 0, \text{ uniformly for } f \in F\,.$$

These last criterions imply easily (3.8) and (3.9) but the converse implication is harder. A vector valued extension of Fréchet-Kolmogorov's characterization will be given in remark 4.3.

4. – Characterization for partial compactness.

The question here is to characterize the sets which are bounded in $L^q(0, T; B)$ and are compact in $L^p(0, T; B)$ with $p < q$. It is called partial compactness since the compactness is not obtained for the larger order p for which the set is bounded.

The main result of this section is the

THEOREM 2. – Let F be a bounded set in $L^q(0, T; B)$ $(1 < q \leqslant \infty$, B Banach space).

([16]) For example see [N] theorem 1.3, p. 59.

Then F is relatively compact in $L^p(0, T; B)$, $\forall p < q$, if and only if:

$\forall 0 < t_1 < t_2 < T$ there holds

$$\left\{\int_{t_1}^{t_2} f(t)\, dt \colon f \in F\right\} \text{ is relatively compact in } B\,, \tag{4.1}$$

$$\|\tau_h f - f\|_{L^1(t_1, t_2; B)} \to 0 \text{ as } h \to 0, \text{ uniformly for } f \in F. \quad \blacksquare \tag{4.2}$$

In comparison with theorem 1, the time criterion in L^p is replaced by the similar one (4.2) in L^1_{loc}.

REMARK 4.1. – The criterion (4.2) can be expressed as: $\forall 0 < t_1 < t_2 < T$, $\forall \varepsilon > 0$, $\exists \eta \leqslant T - t_2$ such that $\forall f \in F$, $\forall h < \eta$, one has $\|\tau_h f - f\|_{L^1(t_1, t_2; B)} \leqslant \varepsilon$. $\blacksquare$

Let us first connect compactness with time-local compactness, it can be connected with partial compactness afterwards, and theorem 2 will follow.

LEMMA 2. – A set F is relatively compact in $L^p(0, T; B)$, $1 \leqslant p < \infty$, if and only if:

$$F \text{ is relatively compact in } L^p_{\text{loc}}(0, T; B)\,, \tag{4.3}$$

$$\int_0^h \|f(t)\|_B^p\, dt + \int_{T-h}^T \|f(t)\|_B^p\, dt \to 0 \text{ as } h \to 0, \text{ uniformly for } f \in F\,. \quad \blacksquare \tag{4.4}$$

The meaning of criterion (4.3) is: $\forall 0 < t_1 < t_2 < T$, F is relatively compact in $L^p(t_1, t_2; B)$ (17).

PROOF OF LEMMA 2. – i) Assume first that F is a relatively compact set of $L^p(0, T; B)$. Then (4.3) is obvious. Let $\bar{f}$ be the extension of f by 0 outside $[0, T]$. Then $\bar{F} = \{\bar{f} \colon f \in F\}$ is relatively compact in $L^p(-T, 2T; B)$ and the time criterion (3.2) for $\bar{F}$ gives (4.4) since, for $h \leqslant T$,

$$\|\tau_h \bar{f} - \bar{f}\|^p_{L^p(-T, 2T-h; B)} = \int_0^h \|f(t)\|_B^p\, dt + \int_0^{T-h} \|f(t+h) - f(t)\|_B^p\, dt + \int_{T-h}^T \|f(t)\|_B^p\, dt\,.$$

ii) Conversely assume that F satisfies (4.3) and (4.4). Set $f_h = 1_{[h, T-h]} f$ and $F_h = \{f_h \colon f \in F\}$. By (4.4), $\forall \varepsilon > 0$, $\exists h$ such that $\|f_h - f\|_{L^p(0, T; B)} \leqslant \varepsilon$ uniformly for $f \in F$. Then F is the uniform limit of the relatively compact sets F_h whence, see (2.2), F is relatively compact in $L^p(0, T; B)$. $\blacksquare$

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(17) Or else: $\{1_{[t_1, t_2]} f \mid f \in F\}$ is relatively compact in $L^p(0, T; B)$.

Now partial compactness can be related with compactness in L^1_{loc}.

LEMMA 3. – Let F be a bounded set in $L^q(0, T; B)$, $1 < q \leq \infty$. If F is relatively compact in $L^1_{loc}(0, T; B)$, then it is relatively compact in $L^p(0, T; B)$, $\forall p < q$. ∎

PROOF OF LEMMA 3. – By Hölder inequality (1.3) there holds, $\forall h \leq T$, $\forall f \in F$,

$$\int_0^h \|f(t)\|_B \, dt + \int_{T-h}^T \|f(t)\|_B \, dt \leq 2h^{1-1/q} \|f\|_{L^q(0, T; B)}$$

then F is relatively compact in $L^1(0, T; B)$ by lemma 2.

Given $1 < p < q$ the lemma 11 in section 10 with $X = Y = B$ and θ defined by $1/p = (1-\theta)/q + \theta/1$ shows that F is relatively compact in $L^p(0, T; B)$. ∎

PROOF OF THEOREM 2. – By theorem 1, the hypothesis (4.1) and (4.2) are equivalent to the compactness of F in $L^1_{loc}(0, T; B)$. By lemma 3 this is equivalent to the compactness in $L^p(0, T; B)$, $\forall p < q$. ∎

OPTIMALITY OF THEOREM 2. – The strict inequality $p < q$ is necessary in theorem 2 (and in lemma 3) since there exists bounded sets F in $L^q(0, T; X)$ which are relatively compact in $L^p(0, T; B)$ for every $p < q$ and yet not for $p = q$: see proposition 1 in section 11.

Remark that for $q = 1$ the theorem would be true but empty since there would be no one $p < q$. ∎

REMARK 4.2. – With the hypothesis of theorem 2, the closure $\overline{F}$ of F in $L^p(0, T; B)$ is included and bounded in $L^q(0, T; B)$.

Indeed let $f \in \overline{F}$ and denote by c the bound of F in $L^q(0, T; B)$; there exists f_n such that $\|f_n\|_{L^q(0,T;B)} \leq c$ and $f_n \to f$ in $L^p(0, T; B)$ then [18] $\|\|f\|_B\|_{L^q(0,T)} \leq c$ and by (1.1) $f \in L^q(0, T; B)$ and $\|f\|_{L^q(0,T;B)} \leq c$. ∎

REMARK 4.3. – Theorem 1 gives, by using lemma 2, a vector valued extension of Fréchet-Kolmogorov's theorem. A set F of $L^p(0, T; B)$ is relatively compact in $L^p(0, T; B)$, $1 \leq p < \infty$, if and only if

$$\forall 0 < t_1 < t_2 < T, \quad \left\{\int_{t_1}^{t_2} f(t) \, dt : f \in F\right\} \text{ is relatively compact in } B,$$

$$\forall a > 0, \quad \int_a^{T-a} \|f(t+h) - f(t)\|_B^p \, dt \to 0 \text{ as } h \to 0, \text{ uniformly for } f \in F,$$

$$\int_0^a \|f(t)\|_B^p \, dt + \int_{T-a}^T \|f(t)\|_B^p \, dt \to 0 \qquad \text{as } a \to 0, \text{ uniformly of } f \in F. \quad ∎$$

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[18] $\|f_n\|_B \to \|f\|_B$ in $L^q(0, T)$ weak if $q < \infty$, and in $L^\infty(0, T)$ weak * if $q = \infty$.

5. – Some estimates by translations.

We search for conditions involving the time criterions (3.2) or (4.2). A first possibility is to consider functions with integrable derivatives or more generally distributions with integrable derivatives.

Let $\mathcal{D}'(]0, T[; B)$ be the space of distributions from $]0, T[$ into B which is defined by L. SCHWARTZ [SC2] as the space of linear continuous maps from $\mathcal{D}(]0, T[)$ [19] into B. The derivative df/dt of a distribution f is defined by $(df/dt)(\varphi) = -f(d\varphi/dt)$, $\forall\varphi \in \mathcal{D}(]0, T[)$.

For every integrable function f, a distribution is defined by $f(\varphi) = \int_0^T f(t)\varphi(t)\, dt$, $\forall\varphi \in \mathcal{D}(]0, T[)$. So $L^p(0, T; B)$ is identified to a sub-space of $\mathcal{D}'(]0, T[; B)$, which allows to define the distributional derivative of any function of $L^p(0, T; B)$.

For the distributions whose derivatives are integrables, the translations are estimated by the (standard)

LEMMA 4. – Let $f \in \mathcal{D}'(]0, T[; B)$ be such that $\partial f/\partial t \in L^r(0, T; B)$ where $1 \leqslant r \leqslant \infty$. Then $f \in C(0, T; B)$ and, $\forall h > 0$,

$$(5.1) \qquad \|\tau_h f - f\|_{L^p(0,T-h,B)} \leqslant \begin{cases} h^{1+1/p-1/r} \left\|\dfrac{\partial f}{\partial t}\right\|_{L^r(0,T;B)} & \text{if } r \leqslant p \leqslant \infty \\ hT^{1/p-1/r} \left\|\dfrac{\partial f}{\partial t}\right\|_{L^r(0,T;B)} & \text{if } 1 \leqslant p \leqslant r\,. \end{cases} \qquad \blacksquare$$

PROOF. – For every $g \in L^1(0, T; B)$ there holds $g - (\partial/\partial t)\int_0^{\bullet} g(s)\, ds = 0$ (it is obvious if g is continuous and the general result follows since $C(0, T; B)$ is dense in $L^1(0, T; B)$ by definition). Setting $g = \partial f/\partial t$ it yields $(\partial/\partial t)\left(f - \int_0^{\bullet}(\partial f/\partial t)(s)\, ds\right) = 0$ then there exists [20] $b \in B$ such that $f - \int_0^{\bullet}(\partial f/\partial t)(s)\, ds = b$.

