An Inverse Problem in Potential Theory (*).

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Summary. – *We consider the classical problem of finding the density* ϱ *of a material body* Ω *embedded into a region* S, *when the potential generated by* Ω *(possibly coinciding with* S*) is known outside (or on the surface of)* S. *In the set of such solutions we look for the density* $\bar{\varrho}$ *which has the smallest* L^2*-norm and we prove that* $\bar{\varrho}$ *belongs to* $L^2_H(\Omega)$, *the space of square summable functions harmonic in* Ω. *However* $\bar{\varrho}$ *is unstable, i.e. its* L^2*-norm does not depend continuously upon the* L^2*-norm of the potential. We show how a continuous dependence may be restored by introducing mild restrictions on the set of admissible solutions.*

1. – Introduction.

The classical inverse problem in potential theory consists in finding the density ϱ of a (charged) material body Ω embedded in a region S by the knowledge of the potential generated by this body outside (or on the surface of) S; incidentally Ω may coincide with S.

As is well-known, this problem has not a unique solution. For example suppose that Ω coincides with S, $\Phi \in C_0^\infty(R^n)$, $\operatorname{supp}\Phi$ (the support of Φ) $\subset \bar{\Omega}$ and take

$$\varrho^+ = \operatorname{Sup}(\Delta\Phi, 0)\,, \qquad \varrho^- = -\operatorname{Inf}(\Delta\Phi, 0)\,.$$

Then ϱ^+ and ϱ^- create outside Ω the same potential U:

$$U(x) = \int_\Omega \varrho^+(y)K(x-y)\,dy = \int_\Omega \varrho^-(y)K(x-y)\,dy\,. \tag{1.1}$$

Here K is the fundamental solution of the Laplace equation, i.e.

$$\begin{aligned} &K(x) = [(n-2)\omega_n]^{-1}|x|^{2-n} \qquad (n\geqslant 3)\,;\ \omega_n = 2\pi^{n/2}/\Gamma(n/2)\,,\\ &K(x) = -(2\pi)^{-1}\log|x| \qquad (n=2)\,. \end{aligned} \tag{1.2}$$

Relation (1.1) follows simply from the identity

$$0 = \int_\Omega K(x-y)\,\Delta\Phi(y)\,dy = \int_\Omega K(x-y)\big(\varrho^+(y) - \varrho^-(y)\big)\,dy\,,$$

which holds for any $x \notin \bar{\Omega}$.

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As the density ϱ cannot be determined uniquely from the values of the potential U outside Ω, a first approach to this theory consists in investigating the structure of the set of solutions. Convexity and compactness properties of this set have been proved in ANGER [1], SCHULZE-WILDENHAIN [9].

However, it is quite obvious that supplementary information have to be provided when the uniqueness of the solution is looked for: e.g. the density ϱ may be supposed not to depend on one co-ordinate x_i. Results in this direction can be found in PRILEPKO [7].

Another classical inverse problem in potential theory consists in looking for the position and shape of the body Ω contained in S, knowing the density ϱ (typically, ϱ is a constant) and the potential U outside S. Several results about this problem can be found in the Soviet literature: see, for instance, LAVRENT'EV [3] and PRILEPKO [8].

Inverse problems for potentials arise in many fields of applied sciences, not only in geophysics, but e.g. in cardiology, biology and solid state physics. A rich bibliography can be found in ANGER [2].

The aim of this paper is to consider the first problem outlined above; in the set of all densities ϱ which give rise to the same potential on the surface of Ω we look for the one which minimizes some functional, e.g. the L^2-norm. Our main interest concerns the stability of this solution, i.e. the dependence of the solution ϱ on the given potential.

Section 2 has a preliminary character: it is devoted to a discussion of some properties of the Newtonian potential.

In section 3 the inverse problem is considered: first this (known) result is stated: among all the densities ϱ which generate the same potential outside Ω, one and only one belongs to $L^2_H(\Omega)$, the space of square summable functions harmonic in Ω [12]. Such a solution is of smallest norm. The L^2-norm of this solution does not depend continuously on the L^2-norm of the potential. This aspect is analyzed and it is shown how, by means of mild restrictions to the set of solutions, it is possible to restore a continuous dependence. The simplest case ($\Omega = R^3_-$) is discussed in detail.

Grounding on the results of section 3, we show in section 4 that our problem can be approximated by a sequence of stable problems. This fact may be useful for numerical computations.

2. – Some properties of the Newtonian potentials.

Although all the following considerations can be carried out in the space R^n ($n \geqslant 3$), we shall work in R^3.

Ω is a simply connected bounded open set in R^3 of class $C^{2,1}$ containing the origin. $\Omega_e = \complement\bar{\Omega}$ is the complementary set of the closure of Ω.

Let A be either Ω or Ω_e. In addition to the ordinary Sobolev spaces $W^{s,2}(\Omega)$ we shall consider also the following:

$$(2.1)\qquad \mathfrak{W}^{s,2}(A) = \{\varrho \in L^2(A, (1+|x|^2)^{-1}) \colon D^\alpha \varrho \in L^2(A),\ 1 \leqslant |\alpha| \leqslant s\} \qquad (s \in N \setminus \{0\})\ (^1)$$

$$(2.2)\qquad L^2_H(A) = \{\varrho \in L^2(A) \colon \Delta\varrho = 0 \text{ in } \mathfrak{D}'(A)\}\ (^2)\,.$$

Consider the «potential operator» U so defined in $L^2(\Omega)$:

$$(2.3)\qquad U\varrho(x) = \int_\Omega |x-y|^{-1}\varrho(y)\,dy\,.$$

We denote by $N(U)$ and $\mathcal{R}(U)$ respectively its kernel and range.

LEMMA 2.1. – i) *operator* $U \colon L^2(\Omega) \to \mathfrak{W}^{2,2}(\Omega_e)$ *is continuous*;

$$(2.4)\qquad \text{ii) } N(U) = L^2_H(\Omega)^\perp\ (^3);$$

$$(2.5)\qquad \text{iii) } \mathcal{R}(U) = \mathfrak{W}^{2,2}(\Omega_e) \cap L^2_H\big(\Omega_e, (1+|x|^2)^{-1}\big)\,.$$

PROOF. – i) It is well-known that $U\varrho \in W^{2,2}_{\text{loc}}(R^3)$, $\forall \varrho \in L^2(R^3)$; moreover there exists a positive constant C_0 (independent of ϱ) such that

$$(2.6)\qquad \|D^\alpha U\varrho\|_{L^2(R^3)} \leqslant C_0 \|\varrho\|_{L^2(R^3)}\,, \qquad \forall \varrho \in L^2(R^3),\ 1 \leqslant |\alpha| \leqslant 2\,.$$

From lemma 2.1 in NIRENBERG-WALKER [6] we deduce also that

$$(2.7)\qquad \big\||x|^{-1} U\varrho\big\|_{L^2(\Omega_e)} \leqslant C_1 \|\varrho\|_{L^2(\Omega)}\,, \qquad \forall \varrho \in L^2(\Omega)\,.$$

Moreover we shall prove now that

$$(2.8)\qquad \|DU\varrho\|_{L^2(\Omega_e)} \leqslant C_2(\Omega)\|\varrho\|_{L^2(\Omega)}\,, \qquad \forall \varrho \in L^2(\Omega)\,,$$

where C_1 and $C_2(\Omega)$ are positive constants, $C_2(\Omega)$ depending only on Ω. Assertion i) easily follows from (2.6), (2.7), (2.8). To prove (2.8) consider first the estimate (which comes again from [6])

$$(2.9)\qquad \big\||x|^{-1} DU\varrho\big\|_{L^2(\Omega_e)} \leqslant C_3 \|\varrho\|_{L^2(\Omega)}\,, \qquad \forall \varrho \in L^2(\Omega)\,,$$

where C_3 is a positive constant independent of ϱ and Ω.

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(1) $L^2(A, \sigma(x))$ (with σ positive) is the weighted Hilbert space normed as follows: $\|\varrho\|_{L^2(A,\sigma(x))} = \int_A \varrho(x)^2\sigma(x)\,dx$.

(2) $\mathfrak{D}'(A)$ is the space of all distributions over A.

(3) $L^2_H(\Omega)^\perp$ is the subspace in $L^2(\Omega)$ orthogonal to $L^2_H(\Omega)$.

