On the Existence of Complete Kähler Metrics of Negative Riemannian Curvature Bounded Away from Zero on Ellipsoidal Domains in C^n (*).

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Summary. – *The purpose of this paper is to prove that every ellipsoidal domain in C^n admits a complete Kähler metric whose Riemannian sectional curvature is bounded from above by a negative constant (Theorem 1). We construct a Kähler metric, in a natural way, as potential of a suitable function defining the boundary (§ 2). Directly we compute the curvature tensor and we find upper and lower bounds for the holomorphic sectional curvature (§ 4, § 5). In order to prove the boundness of Riemannian sectional curvature we use finally a classical pinching argument (§ 6). We also obtain that for certain ellipsoidal domains the curvature tensor is very strongly negative in the sense of [15] (§ 3). Finally we prove that the metric constructed on ellipsoidal domains in C^n is the Bergman metric if and only if the domain is biholomorphic to the ball (Theorem 2). In [8], [9] R. E. Greene and S. G. Krantz gave large families of examples of complete Kähler manifolds with Riemannian sectional curvature bounded from above by a negative constant; they are sufficiently small deformations of the ball in C^n, with the Bergman metric. Before the only known example of complete simply-connected Kähler manifold with Riemannian sectional curvature upper bounded by a negative constant, not biholomorphic to the ball, was the surface constructed by G. D. Mostow and Y. T. Siu in [14], to the best of the author's knowledge, is not known at present if this example is biholomorphic to a domain in C^n.*

1. – Ellipsoidal domains in C^n.

Let C^n be the n-dimensional complex space, we consider coordinates $\{z_1, \ldots, z_n\}$ where $z_\alpha = x_\alpha + \sqrt{-1}\, y_\alpha$, $x_\alpha, y_\alpha \in \boldsymbol{R}$, $\alpha = 1, \ldots, n$.

We define:

$$E = \left\{ z \in C^n \colon \sum_{\alpha,\beta=1}^{n} (a_{\alpha\beta} x_\alpha x_\beta + b_{\alpha\beta} y_\alpha y_\beta + c_{\alpha\beta} x_\alpha y_\beta) < d \right\}. \tag{1.1}$$

E is said to be an *Ellipsoidal Domain* in C^n if the quadratic form in (1.1) is positive definite and d is a positive real number.

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PROPOSITION 1. – *After a complex linear transformation of* $\boldsymbol{C}^n$ *an ellipsoidal domain* E *has the form*:

$$E=\left\{z\in \boldsymbol{C}^n\colon \sum_{\alpha=1}^{n}[A_\alpha(z_\alpha^2+\bar{z}_\alpha^2)+z_\alpha\bar{z}_\alpha]<1\right\} \tag{1.2}$$

where A_α, $\alpha=1,\dots,n$, *are real numbers and* $0\leqslant A_\alpha<\frac{1}{2}$.

PROOF. – In complex coordinates (1.1) becomes:

$$\sum_{\alpha,\beta=1}^{n}(\bar{A}_{\alpha\beta}\bar{z}_\alpha\bar{z}_\beta+A_{\alpha\beta}z_\alpha z_\beta+B_{\alpha\beta}z_\alpha\bar{z}_\beta+\bar{B}_{\alpha\beta}\bar{z}_\alpha z_\beta)<1$$

where

$$A_{\alpha\beta}=\frac{1}{4d}\left(a_{\alpha\beta}-b_{\alpha\beta}-\sqrt{-1}\,c_{\alpha\beta}\right), \qquad B_{\alpha\beta}=\frac{1}{4d}\left(a_{\alpha\beta}+b_{\alpha\beta}+\sqrt{-1}\,c_{\alpha\beta}\right).$$

We define the hermitian form

$$h(z,w)=\sum_{\alpha,\beta=1}^{n}(B_{\alpha\beta}+\bar{B}_{\alpha\beta})z_\alpha\bar{w}_\alpha\,;$$

h is positive definite and so, after a complex linear transformation, E can be written in the following form:

$$E=\left\{z\in \boldsymbol{C}^n\colon \sum_{\alpha=1}^{n}z_\alpha\bar{z}_\alpha+Q(z,z)+\overline{Q(z,z)}<1\right\},$$

where $Q(\cdot,\cdot)$ is a quadratic form. We may suppose $Q(z,z)+\overline{Q(z,z)}$ is maximized at the point $z=(1,0,\dots,0)$ of the unitary sphere. So after a unitary linear transformation we have:

$$E=\left\{z\in \boldsymbol{C}^n\colon \sum_{\sigma=1}^{n}z_\alpha\bar{z}_\alpha+A_1(z_1^2+\bar{z}_1^2)+Q'(z,z)+\overline{Q'(z,z)}<1\right\}$$

where Q' is a quadratic form. We can repeat the same argument on Q'. By induction at least we obtain:

$$E=\left\{z\in \boldsymbol{C}^n\colon \sum_{\alpha=1}^{n}[z_\alpha\bar{z}_\alpha+A_\alpha(z_\alpha^2+\bar{z}_\alpha^2)]<1\right\}.$$

The positive definiteness in (1.1) implies $0\leqslant A_\alpha<\frac{1}{2}$. Q.E.D.

As in [16] we denote an ellipsoidal domain, written in the standard form (1.2), by $E=E(A_1,\dots,A_n)$ and E is said to be *non trivial* if there exists $\alpha\in\{1,\dots,n\}$ such that $A_\alpha\neq 0$.

We conclude by the fundamental S. M. Webster's classification theorem [16]:

THEOREM A. – *Let $E = E(A_1, \dots, A_n)$ and $E' = E(A'_1, \dots, A'_n)$ be ellipsoidal domains in C^2, $n \geq 2$. Then E in biholomorphically equivalent to E' if and only if $A_\alpha = A'_\alpha$ after a reordering. If either E is non trivial then every automorphism of E is a complex linear transformation.*

2. – Kähler metric and curvature tensor.