Then $f \in C(0, T; B)$ and $f(t+h) - f(t) = \int_t^{t+h}(\partial f/\partial t)(s)\, ds$, $0 \leqslant t \leqslant T - h$. So that $\tau_h f - f = hM_h(\partial f/\partial t)$ and (5.1) follows from Young's inequality (1.5). $\blacksquare$

Another possibility in view of verify the time criterions is to consider functions in a Sobolev space. For $0 < \sigma < 1$ and $1 \leqslant p \leqslant \infty$ denote

$$W^{\sigma,p}(0, T; B) = \left\{f\colon f \in L^p(0, T; B) \text{ and } \int_0^T\int_0^T \frac{\|f(t) - f(s)\|_B^p}{|t-s|^{\sigma p+1}}\, dt\, ds < \infty\right\}$$

with the usual change if $p = \infty$.

[19] $\mathcal{D}(]0, T[)$ is the space of real valued C^∞ functions with compact support in $]0, T[$.

[20] L. SCHWARTZ, [SC1] theorem 1, p. 51, proved that every real distribution whose derivative equals 0 is a constant. The proof extends without change to the vector valued distributions.

A semi-norm is defined by $\|f\|_{\dot W^{\sigma,p}} = \big(\int_0^T\int_0^T \ldots\, dt\, ds\big)^{1/p}$ and the norm is defined by $\|f\|_{W^{\sigma,p}} = (\|f\|^p_{L^p} + \|f\|^p_{\dot W^{\sigma,p}})^{1/p}$.

The translations are estimated by the

LEMMA 5. – Let $f \in W^{\sigma,r}(0, T; B)$, $0 < \sigma < 1$, $1 \leqslant r \leqslant \infty$ and let p be such that

$$p \leqslant \infty \text{ if } \sigma > \frac{1}{r}, \ p < \infty \text{ if } \sigma = \frac{1}{r}, \ p \leqslant r_* = \frac{r}{1-\sigma r} \text{ if } \sigma < \frac{1}{r}. \tag{5.2}$$

Then $f \in L^p(0, T; B)$ and there exists c independent of f such that, $\forall h > 0$,

$$\|\tau_h f - f\|_{L^p(0, T-h, B)} \leqslant \begin{cases} c h^{\sigma+1/p-1/r} \|f\|_{\dot W^{\sigma,r}(0,T;B)} & \text{if } r \leqslant p \leqslant \infty \\ c h^\sigma T^{1/p-1/r} \|f\|_{\dot W^{\sigma,r}(0,T;B)} & \text{if } 1 \leqslant r \leqslant p. \end{cases} \tag{5.3}$$ ∎

This lemma may be obtained by interpolation, for some $r \geqslant \boldsymbol{r}$ (21) from lemma 4, and for every r from the fractional Sobolev imbedding theorems of J. PEETRE [P] (22). A direct proof is given in [SI3, remark 8.5], with a known constant c.

The time criterions are also satisfied if

$$\|\tau_h f - f\|_{L^r(0, T; B)} \leqslant M h^\sigma, \quad \forall h > 0 \tag{5.4}$$

with $r \geqslant p$. But it is enough to verify this estimate for some $r < p$, with regard to

LEMMA 6. – Let $f \in L^r(0, T; B)$ satisfy (5.4) with $0 < \sigma \leqslant 1$, $1 \leqslant r \leqslant \infty$ and let p be such that

$$p \leqslant \infty \text{ if } \sigma > \frac{1}{r}, \ p < r_* = \frac{r}{1-\sigma r} \text{ if } \sigma \leqslant \frac{1}{r}. \tag{5.5}$$

Then $f \in L^p(0, T; B)$ and there exists c independent of f such that, $\forall h > 0$,

$$\|\tau_h f - f\|_{L^p(0, T-h; B)} \leqslant \begin{cases} c M h^{\sigma+1/p-1/r} & \text{if } r \leqslant p \leqslant \infty \\ M h^\sigma T^{1/p-1/r} & \text{if } 1 \leqslant p \leqslant r. \end{cases} \tag{5.6}$$ ∎

This result is proved in [SI3, remark 8.4]. The inequality for $p \leqslant r$ is obvious

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(21) One has $(L^p, \dot W^{1,\zeta})_{\sigma,r} = \dot W^{\sigma,r}$ if $1/r = (1-\sigma)/p + \sigma/\zeta$ then lemma 4 imply that $\|\tau_h f - f\|_{L^p} \leqslant h^{\sigma(1+1/p-1/\zeta)} |||f||| = h^{\sigma+1/p-1/r} |||f|||$, where $|||\ |||$ is the interpolation norm and $1 \leqslant \zeta \leqslant p$. This gives (5.3) for $\boldsymbol{r} \leqslant r \leqslant p$ where $1/\boldsymbol{r} = (1-\sigma)/p + \sigma$ and the case $r \geqslant p$ follows from the case $r = p$ by Hölder inequality (1.3).

(22) By [P] theorem 8.1 and 8.2, $W^{\sigma,r} \subset L^{r_*}$ if $\sigma < 1/r$ and $W^{\sigma,r} \subset C^{\sigma-1/r}$ if $\sigma < 1/r$. See equally [BL] theorem 6.5.1, p. 153. All results are proved for real valued functions.

by (1.3). This lemma and the previous one are peculiar cases of imbedding in the spaces $B_q^{\sigma,r}(0, T; B)$ [23].

Let us add an imbedding result in differently valued Sobolev spaces. For m integer and $0 < \sigma < 1$ denote

$$W^{m+\sigma,p}(0, T; B) = \left\{f\colon f, \frac{\partial f}{\partial t}, \dots, \frac{\partial^{m-1}}{\partial t^{m-1}} \in L^p(0, T; B) \text{ and } \frac{\partial^m f}{\partial t^m} \in W^{\sigma,p}(0, T; B)\right\}$$

$$W^{-m+\sigma,p}(0, T; B) = \\ = \left\{f\colon f = g_0 + \dots + \frac{\partial^m g_m}{\partial t^m} \text{ where } g_0, \dots, g_{m-1} \in L^p(0, T; B), g_m \in W^{\sigma,p}(0, T; B)\right\}.$$

The derivation being defined in the distribution sense all these spaces are included in $\mathcal{D}'(]0, T[; B)$; equipped with the standard norms they are Banach spaces.

Let $\dot{W}^{m+\sigma,p}(0, T; B)$, $m \geq 0$, denote the space $W^{m+\sigma,p}(0, T; B)$ equipped with the semi-norm $\|f\|_{\dot{W}^{m+\sigma,p}} = \|\partial^m f/\partial t^m\|_{\dot{W}^{\sigma,p}}$ $(\|\cdot\|_{W^{0,p}} = \|\cdot\|_{L^p})$.

Let B_0, B_1 be two Banach spaces; then the interpolation spaces [24] $(B_0, B_1)_{\theta,q}$ where $0 < \theta < 1$, $1 \leq q \leq \infty$ are intermediate spaces between B_0 and B_1.

LEMMA 7. – Let s_0, s_1 be not integer, $s_1 \geq 0$, $1 \leq r_0 \leq \infty$, $1 \leq r_1 \leq \infty$.
Let $0 < \theta < 1$, $s_\theta = (1-\theta)s_0 + \theta s_1$, $1/r_\theta = (1-\theta)/r_0 + \theta/r_1$. If s_0 is not an integer or if $r_\theta \leq 2$ there holds

$$W^{s_0,r_0}(0, T; B_0) \cap \dot{W}^{s_1,r_1}(0, T; B_1) \subset W^{s_\theta,r_\theta}(0, T; (B_0, B_1)_{\theta,r_\theta}). \quad \blacksquare$$

PROOF. – Let us notice first that $\dot{W}^{s_1,r_1}$ may be replaced by W^{s_1,r_1}. Indeed if f is bounded in $\dot{W}^{s_1,r_1}(0, T; B_1)$ there holds, see [P], $f = f_m + p_m$ where f_m is bounded in $W^{s_1,r_1}(0, T; B_1)$ and p_m is a polynom in t of degree m $(m \leq s_1 < m+1)$ with coefficients in B_1. If f is moreover bounded in $W^{s_0,r_0}(0, T; B)$ then p_m is bounded in $W^{s_0,r_0}(0, T; B_0) + W^{s_1,r_1}(0, T; B_1) \subset W^{\mathbf{s},\mathbf{r}}(0, T; B_1)$, $\mathbf{s} = \inf s_i$, $\mathbf{r} = \inf r_i$, whence its coefficients are bounded in B_1 and finally f is bounded in $W^{s_1,r_1}(0, T; B_1)$.

