L^2-Lower Semicontinuity of Functionals of Quadratic Type (*) (**).

NICOLA FUSCO - GIOCONDA MOSCARIELLO (Napoli) (***)

Summary. – *A representation formula for the L^2-lower semicontinuous envelope of a quadratic integral of Calculus of Variations is given. Some particular cases are explicited in the details.*

Let us consider the functional of the Calculus of Variations

$$F(\Omega, u) = \int_\Omega \sum_{i,j} a_{ij}(x)\, D_i u\, D_j u\, dx \qquad u \in H^{1,2}(\Omega)$$

where $a_{ij} = a_{ji} \in L^\infty(R^n)$, $\sum_{i,j} a_{ij} z_i z_j \geqslant 0$. It is well known that F is weakly sequentially lower semicontinuous (l.s.c.) in $H^{1,2}(\Omega)$. The situation is completely different if we consider topologies such as $L^p(\Omega)$, $p \geqslant 1$. There are classical conditions (see [11], e.g.)

$$\begin{cases} \sum_{i,j} a_{ij}(x) z_i z_j > 0\,, & \forall x,\ z \in R^n \\ a_{ij} \in C^0(R^n)\,, & \end{cases}$$

that ensure F to be L^2-l.s.c., but it is possible to give counterexamples (see [1], [8]) showing that this is not the case at all. In particular it has been proved ([8]) that a necessary and sufficient condition for the functional

$$\int_\Omega a(x) |\dot{u}(x)|^2\, dx$$

to be L^2-l.s.c. on $H^{1,2}(\Omega)$ is that $\forall x$ a.e.

$$a(x) = \lim_{\varepsilon \to 0^+} \left(\frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} \frac{1}{a(t)}\, dt\right)^{-1}.$$

In this paper we extend the preceding results giving a formula for the L^2-l.s.c. envelope of $F(\Omega, u)$.

(*) Entrata in Redazione il 13 maggio 1981.

(**) This paper has been supported by G.N.A.F.A. (Gruppo Nazionale per l'Analisi Funzionale e le sue Applicazioni).

(***) Istituto di Matematica « R. Caccioppoli », Via Mezzocannone, 8 - 80134 Napoli.

From previous results (see [1]) it was known that this envelope is still an integral, namely quadratic, functional. We show that its coefficients $b_{ij}(x)$ are such that $\forall z \in R^n$ and $\forall x$ a.e.

$$\sum_{i,j} b_{ij}(x)\, z_i z_j = \limsup_{|Q|\to 0} \operatorname{Inf}\left\{\frac{1}{\operatorname{mis} Q}\int_Q \sum_{i,j} a_{ij}(x)\, D_i u\, D_j u\, dx: \quad u \in z\cdot x + H^{1,2}(Q)\right\}$$

where Q is any cube in R^n containing x. This formula was proved in [4] in the particular case of coercive functionals in which F is L^2-l.s.c. and so $b_{ij} = a_{ij}$. By this formula we obtain (when $n = 1$) a result proved in [8] and also a general sufficient condition for F to be l.s.c.

The technics we use here are related to the maximum principle for uniformly elliptic operators in divergence form and to the duality, but we often employ the arbitrariness of the open set Ω in which is defined the functional F.

In the particular case that $a_{ij} = \delta_{ij} a_i(x)$ and $a_i(x)$ are products of a measurable function of x_i and of a function of the other variables or in the case $a_{ij} = a_{ij}(x_1)$ we give an explicit calculation of the coefficients b_{ij} of the l.s.c. envelope of F and hence necessary and sufficient conditions for F to be L^2-l.s.c.

All the previous results concern the semicontinuity of the functional $F(\Omega, u)$ on the space $H^{1,2}(\Omega)$. But it is known that the minimum points of $F(\Omega, u)$ do not always belong to $H^{1,2}(\Omega)$ and that the best space where to find them is $T(\Omega) = \{u \in H^{1,1}_{\text{loc}}(\Omega): F(\Omega, u) < +\infty\}$. So it is useful to obtain semicontinuity results for $F(\Omega, u)$ in $T(\Omega)$. At the end of section § 2 we prove a sufficient condition for F to be l.s.c. that is valid up to the space $T(\Omega)$.

Of course, a similar problem arises in considering the Γ-convergence of a sequence of functionals such as

$$F_h(\Omega, u) = \int_\Omega \sum_{i,j} a^h_{ij}(x)\, D_i u D_j u\, dx$$

whith respect to the L^2-topology.

It is known that there exist measurable functions $a_{ij}(x)$, $i, j = 1, \ldots, n$, such that $\forall u \in H^{1,2}(\Omega)$

$$(*) \qquad \Gamma(L^2(\Omega)^-) \lim_h F_h(\Omega, u) = \int_\Omega \sum_{i,j} a_{ij}(x)\, D_i u\, D_j u\, dx\,.$$

In the section § 4 we prove that, with reasonable assumptions on the coefficients a^h_{ij}, the relation $(*)$ is still valid up to the space $T(\Omega)$. This result is shown using some technics introduced by F. C. LIU in [7] and some results on maximal functions in weighted Sobolev spaces proved in [2].

1. – Definitions, notations and preliminary results.

Let be Ap_n the family of all bounded open sets of R^n, $\Omega \in Ap_n$ and $x = (x_1, \dots, x_n) \in R^n$.

We consider functionals of the type:

$$F(\Omega, u) = \int_\Omega \sum_{i,j} a_{ij}(x) D_i u \, D_j u \, dx$$

where $\Omega \in Ap_n$, $u \in H^{1,2}(\Omega)$ and $[a_{ij}]$ is a symmetric $n \times n$ matrix of bounded measurable functions on R^n such that

$$0 \leqslant \sum_{i,j}^{n} a_{ij}(x) z_i z_j \leqslant \Lambda |z|^2 \tag{1.1}$$

$\forall x$ a.e. and $\forall z \in R^n$.

Let us introduce the lower semicontinuous envelope (l.s.e.) of $F(\Omega, u)$ with respect to the strong $L^2(\Omega)$ topology:

$$\bar{F}(\Omega, u) = sc^-(L^2(\Omega)) F(\Omega, u) , \qquad u \in H^{1,2}(\Omega) .$$

It's easy to verfy that:

$$\bar{F}(\Omega, u) = \operatorname{Inf}\left\{\liminf_h F(\Omega, u_h) : u_h \in H^{1,2}(\Omega), \; u_h \to u \text{ in } L^2(\Omega)\right\}.$$

By this formula we deduce that

$$\begin{cases} \forall u \in H^{1,2}(\Omega) \; \exists \{u_h\}_h \subset H^{1,2}(\Omega) : u_h \to u \text{ in } L^2(\Omega) \qquad \text{and} \\ \bar{F}(\Omega, u) = \lim_h F(\Omega, u_h) . \end{cases} \tag{1.2}$$

It is known (see [1], [9]) that if F is a quadratic functional also $\bar{F}$ is a quadratic functional. So there exist some bounded measurable functions $b_{ij}(x)$ such that

$$\bar{F}(\Omega, u) = \int_\Omega \sum_{i,j} b_{ij}(x) D_i u \, D_j u \, dx \qquad \forall u \in H^{1,2}(\Omega) . \tag{1.3}$$

In particular it can be proved that

PROPOSITION 1.1. – *If* $a_{ij} = a_{ij}(x_1)$, *then also the functions* b_{ij} *verifying* (1.3) *depend only on* x_1.

PROOF. – If Ω is a bounded open set, let us consider $u(x) = z\cdot x$, with $z \in R^n$ and $y = (0, \bar{x}_2, \dots, \bar{x}_n)$. From (1.2) we have that $\exists \{u_h\}_h \subset H^{1,2}(\Omega)$: $u_h \to u$ in $L^2(\Omega)$ and

$$\int_\Omega \sum_{i,j} b_{ij}(x) z_i z_j \, dx = \lim_h \int_\Omega \sum_{i,j} a_{ij}(x_1) D_i u_h D_j u_h \, dx .$$

Then, taking $u^y(x) = u(x-y)$, and $u_h^y(x) = u_h(x-y)$

$$\int_{y+\Omega} \sum_{i,j} b_{ij}(x) z_i z_j \, dx = \lim_h \int_{y+\Omega} \sum_{i,j} a_{ij}(x_1) D_i u_h^y(x) D_j u_h^y(x) \, dx .$$

Since $a_{ij} = a_{ij}(x_1)$, the integrals at the right side are equal and so

$$\int_{y+\Omega} \sum_{i,j} b_{ij}(x) z_i z_j \, dx$$

does not depend on y. This gives the proof, provided that z and Ω are arbitrary.

