Singular Points of the Consequent Mapping (*).

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Summary. – *Consider the Ordinary Differential System*

(E) $$\dot{x} = f(t, x)\,, \qquad (t, x) \in \Omega\,, \qquad \Omega \subseteqq [t_0, \infty) \times \boldsymbol{R}^n$$

and a subset ω *of* Ω. *It is known that the consequent mapping* $\mathcal{K}$ *is upper semicontinuous at any point* $P \in \Omega$ *at which* $\mathcal{K}$ *is defined and moreover* $\mathcal{K}(P)$ *is a continuum in* $\partial\omega$. *Here we study the topological properties of the set* $\mathcal{K}(P)$ *in the case where* P *is a singular point of* $\mathcal{K}$, *i.e. there exists a solution of* (E) *through* P *which stays* (*right*) *asymptotic in* ω. *As an application we get an existence result of a general boundary value problem concerning* (E) *and we also prove that the second-order BVP*

$$x'' = f(t, x, x')\,, \qquad a_1 x(t_0) - b_1 x'(t_0) = r_1 \qquad \text{and} \qquad a_2 x(t_1) + b_1 x'(t_1) = r_2$$

has a solution.

0. – Introduction.

We consider a differential equation of the form

(E) $$\dot{x} = f(t, x)\,, \qquad (t, x) \in \Omega$$

where $\Omega \subseteq [t_0, \infty) \times \boldsymbol{R}^n$ is open and $\boldsymbol{R}$ stands for the real line.

Under some assumptions on f among which me refer only to the continuity of f and uniqueness for (E), WAŻEWSKI [13] has proved that there exists a solution x of (E) which remains (right) asymptotic in a certain set $\omega \subseteq \Omega$, namely

$$G(x|\mathrm{Dom}^+ x) \equiv \{(t, x(t)) : t \in \mathrm{Dom}^+ x = [t_0, \infty) \cap \mathrm{Dom}\, x\} \subseteq \omega\,.$$

His method is essentially based on the continuity of the so called consequent mapping $\mathcal{K}_\omega$, i.e. the mapping of ω into its boundary $\partial\omega$ under the action of solutions.

JACKSON and KLAASEN [3] and BEBERNES and SCHUUR [1] have shown that the WAŻEWSKI's result holds without the uniqueness assumption. In this case the consequent mapping $\mathcal{K}_\omega$ is an upper semi-continuous mapping, which sends a point of ω to a continuum (connected and compact) subset of $\partial\omega$. They also have pointed out, that the upper semi-continuity of $\mathcal{K}_\omega$ at the point $P \in \omega$ depends on the point

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and the function f. In [11] these results have been carried out for the case where the function f satisfies the Carathéodory conditions locally in Ω, namely $f \in \mathrm{Car}_{\mathrm{loc}}(\Omega)$ and it has been shown therein, that there is a connection between these results and the well-known [7, 8] Kneser's theorem.

Some Kneser's type topological properties of the cross-section $\mathcal{X}(\alpha; P_0)$ at the extreme (right) point $t = \alpha$ of the common domain of all solutions $\mathcal{X}(P_0)$ of (E) emanating from the point $P_0 \in \Omega$, $P_0 = (t_0, x_0)$, are recently discussed in [7, 8]. More precisely we have proved there that every connected component in $\mathcal{X}(\alpha; P_0)$ is an unbounded set.

In the above mentioned papers, the various versions of the Ważewski's theorem have been proved, by using the fact that there exists a point $P \in \omega$ (called « singular point » of $\mathcal{K}_\omega$) such that the image $\mathcal{K}_\omega(P_0)$ is not a continuum. In the present paper we are discussing the following problem: *If for a certain (singular) point* $P \in \omega$, $P = (\tau, \xi)$ *a solution* $x \in \mathcal{X}(P)$ *remains (right) asymptotic in* ω, *what kind of (topological) properties has the set* $\mathcal{K}_\omega(P)$? In proportion to the Kneser's type property at the extreme point $t = \alpha$ mentioned above, we prove that every connected component in $\mathcal{K}_\omega(P)$ « approaches » the boundary $\partial\Omega$ of Ω, whenever P is a singular point of the mapping $\mathcal{K}_\omega$. Another result of this paper is that for a given interval $[\tau, t^*)$ (depending from ω) and for any point $\hat{t} \in [\tau, t^*)$ there exist an uncountable set of solutions $x \in \mathcal{X}(P)$ whose the restriction on $[\tau, \hat{t}]$ have their graphs in ω, that is

$$G(x|[\tau, \hat{t}]) \equiv \{(t, x(t)) : \tau \leqq t \leqq \hat{t}\} \subseteq \omega .$$

Finally we give sufficient conditions, under which given any subset Z of some cross-section $\omega(t_0) = (\{t_0\} \times \boldsymbol{R}^n) \cap \omega$ of ω, every connected component of the consequent points of Z « approaches » both the sets $\partial\omega(t_0)$ and $\partial\Omega$.

As a first application, we give the following existence result for a boundary value problem of a general type: For a given set $Z \subseteq \omega(t_0)$ and any point $\hat{t}$ in an interval $[t_0, t^*)$ (depending on ω), there exists a solution x of (E) which satisfies the boundary condition

$$x(t_0) \in Z , \quad G(x|[t_0, \hat{t}]) \subseteq \omega \quad \text{and} \quad (\hat{t}, x(\hat{t})) \in \partial\omega .$$

This result for such a BVP, as for as we know, is new. Finally an existence result concerning the second-order scalar boundary value problem

$$x'' = f(t, x, x')$$

$$a_1 x(t_0) - b_1 x'(t_0) = r_1 \quad \text{and} \quad a_2 x(t_1) + b_1 x'(t_1) = r_2 ,$$

which carries out the KAPLAN et al. [4] (cf. also [6], § 3.3) results to the Carathéodory case, is given. Notice that, in this case, our proof is shorter and more formal than the one in [4], though our problem is more general.