On other hand, see [G] (6.9), p. 179 if s_1 are not integers, one has $\big(W^{s_0,r_0}(0, T; B_0), W^{s_1,r_1}(0, T; B_1)\big)_{\theta,r_\theta} = B_{r_\theta}^{s_\theta,r_\theta}\big(0, T; (B_0, B_1)_{\theta,r_\theta}\big)$.

The lemma follows since $B_r^{s,r} = W^{s,r}$ is not an integer, and since $B_r^{s,r} \subset W^{s,r}$ if s is integer and $1 \leq r \leq 2$ from [G, (6.6), p. 178]. $\blacksquare$

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(23) $B_q^{\sigma,r} = \{f \in L^r \mid \int_0^\infty (h^{-\sigma}\|\tau_h f - f\|_{L^r})^q (dh/h) < \infty\}$ for $0 < \sigma < 1$, with usual change for $q = \infty$. Then $B_q^{\sigma,r} \subset B^{\sigma+1/p-1/r,p}$ $\forall p$ satisfying (5.5). For $q = \infty$ one obtains lemma 6 since $B_\infty^{\sigma,r} = \{f$ satisfying (5.4)$\}$, and lemma 5 follows since $W^{\sigma,r} = B_r^{\sigma,r} \subset B_\infty^{\sigma,r}$. These results are proved in [SI3]; they may also be obtained by interpolation from the fractional Sobolev imbedding theorems of [P].

(24) For the definitions see for example [G] whose notations are used here.

REMARK 5.1. – In particular if s_0, s_1 are not integer, $s_0 < 0 < s_1$ and $\theta = -s_0/(s_1 - s_0)$, there holds

$$W^{s_0,1}(0, T; B_0) \cap \dot{W}^{s_1,1}(0, T; B_1) \subset L^1(0, T; (B_0, B_1)_{\theta,1}) . \tag{5.7}$$

Since $W^{s,1}$ increases as s decreases, this imbedding holds for every $s_0 < 0 < s_1$ if $-s_0/(s_1 - s_0) < \theta < 1$. ■

REMARK 5.2. – Denote $\dot{W}_0^{s,p}(0, T; B)$ the closure of $\mathcal{D}(]0, T[; B)$ in $W^{s,p}(0, T; B)$. If B is a reflexive space and if $1 < p < \infty$, $s > 0$ then $W^{-s,p}(0, T; B)$ is the dual space of $\dot{W}_0^{s,p'}(0, T; B')$ where $1/p + 1/p' = 1$, $B' =$ dual space of B, and the norm of $W^{-s,p}$ is equivalent to the dual norm. ■

REMARK 5.3. – The fractional spaces may be characterized by translations since

$$\|f\|_{\dot{W}^{\sigma,r}} = \left(2\int_0^T (h^{-\sigma}\|\tau_h f - f\|_{L^r(0,T-h;B)})^r \frac{dh}{n}\right)^{1/r} \qquad \text{for } 0 < \sigma < 1,\ r < \infty$$

with the usual modification if $r = \infty$. ■

6. – Compactness for functions with values in a compact space X.

Let us now consider another space X, so that

$$X \subset B \text{ with compact imbedding } (X \text{ and } B \text{ are Banach spaces}) . \tag{6.1}$$

The characterization of theorem 1 gives the

THEOREM 3. – Assume (6.1), $F \subset L^p(0, T; B)$ where $1 \leq p \leq \infty$, and

$$F \text{ is bounded in } L^1_{\text{loc}}(0, T; X) , \tag{6.2}$$

$$\|\tau_h f - f\|_{L^p(0,T-h;B)} \to 0 \text{ as } h \to 0, \text{ uniformly for } f \in F . \tag{6.3}$$

Then F is relatively compact in $L^p(0, T; B)$ (and in $C(0, T; B)$ if $p = \infty$). ■

PROOF. – For every $0 < t_1 < t_2 < T$, f is bounded in $L^1(t_1, t_2; X)$. Then $\int_{t_1}^{t_2} f(t)\,dt$ is bounded in X whence it is relatively compact in B, and one uses theorem 1. ■

This result, excepted the localization in (6.2), is announced in [SI2], remark 3.2, and is partially proved in [T2], theorem 13.3.

REMARK 6.1. – The space criterion (6.2) may be replaced by the weaker one: $\forall 0 < t_1 < t_2 < T$, $\int_{t_1}^{t_2} f(t)\, dt$ is bounded in X. ■

REMARK 6.2. – It is not necessary to assume $F \subset L^p(0, T; B)$ in theorem 3. With regard to remark 3.3 it is enough to assume that $\tau_h f - f \in L^p(0, T-h; B)$, $\forall f \in F$.

OPTIMALITY OF THEOREM 3. – The order p is the best one for compactness since it is the best one for boundedness. This is obvious if F is reduced to a single function f such $f(t) = b\varphi(t)$, $b \in X$. ■

The space criterion is obviously verified if F is bounded in $L^1(0, T; X)$, and one may examine whether the localization of this criterion is a useless complication or not. In boundary value problems some estimates, and peculiarly the estimates in the « more regular » spaces, don't hold up to the boundary of the interval $[0, T]$ thus it is usefull to search for local criterions.

REMARK 6.3. – More generally a finite number of singular points $T_0 < T_1 < \dots < T_N$ may be introduced: (6.2) may be replaced by F is bounded in $L^1_{loc}([0, T] \setminus \{T_0, T_1, \dots, \dots, T_N\}; X)$.

Indeed by theorem 3 F is relatively compact in $L^p(T_n, T_{n+1}; B)$, $\forall n$, then in $L^p(0, T; B)$; this extends to $C(0, T; B)$ if $p = \infty$. ■

For differentiable functions it follows:

COROLLARY 1. – Assume (6.1), and let m be any integer.

Let F be bounded in $W^{-m,1}_{loc}(0, T; X)$ and $\partial F/\partial t = \{\partial f/\partial t : f \in F\}$ be bounded in $L^1(0, T; B)$. Then F is relatively compact in $L^p(0, T; B)$, $\forall p < \infty$.

Let F be bounded in $W^{-m,1}_{loc}(0, T; X)$ and $\partial F/\partial t$ be bounded in $L^r(0, T; B)$ where $r > 1$. Then F is relatively compact in $C(0, T; B)$. ■

Remind that F is bounded in $W^{-m,1}_{loc}(0, T; X)$ if $F = \partial^m G/\partial t^m$ where G is bounded in $L^1_{loc}(0, T; X)$. If X is a reflexive space it is particularly satisfied, see remark 5.2, if there exists $\mu > 1$ [25] such that

$$\left| \int_0^T f(t)\varphi(t)\, dt \right| \leq c \left\| \frac{\partial^m \varphi}{\partial t^m} \right\|_{L^\mu(0, T; X')}, \quad \forall \varphi \in \mathcal{D}(]0, T[; X') . \tag{6.4}$$

PROOF OF COROLLARY 1. – The time criterion (6.3) is satisfied by lemma 4. The space criterion (6.2) is satisfied if $m \leq 0$. If $m > 0$ by lemma 7, see remark 5.1, F is bounded in $L^1_{loc}(0, T; \boldsymbol{X})$ where $\boldsymbol{X} = (X, B)_{\theta,1}$, $m/(m+1) < \theta < 1$. The imbedding $\boldsymbol{X} \to B$ is compact by [LP, theorem 2.1, p. 36] [26], then the space criterion is satisfied with $\boldsymbol{X}$. ■

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[25] If $\mu = 1$, F is obviously bounded in $W^{-m-1,1}(0, T_n X)$.

[26] Since $\boldsymbol{X}$ is of class $K_\theta(X, B)$ by [LP] proposition 1.1, p. 27.

OPTIMALITY OF COROLLARY 1. – The restriction $p < \infty$ is necessary since there exists bounded sets F in $L^1(0, T; X)$ with $\partial F/\partial t$ bounded in $L^1(0, T; B)$, which are not relatively compact in $L^\infty(0, T; B)$: see proposition 3 in section 11. ■

REMARK 6.4. – One may avoid the use of intermediate space $\boldsymbol{X}$ as follows: by theorem 2, F is relatively compact in $W^{-m,1}(0, T; B)$, and since F is bounded in $W^{1,1}(0, T; B)$ lemma 10 shows that it is relatively compact in $W^{\sigma,1}(0, T; B)$, $\forall \sigma < 1$ and the corollary 1 follows by the fractional Sobolev imbedding theorem. ■

More generally in the fractional or not Sobolev spaces there hold:

COROLLARY 2. – Assume (6.1). Let F be bounded in $W^{s_0,r_0}_{\mathrm{loc}}(0, T; X) \cap \dot{W}^{s,r}(0, T; B)$ where $s > 0$, $1 \leqslant r \leqslant \infty$ and s_0 is real, $1 \leqslant r_0 \leqslant \infty$.

If $s \leqslant 1/r$ then F is relatively compact in $L^p(0, T; B)$, $\forall p < r_* = r/(1 - sr)$.