On the other hand, by putting $\Omega(2r) = \complement\overline{S(0, 2r)}$ (4), where $r = \operatorname{Sup}_{y\in\Omega} |y|$, we get

$$\|DU\varrho\|_{L^2(\Omega(2r))} \leqslant \left(\int_{\Omega(2r)} \left(\int_{\Omega} |x-y|^{-2}|\varrho(y)|\,dy\right)^2 dx\right)^{1/2} \leqslant \tag{2.10}$$

$$\leqslant \int_{\Omega} |\varrho(y)| \left(\int_{\Omega(2r)} |x-y|^{-4}\,dx\right)^{1/2} dy \leqslant \int_{\Omega} |\varrho(y)| \left(\int_{|x-y|\geqslant r} |x-y|^{-4}\,dx\right)^{1/2} dy \leqslant$$

$$\leqslant (4\pi m(\Omega) r^{-1})^{1/2} \|\varrho\|_{L^2(\Omega)}.$$

Then (2.8) is a consequence of (2.9) and (2.10).

ii) (2.4) follows from the well-known fact that the set of functions

$$x \to |x-y|^{-1}, \qquad y \in \Omega_e$$

is dense in $L^2_H(\Omega)$ (5).

iii) In order to prove (2.5) let us notice first that $U\varrho \in L^2_H(\Omega_e, (1+|x|^2)^{-1})$, so that the inclusion $\mathcal{R}(U) \subset \mathcal{W}^{2,2}(\Omega_e) \cap L^2_H(\Omega_e, (1+|x|^2)^{-1})$ is immediate.

Consider now the equation

$$U\varrho = f, \tag{2.11}$$

where f is any (assigned) function in $\mathcal{W}^{2,2}(\Omega_e) \cap L^2_H(\Omega_e, (1+|x|^2)^{-1})$. We are going to show that equation (2.11) admits a unique solution in $L^2_H(\Omega)$. Thinking of $u = U\varrho$ as a function in $\mathcal{W}^{2,2}(R^3)$, we have to find two functions u and ϱ such that

$$\begin{cases} (u, \varrho) \in \mathcal{W}^{2,2}(R^3) \times L^2(\Omega) \\ \Delta u = -4\pi\varrho & \text{in } \Omega \\ (\varrho, \Delta v)_{L^2(\Omega)} = 0, & \forall v \in W^{2,2}_0(\Omega) \\ u = f & \text{in } \Omega_e. \end{cases} \tag{2.12}$$

That amounts to find the function w (the restriction of u to Ω) such that

$$\begin{cases} w \in W^{2,2}(\Omega) \\ (\Delta w, \Delta v)_{L^2(\Omega)} = 0, & \forall v \in W^{2,2}_0(\Omega) \\ w = f & \text{on } \partial\Omega \\ \dfrac{\partial w}{\partial n} = \dfrac{\partial f}{\partial n} & \text{on } \partial\Omega \ (6). \end{cases}$$

(4) $S(x_0, r)$ is the ball centred at x_0 with radius r.

(5) This assertion holds under very large conditions on $\partial\Omega$, e.g. a cone condition.

(7) $n(x)$ is the outward normal to $\partial\Omega$ at x.

As Ω is of class $C^{2,1}$, such a w is unique and depends continuously on $f \in \mathcal{W}^{2,2}(\Omega_e) \cap \cap L^2_H\big(\Omega_e, (1+|x|^2)^{-1}\big)$. Therefore

$$\varrho = -(4\pi)^{-1}\Delta w \quad \text{in } \Omega . \tag{2.14}$$

Such a function solves eq. (2.11). For, from Green's formula we get for any $x \in \Omega_e$ the equation

$$U\varrho(x) = f(x) + \frac{1}{4\pi}\int_{\partial\Omega}\left\{f(y)\frac{\partial}{\partial n(y)}\left(|x-y|^{-1}\right) - |x-y|^{-1}\frac{\partial f}{\partial n(y)}(y)\right\}d\sigma(y) .$$

But the integral in the last formula vanishes, as one easily verifies by applying Green's formula to f and $y \to |x-y|^{-1}$. ■

The next lemma deals with the «exterior» Dirichlet problem.

LEMMA 2.2. – $\forall f \in L^2(\Omega_e, 1+|x|^2)$, $\forall v \in W^{3/2,2}(\partial\Omega)$, *the problem*

$$\begin{aligned} & u \in \mathcal{W}^{2,2}(\Omega_e) \\ & \Delta u = f \quad \text{in } \Omega_e \\ & \gamma_{\partial\Omega} u = v \end{aligned} \tag{2.15}$$

admits a unique solution satisfying the estimate

$$\|u\|_{\mathcal{W}^{2,2}(\Omega_e)} \leqslant C_5(\Omega_e)\{\|f\|_{L^2(\Omega_e, 1+|x|^2)} + \|v\|_{W^{3/2,2}(\partial\Omega)}\} . \tag{2.16}$$

(*Here* $\gamma_{\partial\Omega}$ *denotes the trace operator on* $\partial\Omega$.)

PROOF OF LEMMA 2.2. – We know that there exists a boundedly supported function $w \in \mathcal{W}^{2,2}(\Omega_e)$ which satisfies the equation $\gamma_{\partial\Omega} w = v$ and the estimate

$$\|w\|_{\mathcal{W}^{2,2}(\Omega_e)} \leqslant C_6(\Omega_e)\|v\|_{W^{3/2,2}(\partial\Omega)} . \tag{2.17}$$

Putting $z = u - w$, problem (2.15) is reduced to the following

$$\begin{aligned} & z \in \mathcal{W}^{2,2}(\Omega_e) \\ & \Delta z = f + \Delta w \in L^2(\Omega_e, 1+|x|^2) \\ & \gamma_{\partial\Omega} z = 0 . \end{aligned} \tag{2.18}$$

Such a problem admits a unique variational solution belonging to $\mathcal{W}^{1,2}_0(\Omega_e) = \{z \in \mathcal{W}^{1,2}(\Omega_e) \colon \gamma_{\partial\Omega} z = 0\}$, according to the following lemma which guarantees that in $\mathcal{W}^{1,2}_0(\Omega_e)$ the norm of the gradient $z \to \|Dz\|_{L^2(\Omega_e)}$ and the norm induced by $\mathcal{W}^{1,2}(\Omega_e)$ are equivalent.

LEMMA 2.3. – *$\forall u \in \mathcal{W}_0^{1,2}(\Omega_e)$ the following estimate holds:*

$$\| |x|^{-1} u \|_{L^2(\Omega_e)} \leq 2 \| Du \|_{L^2(\Omega_e)} . \tag{2.19}$$

By standard regularization procedures we can find a positive constant $C_7(\Omega_e)$ such that the $\mathcal{W}^{2,2}(\Omega_e)$-norm of z can be estimated as follows:

$$\| z \|_{\mathcal{W}^{2,2}(\Omega_e)} \leq C_7(\Omega_e) \{ \| f \|_{L^2(\Omega_e, 1+|x|^2)} + \| \Delta w \|_{L^2(\Omega_e)} \} . \tag{2.20}$$

From this and (2.17) we deduce (2.16). ∎

PROOF OF LEMMA 2.3. – Estimate (2.19) holds for $\Omega_e = R^3$; in fact, take $u \in C_0^\infty(R_3)$ and consider the identity

$$u(x) = -\int_1^{+\infty} \frac{\partial}{\partial t} u(tx)\, dt = -\int_1^{+\infty} Du(tx) \cdot x \, dt .$$

You get

$$\| |x|^{-1} u \|_{L^2(R^3)} \leq \int_1^{+\infty} \left(\int_{R^3} |Du(tx)|^2 \, dx \right)^{1/2} dt = 2 \| Du \|_{L^2(R^3)} .$$

Then the assertion for R^3 follows by density arguments.

In the general case take $u \in \mathcal{W}_0^{1,2}(\Omega_e)$ and define $\tilde{u}$ in the following way: $\tilde{u}(x) = u(x)$ if $x \in \Omega_e$, $\tilde{u}(x) = 0$ if $x \in \complement \Omega_e$. Then $\tilde{u} \in W^{1,2}(R^3)$ and $\widetilde{Du} = D\tilde{u}$, so that the assertion easily follows. ∎

The next lemma is concerned with the restriction of U to $\partial\Omega$, i.e. with the linear operator

$$V = \gamma_{\partial\Omega} U . \tag{2.21}$$

LEMMA 2.4. – i) *Operator V is continuous from $L^2(\Omega)$ onto $W^{3/2,2}(\partial\Omega)$ and*

$$N(V) = N(U) = L_H^2(\Omega)^\perp ; \tag{2.22}$$

ii) *$\forall v \in W^{3/2,2}(\partial\Omega)$ the problem*

$$\begin{cases} \varrho \in L_H^2(\Omega) \\ V\varrho = v \end{cases} \tag{2.23}$$

admits a unique solution satisfying the estimate

$$\| \varrho \|_{L^2(\Omega)} \leq C_8(\Omega) \| v \|_{W^{3/2,2}(\partial\Omega)} . \tag{2.24}$$

PROOF. – Let $\varrho \in L^2(\Omega)$ be a solution to the equation $V\varrho = 0$. Then $u = U\varrho$ solves pb. (2.15) with $f = 0$ and $v = 0$. From lemma 2.2 it follows that $U\varrho = 0$: therefore $N(V) \subset N(U)$ and (2.22) follows immediately.