1. – Preliminaries.

Consider the initial value problem

$$\dot{x} = f(t, x) \tag{E}$$

$$x(t_0) = x_0 \tag{C}$$

where $f \in \mathrm{Car}_{\mathrm{loc}}(\Omega)$ and $P_0 = (t_0, x_0) \in \Omega$. Let $\mathfrak{X}(P, f)$ or simply $\mathfrak{X}(P_0)$ be the family of solutions of (E)–(C), when it is well known that $\mathfrak{X}(P_0) \neq \emptyset$. A topology which associates $\mathrm{Car}_{\mathrm{loc}}(\Omega)$ is introduced by the following convergence: $\{f, f_n\} \subseteq \mathrm{Car}_{\mathrm{loc}}(\Omega)$ and $\lim f_n = f$ means that $\lim \int_I \sup_{x \in K} |f_n(t, x) - f(t, x)|\, dt = 0$, for every compact restangle $I \times K \subseteq \Omega$. Also we assume that the set $\Omega \times \mathrm{Car}_{\mathrm{loc}}(\Omega)$ is endowed with the natural product topology.

THEOREM 1.1 [11]. – *Let* $\{(P_n, f_n\}$ *be a sequence in* $\Omega \times \mathrm{Car}_{\mathrm{loc}}(\Omega)$ *such that* $\lim (P_n, f_n) = (P_0, f)$. *If* $x_n \in \mathfrak{X}(P_n, f_n)$ *for any* $n = 1, 2, \ldots$, *then there exist* $x \in \mathfrak{X}(P, f)$ *and a subsequence* $\{x_{k_n}\}$ *of* $\{x_n\}$ *such that*

$$\lim x_{k_n} = x \quad \textit{uniformly on compact subintervals of } \mathrm{Dom}\, x\,,$$

that is for any compact subinterval I *of* $\mathrm{Dom}\, x$

i) $I \subseteq \mathrm{Dom}\, x_{k_n}$ *for all large* $n = 1, 2, \ldots$;

ii) $\lim x_{k_n} = x$ *uniformly con* I.

Consider now a set $\omega \subseteq \Omega$ such that $\Omega - \bar{\omega} \neq \emptyset$.

A point $P \in \Omega \cap \partial\omega$, $P = (\tau, \xi)$ is a *point of semi-egress* of ω (with respect to the system (E)), iff there exist a solution $x \in \mathfrak{X}(P)$, a point $t_1 \in \mathrm{Dom}^- x = (-\infty, \tau] \cap \cap \mathrm{Dom}\, x$ and an $\varepsilon > 0$ such that

$$G(x|[t_1 - \varepsilon_1, t_1); P) \subseteq \omega^0 \quad \text{and} \quad G(x|[t_1, \tau]; P) \subseteq \partial\omega$$

where $G(x|I) = \{(t, x(t)) : t \in I\}$ denotes the graph of the restriction $x|I$, $I \subseteq \mathrm{Dom}\, x$.

If moreover for any solution $x \in \mathfrak{X}(P)$ there exist a point $t_2 \in \mathrm{Dom}^+ x = [\tau, \infty) \cap \cap \mathrm{Dom}\, x$ and an $\varepsilon > 0$ such that

$$G(x|[\tau, t_2]; P) \subseteq \partial\omega \quad \text{and} \quad G(x|(t_2, t_2 + \varepsilon]; P) \subseteq \Omega - \omega$$

then the point P will be called a *point of strict semi-egress* of ω.

The family of all points of semi-egress or strict semi-egress will be denoted by ω^s or ω^{ss} respectively.

A point $P \in \omega^s$ or $P \in \omega^{ss}$ is a *point of egress* or respectively, a *point of strict egress* of ω iff the above definitions are valid for $t_1 = \tau$ or $t_1 = t_2 = \tau$. Similarly, the set of all points of egress or strict egress of ω will be respectively denoted by ω^e or ω^{se}.

A point $P \in \Omega \cap \partial\omega$ is a point of *semi-ingres*; *strict semi-ingress, ingress* or *strict ingress* of the set ω (with respect the system (E)) iff, it is a point of semi-egress, strict semi-egress, egress or strict egress of the set $\Omega - \omega$, respectively.

A point $Q \in \omega^s$, $Q = (\sigma, \eta)$ with $\sigma \geqq \tau$ is a consequent of the point $P \in \omega$, $P = (\tau, \xi)$, with respect to the set ω (and the system (E)), if there exist a solution $x \in \mathcal{X}(P, Q) \equiv \mathcal{X}(P) \cap \mathcal{X}(Q)$ and a point $t_1 \in [\tau, \sigma]$ such that $G(x|[t_1, \sigma]) \subseteq \partial\omega$ and provided that $\tau < t_1$, $G(x(\tau, t_1)) \subseteq \omega^0$. For such a solution $x \in \mathcal{X}(P)$ we say that it *semi-egresses* from ω (through the point Q) and, in case $Q \in \omega^{ss}$, it *semi-egresses strictly* from ω.

The set of all consequent points of P with respect to ω (and the system (E)) will be denoted by $C(\omega; P)$. We also consider the set $S(\omega) = \{Q \in \omega\colon C(\omega; Q) \neq \emptyset\}$ which is usually called the (left) *shadow* of the set ω and let $\mathcal{K}_\omega$ be the consequent mapping on $S(\omega)$ to ω^s, defined by

$$\mathcal{K}_\omega(Q) = C(\omega; Q)\,, \qquad Q \in S(\omega)\,.$$

Finally, we shall say that a set-valued mapping F, which maps the points of a topological space X into compact subsets of another one Y is upper semi-continuous (usc) at a point $x_0 \in X$ iff for any open subset V in Y with $F(x_0) \subseteq V$ there exists a neighborhood U of x_0 such that $F(x) \subseteq V$ for every $x \in U$ (see, e.g. [2]).