If $s > 1/r$ then F is relatively compact in $C(0, T; B)$. ■

PROOF OF COROLLARY 2. – For $s \geqslant 1$ it follows from corollary 1. For $s < 1$ it follows from theorem 3: the time criterion (6.3) is satisfied by lemma 5, and the space criterion (6.2) is satisfied if $s_0 \geqslant 0$, and if $s_0 < 0$ it is satisfied with $\boldsymbol{X} = (X, B)_{\theta,1}$, $-s_0(s_1 - s_0) < \theta < 1$, by lemma 7 (see remark 5.1). The imbedding $\boldsymbol{X} \to B$ is compact as in the proof of corollary 1. ■

Let us notice that one may obviously conclude with an estimate in a Sobolev space on X:

$$
(6.5) \qquad \left\{
\begin{array}{l}
\text{If } 0 < s \leqslant \dfrac{1}{r},\ W^{s,r}(0, T; X) \subset L^p(0, T; B) \text{ with compact imbedding} \\[2ex]
\forall p < r_* = \dfrac{r}{1 - sr}. \\[2ex]
\text{If } s > \dfrac{1}{r},\ W^{s,r}(0, T; X) \subset C(0, T; B) \text{ with compact imbedding.}
\end{array}
\right.
$$

OPTIMALITY OF COROLLARY 2. – The restriction $p < r_*$ when $s \leqslant 1/r$ is necessary in corollary 2 if $s_0 - 1/r_0 \leqslant s - 1/r$, and in (6.5), since there exists bounded sets in $W^{s,r}(0, T; X)$ which are not relatively compact in $L^{r_*}(0, T; B)$: see (11.4).

The restriction $s > 0$ is necessary if $s_0 \leqslant 0$, whatever be r and p, since there exists bounded sets in $L^\infty(0, T; X)$ which are not relatively compact in $L^1(0, T; B)$: see proposition 2. ■

REMARK 6.5. – With the hypothesis of corollary 2, F is bounded in $L^{r_*}(0, T; B)$ by lemma 5. Then the closure $\overline{F}$ of F in $L^p(0, T; B)$ is bounded in $L^{r_*}(0, T; B)$ by remark 4.2. ■

REMARK 6.6. – *Compactness for the limit coefficients.* F is relatively compact in $L^{r_*}(0, T; B)$ if F is bounded in $W^{s_0,r_0}_{\mathrm{loc}}(0, T; X)$ and if $\|\tau_h f - f\|_{\dot{W}^{s,r}(0, T-h; B)} \to 0$ as $h \to 0$ uniformly for $f \in F$. The proof is similar to the corollary 2 proof.

In the differentiable case F is relatively compact in $C(0, T; B)$ if F is bounded in $W_{\mathrm{loc}}^{-m,1}(0, T; X)$ and if $\|\tau_h(\partial f/\partial t) - \partial f/\partial t\|_{L^1(0, T-h; B)} \to 0$ as $h \to 0$ uniformly for $f \in F$.

Notice that the assumption on $\partial f/\partial t$ way be replaced by $\|(\partial f/\partial t)(t)\|_B \leqslant g(t)$, $\forall t$, where $g \in L^1(0, T)$ is independent of f. ■

7. – Partial compactness for functions with values in a compact space X.

If F is bounded in $L^q(0, T; B)$, the compactness in $L^p(0, T; B)$ for every $p < q$ may be obtained with weaker hypothesis than in the preceding section. Let us consider again

(7.1) $\quad X \subset B$ with compact imbedding (X and B are Banach spaces).

The characterization of theorem 2 gives the

THEOREM 4. – Assume (7.1), $1 < q \leqslant \infty$, and

(7.2) $\quad F$ is bounded in $L^q(0, T; B) \cap L^1_{\mathrm{loc}}(0, T; X)$

(7.3) $\quad \forall 0 < t_1 < t_2 < T$, $\|\tau_h f - f\|_{L^1(t_1, t_2; B)} \to 0$ as $h \to 0$, uniformly for $f \in F$.

Then F is relatively compact in $L^p(0, T; B)$, $\forall p < q$. ■

In comparison with theorem 3 the time criterion which was stated in L^p is replaced here by the similar one (7.3) in L^1_{loc}; it will be weakened again in section 9 (by setting it in any Banach space whatever).

OPTIMALITY OF THEOREM 4. – The strict inequality $p < q$ is necessary from proposition 1 in section 11. ■

For functions bounded in Sobolev spaces it follows:

COROLLARY 3. – Assume (7.1). Let F be bounded in $L^q(0, T; B) \cap W_{\mathrm{loc}}^{-m,1}(0, T; X) \cap$ $\cap\, W_{\mathrm{loc}}^{s,1}(0, T; B)$ where $1 < q \leqslant \infty$, m is integer, $s > 0$. Then F is relatively compact in $L^p(0, T; B)$, $\forall p < q$. ■

PROOF. – The time criterion (7.3) is satisfied by lemma 5. The space criterion (7.2) is satisfied if $s_0 \geqslant 0$; if $s_0 < 0$ F is bounded in $L^1_{\mathrm{loc}}(0, T, \mathbf{X})$ where $\mathbf{X} = (X, B)_{\theta,1}$, $-s_0/(s_1 - s_0) < \theta < 1$, by lemma 7 (see remark 5.1) then the space criterion is satisfied with $\mathbf{X}$ [27]. ■

[27] The imbedding $\mathbf{X} \to B$ is relatively compact as in the proof of corollary 1.

It is possible to avoid the intermediate space $\boldsymbol{X}$, see remark 6.4. Remark that one finds again the corollaries 1 and 2 for $p < \infty$ from corollary 3 and Sobolev imbedding theorem.

OPTIMALITY OF COROLLARY 3. – The restriction $p < q$ is necessary if $s \leqslant 1 - 1/q$: see proposition 4. The restriction $s > 0$ is necessary by proposition 2. ■

REMARK 7.1. – With the hypothesis of theorem 4 or of corollary 3, the closure $\bar{F}$ of F in $L^p(0, T; B)$ is bounded in $L^q(0, T; B)$. It follows from remark 4.2. ■

8. – Compactness for functions with values in an intermediate space.

By using a method due to J. L. LIONS, the time criterion which was stated on translations with values in B may be replaced by the similar one with values in any space Y whatever if the space criterion is strengthened.

So let us consider

(8.1) $\quad X \subset B \subset Y$ with compact imbedding $X \to B$ (X, B and Y are Banach spaces)

The main result of this section is:

THEOREM 5. – Assume (8.1), $1 \leqslant p \leqslant \infty$ and

(8.2) $\quad F$ is bounded in $L^p(0, T; X)$,

(8.3) $\quad \|\tau_h f - f\|_{L^p(0, T-h; Y)} \to 0$ as $h \to 0$, uniformly for $f \in F$.

Then F is relatively compact in $L^p(0, T; B)$ (and in $C(0, T; B)$ if $p = \infty$). ■

In comparison with theorem 3 the time criterion (8.3) is stated with values in Y instead of B, whereas the space criterion (8.2) is stated in L^p instead of L^1_{loc}: Thus for $Y = B$ we don't find again theorem 3; in this case it is better to go back to section 6.

A proof of theorem 5 is given in [SI1], lemma, for reflexive spaces in the case $1 < p < \infty$. The present proof relies on the following estimate of J. L. LIONS [28].

LEMMA 8. – Assume (8.1). Then

(8.4) $\quad \forall \eta > 0, \ \exists N \quad$ such that: $\quad \forall v \in X, \quad \|v\|_B \leqslant \eta \|v\|_X + N\|v\|_Y$. ■

PROOF. – Denote $V_n = \{v \in B: \|v\|_B < \eta + n\|v\|_Y\}$. The sets V_n are open in B, they increase with n and their union covers B. The unit sphere S of X being rela-

[28] See [L1] p. 59 or [L2] lemma 5.1, p. 58.

tively compact in B, there exists a finite N such that $S \subset V_N$. Which yields:

$$\|v\|_B < \eta\|v\|_X + N\|v\|_Y\,, \quad \forall v \in X \quad \text{such that} \quad \|v\|_X = 1\,.$$

The inequality for every $v \in X$ follows by multiplication by any positive number. ■

REMARK 8.1. – If the imbedding $X \to Y$ is compact, then (8.4) characterizes the intermediate spaces B such that the imbedding $X \to B$ is compact. Indeed it is easy to see that (8.4) imply the compactness of the imbedding $X \to B$. ■

In the evolution spaces it follows

LEMMA 9. – Assume (8.1). Let F be bounded in $L^p(0, T; X)$ and be relatively compact in $L^p(0, T; Y)$, where $1 \leqslant p \leqslant \infty$. Then F is relatively compact in $L^p(0, T; B)$. ■

PROOF. – Given $\varepsilon > 0$ there exists a finite subset $\{f_i\}$ of F such that: $\forall f \in F$, $\exists f_i$ such that $\|f - f_i\|_{L^p(0,T;Y)} \leqslant \varepsilon$. The inequality (8.4) implies

$$\|f - f_i\|_{L^p(0,T;B)} \leqslant \eta\|f - f_i\|_{L^p(0,T;X)} + N\|f - f_i\|_{L^p(0,T;Y)} \leqslant \eta c + N\varepsilon$$

where c is the diameter of F in $L^p(0, T; X)$. Given $\varepsilon' > 0$, for $\eta = \varepsilon'/2c$ and $\varepsilon = \varepsilon'/2N$ it yields $\|f - f_i\|_{L^p(0,T;B)} \leqslant \varepsilon'$ which proves that F is relatively compact in $L^p(0, T; B)$. ■

PROOF OF THEOREM 5. – By theorem 1 F is relatively compact in $L^p(0, T; Y)$ and one concludes with lemma 9. ■

OPTIMALITY OF THEOREM 5. – The order p is the best one for compactness since it is the best one for boundedness. It is obvious if F is reduced to a single function f such that $f(t) = b\varphi(t)$, $b \in X$. ■

Let us now give some applications. For the differentiable functions, verifying the time criterion (8.3) by lemma 4, the Aubin's result is extended ([29]) by:

COROLLARY 4. – Assume (8.1).