Let us recall now some results about the duality that we shall use in the following.

If X and X^* are two locally convex topological vector spaces (l.c.t.v.s.) in duality and $\langle\cdot,\cdot\rangle$ is the duality between them, for any $f\colon X \to \bar{R}$, we indicate with f^* the Young-Fenchel transform of f, i.e.

$$f^*(v^*) = \sup_{v\in X} \{\langle v, v^*\rangle - f(v)\} \qquad v^* \in X^*.$$

In the case of integral functionals the expression of the Young-Fenchel transform may be obtained using the following

PROPOSITION 1.2 (see [5]). – *If $f(x, z)\colon \Omega\times R^n \to \bar{R}$ is a Carathéodory function and if $G(u) = \int_\Omega f(x, u(x))\, dx$, $u \in L^p(\Omega)$, is finite on some $u_0 \in L^p(\Omega)$, then $G^*(u^*) = \int_\Omega f^*(x, u^*(x))\, dx$ where $f^*(x, \cdot)$ is the Young-Fenchel transform of $f(x, \cdot)$.*

Given Y and Y^*, other two l.c.t.v.s. in duality and $\Gamma\colon X \to Y$ linear and continuous, we indicate with Γ^* the adjoint mapping of Γ. If $M\colon X \to]-\infty, +\infty]$. $N\colon Y \to]-\infty, +\infty]$ are convex and lower semicontinuous, we can consider the problem:

$$(\mathcal{P}) \qquad \operatorname*{Inf}_{v\in X} [M(v) + N(\Gamma(v))]$$

and its dual problem (see [4])

$$(\mathcal{P}^*) \qquad \operatorname*{Sup}_{p^*\in Y^*} [-M^*(\Gamma^* p^*) - N^*(-p^*)]$$

where M^* and N^* are the Young-Fenchel transform of M and N respectively.

Let us consider also $\forall p \in Y$ the problem:

$$(\mathcal{T}_p) \qquad \operatorname*{Inf}_{v\in X}[M(v) + N(\Gamma(v) - p)]$$

and $\forall p \in Y$ let us define $\mathcal{R}(p) = \operatorname{Inf} \mathcal{T}_p$. Then (see [5])

THEOREM 1.3. – *If $\mathcal{R}(0) < +\infty$ and $\mathcal{R}(p)$ is convex and it is also continuous at* 0, *then* $\operatorname{Inf} \mathcal{T} = \operatorname{Sup} \mathcal{T}^*$ *and the problem* $\operatorname{Sup} \mathcal{T}^*$ *is solvable.*

Let us consider now the particular case that

$$X = H^{1,2}(\Omega)\,, \qquad Y = L^2_n(\Omega)\,, \qquad \Gamma u = Du\,, \qquad N(\Gamma u) = \tfrac{1}{2}F(\Omega, u)$$

and

$$M(u) = \begin{cases} 0 & \text{if } u \in z\cdot x + H_0^{1,2}(\Omega) \\ +\infty & \text{otherwise}\,. \end{cases}$$

Because of continuity and convexity of

$$N(p) = \frac{1}{2}\int_\Omega \sum_{i,j} a_{ij}(x)\, p_i(x)\, p_j(x)\, dx$$

on $L^2_n(\Omega)$, by Theorem 1.3, $\operatorname{Inf} \mathcal{T} = \operatorname{Sup} \mathcal{T}^*$.

It is easy to prove that (see [5])

$$M^*(\Gamma^* p^*) = \begin{cases} \displaystyle\int_\Omega z\cdot p^*\, dx & \text{if } \operatorname{div} p^* = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

So, by Proposition 1.2, we have:

$$\operatorname*{Inf}_{u\in H_0^{1,2}(\Omega)} \frac{1}{2}\, F(\Omega, u + z\cdot x) = \operatorname*{Sup}_{\substack{p^*\in L^2_n(\Omega)\\ \operatorname{div} p^* = 0}} \int_\Omega \left(z\cdot p^* - \frac{1}{2}\sum_{i,j} a^*_{ij} p^*_i p^*_j\right) dx$$

where $[a^*_{ij}]$ is the inverse matrix of $[a_{ij}]$.

2. – Lower semicontinuity properties.

Let $a_{ij}(x)$, $i, j = 1, \dots, n$, be measurable functions on R^n verifying (1.1). For any $\Omega \in Ap_n$ and $z = (z_1, \dots, z_n) \in R^n$, we define:

$$\mu(z, \Omega) = \operatorname{Inf}\left\{\frac{1}{\operatorname{mis}\Omega}\int_\Omega \sum_{i,j} a_{ij}(x)(D_i u + z_i)(D_j u + z_j)\, dx: \quad u \in H_0^{1,2}(\Omega)\right\}$$

and for any $\varepsilon > 0$

$$\mu_\varepsilon(z, \Omega) = \operatorname{Inf}\left\{\frac{1}{\operatorname{mis}\Omega}\int_\Omega \sum_{i,j} (a_{ij}(x) + \varepsilon\delta_{ij})(D_i u + z_i)(D_j u + z_j)\, dx : u \in H_0^{1,2}(\Omega)\right\}.$$

It is easy to show that for any $z \in R^n$ and for any $\Omega \in Ap_n$

$$\lim_{\varepsilon\to 0^+} \mu_\varepsilon(z, \Omega) = \mu(z, \Omega)\,. \tag{2.1}$$

In the following we'll indicate with Q any cube in R^n and with $|Q|$ its measure.

THEOREM 2.1. – *If $F(\Omega, u)$ and $\overline{F}(\Omega, u)$ are defined as in § 1 and $b_{ij}(x)$ are measurable functions in R^n verifying* (1.3), *then for any x a.e. and for any $z \in R^n$:*

$$\sum_{i,j} b_{ij}(x) z_i z_j = \limsup_{|Q|\to 0} \mu(z, Q) \tag{2.2}$$

where Q is a cube containing x.

PROOF. – Let x_0 be a Lebesgue point for the b_{ij}'s; if Q is a cube such that $x_0 \in Q$, let us choose a sequence $\{u_k\}_k$ of functions in $z\cdot x + H_0^{1,2}(Q)$ such that

$$\overline{F}(Q, z\cdot x) = \lim_k F(Q, u_k)\ (^1)\,.$$

Then

$$\mu(z, Q) \leqslant \frac{1}{|Q|}\int_Q \sum_{i,j} b_{ij}(x) z_i z_j\, dx$$

and therefore, letting $|Q| \to 0$

$$\limsup_{|Q|\to 0} \mu(z, Q) \leqslant \sum_{i,j} b_{ij}(x_0) z_i z_j\,.$$

Let us prove now the reverse inequality.

We fix a cube Q_0 with the faces parallel to the coordinate axes; for any $k \in N$ let P_k be a partition of Q_0 whose elements are the cubes congruent to Q_0, contained in Q_0, and whose dimensions are in the rate $1/2^k$ with the corrispondent ones of Q_0 Let us indicate with Q_k^h an element of P_k and with $\chi_{h,k}(x)$ the characteristic function of Q_k^h and define for any $z \in R^n$

$$s_z(x) = \limsup_k \sum_{h=1}^{2^{nk}} \mu(z, Q_k^h)\, \chi_{h,k}(x)\,.$$

[1] It can be shown (see [1]) that the functions u_h in (1.2) can be chosen in $u + H_0^{1,2}(\Omega)$.