The next lemmas give sufficient conditions for the upper semicontinuity of the consequent mapping and some useful properties for a class of usc mappings. We notice that the consequent mapping is included in this class.

LEMMA 1.1 [11]. – *If $P \in S(\omega)$ and every $x \in \mathcal{X}(P)$ semi-egresses strictly from ω, then the consequent mapping $\mathcal{K}_\omega$ is usc at the point P and moreover the image $\mathcal{K}_\omega(P)$ is a continuum in $\partial\omega$.*

LEMMA 1.2 [8]. – *Let X and Y be metric spaces, and let $F\colon X \to 2^Y$ be an usc mapping. If A is a connected subset of X, such that for every $x \in A$ the image $F(x)$ is a continuum, then the image $F(A) = \bigcup\{F(x)\colon x \in A\}$ is also a continuum subset of Y.*

Let A be a subset of Ω and let I be a subset of the projection $\mathrm{pr}_1 A$ of A into the first factor. We will use the notation $A(I) = A \cap (I \times \boldsymbol{R}^n)$ for the *cross-section of A at the set I* and, for brevity we set, $A(\{t\}) = A(t)$, $A([t_1, t_2]) = A[t_1, t_2]$, $A([t_1, t_2)) = A[t_1, t_2)$ and so on. Also let

$$\mathcal{X}(A) \equiv \bigcup\{\mathcal{X}(P)\colon P \in A\} \quad \text{and} \quad \mathcal{X}(P, A) \equiv \mathcal{X}(P) \cap \mathcal{X}(A)$$

and let furthermore $\mathcal{X}(\hat{t}; A) = \{(\hat{t}, x(\hat{t}))\colon x \in \mathcal{X}(A)\}$ represents the cross-section of all solutions $x \in \mathcal{X}(A)$ at the point $t = \hat{t}$. Moreover let $\mathcal{X}_\omega(A)$ be the family of all solutions $x \in \mathfrak{X}(A)$ which remain (*right*) *asymptotic in* ω, namely $x \in \mathfrak{X}_\omega(A) \Leftrightarrow \Leftrightarrow G(x|\mathrm{Dom}^+ x) \subseteq \omega$. Let also

$$\mathfrak{X}_\omega(\hat{t};\, [t_0, t_1], A) = \{(\hat{t}, x(\hat{t})) : x \in \mathfrak{X}(A) \quad \text{and} \quad G(x|[t_0, t_1]) \subseteq \omega\}$$

be the cross-section at the point $\hat{t}$ of the family of all solutions having the graph of their restrictions $x|[t_0, t_1]$ in ω, i.e.

$$\mathfrak{X}_\omega([t_0, t_1];\, A) = \{x \in \mathfrak{X}(A) : G(x|[t_0, t_1]) \subseteq \omega\}\,.$$

Finally the set $A \subseteq \Omega$ is said to be ***R****-almost compact in* Ω (cf. [12]) iff $A(I)$ is a compact subset of Ω for every compact subset I of $\mathrm{pr}_1 A$. It is clear that an $\boldsymbol{R}$-almost compact set is closed.

In the following we shall always assume that ω is an $\boldsymbol{R}$-almost compact set in Ω and moreover $\sup \mathrm{pr}_1 \omega = \sup \mathrm{pr}_1 \Omega = t^*$.

2. – Singular points of the consequent mapping.

Let $P_0 = (t_0, x_0)$ be a point in some cross-section $\omega(t_0)$ of ω. In the case where $\mathfrak{X}(P_0) = \mathfrak{X}_\omega(P_0)$, i.e. all the family $\mathfrak{X}(P_0)$ remains (right) asymptotic in ω, it is clear that the consequent mapping cannot be defined at the point P_0. On the other hand, when $\mathfrak{X}_\omega(P_0) = \emptyset$ the lemma 1.1 contains the main properties of the consequent mapping. So in the following assume both that $\mathfrak{X}_\omega(P_0) \neq \emptyset$ and $\mathfrak{X}_\omega(P_0) \neq \mathfrak{X}(P_0)$ and we are going to study the «image» $\mathcal{K}_\omega(P_0)$ only in the latter case. The main result in this section is that *every connected component* S *of* $\mathcal{K}_\omega(P_0)$ *approaches the boundary* $\partial\Omega$ *of* Ω, i.e. $\bar{S} \cap \partial\Omega \neq \emptyset$ or equivalently $\sup \mathrm{pr}_1 S = t^*$. This result is, in some sense, the dual one for the Kneser's type property of the cross-section $\mathfrak{X}(\alpha; P_0)$ at the extreme (right) point $\alpha = \sup \bigcap \{\mathrm{Dom}\, x : x \in \mathfrak{X}(P_0)\}$ (cf. [8], Th. 2.1).

LEMMA 2.1. – *If* $\omega^s = \omega^{ss}$ *and moreover* $P_0 \in \omega$, $P_0 = (t_0, x_0)$, *and* $\hat{t} \in [t_0, t^*)$ *are points such that the sets* $\mathfrak{X}_\omega(\hat{t};\, [t_0, \hat{t}], P_0)$ *and* $\mathcal{K}_\omega([t_0, \hat{t}];\, P_0) = [\mathcal{K}_\omega(P_0)][t_0, \hat{t}]$ *are nonempty, then they are both compact.*

PROOF. – Consider the compact set $\omega[t_0, \hat{t}]$ in Ω and notice that every point of the cross-section $\omega(\hat{t})$ is a point of strict semi-egress from the set $\omega[t_0, \hat{t}]$, i.e. $\omega(\hat{t}) \subseteq \subseteq (\omega[t_0, \hat{t}])^{ss}$. Consequently

$$(\omega[t_0, \hat{t}])^s = (\omega[t_0, \hat{t}])^{ss}\,.$$

On the other hand, by the compactness of $\omega[t_0, \hat{t}]$, every solution $x \in \mathfrak{X}(P_0)$ semi-egresses (strictly) from this set. Thus by Lemma 1.1, the image $\mathcal{K}_{\omega[t_0, \hat{t}]}(P_0)$, $P_0 \in \omega(t_0)$

is a continuum, where $\mathcal{K}_{\omega[t_0,\hat{t}]}$ stays for the consequent mapping defined with respect to the set $\omega[t_0, \hat{t}]$.