Let F be bounded in $L^p(0, T; X)$ where $1 \leqslant p < \infty$, and $\partial F/\partial t = \{\partial f/\partial f\colon f \in F\}$ be bounded in $L^1(0, T; Y)$. Then F is relatively compact in $L^p(0, T; B)$.

Let F be bounded in $L^\infty(0, T; X)$ and $\partial F/\partial t$ be bounded in $L^r(0, T; Y)$ where $r > 1$. Then F is relatively compact in $C(0, T; B)$. ■

([29]) It was proved in the case $r < 1$, $1 < p < \infty$, see [AU] and [L2] theorem 5.1, p. 68 and theorem 12.1, p. 141.

OPTIMALITY OF COROLLARY 4. – The order p is the best one for compactness for some spaces B satisfying (8.1): see (11.8) and (11.9). On other hand with extra hypothesis on B it is possible to prove compactness for some $q > p$: see corollary 8.

The restriction $p < \infty$ when $r = 1$ is necessary whatever be the space B: see proposition 3. ■

More generally in the Sobolev spaces, verifying the time criterion (8.3) by lemma 5 for fractional orders, one has:

COROLLARY 5. – Assume (8.1) and $1 \leqslant p \leqslant \infty$, $1 \leqslant r \leqslant \infty$. Let F be bounded in $L^p(0, T; X) \cap W^{s,r}(0, T; Y)$ where $s > 0$ if $r \geqslant p$ and where $s > 1/r - 1/p$ if $r \leqslant p$. Then F is relatively compact in $L^p(0, T; B)$ (and in $C(0, T; B)$ if $p = \infty$). ■

For $r = p = 2$ and reflexive spaces one find again the theorem 5.2, p. 61 of [L2].

OPTIMALITY OF COROLLARY 5. – The order p is the best one for compactness for some spaces B as in corollary 4: see (11.8) and (11.9).

The restriction $s > 0$ is necessary whatever be the space B from proposition 2, and the restriction $s > 1/r - 1/p$ when $r \leqslant p$ is necessary from proposition 4. ■

REMARK 8.2. – Some results may be extended to the case where X is not a vector space. Indeed lemma 8 still holds if X is a cone and if $\| \; \|_X$ is replaced a non-negative homogeneous function such that $\{x \in X: \varphi(x) \leqslant 1\}$ is relatively compact in B. This idea is due to J. A. DUBINSKII, see [D] and [L2] section 12, p. 140 where it is applied to a non linear problem.

Remark in particular that it is not necessary that the spaces X and Y be complete. ■

9. – Partial compactness for functions with values in an intermediate space.

If F is bounded in $L^q(0, T; B)$ the compactness in $L^p(0, T; B)$ for every $p < q$ is obtained with weaker hypothesis than in the preceeding section. The boundedness in $L^q(0, T; B)$ may result from estimates in X and Y as it was supposed by J. P. AUBIN [AU], as well as it may be independent (30).

Let us consider again

(9.1) $\quad X \subset B \subset Y$ with compact imbedding $X \to B$ (X, B and Y are Banach spaces).

THEOREM 6. – Assume (9.1), $1 < q \leqslant \infty$ and

(9.2) $\quad F$ is bounded in $L^q(0, T; B) \cap L^1_{\mathrm{loc}}(0, T; X)$,

(9.3) $\quad \forall 0 < t_1 < t_2 < T$, $\|\tau_h f - f\|_{L^1(t_1, t_2; Y)} \to 0$ as $h \to 0$, uniformly for $f \in F$.

(30) For a boundary value problem it may be an « estimate » of the solutions.

Then F is relatively compact in $L^p(0, T; B)$, $\forall p < q$. ■

PROOF. – By theorem 1 F is relatively compact in $L^1_{loc}(0, T; Y)$, then it is relatively compact in $L^1_{loc}(0, T; B)$ by lemma 9, and one conclude with lemma 3. ■

OPTIMALITY IN THEOREM 6. – The restriction $p < q$ is necessary from proposition 1 in section 11. ■

For the differentiable functions it follows, with lemma 4, the

COROLLARY 6. – Assume (9.1) and $1 < q \leqslant \infty$. Let F be bounded in $L^q(0, T; B) \cap L^1_{loc}(0, T; X)$ and $\partial F/\partial t$ be bounded in $L^1_{loc}(0, T; Y)$. Then F is relatively compact in $L^p(0, T; B)$, $\forall p < q$. ■

OPTIMALITY IN COROLLARY 6. – There exists some spaces X, B and Y such that the restriction $p < q$ is necessary: see the proposition 5 with $s_0 = 0$, $s_1 = 1$, $r_0 = r_1 = 1$, $\theta = 1 - 1/q$ then $s_\theta = 1 - 1/q$, $r_\theta = 1$ and $s = 0$, $r = q$. ■

More generally in Sobolev spaces the time criterion (9.3) is obtained for fractional orders by lemma 5 and one has

COROLLARY 7. – Assume (9.1). Let F be bounded in $L^1_{loc}(0, T; X) \cap L^q(0, T; B) \cap W^{s,1}_{loc}(0, T; Y)$ where $1 < q \leqslant \infty$, $s > 0$. Then F is relatively compact in $L^p(0, T; B)$, $\forall p < q$. ■

REMARK 9.1. – With the hypothesis of theorem 6 or of corollary 6 or 7, the closure $\bar{F}$ of F in $L^p(0, T; B)$ is bounded in $L^q(0, T; B)$. It follows from remark 4.2. ■

OPTIMALITY IN COROLLARY 7. – The restriction $p < q$ is necessary (when $s \leqslant 1/q$) whatever be the space B, see proposition 4. When $s > 1/q$ it is necessary for some spaces, see proposition 5 with $\theta = (1/s)(1 - 1/q)$. ■

10. – The case of intermediate spaces of class θ.

Consider a set F satisfying the space and time compactness criterions (8.2) and (8.3) with two different coefficients, say p_0 and p_1.

1) If $p_0 > p_1$ the partial compactness result of theorem 6 gives that F is relatively compact in $L^p(0, T; B)$, $\forall p < p_0$ and this cannot be improved.

2) If $p_0 < p_1$ theorem 5 gives the relative compactness for $p = p_0$. The point now is to improve this result when B is a space of class θ (with respect to X and Y), that is to say if there exists θ and M such that

$$(10.1) \qquad \|v\|_B \leqslant M \|v\|_X^{1-\theta} \|v\|_Y^{\theta}, \quad \forall v \in X \cap Y, \quad \text{where } 0 < \theta < 1 .$$

This definition has been brought by J. L. LIONS and J. PEETRE, [LP] definition 1.1, p. 27 (31), where many properties are given. These spaces are easier to use and are more general than interpolation ones. Indeed every interpolation space $(X, Y)_{\theta,q}$ satisfies (10.1), but B is not necessary an interpolation space. Precisely if $X \subset Y$, (10.1) is equivalent to $B \subset (X, Y)_{\theta,1}$, by [LP] proposition 1.1, p. 27.

As in the preceding sections let us assume that B is intermediate between X and Y:

(10.2) $\quad X \subset B \subset Y \quad$ with compact imbedding $\quad X \to Y$

(X, B and Y being Banach spaces).

A basic result in this section is

THEOREM 7. – Assume (10.1), (10.2), $1 \leqslant p_i \leqslant \infty$ and

(10.3) $\qquad F$ is bounded in $L^{p_0}(0, T; X)$,

(10.4) $\qquad \|\tau_h f - f\|_{L^{p_1}(0,T-h;\,Y)} \to 0$ as $h \to 0$, uniformly for $f \in F$.

Then F is relatively compact in $L^{p_\theta}(0, T; B)$ where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$. ■

The ordre p_θ is intermediate between p_0 and p_1, then this result is fruitful only when $p_1 > p_0$: see the optimality in the following. Remark that in this case p_θ increases with θ then the largest (32) θ satisfying (10.1) is required.

The proof of theorem 7 lies on the following compactness lemma, which is due to J. L. LIONS and J. PEETRE (33).