Let $E = E(A_1, \dots, A_n)$ be an ellipsoidal domain in C^n, we denote

$$\varphi(z, \bar{z}) = 1 - \sum_{\alpha=1}^{n} \left(z_\alpha \bar{z}_\alpha + A_\alpha(z_\alpha^2 + \bar{z}_\alpha^2)\right) \quad \text{and } g = -\log \varphi .$$

We consider $g_{\alpha\bar{\beta}}(z, \bar{z}) = \partial^2 g(z, \bar{z})/(\partial z_\alpha \, \partial \bar{z}_\beta)$, then we define

$$ds^2 = \sum_{\alpha,\beta=1}^{n} g_{\alpha\bar{\beta}} \, dz_\alpha \otimes d\bar{z}_\beta . \tag{2.1}$$

Easily we can prove that ds^2 is a complete Kähler metric on E [3].

In complex coordinates we have:

$$g_{\alpha\bar{\beta}} = \frac{\delta_{\alpha\beta}}{1 - \sum_{\sigma=1}^{n} \left(z_\sigma \bar{z}_\sigma + A_\sigma(z_\sigma^2 + \bar{z}_\sigma^2)\right)} + \frac{(2A_\alpha z_\alpha + \bar{z}_\alpha)(2A_\beta \bar{z}_\beta + z_\beta)}{\left[1 - \sum_{\sigma=1}^{n} \left(z_\sigma \bar{z}_\sigma + A_\sigma(z_\sigma^2 + \bar{z}_\sigma^2)\right)\right]^2} \tag{2.2}$$

where

$$\delta_{\alpha\beta} = \begin{cases} 1 & \alpha = \beta , \\ 0 & \alpha \neq \beta . \end{cases}$$

We are interested in the curvature tensor of the Kähler metric (2.1):

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{\partial^2 g_{\alpha\beta}}{\partial z_\gamma \, \partial \bar{z}_\delta} - \sum_{\mu,\lambda=1}^{n} g^{\lambda\bar{\mu}} \frac{\partial g_{\alpha\bar{\mu}}}{\partial z_\gamma} \frac{\partial z_{\lambda\bar{\beta}}}{\partial \bar{z}_\delta}, \qquad 1 \leq \alpha, \beta, \gamma, \delta \leq n$$

where $g^{\alpha\bar{\beta}} = \overline{(g_{\alpha\bar{\beta}})^{-1}}$.

By a direct, long, calculation we obtain the following expression:

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = g_{\alpha\bar{\beta}} g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}} g_{\gamma\bar{\beta}} + \frac{4A_\alpha A_\beta \delta_{\alpha\beta}\delta_{\delta\beta}}{\varphi^2} \left(\sum_{\sigma=1}^{n} g^{\sigma\bar{\sigma}} \cdot \frac{1}{\varphi} - n + 1\right), \qquad 1 \leq \alpha, \beta, \gamma, \delta \leq n . \tag{2.3}$$

We have the following:

LEMMA 1. – $\sum_{\sigma=1}^{n} g^{\sigma\bar\sigma}\cdot 1/\varphi - (n-1) \geqslant 0$.

PROOF. – Denote

$$\varphi_\alpha = \frac{\partial\varphi}{\partial z_\alpha}, \quad \varphi_{\bar\alpha} = \bar\varphi_\alpha, \quad \varphi_{\alpha\bar\beta} = \frac{\partial^2\varphi}{\partial z_\alpha \partial \bar z_\beta},$$

$$\varphi^{\alpha\bar\beta} = \overline{(\varphi_{\alpha\bar\beta})^{-1}}, \quad \varphi^\alpha = \sum_{\sigma=1}^{n} \varphi^{\alpha\bar\sigma}\varphi_{\bar\sigma}, \quad \varphi^{\bar\sigma} = \overline{\varphi_\sigma}, \qquad |d\varphi|^2 = \sum_{\alpha,\beta=1}^{n} \varphi^{\alpha\bar\beta}\varphi_\alpha\varphi_{\bar\beta}.$$

Then $g^{\sigma\bar\sigma} = \varphi\big(\varphi^{\sigma\bar\sigma} + \varphi^\sigma\varphi^{\bar\sigma}/(-\varphi - |d\varphi|^2)\big)$. Observe $\varphi_{\alpha\bar\beta} = \delta_{\alpha\beta}$, therefore

$$\sum_{\sigma=1}^{n} g^{\sigma\bar\sigma}\frac{1}{\varphi} - (n-1) = \sum_{\sigma=1}^{n} \varphi^{\sigma\bar\sigma}\varphi_{\sigma\bar\sigma} + \frac{\sum_{\sigma=1}^{n}\varphi^\sigma\varphi_{\sigma\bar\sigma}\varphi^{\bar\sigma}}{-\varphi - |d\varphi|^2} - (n-1) = \frac{\varphi}{\varphi + |d\varphi|^2} \geqslant 0\,. \qquad \text{Q.E.D.}$$

First we have:

$$\|\operatorname{grad} g\|^2 = dg(\operatorname{grad} g) = dg\left(4\ \mathrm{Re}\sum_{i,j=1}^{n} g^{i\bar j} g_{\bar j}\frac{\partial}{\partial z_i}\right) = \frac{4|d\varphi|^2}{\varphi + |d\varphi|^2}\,.$$

Second, we observe that $\tilde R_{\alpha\bar\beta\gamma\bar\delta} = g_{\alpha\bar\beta}g_{\gamma\bar\delta} + g_{\alpha\bar\delta}g_{\gamma\bar\beta}$ is the tensor of constant negative holomorphic sectional curvature -4 (see § 4 for definition).

Therefore:

$$R_{\alpha\bar\beta\gamma\bar\delta} = \tilde R_{\alpha\bar\beta\gamma\bar\delta} + \frac{A_\alpha A_\beta \delta_{\alpha\gamma}\delta_{\beta\delta}}{\varphi^2}\left(4 - \|\operatorname{grad}(-\log\varphi)\|^2\right). \tag{2.4}$$

3. – Strong negativity of the curvature of certain ellipsoidal domains.