The results now follows by the closedness of the sets $\partial\omega$ and $\omega(\hat{t})$.

We also need the next lemma from the classical topology.

LEMMA 2.2 ([5], ch. V, § 47, point III, Th. 2). – *If A is an arbitrary proper subset of a continuum C and if S is a connected component of A, then*

$$\bar{S} \cap \overline{(C \setminus A)} \neq \emptyset \quad \text{i.e. } \bar{S} \cap \partial A \neq \emptyset .$$

PROPOSITION 2.1. – *If $\omega^s = \omega^{ss}$ and $P_0 \in \omega(t_0)$, $P_0 = (t_0, x_0)$ is a point such that $\mathcal{X}_\omega(P_0) \neq \emptyset$, then either the family $\mathcal{X}(P_0)$ remains (right) asymptotic in ω, i.e. $\mathcal{X}(P_0) = \mathcal{X}_\omega(P_0)$ or every connected component S of the set $\mathcal{K}_\omega(P_0)$ approaches the boundary $\partial\Omega$ of Ω, i.e. $\bar{S} \cap \partial\Omega \neq \emptyset$.*

PROOF. – Assume that $\mathcal{X}_\omega(P_0) \neq \mathcal{X}(P_0)$, that is $\mathcal{K}_\omega(P_0) \neq \emptyset$. Let S be a connected component of $\mathcal{K}_\omega(P_0)$ and $x \in \mathcal{X}(P_0)$ be such that $(\tau, x(\tau)) \in S$ for some $t \in \operatorname{Dom} x$. (We recall that Ω is open and ω is $\boldsymbol{R}$-almost compact in Ω.)

We are going to prove that $\bar{S} \cap \partial\Omega \neq \emptyset$. By assuming the contrary let $t_1 = \max \operatorname{pr}_1 S$, when $S(t_1) \subseteq \Omega$. Thus, by the $\boldsymbol{R}$-almost compactness of ω, there exists a point $\hat{t} > t_1$ such that the set $\omega[t_0, \hat{t}]$ is a compact subset of Ω. Moreover, Lemma 2.1 ensures that the cross-section $\mathcal{X}_\omega(\hat{t}, [t_0, \hat{t}]; P_0)$ of the family $\mathcal{X}_\omega([t_0, \hat{t}]; P_0) = \{x \in \mathcal{X}(P_0) \colon G(x|[t_0, \hat{t}]) \subseteq \omega\}$ is also a compact subset of the continuum $\mathcal{K}_{\omega[t_0,\hat{t}]}(P_0)$. Now, since clearly

$$\omega^{ss} \cap \mathcal{K}_{\omega[t_0,\hat{t}]}(P_0) = (\omega[t_0, \hat{t}])^{ss} \cap \mathcal{K}_\omega(P_0)$$

the connected component S in $\mathcal{K}_\omega(P_0)$ is also a connected component in $[\mathcal{K}_{\omega[t_0,\hat{t}]}(P_0)] \setminus \omega(\hat{t})$, due to the fact that $\hat{t} > t_1$. Thus in view of Lemma 2.2, the component S approaches the set $[\mathcal{K}_{\omega[t_0,\hat{t}]}(P_0)] \cap \omega(\hat{t}) = \mathcal{X}_\omega(\hat{t}; [t_0, \hat{t}], P_0)$, which by the definition of t_1 is a contradiction.

PROPOSITION 2.2. – *If the assumptions of Proposition 2.1 are fulfilled then, either for any $\hat{t} \in [t_0, t^*)$ the cross-section $\mathcal{X}_\omega(\hat{t}; P_0)$, of all solutions which remain (right) asymptotic in ω, coincides with the continuum $\mathcal{X}(\hat{t}; P_0)$ (and is contained in ω^0), or there exists a point $t_1 \in [t_0, t^*)$ such that for any $\hat{t} \in [t_1, t^*)$ every connected component C of the cross-section $\mathcal{X}_\omega(\hat{t}; [t_0, \hat{t}], P_0)$ approaches the boundary $\partial\omega$, i.e. $\bar{C} \cap \mathcal{K}_\omega(P_0) \neq \emptyset$.*

PROOF. – If $\mathcal{X}_\omega(P_0) = \mathcal{X}(P_0)$ then the result is implied by the Kneser's theory (cf. [7]). So, assume that $\mathcal{X}_\omega(P_0) \neq \mathcal{X}(P_0)$. Set $t_1 = \min \operatorname{pr}_1 \mathcal{K}_\omega(P_0)$ and take $\hat{t} \in [t_1, t^*)$. By Lemmas 1.1 and 2.1, the set $\mathcal{K}_\omega([t_0, \hat{t}]; P_0)$ is compact and the $\mathcal{K}_{\omega[t_0,\hat{t}]}(P_0)$ is a continuum. Thus Lemma 2.2 is applicable and the result follows.

A basic statement of the assumptions in the propositions above is the existence of a point $P_0 \in \omega(t_0)$ such that $\mathcal{X}_\omega(P_0) \neq \emptyset$, i.e. the existence of a solution $x \in \mathcal{X}(\omega(t_0))$

which remains (right) asymptotic in ω. About this fact we refer here to some known results but first we state the following assumptions.

(H_1) *The set* ω *is* $\mathbf{R}$*-almost compact in* Ω, $\sup \mathrm{pr}_1 \omega = \sup \mathrm{pr}_1 \Omega = t^*$ *and* $\partial\omega(t_0, t^*) = \omega^e = \omega^{se}$.