LEMMA 10. – Assume (10.1). Let K be bounded in X and relatively compact in Y. Then K is relatively compact in B. ■

PROOF. – $\forall \varepsilon > 0$ there exists a finite subset $\{v_i\}$ of K such that: $\forall v \in K$, $\exists v_i$ such that $\|v - v_i\|_Y \leqslant \varepsilon$. Then $\|v - v_i\|_B \leqslant M c^{1-\theta} \varepsilon^\theta$ where c is the diameter of K in X, which implies the relative compactness in B. ■

For evolution spaces it follows

LEMMA 11. – Assume (10.1). Let F be bounded in $L^{p_0}(0, T; X)$ and be relatively compact in $L^{p_1}(0, T; Y)$ where $1 \leqslant p_i \leqslant \infty$. Then F is relatively compact in $L^{p_\theta}(0, T; B)$ where $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$. ■

---

(31) With their words it means that $B \in \boldsymbol{K}_\theta(X, Y)$.

(32) If $X \subset Y$ and B is of class θ, then it is of class θ' for every $\theta' \geqslant \theta$.

(33) [LP] theorem 2.3, p. 38.

PROOF. – By the Riesz inequality there holds: $\forall f \in L^{p_0}(0, T; X) \cap L^{p_1}(0, T; Y)$ then $f \in L^{p_\theta}(0, T; B)$ and [34]

$$(10.5) \qquad \|f\|_{L^{p_\theta}(0,T;B)} \leqslant M \|f\|^{1-\theta}_{L^{p_0}(0,T;X)} \|f\|^{\theta}_{L^{p_1}(0,T;Y)} \qquad \text{where} \quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Then one uses lemma 10 with $K = F$. ■

PROOF OF THEOREM 7. – By theorem 1 F is relatively compact in $L^{p_1}(0, T; X)$ and one concludes with lemma 11. ■

REMARK 10.1. – The hypothesis on spaces, (10.1) and (10.2), imply that the imbedding $X \to B$ is compact: it follows from lemma 10 with K = unit ball of X. ■

Remark that the hypothesis on F, (10.3) and (10.4), imply by the Riesz inequality that F is bounded in $L^{p_\theta}(0, T; B)$. Then the partial compactness result of theorem 6 gives again the compactness for every $p < p_\theta$ but it don't allow to get $p = p_\theta$.

OPTIMALITY OF THEOREM 7. – The theorem 7 is not optimal if $p_1 \leqslant p_0$. Indeed if $p_1 < p_0$, the partial compactness result of theorem 6 gives a larger order ($\forall p < p_0$ which is better than $p = p_\theta$); if $p_1 = p_0$ the theorem 5 gives the same order ($p_0 = p_\theta$ which is optimal) for more general spaces B.

At the contrary if $p_1 > p_0$ then p_θ is the best order of compactness as of boundedness for some spaces B: see proposition 6 with $s_0 = s_1 = s_\theta = 0$. It may be proved, whatever be the space B, that the best order is $p_{\theta(B)}$ where $\theta(B)$ is the largest θ satisfying (10.1). ■

For the functions whose derivatives are integrables, the time criterion (10.4) is satisfied for every $p_1 < \infty$, then theorem 7 gives:

$$(10.6) \qquad \left\{ \begin{array}{l} \text{Let } F \text{ be bounded in } L^{p_0}(0, T; X) \text{ and } \dfrac{\partial F}{\partial t} \text{ be bounded in } L^1(0, T; Y). \\ \text{Then } F \text{ is relatively compact in } L^p(0, T; B), \qquad \forall p < \dfrac{p_0}{1-\theta}. \end{array} \right.$$

This improves the corollary 4 in the case $r = 1$. Now if the derivatives are r-integrables one can choose $p_1 = \infty$ in theorem 7, then the compactness hold for $p = p_0/(1-\theta)$. The theorem 7 cannot give a better result, but the result may be improved in the following way.

---

[34] This inequality is obtained by using the Young real inequality $ab \leqslant a^q/q + b^{q'}/q'$ (where $1/q + 1/q' = 1$) for $a = (\|f(t)\|_X / \|f\|_{L^p(X)})^{p(1-\theta)}$, $b = (\|f(t)\|_Y / \|f\|_{L^{p_1}(Y)})^{p\theta}$ and $q = p_0/p(1-\theta)$, $q' = p_1/p\theta$, $p = p_\theta$ and by integrating in t.

COROLLARY 8. – Assume (10.1), (10.2) and $1 \leqslant p_0 \leqslant \infty$, $1 \leqslant r_1 \leqslant \infty$. Let F be bounded in $L^{p_0}(0, T; X)$ and $\partial F/\partial t$ be bounded in $L^{r_1}(0, T; Y)$.

If $\theta(1 - 1/r_1) \leqslant (1 - \theta)/p_0$ then F is relatively compact in $L^p(0, T; B)$, $\forall p < p_*$ where $1/p_* = (1 - \theta)/p_0 - \theta(1 - 1/r_1)$.

If $\theta(1 - 1/r_1) > (1 - \theta)/p_0$ then F is relatively compact in $C(0, T; B)$. ■

If $r_1 = 1$ one finds again (10.6).

PROOF OF COROLLARY 8. – By lemma 4 there holds $\|\tau_h f - f\|_{L^{r_1}(0, T-h; Y)} \leqslant c_1 h$, $\forall f \in F$. Then the Riesz inequality (10.5) gives

$$\|\tau_h f - f\|_{L^{r_\theta}(0, T-h; B)} \leqslant M \|\tau_h f - f\|^{1-\theta}_{L^{p_0}(0, T-h, X)} \|\tau_h f - f\|^{\theta}_{L^{r_1}(0, T-h; Y)} \leqslant M c_0^{1-\theta} c_1^{\theta} h^{\theta}$$

where $1/r_\theta = (1 - \theta)/p_0 + \theta/r_1$ and c_0 is the diameter of F in $L^{p_0}(0, T; X)$.

One concludes with theorem 3, which time criterion (6.3) is verified by lemma 6. ■

More generally it is possible to use assumptions in fractional or not Sobolev spaces. Denote

$$(10.7) \quad \begin{cases} W = W^{s_0, r_0}(0, T; X) \cap W^{s_1, r_1}(0, T; Y) \text{ where } s_i \text{ are reals,}, \quad 1 \leqslant r_i \leqslant \infty \\ s_\theta = (1 - \theta)s_0 + \theta s_1, \quad \dfrac{1}{r_\theta} = \dfrac{1 - \theta}{r_0} + \dfrac{\theta}{r_1} \quad \text{and} \quad s_* = s_\theta - \dfrac{1}{r_\theta}. \end{cases}$$

Then the corollary 8 may be generalized by:

COROLLARY 9. – Assume (10.1), (10.2), (10.7) and $s_\theta > 0$. Let F be bounded in W.

If $s_* \leqslant 0$ then F is relatively compact in $L^p(0, T; B)$, $\forall p < p_* = -1/s_*$.

If $s_* > 0$ then F is relatively compact in $C(0, T; B)$. ■

Remark that the order p_* is intermediate between r_θ and ∞ and that in the fruitful case (see the optimality in the following) where $s_0 - 1/r_0 < s_1 - 1/r_1$, p_* increases as θ does.

PROOF OF COROLLARY 9. – Assume first that s_0, s_1 are not integer and let $a < \theta$ be such that $s_a > 0$, $s_a - 1/r_a > -1/p$ ($p = \infty$ if $s_* > 0$) and such that s_a is not integer. By lemma 7 one has $W \subset W^{s_a, r_a}(0, T; X_a)$ where $X_a = (X, Y)_{a, r_a}$. Since $a < \theta$ it follows from [LP] [35] that $X_a \subset B$ with compact imbedding, and one concludes with (6.5).

If s_0 or s_1 is integer let $s_i' < s_i$ be not integer and such that $s_* > s_*' > -1/p$. Then $W \subset W'$ and one concludes by the non integer case. ■

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[35] The section 1.3, p. 28 of [LP] gives that X_a is of class $K_a(X, Y)$ and $(X, Y)_{\theta,1}$ is of class $K_\theta(X, Y)$. By theorem 2.3, p. 38 the imbedding $X_a \to (X, Y)_{\theta,1}$ is compact and $(X, Y)_{\theta,1} \subset B$ with continuous imbedding by proposition 1.1, p. 27.

The corollary 8 might have been proved so. However the proof which was given has the advantage to avoid the intermediate imbedding result of lemma 7 which full proof is not easy.

Let us remark that, by choosing $s_0 = 0$ and $s_1 = 1$ in corollary 9, one find again the case of differentiable functions which was treated in corollary 8. By choosing $s_0 = 0$ and $s_1 = m$ one treats the case of m times differentiable functions. More generally results on the intermediate derivatives are obtained by applying the corollary 9 to the set $\partial^j F/\partial t^j = \{\partial^j f/\partial t^j : f \in F\}$ with $s_0 = -j$ and $s_1 = m - j$. It yields

COROLLARY 10. – Assume (10.1), (10.2) and $1 \leqslant p_0 \leqslant \infty$, $1 \leqslant r_1 \leqslant \infty$. Let F be bounded in $L^{p_0}(0, T; X)$ and $\partial^m F/\partial t^m$ be bounded in $L^{r_1}(0, T; Y)$, and let $j < \theta m$ and $1/r_\theta = (1-\theta)/p_0 + \theta/r_1$.