First we recall some fundamental definitions. Let M be a complex manifold with Kähler metric

$$ds^2 = \sum_{\alpha\beta} g_{\alpha\bar\beta}\, dz_\alpha \otimes d\bar z_\beta\,.$$

If $X = 2\ \mathrm{Re}\sum_\alpha \xi^\alpha(\partial/\partial z_\alpha)$ and $Y = 2\ \mathrm{Re}\sum_\alpha \eta^\alpha(\partial/\partial z_\alpha)$ are tangent vectors then the (*Riemannian*) *sectional curvature* of the real 2-plane spanned by X and Y is given by:

$$K(X, Y) = -2\cdot\frac{\sum_{\alpha\beta\gamma\delta} R_{\alpha\bar\beta\gamma\bar\delta}(\xi^\alpha\bar\eta^\beta - \eta^\alpha\bar\xi^\beta)\overline{(\xi^\delta\bar\eta^\gamma - \eta^\delta\bar\xi^\gamma)}}{\sum_{\gamma\beta\varkappa\delta} g_{\delta\bar\beta}g_{\gamma\bar\delta}\{(\xi^\alpha\bar\eta^\beta - \eta^\alpha\bar\xi^\beta)\overline{(\xi^\delta\bar\eta^\gamma - \eta^\delta\bar\xi^\gamma)} + (\bar\xi^\alpha\eta^\gamma - \xi^\gamma\eta^\alpha)\overline{(\xi^\beta\eta^\delta - \eta^\beta\xi^\delta)}\}}$$

In particular the sectional curvature is *non positive* if and only if:

$$\sum_{\alpha\beta\gamma\delta} R_{\alpha\bar\beta\gamma\bar\delta}(\xi^\alpha\bar\eta^\beta - \eta^\alpha\bar\xi^\beta)\overline{(\xi^\delta\bar\eta^\gamma - \eta^\delta\bar\xi^\gamma)} \geqslant 0$$

for all complex numbers ξ^α, η^α, and is negative if the equality holds if and only if $(\xi^\alpha\bar{\eta}^\beta - \eta^\alpha\bar{\xi}^\beta) = 0$ for all α, β. In [15] Y. T. SIU introduced the following, stronger definitions:

DEFINITION A. – The curvature tensor $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ is said to be *strongly negative* (*strongly seminegative*) if

$$\sum_{\alpha\beta\gamma\delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}}(A^\alpha\overline{B^\beta} - C^\alpha\overline{D^\beta})\overline{(A^\delta\overline{B^\gamma} - C^\delta\overline{D^\gamma})} > 0 \qquad (\geqslant 0)$$

for all complex numbers A^α, B^α, C^α, D^α, when $(A^\alpha\bar{B}^\beta - C^\alpha\bar{D}^\beta) \neq 0$ for at least one pair of indices (α, β).

DEFINITION B. – The curvature tensor $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ is said to be *very strongly negative* (*very strongly seminegative*) if $\sum_{\alpha\beta\gamma\delta} R_{\alpha\bar{\beta}\gamma\bar{\delta}}\xi^{\alpha\bar{\beta}}\overline{\xi^{\delta\bar{\gamma}}} > 0$ $(\geqslant 0)$ for all complex numbers $\xi^{\alpha\bar{\beta}}$ when $\xi^{\alpha\bar{\beta}} \neq 0$ for at least one pair of indices (α, β).

Obviously if $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ is very strongly negative it is also strongly negative and either conditions imply the negativity of sectional curvature.

Easily we can prove the tensor of constant negative holomorphic sectional curvature, $\tilde{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}$, on the ball is very strongly negative. As a consequence we obtain the following:

PROPOSITION 2. – *Suppose E_σ is an ellipsoidal domain in $\mathbf{C}^n$ of the form*

$$E_\sigma = E(0, \ldots, 0, A_\sigma, 0, \ldots, 0)\,.$$

The curvature tensor of the metric (2.1) *on E_σ is very strongly negative.*

PROOF.

$$\sum_{\alpha,\beta,\gamma,\delta=1}^{n} R_{\alpha\bar{\beta}\gamma\bar{\delta}}\xi^{\alpha\bar{\beta}}\overline{\xi^{\delta\bar{\gamma}}} = \sum_{\alpha,\beta,\gamma,\delta=1}^{n} \tilde{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}\xi^{\alpha\bar{\beta}}\overline{\xi^{\delta\bar{\gamma}}} + \frac{A_\sigma^2}{\varphi^2}|\xi^{\sigma\bar{\sigma}}|^2(4 - \|\mathrm{grad}\,(-\log\varphi)\|^2) \geqslant$$
$$\geqslant \sum_{\alpha,\beta,\gamma,\delta=1}^{n} \tilde{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}\xi^{\alpha\bar{\beta}}\overline{\xi^{\delta\bar{\gamma}}} \geqslant 0\,.$$

The first inequality follows from Lemma 1 while the second from strongly negativity of $\tilde{R}_{\alpha\bar{\beta}\gamma\bar{\delta}}$. In particular equality holds if and only if $\xi^{\alpha\bar{\beta}} = 0$ for all α, β.

Q.E.D.

By direct computation on (2.3) or using more general results [13], we can observe the curvature tensor of the metric (2.1) on $E = E(A_1, \ldots, A_n)$ approaches, near the boundary, the tensor of constant holomorphic sectional curvature -4. So the sectional curvature is bounded from above by a negative constant near the boundary. Using this fact, the continuity of the curvature and Proposition 2 we have:

COROLLARY 1. – *The Riemannian sectional curvature of the metric* (2.1) *on* E_σ *is bounded from above by a negative constant.*

We will see this statement holds for all ellipsoidal domains (§ 6), this is actually the purpose of this paper, but we don't know if Proposition 2 holds for more general ellipsoidal domains than E_σ.

4. – Upper bound of the holomorphic sectional curvature.