Let us show now that pb. (2.23) has a unique solution; as a consequence we shall get $\mathfrak{R}(V) = W^{3/2,2}(\partial\Omega)$. Let ϱ be a solution of (2.23); then $u = U\varrho$ solves (2.15) with $f = 0$. Therefore a positive constant $C_9(\Omega)$ exists such that

$$\|u\|_{\mathcal{W}^{2,2}(\Omega_e)} \leqslant C_9(\Omega)\|v\|_{W^{3/2,2}(\partial\Omega)} . \tag{2.25}$$

Let us put now

$$w(x) = \int_\Omega |x - y|^{-1}\varrho(y)\,dy\,, \qquad \forall x \in \Omega\,. \tag{2.26}$$

The pair (w, ϱ) is a solution to the problem

$$\begin{aligned}
& w \in W^{2,2}(\Omega) \\
& \varrho \in L^2(\Omega) \\
& \Delta w = -4\pi\varrho && \text{in } \Omega \\
& (\varrho, \Delta v)_{L^2(\Omega)} = 0\,, && \forall v \in W_0^{2,2}(\Omega) \\
& \gamma_{\partial\Omega}\frac{\partial^j w}{\partial n^j} = \gamma_{\partial\Omega}\frac{\partial^j u}{\partial n^j}\,, && j = 0, 1\,.
\end{aligned} \tag{2.27}$$

From (2.27) we deduce that w is a solution to the variational problem:

$$\begin{aligned}
& w \in W^{2,2}(\Omega) \\
& (\Delta w, \Delta v)_{L^2(\Omega)} = 0\,, && \forall v \in W_0^{2,2}(\Omega) \\
& \gamma_{\partial\Omega}\frac{\partial^j w}{\partial n^j} = \gamma_{\partial\Omega}\frac{\partial^j u}{\partial n^j}\,.
\end{aligned} \tag{2.28}$$

As Ω is of class $C^{2,1}$ and the norms $u \to \|\Delta u\|_{L^2(\Omega)}$ and $u \to \|u\|_{W^{2,2}(\Omega)}$ are equivalent in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$, we get from (2.25) that

$$\|w\|_{W^{2,2}(\Omega)} \leqslant C_{10}(\Omega)\|v\|_{W^{3/2,2}(\partial\Omega)} . \tag{2.29}$$

Next we deduce that ϱ is given by the formula

$$\varrho = -(4\pi)^{-1}\Delta w\,,$$

so that it satisfies estimate (2.24). As we did at the end of the proof of lemma 2.1 we can now prove that (w, ϱ) solves problem (2.27). ∎

REMARK. – In the sequel we shall use the property: *V is a continuous operator from $W^{s,2}(\Omega)$ to $W^{s+3/2,2}(\partial\Omega)$* $(0 < s < 1/2)$. This assertion is a consequence of the continuity of $R_\Omega U$ (R_Ω is the restriction operator to Ω) from $W^{s,2}(\Omega)$ to $W^{s+2,2}(\Omega)$. To this purpose observe first the equation

$$U\varrho(x) = \int_{R^3} |x - y|^{-1}\tilde{\varrho}(y)\, dy\,, \tag{2.30}$$

where $\tilde{\varrho}(x) = \varrho(x)$ if $x \in \Omega$, $\tilde{\varrho}(x) = 0$ if $x \in \complement\Omega$.

Moreover from the C^1-regularity of Ω we deduce that $W^{s,2}(\Omega) = W^{s,2}_0(\Omega)$ $(0 < s < 1/2)$ and the mapping $\varrho \to \tilde{\varrho}$ is continuous from $W^{s,2}(\Omega)$ to $W^{s,2}(R^3)$. Then, taking lemma 2.1 into account and using representation (2.30), we can easily verify that U maps continuously $W^{j,2}_0(\Omega)$ into $W^{2+j,2}(R^3)$ $(j = 0, 1)$. Therefore $R_\Omega U$ maps continuously $W^{j,2}_0(\Omega)$ into $W^{2+j,2}(\Omega)$ $(j = 0, 1)$. Finally by interpolation we deduce the assertion.

3. – The inverse problem.

Recalling the results of section 2 we can state that the set of functions

$$K(v) = \{\varrho \in L^2(\Omega) \colon V\varrho = v\}\,, \tag{3.1}$$

where $v \in W^{3/2,2}(\partial\Omega)$, is a convex closed unbounded set in $L^2(\Omega)$.

In this section we will consider the problem of finding a function in $K(v)$ minimizing some functional. We restrict our attention to the functional

$$J_0(\varrho) = \|\varrho\|_{L^2(\Omega)}\,. \tag{3.2}$$

REMARK. – Notice that other functionals would be more interesting for applications: e.g. the total mass $\int_\Omega \varrho(x)\, dx$ or the total energy $\int_{\Omega\times\Omega} \varrho(x)\varrho(y)|x - y|^{-1}\, dx\, dy$. But in the first case the knowledge of the potential v on $\partial\Omega$ completely determines the value of the functional (this is a consequence of the Gauss divergence theorem). In the latter case it is known (at least to physicists) that the minimum of the functional does not exist in $L^2(\Omega)$: the whole mass (or the whole charge) concentrates on $\partial\Omega$ [8].

We go on observing that the existence and the uniqueness of the minimum of (3.2) is guaranteed by a well-known theorem of functional analysis. Therefore our

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[8] When Ω is the half-space $x_3 > 0$ we can easily verify this assertion; for the minimal function ϱ must satisfy a non-homogeneous Wiener-Hopf integral equation, whose unique solution is a Dirac mass supported on the plane $x_3 = 0$.

considerations will be mainly concerned with the dependence of the minimizing function on v.

THEOREM 3.1. – *$\forall v \in W^{3/2,2}(\partial\Omega)$ the problem*

$$\varrho \in K(v)\,, \quad J_0(\varrho) = minimum \tag{3.3}$$

admits the unique solution $\varrho = \varrho_0(v)$, where $\varrho_0(v)$ is the unique (harmonic) solution to problem (2.23). It depends linearly on v and satisfies the estimate

$$\|\varrho_0(v_2) - \varrho_0(v_1)\|_{L^2(\Omega)} \leqslant C(\Omega)\|v_2 - v_1\|_{W^{3/2,2}(\partial\Omega)} \tag{3.4}$$

for every $v_1, v_2 \in W^{3/2,2}(\partial\Omega)$.

PROOF. – Notice first that $K(v)$ is a linear affine closed manifold. For, by taking lemma 2.4 into account, $K(v)$ can be decomposed as follows:

$$K(v) = L^2_H(\Omega)^\perp + \varrho_0\,, \tag{3.5}$$

where $\varrho_0 = \varrho_0(v)$ is the unique (harmonic) solution to problem (2.23). Then $\varrho_0 \in L^2_H(\Omega)$ and every $\varrho \in K(v)$ can be represented as $\varrho = \sigma + \varrho_0$ with $\sigma \in L^2_H(\Omega)^\perp$. It follows immediately that $J_0(\varrho) \geqslant J_0(\varrho_0)$, $\forall \varrho \in K(v)$; then ϱ_0 is the unique function minimizing J_0 on $K(v)$. Since ϱ_0 is the solution to pb. (2.23), from (2.24) we get estimate (3.4). ∎

Estimate (3.4) is unsatisfactory from a practical point of view, since the L^2-norm of the density depends on the norm of some derivatives of the potential. We wish now to improve this estimate in order to obtain a dependence of the type

$$\|\varrho_0(v_2) - \varrho_0(v_1)\|_{L^2(\Omega)} \leqslant f(\|v_1 - v_2\|_{L^2(\partial\Omega)})\,, \tag{3.6}$$

where $f\colon \bar{R}_+ \to \bar{R}_+$ is a continuous function vanishing at the origin.

As is known in many similar cases, such a goal may be often achieved by a convenient restriction of the class of admissible solutions ϱ.

LEMMA 3.1. – *Let $\varrho_0 = \varrho_0(v)$ be the solution to pb. (3.3) with $v \in W^{3/2,2}(\partial\Omega)$. Then $\varrho_0(v)$ satisfies the estimate*

$$\|\varrho_0(v)\|_{L^2(\Omega)} \leqslant C(s, \Omega)\|\varrho_0(v)\|^{3/(3+2s)}_{W^{s,2}(\Omega)}\|v\|^{2s/(3+2s)}_{L^2(\partial\Omega)} \tag{3.7}$$

$\forall s \in (0, 1/2)$, where $C(s, \Omega)$ is a positive constant depending only on s and Ω.