If $j \geqslant \theta m - 1/r_\theta$ then $\partial^j F/\partial t^j$ is relatively compact in $L^p(0, T; B)$, $\forall p < p_*$ where $1/p_* = 1/r_\theta - (\theta m - j)$.

If $j < \theta m - 1/r_\theta$ then $\partial^j F/\partial t^j$ is relatively compact in $C(0, T; B)$. ■

OPTIMALITY OF COROLLARIES 8, 9 AND 10. – The restrictions $p < p_*$ are necessary, for some spaces B, from proposition 5. ■

REMARK 10.1. – *The limit cases.* If $\theta = 1$ one has $B = Y$; all the proofs of this section still holds and one finds again almost all the results of section 6 [36].

The assumption that B is of class $\theta = 0$ is satisfied as soon as $X \subset B \subset Y$. But the lemma 10 and 11 are no more valid, then for this case one has to go back to section 8. ■

A consequent class of spaces in boundary value problems is the one of Sobolev spaces built on an open set Ω of R^n. Let us suppose that

$$(10.8)\qquad \begin{cases} \Omega \text{ is open, bounded and it satisfies the cone property } ^{(37)} \text{ in } R^n, \\ X = W^{\alpha_0,\zeta_0}(\Omega)\,, \quad B = W^{\alpha,\zeta}(\Omega) \quad \text{and} \quad Y = W^{\alpha_1,\zeta_1}(\Omega)\,. \end{cases}$$

Then B is intermediate in the sense (10.2) if and only if

$$(10.9)\qquad \alpha_0 > \alpha > \alpha_1 \quad \text{and} \quad \beta_0 > \beta > \beta_1 \quad \text{where} \quad \beta = \alpha - \frac{n}{\zeta}\,, \quad \beta_i = \alpha_i - \frac{n}{\zeta_i}\,.$$

The point now is to find the best, that is the largest, θ. It is given by the

LEMMA 12. – Assume (10.8) and (10.9). Then (10.1) holds $\forall \theta < u$ where

$$(10.10)\qquad u = \inf\left\{\frac{\alpha_0 - \alpha}{\alpha_0 - \alpha_1}\,;\,\frac{\beta_0 - \beta}{\beta_0 - \beta_1}\right\}. \quad ■$$

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(36) The theorem 7 gives again the theorem 3 except the local type in the space criterion (6.2), and the corollary 8 gives again the corollary 2 with the same restriction.

(37) See for example [AD] section 4.3.

PROOF. – By the proposition 1.1 of [LP], (10.1) is equivalent to $(W^{\alpha_0,\zeta_0}, W^{\alpha_1,\zeta_1})_{\theta,1} \subset W^{\alpha,\zeta}$. Now if $\theta < u$ one has [38] $(W^{\alpha_0,\zeta_0}, W^{\alpha_1,\zeta_1})_{\theta,1} \subset W^{\alpha_u,\zeta_u}$ where $\alpha_u = (1-u)\alpha_0 + u\alpha_1$ and $1/\zeta_u = (1-u)/\zeta_0 + u/\zeta_1$.

Denote $a = (\alpha_0 - \alpha)/(\alpha_0 - \alpha_1)$ and $b = (\beta_0 - \beta)/(\beta_0 - \beta_1)$.

If $a \leqslant b$ one has $\alpha_a = \alpha$ and [39] $\zeta_a \geqslant \zeta$ then $W^{\alpha_a,\zeta_a} \subset W^{\alpha,\zeta}$ and (10.1) is satisfied since $u = a$.

If $a \geqslant b$ one has $\alpha_b \geqslant \alpha_a = \alpha$ and $\alpha_b - n/\zeta_b = (1-b)\beta_0 + b\beta_1 = b = \alpha - n/\zeta$ then by the fractional Sobolev theorem there holds $W^{\alpha_b,\zeta_b} \subset W^{\alpha,\zeta}$ and (10.1) is satisfied since $u = b$. ∎

REMARK 10.2. – *The limit coefficients.* (10.1) holds for $\theta = u$ if α_0, α_1 and α are not integer and more generally if (use [G, (6.6) and (6.7), p. 178])

$$\zeta_i \geqslant 2 \text{ if } \alpha_i \text{ is integer}, \quad i = 0, 1, \quad \text{and} \quad \zeta_u \leqslant 2 \text{ if } \alpha \text{ is integer}. \quad ∎$$

11. – Optimality results.

The point now is to verify that the various restrictions on the parameters in the preceding results were necessary. To this end some sets F are built by homotheties from a regular function φ.

Let us begin by some results which are satisfied for every Banach spaces X and B such that $X \subset B$. Denote $(H_\lambda f)(t) = f(\lambda t)$ and let

$$(11.1) \qquad b \in X,\ b \neq 0 \quad \text{and} \quad \varphi \in C^\infty(R),\ \varphi = 0 \quad \text{outside} \quad]0, T[,\ \varphi \not\equiv 0\,.$$

$$(11.2) \qquad g_{n,q}(t) = n^{1/q} b\varphi(nt) \quad \text{and} \quad G_q = \{g_{n,q}\colon n \geqslant 1\}\,.$$

At first let us verify that the partial compactness dont imply the limit compactness.

PROPOSITION 1. – For $1 \leqslant q \leqslant \infty$, G_q is bounded in $L^q(0, T; X)$, is relatively compact in $L^p(0, T; X)$, $\forall p < q$, and is not relatively compact in $L^q(0, T; B)$. ∎

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[38] $(W^{\alpha_0,\zeta_0}, W^{\alpha_1,\zeta_1})_{\theta,1} \subset (B_1^{\alpha_0-e,\zeta_0}, B_1^{\alpha_1-e,\zeta_1})_{\theta,1} = B_1^{\alpha_\theta-e,\zeta_\theta}\ \forall e > 0$. If $\zeta_0 \geqslant \zeta_1$ then $\zeta_\theta \geqslant \zeta_u$ and for $e = \alpha_\theta - \alpha_u$, $B_1^{\alpha_\theta-e,\zeta_\theta} \subset B_1^{\alpha_u,\zeta_u} \subset W^{\alpha_u,\zeta_u}$. If $\zeta_0 < \zeta_1$ then $\zeta_\theta < \zeta_u$ and for $e = (\alpha_\theta - n/\zeta_\theta) - (\alpha_u - n/\zeta_u)$, $B_1^{\alpha_\theta-e,\zeta_\theta} \subset B_1^{\alpha_u,\zeta_u} \subset W^{\alpha_u,\zeta_u}$.

[39] One has

$$0 = (\beta_0 - \beta) - b(\beta_0 - \beta_1) \leqslant (\beta_0 - \beta) - a(\beta_0 - \beta_1) = \alpha_0 - \alpha - n\left(\frac{1}{\zeta_0} - \frac{1}{\zeta}\right) - a(\alpha_0 - \alpha_1) + an\left(\frac{1}{\zeta_0} - \frac{1}{\zeta_1}\right) = n\left(\frac{1}{\zeta} - \frac{1-a}{\zeta_0} - \frac{a}{\zeta_1}\right) = n\left(\frac{1}{\zeta} - \frac{1}{\zeta_a}\right).$$

PROOF. – For $\lambda \geq 1$ one has

$$\|H_\lambda \varphi\|_{L^p(0,T)} = \left(\int_R |\varphi(\lambda t)|^p\, dt\right)^{1/p} = \lambda^{-1/p} \|\varphi\|_{L^p(R)} \tag{11.3}$$

then $\|g_{n,q}\|_{L^p(0,T;X)} = n^{1/q-1/p}\|b\|_X\|\varphi\|_{L^p(R)}$. It follows that G_q is bounded in $L^q(0, T; X)$ and is relatively compact in $L^p(0, T; X)$, $\forall p < q$. Moreover G_q is not relatively compact in $L^q(0, T; X)$ since otherwise there would exist a subsequence such that $g_{m,q} \to g$ as $m \to \infty$ and $\|g\|_{L^q(0,T;X)} = \|b\|_X\|\varphi\|_{L^q(R)}$ which is impossible as $g = 0$ (from the convergence a.e. or from the convergence in $L^p(0, T; X)$).