Let M be a complex manifold with Kähler metric $ds^2 = \sum_{\alpha,\beta} g_{\alpha\bar\beta}\, dz_\alpha \otimes d\bar z_\beta$. Let $X = 2\,\mathrm{Re} \sum_\alpha \xi^\alpha(\partial/\partial z_\alpha)$ be a tangent vector, and J the complex structure tensor on M, we have $JX = 2\,\mathrm{Re}\sum_\alpha \sqrt{-1}\,\xi^\alpha(\partial/\partial z_\alpha)$.

The *holomorphic sectional curvature* of the 1-dimensional complex space spanned by X is given by:

$$H(X) = K(X, JX) = -2\cdot \frac{\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\bar\beta\gamma\bar\delta}\, \xi^\alpha \bar\xi^\beta \xi^\gamma \bar\xi^\delta}{\sum_{\alpha,\beta,\gamma,\delta} g_{\alpha\bar\beta} g_{\gamma\bar\delta}\, \xi^\alpha \bar\xi^\beta \xi^\gamma \bar\xi^\delta}\,.$$

Suppose $E = E(A_1, \ldots, A_n)$ is an ellipsoidal domain in $\mathbf{C}^n$ with Kähler metric (2.1) and $X = 2\,\mathrm{Re} \sum_{\sigma=1}^n \xi^\sigma(\partial/\partial z_\sigma)$ is a tangent vector, then we have:

$$\begin{aligned}
(4.1)\qquad H(X) &= -2\cdot \frac{\sum_{\alpha,\beta,\gamma,\delta=1}^n (g_{\alpha\bar\beta} g_{\gamma\bar\delta} + g_{\alpha\bar\delta} g_{\gamma\bar\beta})\, \xi^\alpha \bar\xi^\beta \xi^\gamma \bar\xi^\delta}{\sum_{\alpha,\beta,\gamma,\delta=1}^n g_{\alpha\bar\beta} g_{\gamma\bar\delta}\, \xi^\alpha \bar\xi^\beta \xi^\gamma \bar\xi^\delta} + \\
&\quad + (-2)\cdot \frac{\sum_{\alpha,\beta=1}^n A_\alpha A_\beta (\xi^\alpha)^2 \overline{(\xi^\beta)^2}}{\sum_{\alpha,\beta=1}^n g_{\alpha\bar\beta} g_{\gamma\bar\delta}\, \xi^\alpha \bar\xi^\beta \xi^\gamma \bar\xi^\delta} \cdot \frac{4 - \|\mathrm{grad}\,(-\log\varphi)\|^2}{\varphi^2} = \\
&= -4 - 2\cdot \frac{\left|\sum_{\sigma=1}^n A_\sigma (\xi^\sigma)^2\right|^2}{\sum_{\alpha,\beta,\gamma,\delta=1}^n g_{\alpha\bar\beta} g_{\gamma\bar\delta}\, \xi^\alpha \bar\xi^\beta \xi^\gamma \bar\xi^\delta} \cdot \frac{4 - \|\mathrm{grad}\,(-\log\varphi)\|^2}{\varphi^2}\,.
\end{aligned}$$

Lemma 1 and (4.1) implies the following:

PROPOSITION 3. – *Suppose* $E = E(A_1, \ldots, A_n)$ *is an ellipsoidal domain in* $\mathbf{C}^n$, *then the holomorphic sectional curvature of the metric* (2.1) *is bounded from above by the negative constant* -4.

5. – Lower bound of the holomorphic sectional curvature.

We are going to find a lower bound for the holomorphic sectional curvature of the Kähler metric (2.1) on ellipsoidal domains. In order to do that we need the following classical result about the decreasing property of holomorphic sectional curvature on submanifolds [11]:

LEMMA A. – *Let M' be a complex submanifold of the Hermitian manifold M. Let G the Hermitian metric of M and G' the induced metric of M'. If X is a tangent vector to M' then $H_{G'}(X) \leqslant H_G(X)$, where $H_{G'}$, H_G are holomorphic sectional curvature on M' and M respectively.*

Let $E = E(A_1, \ldots, A_n)$ be an ellipsoidal domain in $\boldsymbol{C}^n$, let $X = 2 \operatorname{Re} \sum_{\alpha=1}^{n} \xi^\alpha(\partial/\partial z_\alpha)$ be a tangent vector to E at some point $p \in E$. We can consider the 1-dimensional complex submanifold E' of E defined by intersection in $\boldsymbol{C}^n$ of E with the complex plane, π_x, spanned by X, translate at the point p. Then we compute the Gaussian curvature of E' by respect to the induced metric.

Suppose $p = (u_1, \ldots, u_n)$, then:

$$\pi_x = \{z \in \boldsymbol{C}^n \colon z_k = u_k + \xi^k t,\ 1 \leqslant k \leqslant n,\ t \in \boldsymbol{C}\}\,. \tag{5.1}$$

Assume $\xi^i \neq 0$, we have:

$$E' = E \cap \pi_x = \left\{z \in \boldsymbol{C}^n \colon z_k = u_k + \frac{\xi^k}{\xi^i}(z_i - u_i),\ \psi(z_i, \bar z_i) = 0,\ 1 \leqslant k \leqslant n,\ k \neq i\right\} \tag{5.2}$$