Then for any given positive constant E let us define the set

$$H(E) = \{\varrho \in L^2(\Omega)\colon\ \|\varrho\|_{W^{s,2}(\Omega)} \leqslant E\}\,. \tag{3.8}$$

From lemma 3.1 we get immediately the theorem:

THEOREM 3.2. – *For any given $E \in R_+$, let $\varrho_0(v_1)$ and $\varrho_0(v_2)$ be any pair of solutions to problem* (3.3) *belonging to $H(E)$; then the following estimate holds*:

$$\|\varrho_0(v_2) - \varrho_0(v_1)\|_{L^2(\Omega)} \leq C(s, \Omega)(2E)^{3/(3+2s)}\|v_2 - v_1\|_{L^2(\partial\Omega)}^{2s/(3+2s)}, \quad \forall s \in (0, 1/2). \tag{3.9}$$

PROOF OF LEMMA 3.1. – By taking Fourier transforms and using Hölder's inequality (with indexes $p = (3 + 2s)/3$ and $p' = (3 + 2s)/(2s)$) we easily deduce the estimate

$$\|\Phi\|_{W^{3/2,2}(R^2)} \leq C_9(s)\|\Phi\|_{W^{3/2+s,2}(R^2)}^{3/(3+2s)} \cdot \|\Phi\|_{L^2(R^2)}^{2s/(3+2s)} \tag{3.10}$$

which holds for any $\Phi \in W^{3/2+s,2}(R^2)$.

Let now v be an arbitrary function in $W^{3/2+s,2}(\partial\Omega)$ and let v_r $(r = 1, \dots, N)$ be the functions associated with v as in NEČAS [5], pg. 89. Each of these functions is defined on the square $\Delta = \{x \in R^2 \colon 0 \leq x_j \leq \alpha,\ j = 1, 2\}$. Proceeding as in NEČAS [5], th. 3.9, we can prove that there exists an extension operator $\mathcal{E}\colon L^2(\Delta) \to L^2(R^2)$ which is continuous from $W^{t,2}(\Omega)$ into $W^{t,2}(R^2)$ $(0 \leq t \leq 2)$. Then the following chain of estimates holds for any $v \in W^{3/2+s,2}(\partial\Omega)$ $(0 < s < 1/2)$:

$$\begin{aligned}\|v\|_{W^{3/2,2}(\partial\Omega)} &= \left(\sum_{r=1}^{N}\|v_r\|^2_{W^{3/2,2}(\Delta)}\right)^{1/2} \leq \left(\sum_{r=1}^{N}\|\mathcal{E}v_r\|^2_{W^{3/2,2}(R^2)}\right)^{1/2} \leq \\ &\leq C_9(s)\left(\sum_{r=1}^{N}\|\mathcal{E}v_r\|^{6/(3+2s)}_{W^{3/2+s,2}(R^2)}\|\mathcal{E}v_r\|^{4s/(3+2s)}_{L^2(R^2)}\right)^{1/2} \leq \\ &\leq C_9(s)\left(\sum_{r=1}^{N}\|\mathcal{E}v_r\|^2_{W^{3/2+s,2}(R^2)}\right)^{3/2(3+2s)}\left(\sum_{r=1}^{N}\|\mathcal{E}v_r\|^2_{L^2(R^2)}\right)^{s/(3+2s)} \leq \\ &\leq C_{10}(s)\left(\sum_{r=1}^{N}\|v_r\|^2_{W^{3/2+s,2}(\Delta)}\right)^{3/2(3+2s)}\left(\sum_{r=1}^{N}\|v_r\|^2_{L^2(\Delta)}\right)^{s/(3+2s)} = C_{10}(s)\|v\|^{3/(3+2s)}_{W^{3/2+s,2}(\partial\Omega)}\|v\|^{2s/(3+2s)}_{L^2(\partial\Omega)}.\end{aligned} \tag{3.11}$$

Finally, taking lemma 2.4 into account, from (2.24) and (3.11) we get the wanted estimates:

$$\begin{aligned}\|\varrho_0(v)\|_{L^2(\Omega)} &\leq C_8(\Omega)\|v\|_{W^{3/2,2}(\partial\Omega)} \leq C_{11}(s, \Omega)\|V\varrho_0\|^{3/(3+2s)}_{W^{3/2+s,2}(\partial\Omega)}\|v\|^{2s/(3+2s)}_{L^2(\partial\Omega)} \leq \\ &\leq C_{12}(s, \Omega)\|\varrho_0(v)\|^{3/(3+2s)}_{W^{s,2}(\Omega)}\|v\|^{2s/(3+2s)}_{L^2(\partial\Omega)}. \quad \blacksquare\end{aligned}$$

A SPECIAL CASE: $\Omega = R^3_-$. Let us denote by $\hat{}$ the Fourier transformation with respect to $x' = (x_1, x_2)$. Then problem (3.3) may be reduced to minimizing the norm

$$j_0(\hat{\varrho})(\xi) = \|\hat{\varrho}(\xi, \cdot)\|_{L^2(R_-)} \tag{3.12}$$

under the constraint

$$\int_{R_-} \exp(t|\xi|)\hat{\varrho}(\xi, t)\, dt = 2|\xi|\hat{v}(\xi) \tag{3.13}$$

for almost every $\xi \in R^2$. Such a problem obviously admits a unique solution $\hat{\varrho}_0(\xi, \cdot) \in L^2(R_-)$ (for a.e. $\xi \in R^2$). Moreover it can be explicitly determined by the method of Lagrange multipliers or, more simply, by developing $\hat{\varrho}(\xi, \cdot)$ by a series of orthogonal functions (Laguerre functions). One finds the formula

$$\hat{\varrho}_0(\xi, t) = \frac{1}{\pi} |\xi|^2 \exp(t|\xi|)\, \hat{v}(\xi)\,, \tag{3.14}$$

which immediately implies the chain of equations:

$$\|\varrho_0\|^2_{L^2(R^3_-)} = (2\pi)^{-2} \|\hat{\varrho}_0\|^2_{L^2(R^3_-)} = (2\pi^4)^{-1} \int_{R^2} |\xi|^3 |\hat{v}(\xi)|^2 \, d\xi = 2\pi^{-2} \|D^{3/2} v\|^2_{L^2(R^2)} \;(^8)\,. \tag{3.15}$$

In order to get an estimate of the form (3.9) it suffices to restrict the class of admissible solutions of the problem (3.12), (3.13) to the convex set $H(E)$ so defined:

$$H(E) = \{\varrho \in L^2(R^3_-) : \|D^s_{x'} \varrho\|_{L^2(R^3_-)} \leqslant E\} \;(^9)\,, \tag{3.16}$$

where s and E are given positive constants. In fact we have

$$\|D^s_{x'} \varrho_0\|^2_{L^2(R^3_-)} = (2\pi^4)^{-1} \int_{R^2} |\xi|^{3+2s} |\hat{v}(\xi)|^2 \, d\xi\,. \tag{3.17}$$

Applying Hölder's inequality with exponents $(3+2s)/3$ and $(3+2s)/(2s)$ to the integral

$$\int_{R^2} |\xi|^3 |\hat{v}(\xi)|^2 \, d\xi = \int_{R^2} |\xi|^3 |\hat{v}(\xi)|^{6/(3+2s)} |\hat{v}(\xi)|^{s/(3+2s)} \, d\xi$$

and using (3.17) we get

$$\|\varrho_0\|_{L^2(R^3_-)} \leqslant \|D^s_{x'} \varrho_0\|^{3/(3+2s)}_{L^2(R^3_-)} \|v\|^{2s/(3+2s)}_{L^2(R^2)} \leqslant E^{3/(3+2s)} \|v\|^{2s/(3+2s)}_{L^2(R^2)}\,. \tag{3.18}$$

Thus we obtain the final estimate valid for $\varrho_0(v_2)$ and $\varrho_0(v_1)$ belonging to $H(E)$:

$$\|\varrho_0(v_2) - \varrho_0(v_1)\|_{L^2(\Omega)} \leqslant (2E)^{3/(3+2s)} \|v_2 - v_1\|^{2s/(3+2s)}_{L^2(R^2)}\,. \quad \blacksquare \tag{3.19}$$

4. – Reformulation of the problem.

As we saw in the previous section, an estimate of the form (3.6) (namely (3.9)) is obtained by constraining a-priori the solution to some admissible set, in our case by bounding a-priori some derivative of the solution. This suggests to reformulate

---

(8) We put $(D^\alpha v)\hat{}\,(\xi) = (i\xi)^\alpha \hat{v}(\xi)$, $\alpha \in R_+$.

(9) $(D^s_{x'} \varrho_0)\hat{}\,(\xi, x_3) = (i\xi)^s \hat{\varrho}_0(\xi, x_3)$. Notice that (3.16) implies $v \in W^{3/2+s,2}(R^2)$.

problem (3.3) in such a way as to introduce explicity this constraint. On the other hand we shall tolerate some error on the measured data. Thus we will consider a new functional $J(\varrho)$ instead of $J_0(\varrho)$, whose minimum $\bar{\varrho}$, looked for on some closed bounded set, automatically will satisfy the constraint and $V\bar{\varrho}$ will differ from v by a small quantity (in L^2-norm). Now the data of the new problem will be: the measured potential v, the numbers ε (an estimate of the error on the measured potential) and E (an estimate of the a-priori bound for the derivative of ϱ).

In general, without further restrictions on the parameters ε and E, our problem will have no solution. We shall give a compatibility relation between the data in order a solution to exist.

Now, let us pose exactly the new problem: it will be more convenient, for technical reasons, to break it into two steps.