We also indicate with E_0 the set of the Lebesgue points of the b_{ij}'s and with E the subset of R^n whose elements are Lebesgue points for any $s_z(x)$ where z has rational coordinates. If $x_0 \in Q_0 \cap E_0 \cap E$ and Q is a cube with center in x_0 contained in Q_0, we indicate $\forall k$ by $Q_k^1, \ldots, Q_k^{n_k}$ all the cubes of P_k that are contained in the fixed cube Q. From the relation (2.1) it is clear that $\forall k$ it is possible to find $(\varepsilon_k)_k$, $\varepsilon_k \downarrow 0$, such that for any $1 \leqslant h \leqslant n_k$

$$\mu_{\varepsilon_k}(z, Q_k^h) < \mu(z, Q_k^h) + \frac{1}{kn_k}.$$

Then, for any k and for any $1 \leqslant h \leqslant n_k$ we denote u_k the function such that $u_k = u_k(x, h)$, $\forall h$, where $u_k(x, h)$ is the solution in Q_k^h of the Dirichlet problem

$$\operatorname{Min}\left\{\int_{Q_k^h} \sum_{i,j} (a_{ij} + \varepsilon_k \delta_{ij}) D_i u D_j u \, dx: \quad u \in z \cdot x + H_0^{1,2}(Q_k^h)\right\}.$$

From the maximum principle for linear elliptic operators in divergence form (see [6]) we get

$$\sup_{Q_k^h} |u_k - z \cdot x| \leqslant \operatorname*{osc}_{\bar{Q}_k^h} z \cdot x \leqslant |z| \operatorname{diam} Q_k^h$$

and so, fixing $\nu \in N$, $u_k \to z \cdot x$ in $A_\nu = \bigcup_{h=1}^{n_\nu} Q_\nu^h$ in the strong $L^\infty(A_\nu)$ topology, and $u_k = z \cdot x$ on ∂A_ν. Then

$$\bar{F}(A_\nu, z \cdot x) \leqslant \liminf_k \int_{A_\nu} \sum_{i,j} a_{ij} D_i u_k D_j u_k \, dx \leqslant \limsup_k \left\{\int_{A_\nu} \sum_h \mu(z, Q_k^h) \chi_{h,k}(x) \, dx + \frac{1}{k}\right\}$$

where the last summation is extended to all h such that $Q_k^h \subset A_\nu$.

Taking the limit as $\nu \to +\infty$, we have

$$\bar{F}(Q, z \cdot x) \leqslant \int_Q s_z(x) \, dx;$$

dividing both sides by $|Q|$ and taking the limit as $|Q| \to 0$, since $x_0 \in Q_0 \cap E_0 \cap E$, we get

$$\sum_{i,j} b_{ij}(x_0) z_i z_j \leqslant s_z(x_0).$$

Finally, observing that $s_z(x_0) \leqslant \limsup_{|Q| \to 0} \mu(z, Q)$, we get (2.2) for any x a.e. in Q_0 and for any z with rational coordinates.

The general case, $z \in R^n$, follows obviously by the continuity of $\limsup_{|Q| \to 0} \mu(z, Q)$ as function of z.

REMARK. – The previous result was proved by E. DE GIORGI-S. SPAGNOLO (see [4]) in the particular case $\sum_{i,j} a_{ij}(x)z_iz_j \geq |z|^2$, making use of Meyers regularity for the solutions of uniformly elliptic equations. In this case $a_{ij} = b_{ij}$.

COROLLARY 2.2. – *If $a_{ij}(x)$ are such that $\forall z \in R^n$*

$$0 \leq m(z)\,|z|^2 \leq \sum_{i,j} a_{ij}(x) z_i z_j \leq M(x)\,|z|^2$$

with $1/m(x) \in L^1_{\mathrm{loc}}(\Omega)$ and $M(x) \in L^1_{\mathrm{loc}}(\Omega)$, then $F(\Omega, u)$ is L^2-l.s.c. on $H^{1,2}(\Omega)$.

PROOF. – Let us suppose first $M(x) \leq \Lambda$; we choose $x_0 \in \Omega$ such that for this point is verified the relation (2.2). Then, having fixed $z \in R^n$, by the same argument used in § 1 we deduce that $\forall Q: x_0 \in Q$

$$\frac{1}{2}\mu(z, Q) = \sup_{\substack{p^* \in L^2_n(Q)\\ \operatorname{div} p^* = 0}} \frac{1}{|Q|}\int_Q \Big(z\cdot p^* - \frac{1}{2}\sum_{i,j} a^*_{ij}\, p^*_i p^*_j\Big)\, dx\,.$$

Because of the local summability of $1/m(x)$, the a^*_{ij}s are summable in a nhood of x_0. From the convexity of $\sum_{i,j} a_{ij}(x_0) z_i z_j$ as functional of z, there exists (see [5]) $p^*_0 \in R^n$ such that $\frac{1}{2}\sum_{i,j} a_{ij}(x_0) z_i z_j = z\cdot p^*_0 - \frac{1}{2}\sum_{i,j} a^*_{ij}(x_0) p^*_{0,i} p^*_{0,j}$.

Therefore we have:

$$\sup_{\substack{p^* \in L^2_n(Q)\\ \operatorname{div} p^* = 0}} \frac{1}{|Q|}\int_Q \Big(z\cdot p - \frac{1}{2}\sum_{i,j} a^*_{ij}\, p^*_i p^*_j\Big)\, dx \geq z\cdot p^*_0 - \frac{1}{|Q|}\int_Q \frac{1}{2}\sum_{i,j} a^*_{ij}(x)\, p^*_{0,i} p^*_{0,j}\, dx\,.$$

Then if x_0 is also a Lebesgue point of b_{ij}'s and a^*_{ij}'s, taking the limit as $|Q| \to 0$ we obtain

$$\sum_{i,j} b_{ij}(x_0) z_i z_j \geq \sum_{i,j} a_{ij}(x_0) z_i z_j$$

and so, since the inverse inequality is obvious, $F = \overline{F}$.

If $M(x) \in L^1(\Omega)$, let us fix $\Lambda > 0$ and $\Omega_\Lambda = \{x \in \Omega\colon M(x) > \Lambda\}$ and

$$a^\Lambda_{ij}(x) = \begin{cases} a_{ij}(x) & \text{if } x \notin \Omega_\Lambda \\ \min\{\Lambda\delta_{ij}, m(x)\} = m^\Lambda(x) & \text{if } x \in \Omega_\Lambda\,. \end{cases}$$

Then

$$0 \leq m^\Lambda(x)\,|z|^2 \leq \sum_{i,j} a^\Lambda_{ij}(x) z_i z_j \leq \Lambda\,|z|^2$$

and $1/m^{\Lambda}(x) \in L^1_{\text{loc}}(\Omega)$. So, if $u_k \to u$ in $L^2(\Omega)$ with $u \in H^{1,2}(\Omega)$:

$$\liminf_k F(\Omega, u_k) \geqslant \liminf_k \int_\Omega \sum_{i,j} a^{\Lambda}_{ij}(x)\, D_i u_k\, D_j u_k\, dx \geqslant \int_\Omega \sum_{i,j} a^{\Lambda}_{ij}(x)\, D_i u\, D_j u\, dx\,.$$

Then the result follows, taking the limit as $\Lambda \to +\infty$ and observing that $\left\{\sum_{i,j} a^{\Lambda}_{ij}(x)\, D_i u\, D_j u\right\}$ increases as Λ increases to $+\infty$. Finally the case $M(x) \in L^1_{\text{loc}}(\Omega)$ is transled by a standard argument from the case $M(x) \in L^1(\Omega)$.

REMARK. – From the previous result it's clear that if the coefficients a_{ij} are continuous and $m(x) > 0$, then F is L^2-l.s.c. on $H^{1,2}(\Omega)$. This is also a particular case of semicontinuity theorem proved by J. SERRIN, quoted in [10].