(H_2) $\omega^s = \omega^{ss}$ *and there exist a subset* Z *of* $\omega^0 \cap \omega^s$ *and a retraction of* ω^s *onto* $\omega^s \cap Z$ *but there does not exist retraction of* Z *onto the set* $\omega^s \cap Z$.

(H_3) i) $f(t, 0) = 0$, *for any* $t \geqq t_0$ *and the zero function is the only solution of the initial value problem* (E), $x(t_1) = 0$, *for any* $t_1 \geqq t_0$.

ii) $\omega^s = \omega^{ss}$ *and* $(\Omega - \omega)^s = (\Omega - \omega)^{ss}$.

iii) *There exist closed sets* $Q_1, Q_2, \ldots, Q_p$ *such that* $\bigcup_{i=1}^{p} Q_i = \omega^{ss}$ *and* $Q_i \cap Q_j = \emptyset$ *or* $Q_i \cap Q_j \subseteq Q_0$ *for* $i \neq j$, *where* $Q_0 = \{(t, x) \in \Omega : x = 0\}$.

iv) *There exist a continuum* Z *in* $\omega(t_0)$ *and indeces* $i_1, i_2, \ldots, i_k$, $2 \leqq k \leqq p$ *such that* $Q_{i_n} \cap Z \neq \emptyset$, $n = 1, 2, \ldots, k$.

LEMMA 2.3 [10]. – *Under the assumption* (H_1), *there exists a solution* $x \in \mathfrak{X}(P_0)$, $P_0 \in \omega(t_0)$ *which remains (right) asymptotic in* ω.

LEMMA 2.4 [11]. – *Under the assumption* (H_2) *there exist a point* $P_0 \in Z$ *and a solution* $x \in \mathfrak{X}(P_0)$ *remaining (right) asymptotic in* ω.

LEMMA 2.5 [9]. – *Suppose that* (H_3) *holds. Then the result of Lemma* 2.4 *remains valid.*

Now we are ready to formulate the first of our main results of this section. It is not hard to see that its proof follows by Proposition 2.1, 2.2 and Lemmas 2.3, 2.4 and 2.5.

THEOREM 2.1. – *Assume that the set* ω *is* $\mathbf{R}$*-almost compact in* Ω, $\sup \mathrm{pr}_1 \omega = \sup \mathrm{pr}_1 \Omega = t^*$ *and at least one from the assumptions* (H_1), (H_2) *or* (H_3) *is valid. Then there exists a point* $P_0 \in \omega(t_0)$ *such that, either the continuum* $\mathfrak{X}(P_0)$ *remains (right) asymptotic in* ω *(thus for any* $\hat{t} \in [t_0, t^*)$ $\mathfrak{X}(\hat{t}; P_0)$ *is also a continuum in* ω^0*) or for any* $\hat{t} \in \mathrm{pr}_1 \mathcal{K}_\omega(P_0)$,

$$\bar{S} \cap \partial\Omega \neq \emptyset \quad \text{and} \quad \bar{C} \cap \mathcal{K}_\omega(P_0) \neq \emptyset$$

for all connected components S *and* C *in* $\mathcal{K}_\omega(P_0)$ *and* $\mathfrak{X}_\omega(\hat{t}; [t_0, \hat{t}]; P_0)$ *respectively.*

In the special case where $\Omega = [t_0, \infty) \times \mathbf{R}$ *we get the following result:*

COROLLARY 2.1. – *In addition to the assumptions of the previous theorem we assume* *that* $\mathcal{X}_\omega(P_0) \neq \mathcal{X}(P_0)$, *where* P_0 *is guaranteed by the theorem. Then the set* $\mathcal{K}_\omega(P_0)$ *has two connected components* S_1 *and* S_2. *Furthermore if* S_1 *and* S_2 *are both nonempty then*

$$\sup \operatorname{pr}_1 S_1 = \sup \operatorname{pr}_1 S_2 = \infty \quad \textit{and} \quad \partial\omega[t_1, \infty) = \omega^{ss}[t_1, \infty) \subseteq S_1 \cup S_2$$

where $t_1 = \max\{\min \operatorname{pr}_1 S_1, \ \min \operatorname{pr}_1 S_2\}$.

Up to now, we have studied the behavior of the connected components of $\mathcal{K}_\omega(P_0)$ to the « right ». Next we deal with the following problem: « *Is there a nonempty set* $Z \subseteq \omega(t_0)$ *such that*

$$S \cap \partial\omega(t_0) \neq \emptyset \quad \textit{and} \quad \bar{S} \cap \partial\Omega \neq \emptyset$$

for every connected component S *of* $\mathcal{K}_\omega(Z)$? » We give a positive answer to this problem but first we give the following lemma:

LEMMA 2.6. – *Suppose that* Z *is a compact subset of the cross-section* $\omega(t_0)$ *and* $\omega^s = \omega^{ss}$. *Then the set* $\mathcal{X}_\omega(Z) = \bigcup\{\mathcal{X}_\omega(P) : P \in Z\}$ *of solutions which start from* Z *and remain asymptotic in* ω *is compact* (*hence its cross-section* $Z_\omega = Z_\omega(t_0; Z)$ *is also compact*).

PROOF. – Let $\{x_n\}$ be a sequence in $\mathcal{X}_\omega(Z)$. In view of Theorem 1.1, we may assume that

$$\lim x_n = x \quad \text{uniformly on every compact subset of } \operatorname{Dom} x\,.$$

Now, if $x \notin \mathcal{X}_\omega \equiv \mathcal{X}_\omega(\omega(t_0))$ then $(\hat{t}, x(\hat{t})) \in \Omega - \omega$ for some $\hat{t} \in \operatorname{Dom} x$. But then, we would have $(\hat{t}, x_n(\hat{t})) \in \Omega - \omega$, for all large $n = 1, 2, \ldots$ which is a contradiction. Finally the compactness of Z implies that $x \in \mathcal{X}_\omega(Z)$.