where

$$\begin{aligned}\psi(z, \bar z) = 1 - \Biggl\{ &\frac{\sum_{\alpha=1}^{n} |\xi^\alpha|^2}{|\xi^i|^2} z\bar z + 2 \operatorname{Re}\left[\left(\frac{z}{\xi^i}\right)^2 \sum_{\alpha=1}^{n} A_\alpha (\xi^\alpha)^2\right] + \\ &+ 2 \operatorname{Re}\left[\left(\frac{z}{\xi^i}\right)^2 \sum_{\substack{\alpha=1 \\ \alpha\neq i}}^{u} \xi^\alpha \left(\bar u_\alpha + 2A_\alpha u_\alpha - \frac{\bar\xi^\alpha}{\bar\xi^i}\bar u_i - 2A_\alpha \frac{\xi^\alpha}{\xi} u_i\right)\right] + \\ &+ \sum_{\substack{\alpha=1 \\ \alpha\neq i}}^{n} \left(|u_\alpha|^2 + A_\alpha(u_\alpha^2 + \bar u_\alpha^2)\right) + 2 \operatorname{Re}\left(\sum_{\substack{\alpha=1 \\ \alpha\neq i}}^{n} A_\alpha (\xi^\alpha)^2 \left(\frac{u_i}{\xi^i}\right)^2\right) + \\ &- 2 \operatorname{Re}\left[\left(\frac{u_i}{\xi^i}\right)^2 \sum_{\substack{\alpha=1 \\ \alpha\neq i}}^{\alpha} \xi^\alpha \left(\bar u_\alpha + 2A_\alpha u_\alpha - \frac{\bar\xi^\alpha}{\bar\xi^i}\bar u_i\right)\right]\Biggr\}.\end{aligned} \tag{5.3}$$

For all tangent vectors and all points E' is a 1-dimensional ellipsoidal domain linearly embedded in E so the metric induced on E' by the Kähler metric (2.1) is given by:

$$ds^2 = -\frac{\partial^2 \log \psi(z, \bar z)}{\partial z\, \partial \bar z}\, dz \otimes d\bar z\,. \tag{5.4}$$

We recall if S is a Riemann surface with Kähler metric $ds^2 = h\, dz \otimes d\bar{z}$, then the Gaussian curvature K is given by:

$$K = \frac{-2}{h}\,\frac{\partial^2 \log h}{\partial z\, \partial \bar{z}}\,.$$

In particular if $S = \{z \in \boldsymbol{C}\colon\ \varrho(z, \bar{z}) > 0\}$ by direct computation we obtain:

$$K = -4 + \frac{2\varrho^3}{(|\partial\varrho/\partial z|^2 - \varrho(\partial^2\varrho/\partial z\,\partial\bar{z}))^3}\cdot \left\{\left|\frac{\partial^2\varrho}{\partial z^2}\right|^2 \frac{\partial^2\varrho}{\partial z\,\partial\bar{z}} - \varrho\,\frac{\partial^4\varrho}{\partial z^2\,\partial\bar{z}^2}\,\frac{\partial^2\varrho}{\partial z\,\partial\bar{z}} + \frac{\partial^4\varrho}{\partial z^2\,\partial\bar{z}^2}\left|\frac{\partial\varrho}{\partial z}\right|^2 + \varrho\left|\frac{\partial^3\varrho}{\partial z^2\,\partial\bar{z}}\right|^2 - 2\,\mathrm{Re}\left(\frac{\partial\varrho}{\partial\bar{z}}\,\frac{\partial^2\varrho}{\partial z^2}\,\frac{\partial^3\varrho}{\partial z\,\partial\bar{z}^2}\right)\right\}. \tag{5.5}$$

LEMMA 2. – *The Gaussian curvature of the metric* (5.4) *on* E' *is bounded from below by the negative constant* -6.

PROOF. – From (5.3) and (5.5) we obtain:

$$K = -4 + 2\cdot\frac{\psi^3|\partial^2\psi/\partial z^2|^2\cdot\partial^2\psi/\partial z\,\partial\bar{z}}{(|\partial\psi/\partial z|^2 - \psi(\partial^2\psi/\partial z\,\partial\bar{z}))^3}\,. \tag{5.6}$$

Therefore is enough to prove:

$$\frac{\psi^3|\partial^2\psi/\partial z^2|^2\cdot\partial^2 z/\partial z\,\partial\bar{z}}{(|\partial\psi/\partial z|^2 - \psi(\partial^2\psi/\partial z\,\partial\bar{z}))^3} \geqslant -1$$

or equivalently:

$$\frac{-4(\psi^3/|\xi^i|^6)\left|\sum_{\alpha=1}^{n} A_\alpha(\xi^\alpha)^2\right|^2 \sum_{\alpha=1}^{n}|\xi^\alpha|^2}{\left(|\partial\psi/\partial z|^2 + (\psi/|\xi^i|^2)\sum_{\alpha=1}^{n}|\xi^\alpha|^2\right)^3} \geqslant -1\,.$$

By using Cauchy-Schwarz inequality and the fact $A_\alpha < \frac{1}{2}$, $1 \leqslant \alpha \leqslant n$, we have:

$$4\left|\sum_{\alpha=1}^{n} A_\alpha(\xi^\alpha)^2\right|^2 = 4\left|\sum_{\alpha=1}^{n}(\sqrt{A_\alpha}\,\xi^\alpha)(\sqrt{A_\alpha}\,\xi^\alpha)\right|^2 \leqslant 4\left(\sum_{\alpha=1}^{n}|\sqrt{A_\alpha}\,\xi^\alpha|^2\right)^2 = 4\left(\sum_{\alpha=1}^{n} A_\alpha|\xi^\alpha|^2\right)^2 < \left(\sum_{\alpha=1}^{n}|\xi^\alpha|^2\right)^2. \tag{5.7}$$

Then:

$$\frac{-4(\psi^3/|\xi^i|^6)\left|\sum_{\alpha=1}^{n} A_\alpha(\xi^\alpha)^2\right|^2\sum_{\alpha=1}^{n}|\xi^\alpha|^2}{\left(|\partial\psi/\partial z|^2 + (\psi/|\xi^i|^2)\sum_{\alpha=1}^{n}|\xi^\alpha|^2\right)^3} > \frac{-\psi^3\,|\xi^i|^6\left(\sum_{\alpha=1}^{n}|\xi^\alpha|^2\right)^3}{\left(|\partial\psi/\partial z|^2 + (\psi/|\xi^i|^2)\sum_{\alpha=1}^{n}|\xi^\alpha|^2\right)^3} \geqslant -1 \tag{5.8}$$

which is the statement. Q.E.D.