If $1/m(x) \in L^1_{\text{loc}}(\Omega)$ and Ω is a open subset of R^n let us denote with $W(m; \Omega)$ the space of all functions $u(x) \in H^{1,1}_{\text{loc}}(\Omega)$ such that

$$[u]_{m;\Omega} = \int_\Omega |u(x)|^2\, dx + \int_\Omega m(x)\, |Du|^2\, dx < +\infty$$

equipped with the norm $\|u\|_{m;\Omega} = [u]^{\frac{1}{2}}_{m;\Omega}$.

Then (see [12])

THEOREM 2.3. – *If $1/m(x) \in L^1(\Omega)$, then $C^1(\Omega) \cap W(m; \Omega)$ is dense in $W(m; \Omega)$ with respect to the norm $\|\cdot\|_{m;\Omega}$.*

By this result and by Corollary 2.2 follows easily:

THEOREM 2.4. – *If $a_{ij}(x)$ are such that $\forall z \in R^n$*

$$0 \leqslant m(x)\, |z|^2 \leqslant \sum_{i,j} a_{ij}(x)\, z_i z_j \leqslant M(x)\, |z|^2$$

with $1/m(x) \in L^1_{\text{loc}}(\Omega)$ and $M(x)/m(x) \in L^\infty_{\text{loc}}(\Omega)$, then $F(\Omega, u)$ is L^2-l.s.c. on $W(m; \Omega)$.

3. – Some examples.

In this section we give some applications of the formula (2.2) in particular cases. Let us begin with the one-dimensional case.

PROPOSITION 3.1 (see also [8]). – *In the same hypothesis of Theorem 2.1, if $n = 1$, we have for any x a.e. in R^n:*

$$b(x) = \lim_{\varepsilon \to 0^+} \big(\varphi_\varepsilon(x)\big)^{-1}$$

where

$$\varphi_\varepsilon(x) = \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} \frac{1}{a(t)}\,dt \;(^2)\,.$$

PROOF. – If $n = 1$, from the formula (2.2) we have for any x a.e.:

$$b(x) = \limsup_{|I|\to 0} \mu(1, I) \tag{3.1}$$

where I is any open interval containing x.

A direct calculation shows that for any $\varepsilon > 0$

$$\mu_\varepsilon(1, I) = \left(\frac{1}{|I|}\int_I \frac{1}{a(t)+\varepsilon}\,dt\right)^{-1}$$

and so the result follows at once from (2.1) and (3.1).

From this Proposition follows also

PROPOSITION 3.2. – *Let be*

$$F(\Omega, u) = \int_\Omega \sum_{i=1}^n a_i(x)|D_i u|^2\,dx$$

with $a_i(x) = a_i'(x_i)a_i''(\hat{x}_i)$ $(\hat{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n))$. *Then*

$$\bar{F}(\Omega, u) = \int_\Omega \sum_{i=1}^n b_i(x)|D_i u|^2\,dx$$

where for any x_i *a.e. in* R $b_i(x) = b_i'(x_i)a_i''(\hat{x}_i)$ *and*

$$b_i'(x_i) = \left(\lim_{\varepsilon\to 0^+}\frac{1}{2\varepsilon}\int_{x_i-\varepsilon}^{x_i+\varepsilon}\frac{1}{a_i'(t)}\,dt\right)^{-1}.$$

PROOF. – We can suppose $\Omega = \prod_{k=1}^n]\alpha_k, \beta_k[$ and $u = z\cdot x$.

By the previous Proposition and by (1.2) we can say that $\forall i$ there exists a sequence $(u_h^i(x_i))$ converging to $z_i x_i$ in $L^2(]\alpha_i, \beta_i[)$ such that

$$\int_{\alpha_i}^{\beta_i} b_i'(x_i)\,z_i^2\,dx_i = \lim_h \int_{\alpha_i}^{\beta_i} a_i'(x_i)\,|\dot{u}_h^i(x_i)|^2\,dx\,.$$

(2) If the $\lim_{\varepsilon\to 0}\varphi_\varepsilon(x) = +\infty$, $b(x)$ is intended to be egual to 0.

But the sequence

$$u_h(x) = \sum_{i=1}^{n} u_h^i(x_i)$$

converges to $z\cdot x$ in $L^2(\Omega)$. So

$$\overline{F}(\Omega, u) \leqslant \lim_h F(\Omega, u_h) = \int_\Omega \sum_{i=1}^{n} b_i(x)|D_i u|^2\, dx .$$

To obtain the reverse inequality it's enough to observe that if $u_h \to z\cdot x$ in $L^2(\Omega)$

$$\liminf_h F(\Omega, u_h) \geqslant \sum_{i=1}^{n}\left(\int_{\prod_{k\neq i}]\alpha_k,\beta_k[} a_i''(\hat{x}_i)\, d\hat{x}_i \cdot \liminf_h \int_{\alpha_i}^{\beta_i} a_i'(x_i)|D_i u_h|^2\, dx\right) \geqslant$$

$$\geqslant \sum_{i=1}^{n}\left(\int_{\prod_{i\neq k}]\alpha_k,\beta_k[} a_i''(\hat{x}_i)\, d\hat{x}_i \int_{\alpha_i}^{\beta_i} b_i'(x_i)\, z_i^2\, dx_i\right).$$

REMARK. – In the previous cases continuity assumptions on $a(x)$ and $a_i'(x_i)$ respectively are sufficient to ensure the l.s.c. of $F(\Omega, u)$ if $a(x)$ and $a_i'(x_i)$ are also greater than 0, a.e.

Now we intend to give an explicit calculation of the coefficients b_{ij} in the case that a_{ij} depend only on one variable. To this aim, if I is an open interval of R and $a_{ij}(x_1)$ are measurable functions of R verifying (1.1), for any $z \in R^n$ we define:

$$\gamma(z, I) = \operatorname{Inf}\left\{\frac{1}{\operatorname{mis} I}\int_I \left[a_{11}(x_1)\,\dot{u}^2 + 2\sum_{i=2}^{n} a_{1i}(x_1)\,\dot{u}z_i + \right.\right.$$
$$\left.\left. + \sum_{i,j=2}^{n} a_{ij}(x_1)\, z_i z_j\right] dx_1 \colon u \in z_1 x_1 + H_0^{1,2}(I)\right\},$$

$$\gamma_\varepsilon(z, I) = \operatorname{Inf}\left\{\frac{1}{\operatorname{mis} I}\int_I \left[(a_{11} + \varepsilon)\,\dot{u}^2 + 2\sum_{i=2}^{n} a_{1i}\,\dot{u}z_i + \right.\right.$$
$$\left.\left. + \sum_{i,j=2}^{n} (a_{ij} + \varepsilon\delta_{ij})\, z_i z_j\right] dx_1 \colon u \in z_1 x_1 + H_0^{1,2}(I)\right\}.$$

These quantities are closed related to the ones defined in section § 2.

Again, we have that for any interval I and for any $z \in R^n$

$$\lim_{\varepsilon\to 0^+} \gamma_\varepsilon(z, I) = \gamma(z, I) . \tag{3.2}$$

In the following, every time we fix a point $x_1 \in R$, we shall denote by $Q = \prod_{i=1}^{n} I_i$ a cube in R^n such that $x_1 \in I_1$.

THEOREM 3.3. – *If $a_{ij}(x_1)$ are measurable functions of R verifying* (1.1), *for any $z \in R^n$ and for any x_1 a.e. in R we have:*

$$\sum_{i,j} b_{ij}(x_1)\, z_i z_j = \limsup_{|Q|\to 0} \mu(z, Q) = \limsup_{|I_1|\to 0} \gamma(z, I_1)$$

and $b_{ij} = \bar{b}_{ij}(x_1)$ *a.e. in* R, *where*

$$\bar{b}_{ij}(x) = \begin{cases} a_{ij}(x_1) & \text{if } (\lim_{\varepsilon\to 0^+} \varphi_\varepsilon(x_1))^{-1} = a_{11}(x_1) \\ a_{ij}(x_1) - \bar{a}_{ij}(x_1) & \text{otherwise} \end{cases}$$

where:

$$\bar{a}_{ij}(x_1) = \begin{cases} \dfrac{a_{1i}(x_1)\, a_{1j}(x_1)}{a_{11}(x_1)} & \text{if } a_{11}(x_1) \neq 0 \\ 0 & \text{if } a_{11}(x_1) = 0\,. \end{cases}$$

From the definition of $\bar{b}_{ij}(x_1)$ it is clear that in the points x_1 such that $1/a_{11}$ is summable in some nhood of the point, the quadratic form $\sum_{i,j} b_{ij} z_i z_j$ coincides with $\sum_{i,j} a_{ij} z_i z_j$. Otherwise it reduces to the form $\sum_{i,j=2}^{n} (a_{ij} - \bar{a}_{ij}) z_i z_j$.