First we consider the problem (P1):

(P1) *for every fixed*

$$(\lambda, v, \varepsilon, E) \in \bar{R}_+ \times L^2(\Gamma, \mu) \times R_+ \times R_+ \tag{4.1}$$

minimize the functional

$$J(\varrho) = \|\varrho\|^2_{L^2(\Omega)} + \lambda E^{-2}|\varrho|^2_{W^{s,2}(\Omega)} + \lambda\varepsilon^{-2}\|V\varrho - v\|^2_{L^2(\Gamma,\mu)} \tag{4.2}$$

on the whole space $W^{s,2}(\Omega)$ $(0 < s < 1/2)$.

Here we have put

$$|\varrho|_{W^{s,2}(\Omega)} = \left(\int_{\Omega\times\Omega} |x-y|^{-3-2s}|\varrho(x) - \varrho(y)|^2\,dx\,dy\right)^{1/2} \tag{4.3}$$

and we have denoted by Γ any submanifold of $\partial\Omega$ whose dimension may be 2, 1 (in this case we suppose Γ is a Jordan curve of class C^1) or 0 $\left(\text{in this case we suppose } \Gamma \text{ consists of a finite number of points, } \Gamma = \bigcup_{j=0}^{n}\{x_j\}\right)$; μ is the measure induced on Γ by the three-dimensional Lebesgue measure if $\dim\Gamma = 1$ or 2; if $\dim\Gamma = 0$, we put, for instance, $\mu\{x_j\} = 1$ $(j = 1, \ldots, n)$.

Notice that we allow here the experimental measurements $v(x)$ to be performed on a submanifold Γ of $\partial\Omega$; moreover $v(x)$ does not coincide, in general, with the value of the potential $V\varrho(x)$ on Γ.

We shall prove the existence and uniqueness of a solution $\varrho = \bar{\varrho}(\lambda, v, \varepsilon, E)$ to the problem (P1). Of course this solution, depending on the given parameter λ, might be such that $V\varrho$ is markedly far from the experimental value v. Thus we are led to consider the second step.

(P2) *for every fixed*

$$(v, \varepsilon, E) \in L^2(\Gamma, \mu) \times R_+ \times R_+ \tag{4.4}$$

minimize the function $\lambda \to \|\bar{\varrho}(\lambda, v, \varepsilon, E)\|_{L^2(\Omega)}$ *on the set*

$$(4.5)\quad \Lambda(v, \varepsilon, E) = \{\lambda \in \bar{R}_+ : E^{-2}|\bar{\varrho}(\lambda, v, \varepsilon, E)|^2_{W^{s,2}(\Omega)} + \varepsilon^{-2}\|V\bar{\varrho}(\lambda, v, \varepsilon, E) - v\|^2_{L^2(\Gamma,\mu)} \leqslant 1\}.$$

Now, if there exists a minimal function $\bar{\varrho}(\lambda_0, v, \varepsilon, E)$, it necessarily satisfies the estimate

$$(4.6)\qquad \|V\bar{\varrho}(\lambda_0, v, \varepsilon, E) - v\|_{L^2(\Gamma,\mu)} \leqslant \varepsilon ,$$

i.e. we allow the experimental measurements to be far from the potential $V\varrho$ no more than a given quantity ε in L^2-norm. Moreover the given parameter E is a measure of the a-priori bound for the derivative of the solution.

We shall prove that, under suitable restriction on the data (v, ε, E), problem (P2) admits a unique solution continuously depending on v in the L^2-norm.

THEOREM 4.1. – *Problem* (P1) *admits a unique solution* $\varrho = \bar{\varrho}(\lambda, v, \varepsilon, E)$, *which depends linearly on* v *and satisfies the estimates*

$$(4.7)\qquad \|V\bar{\varrho}(\lambda, v, \varepsilon, E)\|_{L^2(\Gamma,\mu)} \leqslant \|v\|_{L^2(\Gamma,\mu)}$$

$$(4.8)\qquad \|\bar{\varrho}(\lambda, v, \varepsilon, E)\|_{W^{s,2}(\Omega)} \leqslant \alpha(\lambda, \varepsilon, E, s, \Omega)\|v\|_{L^2(\Gamma,\mu)}\ \text{[10]} .$$

REMARK. – The solution $\bar{\varrho}(\lambda, v, \varepsilon, E)$ is also continuously differentiable with respect to its arguments. The estimates for the derivatives can be obtained by straightforward calculations.

PROOF OF THEOREM 4.1. – Notice that, if $\lambda = 0$ or $v = 0$, the functional J obviously attains its minimum at $\varrho = 0$. Thus let us assume $\lambda > 0$ and $v \neq 0$. From definition (4.2) we get

$$(4.9)\qquad J(\varrho) \geqslant \min(1, \lambda/E^2)\|\varrho\|^2_{W^{s,2}(\Omega)}, \qquad \forall \varrho \in W^{s,2}(\Omega) .$$

Then there exists a closed ball $\overline{S(0, r)} \subset W^{s,2}(\Omega)$ with centre at $\varrho = 0$ and radius $r > 0$ such that

$$(4.10)\qquad \operatorname*{Inf}_{\varrho \in \overline{S(0,r)}} J(\varrho) = \operatorname*{Inf}_{\varrho \in W^{s,2}(\Omega)} J(\varrho) .$$

The existence of the minimum now follows from the property: if $\{\varrho_n\} \subset \overline{S(0, r)}$ weakly converges to a function $\varrho \in \overline{S(0, r)}$, then there exists a subsequence $\{\varrho_{n_k}\}$ strongly converging to ϱ in $L^2(\Omega)$ for which

$$(4.11)\qquad \varliminf_{k \to +\infty} J(\varrho_{n_k}) \geqslant J(\varrho) .$$

[10] The dependence of α on λ, ε, E will be detailed in the proof of the theorem.

The uniqueness of the minimum and its properties follow from the fact that $J \in C^1(W^{s,2}(\Omega), R)$.

The minimal function ϱ satisfies the equation

$$J'(\varrho)h = 0\,, \qquad \forall h \in W^{s,2}(\Omega)\,. \tag{4.12}$$

Eq. (4.12) may be written in the form

$$\mathcal{B}(h, \varrho, \lambda) \equiv (h, \varrho)_{L^2(\Omega)} + \lambda B(h, \varrho) = \lambda\varepsilon^{-2}(\nabla h, v)_{L^2(\Gamma,\mu)}\,, \qquad \forall h \in W^{s,2}(\Omega)\,. \tag{4.13}$$

The bilinear form B (defined below) is continuous on $W^{s,2}(\Omega)$ and $W^{s,2}(\Omega)$-elliptic:

$$B(h, \varrho) = E^{-2}(h, \varrho)_{W^{s,2}(\Omega)} + \varepsilon^{-2}(\nabla h, \nabla\varrho)_{L^2(\Gamma,\mu)} \tag{4.14}$$

where

$$(h, \varrho)_{W^{s,2}(\Omega)} = \int_{\Omega\times\Omega} |x-y|^{-3-2s}[h(x) - h(y)]\cdot[\varrho(x) - \varrho(y)]\,dx\,dy\,. \tag{4.15}$$

From the equation, valid for every $h \in W^{s,2}(\Omega)$,

$$\mathcal{B}(h, h, \lambda) = \|h\|^2_{L^2(\Omega)} + \lambda E^{-2}|h|^2_{W^{s,2}(\Omega)} + \lambda^{-2}\|\nabla h\|^2_{L^2(\Gamma,\mu)} \tag{4.16}$$

we immediately deduce estimate (4.7).

To prove (4.8) notice first that the norms

$$\|h\|_1 = \|h\|_{W^{s,2}(\Omega)}$$
$$\|h\|_2 = \left(\|\nabla h\|^2_{L^2(\Gamma,\mu)} + |h|^2_{W^{s,2}(\Omega)}\right)^{1/2}$$

are equivalent in $W^{s,2}(\Omega)$ (see e.g. NEČAS [5] for similar arguments). Then from (4.16) we get the estimate

$$\mathcal{B}(h, h, \lambda) \geqslant \|h\|^2_{L^2(\Omega)} + \lambda \min(E^{-2}, \varepsilon^{-2})c(s, \Omega)\|h\|^2_{W^{s,2}(\Omega)}\,, \qquad \forall h \in W^{s,2}(\Omega)\,. \tag{4.17}$$

From this inequality and (4.13) we deduce

$$\begin{aligned}[1 + \lambda \min(E^{-2}, \varepsilon^{-2})c(s, \Omega)]\|\bar\varrho(\lambda, v, \varepsilon, E)\|^2_{L^2(\Omega)} &\leqslant \\ \leqslant \lambda\varepsilon^{-2}\|\nabla\bar\varrho(\lambda, v, \varepsilon, E)\|_{L^2(\Gamma,\mu)}\|v\|_{L^2(\Gamma,\mu)} &\leqslant \lambda\varepsilon^{-2}\|v\|^2_{L^2(\Gamma,\mu)}\,.\end{aligned} \tag{4.18}$$