REMARK. – In (5.7), and therefore in (5.8), we have strong inequality, depending on geometric constant of E.

We can choose $\sigma = \min_{i=1,\dots,n}\{\sigma_i\colon A_i \leqslant \frac{1}{2} - \sigma_i\}$ so that

$$K \geqslant -6 + 2(4\sigma - 4\sigma^2)\cdot\left(1 - \frac{|\xi^i|^2\,|\partial\psi/\partial z|^2}{|\xi^i|^2\,|\partial\psi/\partial z|^2 + \psi\sum_{\alpha=1}^{n}|\xi^\alpha|^2}\right)^3. \tag{5.9}$$

Observe the second addend in (5.9) is positive in E' and it approaches zero near the boundary of E', therefore the Gaussian curvature is strictly greater than -6. Then from Lemma A, Lemma 2 and Remark we obtain the following:

PROPOSITION 4. – *The holomorphic sectional curvature of the Kähler metric* (2.1) *on E is greater than the negative constant* -6.

COROLLARY 2. – *Let $E = E(A_1, \dots, A_n)$ be an ellipsoidal domain in $\mathbf{C}^n$, there exists a positive constant ε such that the holomorphic sectional curvature of the Kähler metric* (2.1) *on E is bounded from below by the negative constant* $-6 + \varepsilon$.

PROOF. – We know the curvature tensor (2.3) approaches, near the boundary of E, the tensor of constant negative holomorphic sectional curvature -4, therefore we can choose an open neighbourhood U of the boundary such that in U the holomorphic sectional curvature is bounded from below by the negative constant -5. Let p be a point belonging to $E\setminus U$ and let X be a tangent vector at the point p; there exists a positive constant ε' such that $H(X)(p) > -6 + \varepsilon'$. By varying X in the unit ball of T_pE (tangent space to E at the point p) and p in the compact set $E\setminus U$, by using again the continuity of the curvature, we are able to obtain ε. Q.E.D.

6. – Negativity of the Riemannian sectional curvature.

We need the following classical result [2]:

LEMMA B. – *Let M be a complex manifold with Kähler metric G, let X and Y be orthogonal tangent vectors, and* $\operatorname{Re} G(X, J, Y) = \cos\theta \geqslant 0$. *Then,*

$$K(X, Y) = \tfrac{1}{8}[3(1+\cos\theta)^2 H(X + JY) + 3(1-\cos\theta)^2 H(X - JX) + \\ - H(X+Y) - H(X-Y) - H(X) - H(Y)] \tag{6.1}$$

when K (resp. H) *is the Riemannian sectional curvature* (*resp. holomorphic*) *of G and J is the complex structure tensor of M.*

PROPOSITION 5. – *Let M be a complex manifold with Kähler metric G such that the holomorphic sectional curvature of G is bounded from above and from below by negative constants: $-A \leqslant H \leqslant -B$. Then the Riemannian sectional curvature of G, K, is bounded from below by a negative constant. Moreover if $A/B \leqslant \frac{3}{2} - \bar{\varepsilon}$ holds, for some positive constant $\bar{\varepsilon}$, then K is bounded from above by a negative constant.*

PROOF. – Let π be a real 2-plane tangent to M at some point p. Let X and Y be tangent vectors spanning π, we can suppose X and Y are orthonormal and $\cos(\widehat{X, JK}) = \cos\theta \geqslant 0$. It follows directly from (6.1)

$$-\frac{3}{2}A + \frac{1}{2}B \leqslant K(X, Y) \leqslant -\frac{1}{2}B\left(\frac{3}{2} - \frac{A}{B}\right). \tag{6.2}$$

This is exactly the statement. Q.E.D.

Let $E = E(A_1, \ldots, A_n)$ be an ellipsoidal domain in $\boldsymbol{C}^n$ with Kähler metric (2.1), then E satisfies Proposition 3, Corollary 2 and therefore Proposition 5, so we have:

THEOREM 1. – *Every ellipsoidal domain $\boldsymbol{C}^n$ admits a complete Kähler metric whose Riemannian sectional curvature is bounded from above and from below by two negative constants depending on the boundary geometry only.*

7. – Relations with the Bergman metric.

In this paragraph we are interested in the relations between the metric (2.1) constructed in § 2 and the Bergman metric of ellipsoidal domains. The problem arises in a natural way from the following remarks.

Let $E = E(A_1, \ldots, A_n)$ be an ellipsoidal domain in $\boldsymbol{C}^n$, define the metric:

$$d\tilde{s}^2 = (n+1)\, ds^2 ,$$

where ds^2 is the Kähler metric (2.1) on E.

REMARK 1. – Suppose $A_\alpha = 0$ for $1 \leqslant \alpha \leqslant n$, then $E = E(0, \ldots, 0)$ is the unit ball in $\boldsymbol{C}^n$, $\boldsymbol{B}^n$, and $d\tilde{s}^2$ is the Bergman metric of $\boldsymbol{B}^n$:

$$d\tilde{s}^2 = \sum_{\alpha,\beta=1}^{n} \left(\partial^2 \log \left(1 - \sum_{\sigma=1}^{n} z_\sigma \bar{z}_\sigma\right)^{-(n+1)} \Big/ \partial z_\alpha\, \partial \bar{z}_\beta\right) dz_\alpha \otimes d\bar{z}_\beta .$$

REMARK 2. – For every ellipsoidal domain in $\boldsymbol{C}^n$, $E = E(A_1, \ldots, A_n)$, the curvature tensor of the metric $d\tilde{s}^2$ approaches near the boundary the curvature tensor of the Bergman metric of E [13].

REMARK 3. – From S. M. Webster's classification theorem (Theorem A, § 1) it follows easily that every automorphism of $E = E(A_1, \ldots, A_n)$ $(n \geqslant 2)$ is an isometry of the metric $d\tilde{s}^2$.