Before giving the proof of this result let us prove the following:

LEMMA 3.4. – *In the same hypothesis of the previous Theorem, for any cube* $Q = \prod_{i=1}^{n} I_i$ *and for any* $z \in R^n$ *we have*:

$$\mu(z, Q) \geq \gamma(z, I_1)\,. \tag{3.3}$$

PROOF. – Let us fix $z \in R^n$ and Q by the same argument used in § 1 we get, for any $\varepsilon > 0$:

$$\frac{1}{2}\mu_\varepsilon(z, Q) = \sup_{\substack{p^* \in L^2_n(Q) \\ \operatorname{div} p^* = 0}} \frac{1}{|Q|} \int_Q \left(z \cdot p^* - \frac{1}{2}(a_{ij} + \varepsilon\delta_{ij})^* p_i^* p_j^* \right) dx$$

where the matrix $[(a_{ij} + \varepsilon\delta_{ij})^*]$ is the inverse matrix of $[a_{ij} + \varepsilon\delta_{ij}]$. Similary it can be shown that

$$\frac{1}{2}\gamma_\varepsilon(z, I_1) = \sup_{\substack{p_1^* \in L^2(I_1) \\ \operatorname{div} p_1^* = 0}} \frac{1}{|I_1|} \left\{ \int_{I_1} p_1^* z_1 - \frac{1}{2} \frac{p_1^{*2}}{a_{11} + \varepsilon} + \left(\sum_{j=2}^{n} \frac{a_{1j}}{a_{11} + \varepsilon} z_j \right) p_1^* + \frac{1}{2} \sum_{i,j=2}^{n} \left((a_{ij} + \varepsilon\delta_{ij}) - \frac{a_{1i} a_{1j}}{a_{11} + \varepsilon} \right) z_i z_j \right\}.$$

Actually the supremum at the right side is taken up to all constants p_1^*. For any constant $\bar{p}_1^*$ let us choose

$$\bar{p}^*(x_1) = \left\{ \bar{p}_1^*,\ \frac{a_{1i}\bar{p}_1^*}{a_{11} + \varepsilon} + \sum_{r=2}^{n} \left((a_{ir} + \varepsilon\delta_{ir}) - \frac{a_{1i} a_{1r}}{a_{11} + \varepsilon} \right) z_r \quad i = 2, \ldots, n \right\}.$$

Since a_{ij} depend only on x_1, $\operatorname{div}\bar{p}^* = 0$. So, by mean of easy (although tedious) calculation, we get:

$$\frac{1}{2}\mu_\varepsilon(z, Q) \geqslant \frac{1}{|Q|}\int_Q\left(z\cdot\bar{p}^* - \frac{1}{2}\sum_{i,j=1}^{n}(a_{ij}+\varepsilon\delta_{ij})^*\bar{p}_i^*\bar{p}_j^*\right)dx =$$

$$= \frac{1}{|I_1|}\int_{I_1}\bar{p}_1^* z_1 - \frac{1}{2}\frac{\bar{p}_1^{*2}}{a_{11}+\varepsilon} + \left(\sum_{i=2}^{n}\frac{a_{1i}}{a_{11}+\varepsilon}z_i\right)\bar{p}_1^* + $$
$$+ \frac{1}{2}\sum_{i,j=2}^{n}\left((a_{ij}+\varepsilon\delta_{ij}) - \frac{a_{1i}a_{1j}}{a_{11}+\varepsilon}\right)z_i z_j\, dx_1 .$$

Then, taking the supremum up to all constants $\bar{p}_1^*$ we get

$$\mu_\varepsilon(z, Q) \geqslant \gamma_\varepsilon(z, I_1)$$

and so, using (2.1) and (3.2), (3.3) is proved.

PROOF OF THEOREM 3.3. - From Theorem 2.1 and from the previous Lemma it follows:

$$\sum_{i,j} b_{ij}(x_1)z_i z_j = \limsup_{|Q|\to 0}\mu(z, Q) \geqslant \limsup_{|I_1|\to 0}\gamma(z, I_1) . \tag{2.4}$$

But, as it can be easily checked by direct calculation,

$$\gamma_\varepsilon(z, I_1) = z_1^2\left(\frac{1}{|I_1|}\int_{I_1}\frac{dt}{a_{11}+\varepsilon}\right)^{-1} + 2\sum_{i=2}^{n} z_1 z_i \frac{\int_{I_1} a_{1i}(t)/(a_{11}+\varepsilon)}{\int_{I_1} dt/(a_{11}+\varepsilon)} +$$
$$+ \sum_{i,j=2}^{n}\frac{z_i z_j}{|I_1|}\left\{\int_{I_1}(a_{ij}+\varepsilon\delta_{ij}) - \int_{I_1}\frac{a_{1i}a_{1j}}{a_{11}+\varepsilon} + \frac{\left(\int_{I_1} a_{1i}/(a_{11}+\varepsilon)\right)\left(\int_{I_1} a_{1j}/(a_{11}+\varepsilon)\right)}{\int_{I_1} 1/(a_{11}+\varepsilon)}\right\}.$$

In the points x_1 such that $1/a_{11}$ is summable in some nhood of the point we have a.e.

$$\lim_{|I_1|\to 0}\lim_{\varepsilon\to 0^+}\gamma_\varepsilon(z, I_1) = \sum_{i,j} a_{ij}(x_1)z_i z_j .$$

Otherwise,

$$\lim_{\varepsilon\to 0^+}\frac{1}{|I_1|}\int_{I_1}\frac{dt}{a_{11}+\varepsilon} = +\infty$$

for any nhood I_1 of x_1 and so

$$\limsup_{|I_1|\to 0}\gamma(z, I_1) = \sum_{i,j=2}^{n}\left(a_{ij}(x_1) - \bar{a}_{ij}(x_1)\right) z_i z_j +$$

$$+ \limsup_{|I_1|\to 0}\frac{1}{|I_1|}\lim_{\varepsilon\to 0^+}\frac{\sum_{i,j=2}^{n}\left(\int_{I_1} a_{1i}/(a_{11}+\varepsilon)\right)\left(\int_{I_1} a_{1j}/(a_{11}+\varepsilon)\right)}{\int_{I_1} 1/(a_{11}+\varepsilon)} z_i z_j \geq \sum_{i,j=2}^{n}(a_{ij} - \bar{a}_{ij})\, z_i\, z_j\,.$$

So in both cases:

$$\sum_{i,j=1}^{n} b_{ij}(x_1) z_i z_j \geq \sum_{i,j=1}^{n} \bar{b}_{ij}(x_1) z_i z_j\,.$$

To prove the reverse inequality it is enough, since $\bar{F} \leq F$, to confine ourselves at the points x_1 such that

$$a_{11}(x_1) \neq \lim_{\varepsilon\to 0^+}\frac{1}{2\varepsilon}\int_{x_1-\varepsilon}^{x_1+\varepsilon}\frac{1}{a_{11}(t)}\,dt\,.$$

If x_1 is one of such points we can suppose that $1/a_{11}$ is not summable in any nhood of x_1.

So, if we fix $I_1 = (a, b) \ni x_1$, $1/a_{11} \notin L^1(I_1)$.