Now consider the following mapping:

$$I: P \to \operatorname{pr}_1 \mathcal{K}_\omega(P)\,, \qquad P \in S(\omega)$$

where recall that $S(\omega)$ is the (left) shadow of the set ω.

Moreover, by Lemma 1.1, this multi-valued mapping is usc at any point P_0 for which $\mathcal{X}_\omega(P_0) = \emptyset$ and hence clearly the image $I(P_0)$ is then a closed subinterval of the set $[t_0, t^*)$. Now, our purpose is to study the case where $\mathcal{X}_\omega(P_0) \neq \emptyset$. Then by Proposition 2.1, the image $I(P_0)$ is an interval of the form $[t_1, t^*)$ where $t_1 = \min \operatorname{pr}_1 \mathcal{K}_\omega(P_0)$.

THEOREM 2.2. – *Suppose that the set* ω *is* $\boldsymbol{R}$*-almost compact in* Ω *and* (H_3) (*or* (H_2) *for some continuum* $Z \subseteq \omega(t_0)$) *is satisfied. Then every connected component* S *of* $\mathcal{K}_\omega(Z)$ *approaches the boundary* $\partial\Omega$ *of* Ω, *i.e.* $\bar{S} \cap \partial\Omega \neq \emptyset$. *Moreover*

$$I(Z) = \operatorname{pr}_1 S = [t_0, t^*)$$

where $t^* = \sup \operatorname{pr}_1 \omega = \sup \operatorname{pr}_1 \Omega$.

PROOF. – Let $Z \subseteq \omega(t_0)$ be a continuum which satisfies (H_3) (or H_2). By Theorem 2.1, we have $\mathcal{X}_\omega(Z) \neq \emptyset$ and so by Lemma 2.6, the cross-section $Z_\omega = \mathcal{X}_\omega(t_0; Z)$ is a compact subset of ω^0.

Let S be any connected component in $\mathcal{K}_\omega(Z)$ and let S_0 be the connected component in $Z \setminus Z_\omega$ such that $S \cap \mathcal{K}_\omega(S_0) \neq \emptyset$. Lemma 2.2 and the assumption (H_1) (or H_2) imply then

$$\bar{S}_0 \cap Z_\omega \neq \emptyset \quad \text{and} \quad S_0 \cap \omega^{ss} \neq \emptyset . \tag{1}$$

Clearly $\mathcal{X}_\omega(S_0) \neq \emptyset$ and thus, by Lemma 1.2, the set $\mathcal{K}_\omega(S_0)$ is a connected subset of S. By (1) it is clear that $\min \mathrm{pr}_1 \mathcal{K}_\omega(S_0) = \min \mathrm{pr}_1 S = t_0$. We examine now the following two possible cases:

i) Every solution $x \in \mathcal{X}([\partial S_0] \cap Z_\omega)$ remains (right) asymptotic in ω, that is

$$\mathcal{X}_\omega(\omega^0 \cap \partial S_0) = \mathcal{X}(\omega^0 \cap \partial S_0) . \tag{2}$$

In order to prove that S approaches the boundary $\partial\Omega$ of Ω it is enough to prove that $\mathcal{K}_\omega(S_0)$ has the same property, since $\mathcal{K}_\omega(S_0) \subseteq S$. By assuming that $\overline{[\mathcal{K}_\omega(S_0)]} \cap \partial\Omega = \emptyset$, and setting $t_1 = \max \mathrm{pr}_1 \mathcal{K}_\omega(S_0)$ we have $t_1 < t^*$ and observe that the set $\omega[t_0, t_1]$ is a compact subset of Ω. In view of (1) and Theorem 1.1, there exist a sequence $\{x_n\}$ of solutions in $\mathcal{X}(S_0)$ and an $x \in \mathcal{X}([\partial S_0] \cap Z_\omega)$ such that $\lim x_n = x$ uniformly on every compact subset of Dom x. But then, by virtue of (2), x remains asymptotic in ω and so we would have $(t_1, x(t_1)) \in \omega^0$. Consequently we would get $(t_1, x_n(t_1)) \in \omega^0$ for all large $n = 1, 2, \ldots$ which, by the choise of S_0, is a contradiction.

ii) There exists a point $P_0 \in [\partial S_0] \cap Z_\omega$ such that $\mathcal{K}_\omega(P_0) \neq \emptyset$. Then, by Theorem 2.1, any connected component S^* of $\mathcal{K}_\omega(P_0)$ satisfies $\bar{S}^* \cap \partial\Omega \neq \emptyset$. Now, if we prove that $[\mathcal{K}_\omega(S_0)] \cap S^* \neq \emptyset$, then S will have the desired properties, since clearly $[\mathcal{K}_\omega(S_0)] \cup S^* \subseteq S$. By assuming the contrary, we get $t_1 = \max \mathrm{pr}_1 \mathcal{K}_\omega(S_0) < < \min \mathrm{pr}_1 S^*$. Let $\{x_n\}$ be a sequence in $\mathcal{X}(S_0)$ and x be a solution in $\mathcal{X}(P_0)$ such that $\lim x_n = x$ uniformly on compact subintervals of Dom x. Then again we get easily the same contradiction as in the first case.

THEOREM 2.3. – *If the assumption* (H_1) *is satisfied and in addition the cross-section* $\omega(t_0)$ *of* ω *is a continuum, then*

$$S \cap \partial\omega(t_0) \neq \emptyset \quad \textit{and} \quad \bar{S} \cap \partial\Omega \neq \emptyset \tag{S_2}$$

for every connected component S *of the set* $\mathcal{K}_\omega(\omega(t_0))$.

PROOF. – The proof goes along the lines of the proof of the previous theorem, under obvious modifications.

3. – Applications.