Then we have

$$\|\bar\varrho(\lambda, v, \varepsilon, E)\|_{L^2(\Omega)} \leqslant \varepsilon^{-1}\lambda^{1/2}[1 + \lambda \min(E^{-2}, \varepsilon^{-2})\,c(s, \Omega)]^{-1/2}\|v\|_{L^2(\Gamma,\mu)}\,. \tag{4.19}$$

From (4.13), (4.17), (4.18), (4.19) one obtains the chain of inequalities

$$
\begin{aligned}
(4.20)\qquad & \|\bar{\varrho}(\lambda, v, \varepsilon, E)\|^2_{W^{s,2}(\Omega)} \leqslant \max(E^2\varepsilon^{-2}, 1)\, c(s, \Omega)^{-1}\|v\|_{L^2(\Gamma,\mu)} \times \\
& \times \|V\bar{\varrho}(\lambda, v, \varepsilon, E)\|_{L^2(\Gamma,\mu)} \leqslant \max(E^2\varepsilon^{-2}, 1)\, c(s, \Omega)^{-1}\|V\|\,\|v\|_{L^2(\Gamma,\mu)}\|\bar{\varrho}(\lambda, v, \varepsilon, E)\|_{L^2(\Omega)} \leqslant {}^{(11)} \\
& \leqslant \varepsilon^{-1}\max(E^2\varepsilon^{-2}, 1)\, c(s, \Omega)^{-1} c_0(\Omega)\, \lambda^{1/2}[1 + \lambda \min(E^{-2}, \varepsilon^{-2})\, c(s, \Omega)]^{-1/2}\|v\|^2_{L^2(\Gamma,\mu)}\,.
\end{aligned}
$$

We conclude by observing that (4.8) follows immediately from (4.20). ■

Let us consider now problem (P2). It will be convenient to study first the properties of the function $\Phi\colon \bar{R}_+ \times L^2(\Gamma, \mu) \times R_+ \times R_+ \to \bar{R}_+$ defined as follows

$$
(4.21)\qquad \Phi(\lambda, v, \varepsilon, E) = E^{-2}|\bar{\varrho}(\lambda, v, \varepsilon, E)|^2_{W^{s,2}(\Omega)} + \varepsilon^{-2}\|V\bar{\varrho}(\lambda, v, \varepsilon, E) - v\|^2_{L^2(\Gamma,\mu)}\,.
$$

The main properties of Φ are listed in the next lemma 4.1. In particular it is proved that Φ, as a function of λ, is decreasing from the value $\varepsilon^{-2}\|v\|^2_{L^2(\Gamma,\mu)}$ (attained at $\lambda = 0$) to some limiting value called $\Psi(v, \varepsilon, E)$.

Notice that the equation $\Phi(\lambda, v, \varepsilon, E) = 1$ is just the equation of the border of the set $\Lambda(v, \varepsilon, E)$ (defined by (4.5)) on which we are looking for the minimum of the function $\lambda \to \|\bar{\varrho}(\lambda, v, \varepsilon, E)\|_{L^2(\Omega)}$. Since this minimum, if it exists, it is attained at the border of this set, from the stated properties of Φ we immediately derive the conditions on (v, ε, E) which guarantee the existence of the minimum.

LEMMA 4.1. – *Function Φ, defined by (4.21), is continuously differentiable in its arguments and has the properties:*

i) $\Phi(\lambda, v, \varepsilon, E) > 0 \Leftrightarrow v \neq 0$;

ii) $\dfrac{\partial\Phi}{\partial\lambda}(\lambda, v, \varepsilon, E) = -2\mathcal{B}\left(\dfrac{\partial\bar{\varrho}}{\partial\lambda}(\lambda, v, \varepsilon, E), \dfrac{\partial\bar{\varrho}}{\partial\lambda}(\lambda, v, \varepsilon, E), \lambda\right)$ *and*

$\dfrac{\partial\Phi}{\partial\lambda}(\lambda, v, \varepsilon, E) < 0 \Leftrightarrow v \neq 0$;

iii) $\Phi(0, v, \varepsilon, E) = \varepsilon^{-2}\|v\|^2_{L^2(\Gamma,\mu)}$;

iv) $\Psi(v, {}_L, E) = \lim\limits_{\lambda\to+\infty} \Phi(\lambda, v, \varepsilon, E) =$

$$
= E^{-2}|\bar{\varrho}(+\infty, v, \varepsilon, E)|^2_{W^{s,2}(\Omega)} + \varepsilon^{-2}\|V\bar{\varrho}(+\infty, v, \varepsilon, E) - v\|^2_{L^2(\Gamma,\mu)},
$$

where $\bar{\varrho}(+\infty, v, \varepsilon, E)$ is the solution to the variational equation

$$
(4.22)\qquad B\big(h, \bar{\varrho}(+\infty, v, \varepsilon, E)\big) = \varepsilon^{-2}(Vh, v)_{L^2(\Gamma,\mu)}\,, \qquad \forall h \in W^{s,2}(\Omega)\,.
$$

---

(11) $\|V\| = \sup\limits_{h\in L^2(\Omega)\setminus\{0\}} \|h\|^{-1}_{L^2(\Omega)}\|Vh\|_{L^2(\Gamma,\mu)} = c_0(\Omega)$.

Moreover the limiting value Ψ has the following properties:

v) $\dfrac{\partial \Psi}{\partial \varepsilon}(v, \varepsilon, E) = -2\varepsilon^{-3}\| V\bar{\varrho}(+\infty, v, \varepsilon, E) - v\|^2_{L^2(\Gamma,\mu)}$;

vi) $\dfrac{\partial \Psi}{\partial \varepsilon}(v, \varepsilon, E) < 0 \Leftrightarrow v \neq \textit{constant}\cdot V\mathbf{1}$ (12);

vii) $\dfrac{\partial \Psi}{\partial \varepsilon}(v, \varepsilon, E) = -2E^{-3}|\bar{\varrho}(+\infty, v, \varepsilon, E)|^2_{W^{s,2}(\Omega)}$;

viii) $\dfrac{\partial \Psi}{\partial \varepsilon}(v, \varepsilon, E) < 0 \Leftrightarrow v \neq \textit{constant}\cdot V\mathbf{1}$.

PROOF. – Most of the above statements i) to viii) are readily verifiable: thus the proofs of some assertions are omitted.

i): $\Phi(\lambda, v, \varepsilon, E) = 0 \Rightarrow \bar{\varrho}(\lambda, v, \varepsilon, E) = c = \text{constant}, \quad v = cV\mathbf{1} \Rightarrow c(h, \mathbf{1})_{L^2(\Omega)}$, $\forall h \in W^{s,2}(\Omega) \Rightarrow c = 0 \Rightarrow v = 0$. Conversely $v = 0 \Rightarrow \bar{\varrho}(\lambda, 0, \varepsilon, E) = 0 \Rightarrow \Phi(\lambda, 0, \varepsilon, E) = 0$.

ii): it is a consequence of (4.13) and (4.14) (substitute there $h = (\partial\bar{\varrho}/\partial\lambda)(\lambda, v, \varepsilon, E)$, $\varrho = \bar{\varrho}(\lambda, v, \varepsilon, E)$) and of the following equations where we replace h by $\partial\bar{\varrho}/\partial\lambda\ (\lambda, v, \varepsilon, E)$

$$\mathcal{B}\left(h, \frac{\partial\bar{\varrho}}{\partial\lambda}(\lambda, v, \varepsilon, E), \lambda\right) = -B\big(h, \bar{\varrho}(\lambda, v, \varepsilon, E)\big) + \varepsilon^{-2}(Vh, v)_{L^2(\Gamma,\mu)} = \lambda^{-1}\big(h, \bar{\varrho}(\lambda, v, \varepsilon, E)\big)_{L^2(\Omega)}, \quad \forall h \in W^{s,2}(\Omega) . \tag{4.23}$$

Moreover we have

$$\frac{\partial\Phi}{\partial\lambda}(\lambda, v, \varepsilon, E) = 0 \Leftrightarrow (\text{by } (4.17)), \qquad \frac{\partial\varrho}{\partial\lambda}(\lambda, v, \varepsilon, E) = 0 \Leftrightarrow (\text{by } (4.23)),$$

$$B\big(h, \bar{\varrho}(\lambda, v, \varepsilon, E)\big) = \varepsilon^{-2}(Vh, v)_{L^2(\Gamma,\mu)}, \quad \forall h \in W^{s,2}(\Omega) \Leftrightarrow (\text{by } (4.13)),$$

$$\big(h, \bar{\varrho}(\lambda, v, \varepsilon, E)\big)_{L^2(\Omega)} = 0, \quad \forall h \in W^{s,2}(\Omega) \Leftrightarrow \bar{\varrho}(\lambda, v, \varepsilon, E) = 0 \Leftrightarrow (Vh, v)_{L^2(\Gamma,\mu)} = 0,$$

$\forall h \in W^{s,2}(\Omega) \Leftrightarrow v = 0$ (remember that $VW^{s,2}(\Omega)$ is dense in $L^2(\Gamma, \mu)$).

iv): as $\lambda \to \Phi(\lambda, v, \varepsilon, E)$ strictly decreases if $v \neq 0$, then the limit $\Psi(v, \varepsilon, E)$ exists and is finite. To prove the assertion it suffices to show that $\bar{\varrho}(\lambda, v, \varepsilon, E) \to \bar{\varrho}(+\infty, v, \varepsilon, E)$ in $W^{s,2}(\Omega)$ as $\lambda \to +\infty$. By substracting (4.14) from (4.22) one gets the equation

$$B\big(h, \bar{\varrho}(\lambda, v, \varepsilon, E) - \bar{\varrho}(+\infty, v, \varepsilon, E)\big) = -\lambda^{-1}\big(h, \bar{\varrho}(\lambda, v, \varepsilon, E)\big)_{L^2(\Omega)}, \quad \forall h \in W^{s,2}(\Omega) . \tag{4.24}$$

(12) $\mathbf{1}$ stands for the constant function which equals 1 in Ω.