Then let us consider a sequence of partitions P_k of I_1, such that $P_k = \{x_{0,k} = a < x_{1,k} < \ldots < x_{n_k,k} = b\}$ and $\forall k$, P_k refines P_{k-1} and the sizes of the intervals in P_k tend to 0 as $k \to +\infty$. For any k we can choose $\varepsilon_k > 0$ such that:

$$\frac{1}{x_{h,k} - x_{h-1,k}}\int_{x_{h-1,k}}^{x_{h,k}}\frac{dt}{a_{11}+\varepsilon_k} = \fint_{x_{h-1,k}}^{x_{h,k}}\frac{dt}{a_{11}+\varepsilon_k} > k \tag{3.5}$$

for any $1 \leq h \leq n_k$. Let us consider the functions:

$$u_k(x) = \sum_{h=1}^{n_k}\left\{\frac{\displaystyle\int_{x_{h-1,k}}^{x_{h,k}}\frac{\sum_{i=2}^{n} a_{1i}(t) z_i\,dt}{a_{11}(t)+\eta}\int_{x_{h-1,k}}^{x}\frac{dt}{a_{11}(t)+\varepsilon_k}}{\displaystyle\int_{x_{h-1,k}}^{x_{h,k}}\frac{dt}{a_{11}(t)+\varepsilon_k}} + \right.$$

$$\left. + \frac{\displaystyle\int_{x_{h-1,k}}^{x}\frac{z_1\,dt}{a_{11}(t)+\varepsilon_k}}{\displaystyle\fint_{x_{h-1,k}}^{x_{h,k}}\frac{dt}{a_{11}(t)+\varepsilon_k}} - \int_{x_{h-1,k}}^{x}\frac{\sum_{i=2}^{n} a_{1i}(t) z_i}{a_{11}(t)+\eta}\,dt + z_1 x_{h-1,k}\right\}\chi_{h,k}(x)$$

where η is a fixed positive number and $\chi_{h,k}$ is the characteristic function of the interval $[x_{h-1,k}, x_{h,k}]$. It is easy to prove that $u_k \to z_1 x_1$ in $L^2(I_1)$ and so the functions

$$v_k(x) = u_k(x_1) + \sum_{i=2}^{n} z_i x_i$$

converge to $z\cdot x$ in $L^2(Q)$.

Finally we get, by some calculation, by (3.5)

$$\int_Q \sum_{i,j} b_{ij}(x)\, z_i\, d_j\, dx \leq \liminf_k \int_Q \Big[a_{11}\dot{u}_k^2 + 2\sum_{i=2}^{n} a_{1i}\dot{u}_k z_i + \sum_{i,j=2}^{n} a_{ij} z_i z_j\Big]\, dx \leq$$
$$\leq \liminf_k \Big[c(\eta)\, O\Big(\frac{1}{k}\Big) + \sum_{i,j=2}^{n} z_i z_j \int_Q \frac{a_{1i}a_{1j}a_{11}}{(a_{11}+\eta)^2} -$$
$$-2\sum_{i,j=2}^{n} z_i\, z_j \int_Q \frac{a_{1i}a_{1j}}{a_{11}+\eta}\, dx + \sum_{i,j=2}^{n} \int_Q a_{ij} z_i z_j\, dx\Big]$$

where $c(\eta)$ is a constant depending only on η, $|Q|$, z, Λ.

And then, taking the limit as $k \to +\infty$, $\eta \to 0$ we get:

$$\int_Q \sum_{i,j} b_{ij}(x)\, z_i z_j\, dx \leq \sum_{i,j=2}^{n} \int_Q \Big(a_{ij} - \frac{a_{1i}a_{1j}}{a_{11}}\Big) z_i z_j\, dx$$

and so Theorem 3.1 is completely proved.

COROLLARY 3.3. – *In the same hypothesis of Theorem* 3.1 *if* $1/a_{11} \in L^1_{\mathrm{loc}}(R)$ *or if* $a_{11}(x_1)$ *is continuous the functional* $F(\Omega, u)$ *is* L^2*-l.s.c.*

4. – Γ-convergence of quadratic functionals.

In this section we want to study the Γ-convergence of quadratic functionals with respect to the L^2-topology, giving an integral representation theorem for the Γ-limit up to the space $W(m, \Omega)$.

Let us begin with the definition of Γ-convergence.

DEFINITION 4.1. – If (X, τ) is a first countable topological space and $(F_h(u))_h$ is a sequence of functionals such that $u \in X \to F_h(u) \in R \cup \{+\infty\}$ we shall say that $F(u) = \Gamma(\tau^-)\lim_h F_h(u)$ iff

i) $\forall u \in X$ and $\forall (u_h)_h \subset X$ such that $u_h \xrightarrow{\tau} u$, $F(u) \leq \liminf_h F_h(u_h)$;

ii) $\forall u \in X$, $\exists (u_h)_h \subset X$ such that $u_h \xrightarrow{\tau} u$ and $F(u) = \lim_h F_h(u_h)$.

For the basic properties and applications to the Calculus of Variation of such convergence see [3] and the bibliography listed in [1].

In the last years have been proved many theorems insuring when, starting from a sequence of integral functionals, also the Γ-limit is an integral functional. In particular from Theorem 3.2 in [1] it can be proved the following

THEOREM 4.2. – *If* $([a_{ij}^h])_h$ *is a sequence of* $n\times n$ *matrices of bounded measurable functions* $a_{ij}^h(x)$ *on* R^n *such that*

$$m(x)|\zeta|^2 \leqslant \sum_{i,j} a_{ij}^h(x)\zeta_i\zeta_j \leqslant M(x)|\zeta|^2 \tag{4.1}$$

$\forall\zeta\in R^n$, $\forall x$ *a.e. in* R^n, *where*

$$m(x)\in L^1(R^n); \qquad \frac{1}{m(x)}\in L^1_{\text{loc}}(R^n) \tag{4.2}$$

$$M(x)\in L^1_{\text{loc}}(R^n) \tag{4.3}$$

then there exist an increasing sequence $(h_k)_k$ *of positive integers and a matrix* (a_{ij}) *of bounded measurable functions such that*

$$m(x)|\zeta|^2 \leqslant \sum_{i,j} a_{ij}(x)\zeta_i\zeta_j \leqslant M(x)|\zeta|^2 \tag{4.4}$$

$\forall\zeta\in R^n$, $\forall x$ *a.e. in* R^n; $\forall\Omega\in Ap_n$, $\int_{\partial\Omega} M(x)\,dx = 0$ *and* $\forall u\in C^1(R^n)$

$$\int_\Omega \sum_{i,j} a_{ij}(x)\,D_iu\,D_ju\,dx = \Gamma(L^2(\Omega)^-)\lim_k \int_\Omega \sum_{i,j} a_{ij}^{h_k}(x)\,D_iu\,D_ju\,dx\,. \tag{4.5}$$

If we suppose, as in Theorem 2.4

$$\frac{M(x)}{m(x)}\in L^\infty(R^n) \tag{4.6}$$

we may ask if it is still possible to give (4.5) also, for functions $u\in W(m;\Omega)$, or for functions $u\in W_{\text{loc}}(m;R^n) = \{u\in W^{1,1}_{\text{loc}}(R^n)\colon \forall\Omega\in Ap_n,\ u\in W(m;\Omega)\}$.

So, let us define:

$$F_h(\Omega,u) = \begin{cases} \displaystyle\int_\Omega \sum_{i,j} a_{ij}^h(x)\,D_iu\,D_ju\,dx & \text{if } u\in C^1(R^n)\,,\\ +\infty & \text{if } u\in W_{\text{loc}}(m;R^n) - C^1(R^n)\,. \end{cases}$$

Then we can prove:

THEOREM 4.3. – *If are verified the hypothesis of Theorem* 4.2, *the* (4.6) *and the* *following*

$$\text{(4.7)}\qquad \operatorname*{Sup}_{Q\subset R^n}\left(\fint_Q m(x)\,dx\right)\left(\fint_Q \frac{dx}{m(x)}\right) < +\infty$$

where Q is any cube in R^n, then $\forall\Omega\in Ap_n$ such that $\int_{\partial\Omega} M(x)\,dx = 0$ we have, together with (4.5), $\forall u\in W_{\text{loc}}(m; R^n)$

$$\text{(4.8)}\qquad \int_\Omega \sum_{i,j} a_{ij}(x)\,D_i u\,D_j u\,dx = \Gamma(L^2(\Omega)^-)\lim_k F_{h_k}(\Omega, u)\,.$$

Before giving the proof of this theorem we need two preliminary results.