On the base of the results of the previous section we give here two existence results for boundary value problems. The first of these is new even for the particular case where f is continuous and it is of a general type. The second one refers to Carathéodory systems though we mention that Kaplan, Lasota and Yorke have discussed the same subject in [4] (cf. also [6], § 3.3) for the continuous case.

Consider an $\boldsymbol{R}$-almost compact set ω such that $\sup \mathrm{pr}_1 \omega = \sup \mathrm{pr}_1 \Omega = t^*$ and a subset Z of its cross-section $\omega(t_0)$. We are going to show that for any point $\hat{t} \in [t_0, t^*)$, the BVP

$$\dot{x} = f(t, x) \tag{E}$$

$$x(t_0) \in Z\,, \qquad (\hat{t}, x(\hat{t})) \in \partial\omega \tag{P}$$

has a solution x such that $G(x|[t_0, \hat{t}]) \subseteq \omega$.

Proposition 3.1. – *Assume that* $\mathcal{X}_\omega(P_0) \neq \emptyset$ *and* $\mathcal{X}_\omega(P_0) \neq \mathcal{X}(P_0)$ *for some* $P_0 \in \omega$, $P_0 = (t_0, x_0)$. *If moreover* $\omega^s[t_0, t^*) = \omega^{ss}[t_0, t^*)$, *then for every* $\hat{t} \in I(P_0) = [t_1, t^*)$ *there exists a solution* $x \in \mathcal{X}(P)$ *such that* $G(x|[t_0, \hat{t}]) \subseteq \omega$ *and* $(\hat{t}, x(\hat{t})) \in \partial\omega$. *Moreover there exists a solution* $x \in \mathcal{X}_\omega(P_0)$ *such that*

$$\mathrm{dist}\big(\partial\omega, G(x|[t_0, t^*))\big) = \lim_{t \to t^*} \mathrm{dist}\,[\partial\Omega, (t, x(t))] = 0\,.$$

Proof. – The first part of the conclusion follows at once by Proposition 2.1, in view of the definition of the interval $I(P_0) = [t_1, t^*)$. Let now a sequence $\{x_n\}$ where x_n is a solution of the BVP (E)–(P) with $\hat{t} = t_n$, $n = 1, 2, \dots$ and such that $\lim t_n = t^*$. Now, by Theorem 1.1 and a diagonalization argument (cf. [10]) we obtain a limit function of $\{x_n\}$, say $x \in \mathcal{X}(P_0)$ which clearly has the desired property.

A consequence of Proposition 2.2 is the next one.

Proposition 3.2. – *If the assumptions of the previous proposition are satisfied then for every* $\hat{t} \in (t_1, t^*)$ *there exist an uncountable set of solutions* $x \in \mathcal{X}(P_0)$ *with* $G(x|[t_0, \hat{t}]) \subseteq \omega^0$.

The following two theorems (and especially the second one) constitute the existence result of the BVP (E)–(P). Notice that their proofs are a direct consequent of the Propositions 3.1, 3.2 and Theorems 2.1 and 2.2.

Theorem 3.1. – *Assume that the set* ω *is* $\boldsymbol{R}$*-almost compact in* Ω, $\sup \mathrm{pr}_1 \omega = \sup \mathrm{pr}_1 \Omega = t^*$ *and at least one from the assumptions* (H_1), (H_2) *or* (H_3) *is valid. Then there exists a point* $P_0 \in \omega(t_0)$ *such that either* $\mathcal{K}_\omega(P_0) = \emptyset$, *or for every* $\hat{t} \in I(P_0)$

there exist solutions $x \in \mathfrak{X}(P_0)$ *and* $\hat{x} \in \mathfrak{X}_\omega(P_0)$ *such taht*

$$G(x|[t_0, \hat{t}]) \subseteq \omega\,, \quad (\hat{t}, x(\hat{t})) \in \partial\omega \quad \text{and} \quad \lim_{t\to t^*} \operatorname{dist}\big(\partial\Omega, (t, x(t))\big) = 0\,.$$

THEOREM 3.2. – *Suppose that the set* ω *is* $\boldsymbol{R}$*-almost compact,* $\sup \operatorname{pr}_1 \omega = \sup \operatorname{pr}_1 \Omega = t^*$ *and* (H_3) *(or* (H_2) *for some continuum* $Z \subseteq \omega(t_0)$*) is satisfied. Then for every* $\hat{t} \in [t_0, t^*)$.

i) *There exists a solution of the boundary value problem* (E)–(P).

ii) *The set,* $\mathfrak{X}_\omega([t_0, \hat{t}]; Z) = \{x \in \mathfrak{X}(Z) \colon G(x|[t_0, \hat{t}]) \subseteq \omega\}$ *of all solutions* $x \in \mathfrak{X}(Z)$ *whose the restrictions on* $[t_0, \hat{t}]$ *have their graphs in* ω*, is uncountable and* $\lim_{t\to t^*} \operatorname{dist}\big(\partial\Omega, (t, x(t))\big) = 0$ *for a certain solution* $x \in \mathfrak{X}_\omega(Z) \cap \mathfrak{X}_\omega([t_0, \hat{t}]; Z)$.

As a second application we shall proove that the BVP

$$x'' = f(t, x, x') \tag{3}$$

$$L_1(x) = r_1\,, \quad L_1(x) \equiv a_1 x(t_1) - b_1 x'(t_1) \tag{4}$$

$$L_2(x) = r_2\,, \quad L_2(x) \equiv a_2 x(t_2) + b_2 x'(t_2) \tag{5}$$

has a solution x bounded on $[t_1, t_2]$ by two functions α and β given below and $f \in \mathrm{Car}_{\mathrm{loc}}([t_1, \infty) \times \boldsymbol{R}^2)$.