From (4.24), (4.14) and (4.8) one derives the inequality

$$(4.25)\qquad \|\bar{\varrho}(\lambda, v, \varepsilon, E) - \bar{\varrho}(+\infty, v, \varepsilon, E)\|_{W^{s,2}(\Omega)} \leqslant \max(E^2, \varepsilon^2)\lambda^{-1}\alpha(\lambda, \varepsilon, E, s, \Omega)\|v\|_{L^2(\Gamma,\mu)}.$$

It implies the assertion, since $\alpha(\cdot, \varepsilon, E, s, \Omega)$ is bounded. ∎

REMARK. – Notice that from (4.22) it follows that

$$(4.26)\qquad \bar{\varrho}(+\infty, V1, \varepsilon, E) = 1\,, \quad \forall(\varepsilon, E) \in R_+ \times R_+$$

and, as a consequence,

$$(4.27)\qquad \Psi(\textit{constant}\cdot V1, \varepsilon, E) = 0\,, \quad \forall(\varepsilon, E) \in R_+ \times R_+\,.$$

Let us consider now the equation

$$(4.28)\qquad \Phi(\lambda, v, \varepsilon, E) = 1\,.$$

LEMMA 4.2. – i) *If* $\varepsilon > \|v\|_{L^2(\Gamma,\mu)}$ *there is no value of* λ *satisfying eq.* (4.28);

ii) *if* $\varepsilon = \|v\|_{L^2(\Gamma,\mu)}$ *eq.* (4.28) *is satisfied only for* $\lambda = 0$;

iii) *if* $0 < \varepsilon < \|v\|_{L^2(\Gamma,\mu)}$ *there exists one (and only one) value of* λ *satisfying eq.* (4.28) *iff*

$$(4.29)\qquad \Psi(v, \varepsilon, E) < 1\,.$$

In other words eq. (4.28) *is uniquely solvable with respect to* λ *iff the triplet* (v, ε, E) *is an element of the set* D *so defined*

$$(4.30)\quad D = \{(v, \varepsilon, E) \in L^2(\Gamma, \mu) \times R_+ \times R_+ : 0 < \varepsilon \leqslant \|v\|_{L^2(\Gamma,\mu)}, \quad \Psi(v, \varepsilon, E) < 1\}\,.$$

Moreover the function $\varphi(v, \varepsilon, E)$, *implicitly defined by* (4.28), *belongs to* $C^0(D) \cap C^1(\mathring{D})$ (13).

PROOF. – It derives obviously from properties i) to v) of the previous lemma. ∎

In the next lemma we want to study more closely the compatibility relation (4.29). Therefore let us consider the equation

$$(4.31)\qquad \Psi(v, \varepsilon, E) = 1\,.$$

LEMMA 4.3. – *Eq.* (4.31) *is uniquely solvable with respect to* E *iff the pair* (v, ε) *is an element of the set* $\mathcal{A}$, *where*

$$(4.32)\qquad \mathcal{A} = \{(v, \varepsilon) \in L^2(\Gamma, \mu) \times R_+ : 0 < \varepsilon < \bar{\varepsilon}(v)\}$$

(13) $\mathring{D}$ stands for the interior of D.

and

(4.33) $$\bar{\varepsilon}(v) = \|(V1, v)_{L^2(\Gamma,\mu)} \cdot \|V1\|^{-2}_{L^2(\Gamma,\mu)} \cdot V1 - v\|_{L^2(\Gamma,\mu)}\,.$$

Moreover the function $\psi(v, \varepsilon)$, *implicitly defined by* (4.31), *belongs to* $C^0(\mathcal{A}) \cap C^1(\mathring{\mathcal{A}})$ *and the function* $\varepsilon \to \psi(v, \varepsilon)$ *is strictly decreasing in* $(0, \bar{\varepsilon}(v))$ *and vanishes when* $\varepsilon = \bar{\varepsilon}(v)$.

REMARK. – The general behaviour of the function $\varepsilon \to \psi(v, \varepsilon)$ is illustrated in fig. 1.

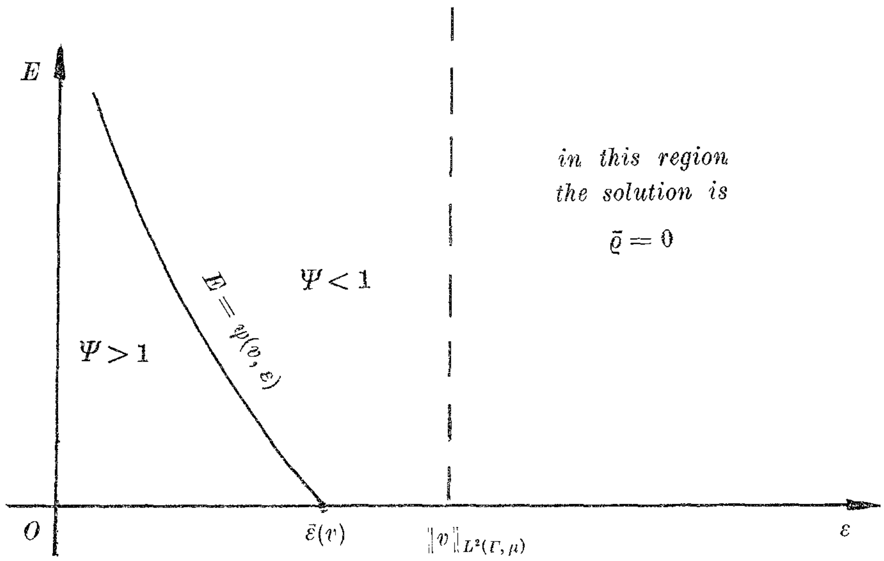

Fig. 1.

Notice that, as $\varepsilon \to 0^+$, then $\psi \to +\infty$; however it may happen that there exists a finite limit for special (« very regular ») v (see the example at the end of the section). In particular, when $v = V1$, then $\bar{\varepsilon}(v) = 0$ and the curve in fig. 1 reduces to the positive half-axis $\varepsilon = 0$, $E > 0$.

In general notice that $\bar{\varepsilon}(v)$ is the norm of the projection of v along the direction of the vector orthogonal to $V1$ belonging to the plane containing v and $V1$. Thus we have

$$\bar{\varepsilon}(v) \leqslant \|v\|_{L^2(\Gamma,\mu)}\,.$$

In particular, when Ω is a ball and $\Gamma = \partial\Omega$, we get

$$\bar{\varepsilon}(v) = \|\text{arithmetic mean of } v \text{ on the surface of the ball} - v\|_{L^2(\Gamma,\mu)}\,.$$

PROOF OF LEMMA 4.3. – First we prove the relation

(4.34) $$V\bar{\varrho}(+\infty, v, \varepsilon, E) \to v \quad \text{in } L^2(\Gamma, \mu) \text{ as } E \to +\infty\,, \quad \forall (v, \varepsilon) \in L^2(\Gamma, \mu) \times R_+\,.$$

To this purpose observe that it is easy to show (arguing as in the proof of theorem 4.1) that the solution $\bar{\varrho}(+\infty, v, \varepsilon, E)$ to eq. (4.14) satisfies the following estimates valid for any $(v, \varepsilon, E) \in L^2(\Gamma, \mu) \times R_+ \times R_+$:

$$\| V\bar{\varrho}(+\infty, v, \varepsilon, E)\|_{L^2(\Gamma,\mu)} \leqslant \|v\|_{L^2(\Gamma,\mu)} \tag{4.35}$$

$$\|\bar{\varrho}(+\infty, v, \varepsilon, E)\|_{W^{s,2}(\Omega)} \leqslant C(s, \Omega)(1 + E^2\varepsilon^{-2})^{1/2}\|v\|_{L^2(\Gamma,\mu)} \tag{4.36}$$

$$|\bar{\varrho}(+\infty, v, \varepsilon, E)|_{W^{s,2}(\Omega)} \leqslant E\varepsilon^{-1}\|v\|_{L^2(\Gamma,\mu)}\,. \tag{4.37}$$