LEMMA 4.4 (see [2]). – *If $m(x)$ verifies* (4.2) *and* (4.7), *then for any $f\in L^1_{\text{loc}}(R^n)$, such that $\int_{R^n}|f(x)|^2 m(x)\,dx < +\infty$,*

$$\int_{R^n}|(Mf)(x)|^2 m(x)\,dx \leqslant C\int_{R^n}|f(x)|^2 m(x)\,dx$$

where

$$(Mf)(x) = \operatorname*{Sup}_{Q\subset R^n}\fint_Q |f(x)|\,dx \qquad \textit{and} \qquad C = C(n; m(x)) > 0\,.$$

If $u(x)\in W^{1,1}(R^n)$, let us define

$$(M^* u)(x) = (Mu)(x) + \sum_{|\alpha|=1}(MD^\alpha u)(x)\,,$$

and if $F\subset R^n$ is a closed set

$$R(u; F) = \operatorname*{Sup}_{x,y\in F}\left\{\operatorname*{Sup}_{|\alpha|=1}|D^\alpha u(x) - D^\alpha u(y)|;\ \frac{|u(y) - u(x) - \sum_{|\alpha|=1} D^\alpha u(x)(y_\alpha - x_\alpha)|}{|x-y|}\right\}.$$

Then we have

LEMMA 4.5 (see [7]). – *If $u(x)\in W^{1,1}(R^n)$, $\forall\varepsilon>0$ there exist a closed set $F\subset R^n$ and a function $g(x)\in C^1(R^n)$ such that $|F^c|<\varepsilon$; $u = g$ and $D^\alpha u = D^\alpha g$ on F; $R(g; R^n)\leqslant \leqslant CR(u; F)$* where $C = C(n) > 0$.

From the previous lemma we can prove

LEMMA 4.6. – *If $m(x)$ verifies* (4.2) *and* (4.7) *and if $u(x)\in W^{1,1}(R^n)\cap W(m; R^n)$, then $\forall\varepsilon>0$ there exist a closed set $F\subset R^n$ and a function $g(x)\in C^1(R^n)$ such that $|F^c|<\varepsilon$; $u = g$ and $D^\alpha u = D^\alpha g$ on F; $\|g-u\|_{m;R^n}<\varepsilon$.*

PROOF. – If $\delta > 0$, let us denote $Z_\delta = \{x \in R^n : (M^*u)(x) < \delta\}$. Then, see [6], page 650,

$$\forall x,\ y \in Z_\delta\,, \qquad I(x, y) \leqslant c_1(n)\,\delta$$

where

$$I(x, y) = \sum_{|\alpha|\leqslant 1} |D^\alpha u(x) - D^\alpha u(y)| + \frac{|u(y) - u(x) - \sum_{|\alpha|=1} D^\alpha u(x)(y_\alpha - x_\alpha)|}{|x - y|}\,.$$

Then, by Lemma 4.5, there exist a closed set $F \subset Z_\delta$ and a function $g(x) \in C^1(R^n)$ such that $u = g$ and $D^\alpha u = D^\alpha g$ on F, $|F^c| < 2|Z_\delta^c|$ and

$$\int_{F^c} m(x)\, dx < 2\int_{Z_\delta^c} m(x)\, dx\,,$$

$E(g; R^n) \leqslant c_2(n) E(u; F) \leqslant c_2 c_1 \delta$.

Then:

$$\begin{aligned}\int_{R^n} |Dg - Du|^2 m(x)\, dx = \int_{F^c} |Dg - Du|^2 m(x)\, dx \\ &\leqslant c_3(n)\ \Big[\delta^2 \int_{F^c} m(x)\, dx + \int_{F^c} |Du|^2 m(x)\, dx\Big] \\ &\leqslant 2c_3(n)\Big[\delta^2 \int_{Z_\delta^c} m(x)\, dx + \int_{F^c} |Du|^2 m(x)\, dx\Big] \\ &\leqslant 2c_3(n)\Big[\int_{Z_\delta^c} m(x)\,|(M^*u)(x)|^2\, dx + \int_{F^c} |Du|^2 m(x)\, dx\Big]\,.\end{aligned}$$

And by Lemma 4.4, letting $\delta \to +\infty$, we get the proof.

Now we can give the

PROOF OF THEOREM 4.3. – Let us suppose that $\forall \Omega \in Ap_n$ and $\forall u \in C^1(R^n)$

$$\int_\Omega \sum_{i,j} a_{ij}(x)\, D_i u\, D_j u\, dx = \Gamma(L^2(\Omega)^-) \lim_h \int_\Omega \sum_{i,j} a_{ij}^h(x)\, D_i u\, D_j u\, dx\,.$$

By a general compactness result we can suppose that there exist

$$F(\Omega, u) = \Gamma(L^2(\Omega)) \lim_h F_h(\Omega, u) \qquad \text{on } W_{\text{loc}}(m; R^n)\,. \tag{4.9}$$

Actually, we have to prove that

$$F(\Omega, u) = \int_\Omega \sum_{i,j} a_{ij}(x)\, D_i u\, D_j u\, dx \qquad \forall u \in W_{\text{loc}}(m; R^n)\,. \tag{4.10}$$

If $u \in W_{\text{loc}}(m; R^n)$, by Lemma 4.5, there exist a sequence $(w_h)_h \subset C^1(R^n)$ such that $w_h \to u$ in $L^2(\Omega)$ and

$$\int_\Omega \sum_{i,j} a_{ij}(x)\, D_i u\, D_j u\, dx = \lim_h \int_\Omega \sum_{i,j} a_{ij}(x)\, D_i w_h\, D_j w_h\, dx\,.$$

Since Γ-limits are l.s.c. (see [3]) we have:

$$F(\Omega, u) \leqslant \lim_h F(\Omega, w_h) = \lim_h \int_\Omega \sum_{i,j} a_{ij}(x)\, D_i w_h\, D_j w_h\, dx$$

and so

$$F(\Omega, u) \leqslant \int_\Omega \sum_{i,j} a_{ij}(x)\, D_i u\, D_j u\, dx \qquad \forall u \in W_{\text{loc}}(m; R^n)\,. \tag{4.11}$$

Let us prove the reverse inequality. Fixed $u \in W_{\text{loc}}(m; R^n)$, by (ii), let $(u_h)_h \subset C^1(R^n)$ such that

$$\begin{cases} u_h \to u \quad \text{in } L^2(\Omega) \quad \text{and} \\ F(\Omega, u) = \lim_h F_h(\Omega, u_h)\,. \end{cases} \tag{4.12}$$

We can suppose $F(\Omega, u) < +\infty$, otherwise (4.10) follows from (4.12). Let us fix $\varepsilon > 0$; then by Lemma 4.6 there exist a closed subset $F \subset\subset \Omega$ and a function $w^\varepsilon \in C^1(R^n)$ such that

$$\begin{cases} |\Omega - F_\varepsilon| < \varepsilon; \quad w^\varepsilon = u \quad \text{and} \quad Dw^\varepsilon = Du \quad \text{on } F_\varepsilon\,, \\ \|u - w_\varepsilon\|^2_{m;\Omega} < \varepsilon\,. \end{cases} \tag{4.13}$$

Let us denote by $(\Omega_h)_h$ an decreasing sequence of open sets such that

$$\begin{cases} F_\varepsilon \subset\subset \Omega_h \subset\subset \Omega \\ \lim_h \operatorname{dist}(F_\varepsilon, \partial\Omega_h) = 0 \end{cases} \tag{4.14}$$

and by $(\varphi_h)_h \subset C^1_0(\Omega_h)$ a sequence of functions such that

$$\begin{cases} 0 \leqslant \varphi_h \leqslant 1 \quad \text{on } \Omega_h; \quad \varphi_h = 1 \quad \text{and} \quad D\varphi_h = 0 \quad \text{on } F_\varepsilon\,, \\ |D\varphi_h(x)| \leqslant \dfrac{2}{\operatorname{dist}(F_\varepsilon, \partial\Omega_h)}\,. \end{cases} \tag{4.15}$$