Let α and β be two real functions in $C^2([t_1, t_2], \boldsymbol{R})$ $\big(C^2([t_1, t_2], \boldsymbol{R})$ is the space of twice differentiable on $[t_1, t_2]$ real functions$\big)$ such that

$$\alpha(t) \leqq \beta(t)\,, \quad \alpha''(t) \geqq f\big(t, \alpha(t), \alpha'(t)\big) \quad \text{and} \quad \beta''(t) \leqq f\big(t, \beta(t), \beta'(t)\big)\,, \quad t_1 \leqq t \leqq t_2 \tag{6}$$

and

$$L_i(\alpha) \leqq r_i \leqq L_i(\beta)\,, \quad i = 1, 2\,. \tag{7}$$

We consider now the set

$$\omega \equiv \{(t, x, y) \in \Omega \colon t_1 \leqq t \leqq t_2 \quad \text{and} \quad \alpha(t) \leqq x \leqq \beta(t)\}$$

and its faces

$$Q_\alpha \equiv \{(t, x, y) \in \partial\omega \colon x = \alpha(t)\} \quad \text{and} \quad Q_\beta = \{(t, x, y) \in \partial\omega \colon x = \beta(t)\}$$

as well as their subsets

$$Q'_\alpha = \{(t, x, y) \in Q_\alpha \colon y \leqq \alpha'(t)\} \quad \text{and} \quad Q_\beta = \{(t, x, y) \in Q_\beta \colon y \geqq \beta'(t)\}\,.$$

Finally, assume that f is continuous on some neighborhoods V_α and V_β of the « curves »

$$\gamma_\alpha = \{(t, \alpha(t), \alpha'(t)) : t_1 \leqq t \leqq t_2\} \quad \text{and} \quad \gamma_\beta = \{(t, \beta(t), \beta'(t)) : t_1 \leqq t \leqq t_2\}$$

respectively and furthermore f satisfies a Nagumo condition on ω, that is there exists a positive and locally measurable function φ on $[0, \infty)$ such that the function $s/\varphi(s)$ is locally bounded on $[0, \infty)$, $\int_0^\infty [s/\varphi(s)]\, ds = \infty$ and $|(h(t, x, y))| \leqq \varphi(|y|)$ for almost all $(t, x, y) \in \omega$.

THEOREM 3.3. – *Assume that there exist two real functions α and β which satisfy* (6) *and* (7). *If f satisfies on ω the above Nagumo condition, then there exists a solution x of the* BVP (3), (4) *and* (5) *such that $\alpha(t) \leqq x(t) \leqq \beta(t)$, for all $t \in [t_1, t_2]$.*

PROOF. – Consider the set

$$S_1 = \{(t_1, \xi, \eta) : a_1\xi - b_1\eta = r_1 \quad \text{and} \quad \alpha(t_1) \leqq \xi \leqq \beta(t_1)\}$$

which is clearly a continuum in the cross-section $\omega(t_1)$ of ω. Since, $x''(t)$ exists a any point $t \in [t_1, t_2]$ such that $x(t) = \alpha(t)$ and $x'(t) = \alpha'(t)$ or $x(t) = \beta(t)$ and $x'(t) = \beta'(t)$ (by the continuity of f in the neighborhoods V_α and V_β) it is easy to show (cf. also [3, 11]) that

$$(8) \qquad Q'_\alpha \cup Q'_\beta \cup \omega(t_2) = \omega^s = \omega^{ss} \quad \text{and} \quad (Q_\alpha - Q'_\alpha) \cup (Q_\beta - Q'_\beta) = (\Omega - \omega)^s = (\Omega - \omega)^{ss}\,.$$

Consequently, by the Nagumo condition, every solution $x \in \mathfrak{X}(S_1)$ has its derivative bounded and so it semi-egresses (strictly) from the set ω. Thus, by Lemmas 1.1 and 1.2, the set $\mathcal{K}_\omega(S_1)$ is also a continuum and hence there exists a solution $x_\alpha \in \mathfrak{X}(S_1)$ such that

$$\alpha(t) \leqq x_\alpha(t) \leqq \beta(t)\,, \quad t_1 \leqq t \leqq t_2 \quad \text{and} \quad x_\alpha(t_2) = \alpha(t_2)\,.$$

By Lemma 2.2 now, the connected component S_2 in $[\omega(t_2) \cap \mathcal{K}_\omega(S_1)]Q$ containing the point $(t_2, x_\alpha(t_2), x'_\alpha(t_2))$ reaches the set Q_β, that is there exists a solution $x_\beta \in \mathfrak{X}(S_1)$ such that

$$\alpha(t) \leqq x_\beta(t) \leqq \beta(t)\,, \quad t_1 \leqq t \leqq t_2\,, \quad x_\beta(t_2) = \beta(t_2) \quad \text{and} \quad (t_2, x_\beta(t_2), x'_\beta(t_2)) \in S_2\,.$$

(In the case where $\alpha(t_2) = \beta(t_2)$ we set $x_\alpha \equiv x_\beta$ and so we obtain $S_2 = \{(t_2, x_\alpha(t_2) x'_\alpha(t_2))\}$). In view of (8) and the definition of the consequent mapping we have

$(t_2, x_\alpha(t_2), x'_\alpha(t_2)) \in Q'_\alpha$ and $(t_2, x_\beta(t_2), x'_\beta(t_2) \in Q'_\beta$, that is

$$x'_\alpha(t_2) \leqq \alpha'(t_2) \quad \text{and} \quad x'_\beta(t_2) \geqq \beta'(t_2)\,. \tag{9}$$

Now without loss of generality we may assume that $b_i > 0$ $(i = 1, 2)$ and then by (7) and (9) we obtain

$$L_2(x_\alpha) \leqq r_2 \leqq L_2(x_\beta)\,.$$

Thus, by the connectedness of the component S_2 and the continuity of the function L_2 we get a solution $x \in \mathfrak{X}(S_1)$ such that $L_2(x) = 0$ and $\alpha(t) \leqq x(t) \leqq \beta(t)$ for all $t \in [t_1, t_2]$.

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