However we are able to prove the following:

THEOREM 2. – *The complete Kähler metric $d\tilde{s}^2$ constructed on every ellipsoidal domain in $\boldsymbol{C}^n$ $(n \geqslant 2)$ is the Bergman metric if and only if the domain is biholomorphic to the ball.*

In order to prove the theorem we need some general results about Bergman kernel of strictly pseudoconvex domains in $\boldsymbol{C}^n$ and about Cauchy-problem for $\partial\bar{\partial}$-operator.

a) Suppose $D \subset\subset \boldsymbol{C}^n$ is a strictly pseudoconvex domain in $\boldsymbol{C}^n$ with smooth boundary (bD) and φ is a strictly plurisubharmonic function defining bD. It is well known that there exist C^∞ functions on $\boldsymbol{C}^n$, F, G, such that on $\bar{D} \times \bar{D}$ $(\bar{D} = D \cup bD)$ the Bergman kernel of D, K_D, has the form:

$$K_D(z, z) = K_D(z) = F(z)/\big(\varphi^{n+1}(z)\big) + G(z) \log \varphi(z) \quad [6] \quad [7] \quad [12]$$

While almost nothing is known about function G, F is completely determined on bD by:

$$F(z)|_{bD} = \big((\det \mathfrak{L}_z)\|\operatorname{grad} \varphi(z)\|^2\big)|_{bD}$$

where $\det \mathfrak{L}_z$ is the determinant of the Levi form of bD at the point $z \in bD$.

Suppose D is an ellipsoidal domain in $\boldsymbol{C}^n$: $D = E = E(A_1, \ldots, A_n)$, then we can easily compute $F(z)|_{bE}$ and we obtain:

$$F(z)|_{bE} = \left\{2 + \sum_{\alpha=1}^{n} (4A_\alpha^2 - 1) z_\alpha \bar{z}_\alpha\right\}_{bE}. \tag{7.1}$$

b) Let $D = \{\varphi > 0\}$ be a domain in $\boldsymbol{C}^n$ $(n \geqslant 2)$ with C^k $(k \geqslant 3)$ smooth boundary $bD = \{\varphi = 0\}$. The characterization of functions $f \in C^k(bD)$ which are the restrictions of pluriharmonic functions on D has been studied by many authors [1], [4], [5]; in particular will be useful the following characterization:

LEMMA C [1]. – *Suppose the Levi form of bD is non vanishing and suppose bD is connected. If $f \in C^k(bD)$ f may be extended to a pluriharmonic function in D if and only if there exists a function $\Phi \in C^1(bD)$ such that:*

(I) $d\varphi \wedge d^c\varphi \wedge dd^c f = \Phi\, d\varphi \wedge d^c\varphi \wedge dd^c\varphi$,

(II) $d\varphi \wedge dd^c f = d\varphi \wedge d\Phi \wedge d^c\varphi + \Phi\, d\varphi \wedge dd^c\varphi$,

where $d^c = 1/(2\sqrt{-1})(\bar{\partial} - \partial)$, $d = \frac{1}{2}(\bar{\partial} + \partial)$.

Now we are ready to prove Theorem 2:

PROOF (Theorem 2). – Let $E = E(A_1, \dots, A_n)$ be an ellipsoidal domain in $\mathbf{C}^n$ $(n \geqslant 2)$. Suppose $d\tilde{s}^2$ is the Bergman metric on E, that is:

$$d\tilde{s}^2 = \sum_{\alpha,\beta=1}^{n} \left(\partial^2 \log K_E(z)/\partial z_\alpha \, \partial \bar{z}_\beta\right) dz_\alpha \otimes d\bar{z}_\beta \, .$$

Thus we have:

$$-(n+1)\, \partial\bar{\partial} \log \varphi(z) = \partial\bar{\partial} \log K_E(z) \, ,$$

where

$$\varphi(z) = 1 - \sum_{\sigma=1}^{n} \left(A_\sigma(z_\sigma^2 + \bar{z}_\sigma^2) + z_\sigma \bar{z}_\sigma\right) .$$

Necessarily:

$$\partial\bar{\partial} \log\left(\varphi^{n+1}(z) \cdot K_E(z)\right) = 0 \qquad \text{on } E \, . \tag{7.2}$$

Condition (7.2) means that

$$f(z) = \log\left(\varphi^{n+1}(z) \cdot K_E(z)\right) \tag{7.3}$$

is a pluriharmonic function on E. From (7.1) we known the boundary value of f, that is

$$f(z)_{|bE} = \log\left\{\left(2 + \sum_{\sigma=1}^{n} (4A_\sigma^2 - 1)\, z_\sigma \bar{z}_\sigma\right)\right\}_{|bE} .$$

So $d\tilde{s}^2$ is the Bergman metric of E if and only if the following Cauchy-problem for $\partial\bar{\partial}$-operator

$$\begin{cases} \partial\bar{\partial} f = 0 & \text{on } E \, , \\ f(z) = \log\left\{\left(2 + \sum_{\sigma=1}^{n} (4A_\sigma^2 - 1)\, z_\sigma \bar{z}_\sigma\right)\right\} & \text{on } bE \, . \end{cases} \tag{7.5}$$

has a solution.

Observe that for $A_1 = \dots = A_n = 0$ $f|_{bE} = 0$, then (7.5) has, in this case, the trivial solution $f = 0$, and $K_{B^n}(z) = \left(1 - \sum_{\sigma=1}^{n} z_\sigma \bar{z}_\sigma\right)^{-(n+1)}$.

In the general case we can use Lemma C. Conditions (I), (II) can be written in terms of ∂ and $\bar{\partial}$. Selecting the terms in (II) of type (1, 2) we have:

(I′) $\bar{\partial}\varphi \wedge \partial\varphi \wedge \partial\bar{\partial} f = \Phi\, \bar{\partial}\varphi \wedge \partial\varphi \wedge \partial\bar{\partial}\varphi$,

(II′) $\bar{\partial}\varphi \wedge \partial\bar{\partial} f = + \, \bar{\partial}\varphi \wedge \bar{\partial}(\Phi\, \partial\varphi)$.