Then let us define

$$\begin{cases} \tilde\varphi_h(x) = \varphi_h(x) \operatorname{dist}(F_\varepsilon, \partial\Omega_h)\,, \\ v_h(x) = u_h \tilde\varphi_h + (1 - \tilde\varphi_h)(u_h - u + w^\varepsilon) \end{cases} \tag{4.16}$$

and $w_h(x) \in C^1(R^n)$ such that

$$\|w_h - v_h\|^2_{m;\Omega} \to 0 \quad \text{as } h \to +\infty . \tag{4.17}$$

Then:

$$\int_{F_\varepsilon} \sum_{i,j} a_{ij}(x) D_i u D_j u\, dx \leqslant F(\Omega, w^\varepsilon) \leqslant \liminf_h F_h(\Omega, w_h) = \liminf_h \int_\Omega \sum_{i,j} a^h_{ij}(x) D_i w_h D_j w_h\, dx$$

and so, by (4.17)

$$\int_{F_\varepsilon} \sum_{i,j} a_{ij}(x) D_i u D_j u\, dx \leqslant \liminf_h \int_\Omega \sum_{i,j} a^h_{ij}(x) D_i v_h D_j v_h\, dx . \tag{4.18}$$

If we fix $\tau, \sigma \in]0, 1[$, then $\forall h$

$$\int_\Omega \sum_{i,j} a^h_{ij}(x) D_i(\tau\sigma v_h) D_j(\tau\sigma v_h)\, dx = a_h + b_h + c_h \tag{4.19}$$

where

$$a_h = \int_{\Omega - \Omega_h} \sum_{i,j} a^h_{ij}(x) D_i(\tau\sigma v_h) D_j(\tau\sigma v_h)\, dx$$

$$b_h = \int_{\Omega_h - F_\varepsilon} \sum_{i,j} a^h_{ij}(x) D_i(\tau\sigma v_h) D_j(\tau\sigma v_h)\, dx$$

$$c_h = \int_{F_\varepsilon} \sum_{i,j} a^h_{ij}(x) D_i(\tau\sigma v_h) D_j(\tau\sigma v_h)\, dx .$$

Observing that

$$Dv_h(x) = Du_h \tilde{\varphi}_h + D(u_h - u + w^\varepsilon)(1 - \tilde{\varphi}_h) + D\tilde{\varphi}_h(w^\varepsilon - u)$$

we have, from (4.15)

$$\begin{aligned} a_h = \int_{\Omega-\Omega_h} \sum_{i,j} a^h_{ij}(x) D_i\big(\sigma\tau(u_h - u + w^\varepsilon)\big) D_j\big(\sigma\tau\, (u_h - u + w^\varepsilon)\big)\, dx \leqslant \\ \leqslant \sigma^2 \tau F_h(\Omega - \Omega_h, u_h) + \frac{\sigma^2\tau^2}{1-\tau} \int_{\Omega-\Omega_h} \sum_{i,j} a^h_{ii}(x) D_i(w^\varepsilon - u) D_j(w^\varepsilon - u)\, dx \leqslant \\ \leqslant \sigma^2 \tau F_h(\Omega - \Omega_h, u_h) + \frac{\sigma^2\tau^2}{1-\tau}\, \varepsilon \operatorname*{Sup}_\Omega \frac{M(x)}{m(x)} ; \end{aligned}$$

$$\begin{aligned} b_h = \int_{\Omega_h - F_\varepsilon} \sum_{i,j} a^h_{ij}(x) D_i(\sigma\tau v_h) D_j(\sigma\tau v_h)\, dx \leqslant \\ \leqslant \sigma^2 \tau \int_{\Omega_h - F_\varepsilon} \sum_{i,j} a^h_{ij}(x) [\tilde{\varphi}_h D_i u_h + (1 - \tilde{\varphi}_h) D_i(u_h - u + w^\varepsilon)] \end{aligned}$$

$$[\tilde{\varphi}_h D_j u_h + (1-\tilde{\varphi}_h) D_j(u_h - u + w^\varepsilon)]\, dx +$$

$$+ \frac{\sigma^2\tau^2}{1-\tau} \int_{\Omega_h - F_\varepsilon} \sum_{i,j} a_{ij}^h(x)(w^\varepsilon - u)^2 D_i\tilde{\varphi}_h D_j \tilde{\varphi}_h\, dx \leqslant$$

$$\leqslant \sigma^2\tau \int_{\Omega_h - F_\varepsilon} \tilde{\varphi}_h(x) \sum_{i,j} a_{ij}^h(x) D_i u_h D_j u_h\, dx + \frac{4\sigma^2\tau^2}{1-\tau}\varepsilon +$$

$$+ \tau \int_{\Omega_h - F_\varepsilon} (1-\tilde{\varphi}_h) \sum_{i,j} a_{ij}^h(x) D_i[\sigma u_h + \sigma(w^\varepsilon - u)] D_j[\sigma u_h + \sigma(w^\varepsilon - u)]\, dx \leqslant$$

$$\leqslant \sigma^2\tau \int_{\Omega_h - F_\varepsilon} \tilde{\varphi}_h(x) \sum_{i,j} a_{ij}^h(x) D_i u_h D_j u_h\, dx + \frac{4\sigma^2\tau^2}{1-\tau}\varepsilon +$$

$$+ \sigma\tau \int_{\Omega_h - F_\varepsilon} (1-\tilde{\varphi}_h) \sum_{i,j} a_{ij}^h(x) D_i u_h D_j u_h\, dx +$$

$$+ \frac{\sigma^2\tau}{1-\sigma} \int_{\Omega_h - F_\varepsilon} (1-\tilde{\varphi}_h) \sum_{i,j} a_{ij}^h(x) D_i(w^\varepsilon - u) D_j(w \; - u)\, dx \leqslant$$

$$\leqslant \sigma\tau \int_{\Omega_h - F_\varepsilon} \sum_{i,j} a_{ij}^h(x) D_i u_h D_j u_h\, dx + \frac{\sigma^2\tau}{1-\sigma}\varepsilon \operatorname*{Sup}_{\Omega} \frac{M(x)}{m(x)} + \frac{4\sigma^2\tau^2}{1-\tau}\varepsilon\,.$$

Finally

$$c_h = \sigma^2\tau^2 F_h(F_\varepsilon; u_h)\,.$$

From (4.19) and from the following inequalities we have

$$\tau^2\sigma^2 F_h(\Omega, v_h) \leqslant \sigma^2\tau F_h(\Omega - \Omega_h, u_h) + \sigma\tau F_h(\Omega_h - F_\varepsilon, u_h) +$$

$$+ \sigma^2\tau^2 F_h(F_\varepsilon; u_h) + \varepsilon \operatorname*{Sup}_{\Omega} \frac{M(m)}{m(x)} \left(\frac{\sigma^2\tau^2}{1-\tau} + \frac{\sigma^2\tau}{1-\sigma}\right) + \frac{4\sigma^2\tau^2}{1-\tau}\varepsilon \leqslant$$

$$\leqslant \sigma\tau F_h(\Omega, u_h) + \varepsilon \operatorname*{Sup}_{\Omega} \frac{M(x)}{m(x)} \left(\frac{\sigma^2\tau^2}{1-\tau} + \frac{\sigma^2\tau}{1-\sigma}\right) + \frac{4\sigma^2\tau^2}{1-\tau}\varepsilon\,.$$

Letting $h \to +\infty$ and then $\varepsilon \to 0$ we have, from (4.12) and (4.18)

$$\sigma^2\tau^2 \int_{\Omega} \sum_{i,j} a_{ij}(x) D_i u D_j u\, dx \leqslant \sigma\tau F(\Omega, u)$$

and so, letting $\sigma, \tau \to 1^-$ we get

$$\int_{\Omega} \sum_{i,j} a_{ij}(x) D_i u D_j u\, dx \leqslant F(\Omega, u)$$

and so, by (4.11), (4.10) is proved.

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