In these results X may be replaced by B, since $b \in B$, which ends the proof. ■

REMARK 11.1. – The compactness properties may be proved by the characterization of theorem 1: the space criterion (3.1) is satisfied since

$$\int_{t_1}^{t_2} g_{n,q}(t)\, dt \in \{\mu b\colon |\mu| \leq |\varphi|_{L^1(R)}\} \text{ which is compact in } X\,.$$

If $p < q$ the time criterion (3.2) is satisfied since (use lemma 4 if $1 \leq h \leq 1/n$)

$$\|\tau_h g_{n,q} - g_{n,q}\|_{L^p(0,T-h;X)} \leq n^{1/q-1/p}\|b\|_X\|\tau_{hn}\varphi - \varphi\|_{L^p(R)} \leq$$
$$\leq h^{1/q-1/p}\|b\|_X \operatorname{Sup}\left\{\left\|\frac{\partial\varphi}{\partial t}\right\|_{L^p(R)};\ 2^{1/p}\|\varphi\|_{L^p(R)}\right\} \to 0 \text{ as } h \to 0\,.$$

For $p = q$ the time criterion is not satisfied since, for $h \leq T/2$ and $n \geq T/h$

$$\|\tau_h g_{n,q} - g_{n,q}\|_{L^q(0,T-h;X)} = \|b\|_X\|\varphi\|_{L^q(R)} \nrightarrow 0 \text{ as } h \to 0\,. \quad ■$$

Now let us verify that the compactness in the Sobolev imbedding theorem holds only for positive regularity order s. Let

$$k_n(t) = b \sin\left(nt\frac{\pi}{T}\right) \quad \text{and} \quad K = \{k_n\colon n \geq 1\}\,.$$

PROPOSITION 2. – K is bounded in $L^\infty(0, T; X)$ and is not relatively compact in $L^1(0, T; B)$. ■

PROOF. – One has $\|k_n\|_{L^\infty(0,T;X)} = \|b\|_X$. On other hand, if $h = T/n$,

$$\|\tau_h k_n - k_n\|_{L^1(0,T-h;B)} = 4\,\frac{n-1}{n}\,\frac{T}{\pi}\|b\|_B \nrightarrow 0 \quad \text{as} \quad h \to 0$$

then the time criterion (3.2) of theorem 1 is not satisfied, so that K is not relatively compact in $L^1(0, T; B)$. ■

Then let us verify that the imbedding in the Sobolev theorem is not compact for the limit coefficients. Begin by the integer case:

PROPOSITION 3. – G_∞ is bounded in $C(0, T; X)$, $\partial G_\infty/\partial t$ is bounded in $L^1(0, T; X)$ and G_∞ is not relatively compact in $L^\infty(0, T; X)$. ■

PROOF. – From proposition 1 it remains to bound $\partial G_\infty/\partial t$. Now

$$\left\|\frac{\partial g_{n,\infty}}{\partial t}\right\|_{L^1(0,T;X)} = \|b\|_X \int_R \left|n\frac{\partial\varphi}{\partial t}(nt)\right| dt = \|b\|_X \left\|\frac{\partial\varphi}{\partial t}\right\|_{L^1(R)}. \quad ■$$

In the fractionnary case one has:

PROPOSITION 4. – Let $1 \leqslant r \leqslant q \leqslant \infty$. Then G_q is bounded in $W^{1/r-1/q,r}(0, T, X) \cap \cap L^q(0, T; X)$ and is not relatively compact in $L^q(0, T; B)$. ■

Changing the notations it yields:

(11.4) let $0 \leqslant s \leqslant \dfrac{1}{r}$ and $r_* = \dfrac{r}{1-rs}$. Then G_{r_*} is bounded in

$W^{s,r}(0, T; X) \cap L^{r_*}(0, T; X)$ and is not relatively compact in $L^{r_*}(0, T; B)$.

PROOF OF PROPOSITION 4. – From proposition 1 it remains to bound G_q in $\mathring{W}^{s,r}$ where $s = 1/r - 1/q$. Now by remark 5.2 one has

$$(11.5) \quad \|H_\lambda\varphi\|_{\dot{W}^{s,r}(0,T;X)} \leqslant \left(2\int_0^\infty (h^{-s}\|\tau_h H_\lambda\varphi - H_\lambda\varphi\|_{L^r(R)})^r \frac{dh}{h}\right)^{1/r} = \lambda^{s-1/r}\|\varphi\|_{W^{s,r}(R)}$$

then $\|g_{n,q}\|_{\dot{W}^{s,r}(0,T;X)} \leqslant n^{1/q+s-1/r}\|b\|_X\|\varphi\|_{\dot{W}^{s,r}(R)}$. ■

Now let us give result in intermediate spaces, with coefficients depending on the spaces in which the functions are valued.

Let $1 \leqslant a \leqslant \infty$ be given and $X_\theta = W^{1-\theta,a}(0, T)$. Then [40] the imbedding $X_0 \to X_1$ is compact and X_θ is of class θ with respect to X_0 and X_1. Denote

$$W = W^{s_0,r_0}(0, T, X_0) \cap W^{s_1,r_1}(0, T; X_1) \text{ where } s_i \text{ are reals, } 1 \leqslant r_i \leqslant \infty$$

$$f_n(t, x) = n^{-(s_1-1/r_1)}\mu^{1/a}\varphi(nt)\varphi(\mu x) \quad \text{where} \quad \mu = n^{(s_1-1/r_1)-(s_0-1/r_0)}$$

$$F = \{f_n: n \geqslant 1\}, \quad s_\theta = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{r_\theta} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}.$$

PROPOSITION 5. – Assume $s_1 - 1/r_1 \geqslant s_0 - 1/r_0$: Then F is bounded and not relatively compact in W and in $W^{s,r}(0, T; X_\theta)$ if $s - 1/r = s_\theta - 1/r_\theta$. ■

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[40] For example by corollary 1 and by the application in section 10.

PROOF. – For every s and r there exists $0 < m < M$ such that, $\forall \lambda \geq 1$ [41],

$$(11.7) \qquad \|H_\lambda \varphi\|_{W^{s,r}(0,T)} = c\lambda^{s-1/r}\|\varphi\|_{W^{s,r}(R)} \quad \text{where} \quad m \leq c = c(\lambda, \varphi) \leq M .$$

Since $s_1 - 1/r_1 \geq s_0 - 1/r_0$ one has $\mu \geq 1$ then (11.7) imply

$$\|f_n\|_{W^{s,r}(0,T;\, W^{1-\theta,q}(0,T))} = c(n, \varphi)\, n^{(s-1/r)-(s_1-1/r_1)}\mu^{1-\theta} = c(n, \varphi)\, n^{(s-1/r)-(s_\theta-1/r_\theta)} .$$

One concludes by the same argument as in the proof of proposition 1, since $W = \bigcap_{\theta=0,1} W^{s_\theta, r_\theta}(0, T; X_\theta)$. ∎

Let us complete the proposition 5 by a result for a compact set in W: denote $F = \{f_n/\log n : n \geq 1\}$.

PROPOSITION 6. – Assume $s_1 - 1/r_1 \geq s_0 - 1/r_0$. Then F is relatively compact in W and, if $s - 1/r > s_\theta - 1/r_\theta$, F is not bounded in $W^{s,r}(0, T; X_\theta)$. ∎

PROOF. – It is similar to the preceding one. ∎

Let us give an application. Let $1 \leq r_i \leq \infty$ and $s_1 \geq 0$. Then [42]

(11.8) $\forall r > r_0$ there exists $0 < \theta < 1$ and a compact set F in $L^{r_0}(0, T; X_0) \cap W^{s_1, r_1}(0, T; X)$ which is not bounded in $L^r(0, T; X_\theta)$.

REMARK 11.1. – There exists some intermediate spaces B, with compact imbedding $X_0 \to B$, which are suitable for all $r > r_0$, that is to say such that

(11.9) there exists a bounded set F in $L^{r_0}(0, T; X_0) \cap W^{s_1, r_1}(0, T; X_1)$ which is compact for no one space $L^r(0, T; B)$, $r > r_0$.

For example

$$B = \left\{ f \in L^\infty(0, 1) :\ \operatorname*{Sup}_{h>0} \frac{|\log h|}{h} \|\tau_h f - f\|_{L^\infty(0,1-h)} < \infty \right\},$$

this will not be stated here. Remark that no one space of class $\theta > 0$ may be suitable for all $r > r_0$ (it follows from corollary 9). ∎

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[41] If $s = 0$ see (11.3). For the function ψ with compact support in $]0, T[$ which is the case for $H_\lambda \varphi$, one has if $0 < s < 1$, $\|\psi\|_{W^{s,r}(0,T)} \simeq \|\psi\|_{W^{s,r}(R)} \simeq \|\psi\|_{\dot W^{s,r}(R)}$ and (11.7) follows from a change of variable, see (11.5). In the general case one has

$$\|\psi\|_{W^{m+\sigma,r}(0,T)} \simeq \left\|\frac{\partial^m \psi}{\partial t^m}\right\|_{W^{\sigma,r}(0,T)} \qquad \text{and} \qquad \|\psi\|_{W^{-m+\sigma,r}(0,T)} \simeq \|\psi\|_{W^{\sigma,r}(0,T)}$$

where $\partial^m \psi_m/\partial t^m = \psi$, $\psi_m(t) = 0$ if $t \leq 0$, and (11.7) follows since $(\partial^m H_\lambda \varphi)/\partial t^m = \lambda^m H_\lambda(\partial^m \varphi/\partial t^m)$.

[42] If $s_1 - 1/r_1 \leq -1/r_1$ it follows from proposition 4, $\forall \theta$. If $s_1 - 1/r_1 \geq -1/r_0$ it follows from proposition 6 with $s_0 = s = 0$ and θ small enough $s_\theta - 1/r_\theta < -1/r$.

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