First suppose $n = 2$, so condition (I′) is automatically satisfied. In particular one obtains that

$$\Phi = \frac{\bar\partial\varphi\wedge\partial\varphi\wedge\partial\bar\partial f}{\bar\partial\varphi\wedge\partial\varphi\wedge\partial\bar\partial\varphi}$$

and by direct computation:

$$\begin{aligned}(7.6)\qquad \Phi(z_1, z_2) = \{&|(2A_1\bar z_1 + z_1)z_1 + (2A_2\bar z_2 + z_2)z_2|^2(4A_1^2 - 1)(4A_2^2 - 1) +\\ &+ 2\left((4A_1^2 - 1)|2A_2\bar z_2 + z_2|^2 + (4A_2^2 - 1)|2A_1 z_1 + \bar z_1|^2\right)\}\cdot\\ &\cdot\left(2 + (4A_1^2 - 1)z_1\bar z_1 + (4A_2^2 - 1)z_2\bar z_2\right)^{-3}.\end{aligned}$$

Again a direct computation shows that condition (II′) is satisfied if and only if $A_1 = A_2 = 0$.

So Theorem 2 is true for $n = 2$.

Now suppose $n\geqslant 3$. Condition (I′) is not in general satisffied. Write

$$\bar\partial\varphi\wedge\partial\varphi\wedge(\partial\bar\partial f - \Phi\,\partial\bar\partial\varphi) = \sum_{i,j,k,l=1}^{n}\omega_{\bar i j k \bar l}\,d\bar z_i\wedge dz_j\wedge dz_k\wedge d\bar z_l$$

and consider the condition:

$(\mathrm{I}')_{\bar i i j \bar j}\quad \omega_{\bar i i j \bar j} = 0.$

Suppose $A_\alpha \neq 0$ for some $\alpha\in\{1, \dots, n\}$. We have

$$\partial\bar\partial f = \frac{\sum\limits_{\sigma=1}^{n}(4A_\sigma^2 - 1)\,dz_\sigma\wedge d\bar z_\sigma}{2 + \sum\limits_{\gamma=1}^{n}(4A_\gamma^2 - 1)z_\gamma\bar z_\gamma} - \frac{\sum\limits_{\sigma,\delta=1}^{n}(4A_\sigma^2 - 1)(4A_\delta^2 - 1)z_\sigma\bar z_\delta\,dz_\delta\wedge d\bar z_\sigma}{\left(2 + \sum\limits_{\gamma=1}^{n}(4A_\gamma^2 - 1)z_\gamma\bar z_\gamma\right)^2}$$

and condition $(\mathrm{I}')_{iijj}$ becomes:

$$(7.8)\qquad |2A_i z_i + \bar z_i|^2\cdot\left\{\frac{4A_j^2 - 1}{2 + \sum\limits_{\gamma=1}^{n}(4A_\gamma^2 - 1)z_\gamma\bar z_\gamma} - \frac{(4A_j^2 - 1)^2 z_j\bar z_j}{\left(2 + \sum\limits_{\gamma=1}^{n}(4A_\gamma^2 - 1)z_\gamma\bar z_\gamma\right)^2} - \Phi\right\} = 0.$$

This implies:

$$\Phi(z) = \frac{4A_j^2 - 1}{2 + \sum\limits_{\sigma=1}^{n}(4A_\sigma^2 - 1)z_\sigma\bar z_\sigma}\cdot\left\{1 - \frac{(4A_j^2 - 1)z_j\bar z_j}{2 + \sum\limits_{\sigma=1}^{n}(4A_\sigma^2 - 1)z_\sigma\bar z_\sigma}\right\}.$$

First for $i\neq j\neq k$ $(\mathrm{I}')_{\bar i i j \bar j}$ and $(\mathrm{I}')_{\bar i i k \bar k}$ at the point $P_i = (0, \dots, z_i, \dots, 0)$ give $A_j = A_k$, so $A_1 = \dots = A_n$.

Second for all $i\neq j\neq k$ $(\mathrm{I}')_{\bar i i j \bar j}$ and $(\mathrm{I}')_{\bar i i k \bar k}$ give $z_j\bar z_j = z_k\bar z_k$ which is absurd.

So compatibility condition (I′) is not satisfied if $n\geqslant 3$ and $A_\alpha\neq 0$ for some $\alpha\in\{1, \dots, n\}$, and the proof is complete. Q.E.D.

REMARK. – If $n = 1$, by using the « Uniformization Theorem for Riemann surfaces », we have trivially that the metric $d\tilde{s}^2$ on $E = E(A) \subset \boldsymbol{C}$ is the Bergman metric of E if and only if $A = 0$. In fact the Gaussian curvature of $d\tilde{s}^2$ is constant if and only if $A = 0$. Therefore for $n = 1$ the Bergman metric and $d\tilde{s}^2$ are uniformly equivalent on $E = E(A)$. In fact, follows immediately from Schwarz Lemma [17], that two complete Kähler metrics, G_1, G_2, with Riemannian sectional curvature bounded between two negative constants are uniformly equivalent, that is there exist two real positive numbers m, n, such that $mG_1 \geqslant G_2 \geqslant nG_1$.

Unfortunately almost nothing is known about Bergman metric of ellipsoidal domains in $\boldsymbol{C}^n$ $(n \geqslant 2)$; we think it is interesting to conclude by the following:

THEOREM B [10] – *Let M be a complete simply-connected Kähler manifold with Kähler metric G. Suppose the Riemannian sectional curvature of G is bounded from above and from below by two negative constants. Then there exists on M the Bergman metric, β, is complete and $\beta \geqslant A'G$ holds for some positive constant A'. Moreover the Bergman kernel form K_M satisfies $c_1 \Omega \geqslant K_M \geqslant c_2 \Omega$ where Ω is the volume form of G and c_1, c_2, are positive constants.*

CONJECTURE [10] – Under the hypothesis of Theorem B the Bergman metric and G are uniformly equivalent.

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