From (4.14) and (4.37) we immediately deduce the relation

$$\lim_{E\to+\infty} \big(Vh, V\bar{\varrho}(+\infty, v, \varepsilon, E) - v\big)_{L^2(\Gamma,\mu)} = 0\,, \qquad \forall (h, v, \varepsilon) \in W^{s,2}(\Omega) \times L^2(\Gamma, \mu) \times R_+\,. \tag{4.38}$$

Recall then that $VW^{s,2}(\Omega)$ is strongly dense in $L^2(\Gamma, \mu)$ and the set $\{V\bar{\varrho}(+\infty, v, \varepsilon, E)\colon (\varepsilon, E) \in R_+ \times R_+\}$ is strongly bounded in $L^2(\Gamma, \mu)$ for every (fixed) $v \in L^2(\Gamma, \mu)$, owing to estimate (4.35). By virtue of a well-known theorem in Functional Analysis (see for instance YOSIDA [11], theor. 3, p. 121) from (4.38) we obtain that

$$V\bar{\varrho}(+\infty, v, \varepsilon, E) \rightharpoonup v \quad \text{in } L^2(\Gamma, \mu) \text{ as } E \to +\infty\,, \quad \forall (v, \varepsilon) \in L^2(\Gamma, \mu) \times R_+\,. \tag{4.39}$$

From (4.35) and (4.39) we easily get the following estimate, which proves (4.34):

$$\| V\bar{\varrho}(+\infty, v, \varepsilon, E) - v\|^2_{L^2(\Gamma,\mu)} \leqslant 2\big(v - V\bar{\varrho}(+\infty, v, \varepsilon, E), v\big)_{L^2(\Gamma,\mu)}\,.$$

Thus from (4.14) (with $h = \bar{\varrho}(+\infty, v, \varepsilon, E)$), (4.34), (4.35) we deduce the relation

$$\lim_{E\to+\infty} E^{-2}|\bar{\varrho}(+\infty, v, \varepsilon, E)|^2_{W^{s,2}(\Omega)} = 0\,, \quad \forall (v, \varepsilon) \in L^2(\Gamma, \mu) \times R_+\,. \tag{4.40}$$

Finally (4.34) and (4.40) imply that

$$\lim_{E\to+\infty} \Psi(v, \varepsilon, E) = 0\,, \quad \forall (v, \varepsilon) \in L^2(\Gamma, \mu) \times R_+\,. \tag{4.41}$$

The next step consists in proving that the relation

$$\lim_{E\to0+} \Psi(v, \varepsilon, E) = \varepsilon^{-2}\bar{\varepsilon}(v)^2 \tag{4.42}$$

holds for any $(v, \varepsilon) \in L^2(\Gamma, \mu) \times R_+$.

First we show that there exists a sequence $\{E_n\} \subset R_+$ decreasing monotonically to 0 as $n \to +\infty$ such that for any $(v, \varepsilon) \in L^2(\Gamma, \mu) \times R_+$

$$\bar{\varrho}(+\infty, v, \varepsilon, E_n) \to (V1, v)_{L^2(\Gamma,\mu)} \cdot \|V1\|^{-1}_{L^2(\Gamma,\mu)} \cdot 1 \quad \text{in } L^2(\Omega) \text{ as } n \to +\infty \tag{4.43}$$

$$|E_n^{-1}\bar{\varrho}(+\infty, v, \varepsilon, E_n)|_{W^{s,2}(\Omega)} \to 0 \qquad \text{as } n \to +\infty\,. \tag{4.44}$$

We postpone for a moment the proofs of (4.43) and (4.44) and observe that they, together with the monotonicity of $E \to \Psi(v, \varepsilon, E)$ (see ix) in lemma 4.1), imply immediately relation (4.42).

To prove (4.43) we recall that (4.36) (with a fixed pair (v, ε)) and the boundedness of Ω assure the existence of a function $\varrho_0 \in W^{s,2}(\Omega)$ and of a sequence $\{E_n\} \subset R_+$, $E_n \downarrow 0$ as $n \to +\infty$ such that

$$\bar{\varrho}(+\infty, v, \varepsilon, E_n) \rightharpoonup \varrho_0 \quad \text{in } W^{s,2}(\Omega) \tag{4.45}$$

$$\bar{\varrho}(+\infty, v, \varepsilon, E_n) \to \varrho_0 \quad \text{in } L^2(\Omega) \tag{4.46}$$

as $n \to +\infty$, $\forall (v, \varepsilon) \in L^2(\Gamma, \mu) \times R_+$.

Hence from (4.14) and (4.35) we deduce that

$$(h, \varrho_0)_{W^{s,2}(\Omega)} = 0\,, \qquad \forall h \in W^{s,2}(\Omega)\,.$$

This means that

$$\varrho_0 = c\mathbf{1} \tag{4.47}$$

for some $c \in R$. Moreover from (4.14) (with $h = \mathbf{1}$ and $E = E_n$) it is easy to infer the equation $(V\mathbf{1}, cV\mathbf{1} - v)_{L^2(\Gamma,\mu)} = 0$, which, in turn, implies

$$c = (V\mathbf{1}, v)_{L^2(\Gamma,\mu)} \cdot \|V\mathbf{1}\|^{-1}_{L^2(\Gamma,\mu)}\,. \tag{4.48}$$

Observe now that (4.43) is an immediate consequence of (4.46), (4.47), (4.48).

It remains to prove relation (4.44); it is implied by (4.14) (with $h = \bar{\varrho}(+\infty, v, \varepsilon, E_n)$) and (4.43).

Taking now relations (4.41) and (4.42) into account and recalling the monotonicity of Φ with respect to E, it is immediate to realize that eq. (4.31) is solvable for $E \in R_+$ iff the pair (v, ε) belongs to the set $\mathcal{A}$ defined by (4.32).

Moreover, since $(v, \varepsilon) \in \mathcal{A}$ implies $v \neq c\mathbf{1}$, $\forall c \in R$, from viii) and x) in lemma 4.1 we get the inequalities $(\partial\Psi/\partial\varepsilon)(v, \varepsilon, E) < 0$ and $(\partial\Psi/\partial E)(v, \varepsilon, E) < 0, \forall (v, \varepsilon, E) \in \mathcal{A} \times R_+$. The previous relations imply that eq. (4.31) implicitly defines a unique function $\psi: \mathcal{A} \to R_+$. Owing to lemma 4.1 and to the implicit function theorem, it has the property stated in the lemma. ■

We can now solve problem (P2) stated at the beginning of this section. First we observe that the set $\Lambda(v, \varepsilon, E)$ defined by (4.5) is, due to lemma 4.2, a closed half-line in $\bar{R}_+$, $\forall (v, \varepsilon, E) \in D \cup \tilde{D}$, $\tilde{D}$ being defined by

$$\tilde{D} = \{(v, \varepsilon, E) \in L^2(\Gamma, \mu) \times R_+ \times R_+ : 0 \leqslant \|v\|_{L^2(\Gamma,\mu)} \leqslant \varepsilon\}\,. \tag{4.49}$$

More particularly we get the following equations:

$$\Lambda(v,\varepsilon,E)=\begin{cases}[\varphi(v,\varepsilon,E),+\infty)\,, & \forall(v,\varepsilon,E)\in D\,,\\ \bar{R}_+\,, & \forall(v,\varepsilon,E)\in\tilde{D}\,.\end{cases}\tag{4.50}$$

We can now prove the following existence-uniqueness theorem for problem (P2):

THEOREM 4.2. – *For every $(v,\varepsilon,E)\in D\cup\tilde{D}$ the function $\lambda\to\|\bar{\varrho}(\lambda,v,\varepsilon,E)\|_{L^2(\Omega)}$ attains its least value $m(v,\varepsilon,E)$ on $\Lambda(v,\varepsilon,E)$ at a unique point λ_0, the pair $(\lambda_0,\,m(v,\varepsilon,E))$ being given by the formulas*

$$\lambda_0=\begin{cases}\varphi(v,\varepsilon,E)\,, & \forall(v,\varepsilon,E)\in D\\ 0 & \forall(v,\varepsilon,E)\in\tilde{D}\end{cases}\tag{4.51}$$

$$m(v,\varepsilon,E)=\begin{cases}\bar{\varrho}(\varphi(v,\varepsilon,E),v,\varepsilon,E)\,, & (v,\varepsilon,E)\in D\,,\\ 0 & (v,\varepsilon,E)\in\tilde{D}\,.\end{cases}\tag{4.52}$$

Moreover the function $(v,E)\to m(v,\varepsilon,E)$ belongs to $C^0(D_\varepsilon\cup\tilde{D}_\varepsilon;\,W^{s,2}(\Omega))\cap C^1(\mathring{D}_\varepsilon\cup\mathring{\tilde{D}}_\varepsilon;\,W^{s,2}(\Omega))$ (14).

PROOF OF THEOREM 4.2. – The statement relative to λ_0 is an obvious consequence of lemma 4.2 and eq. (4.23) (15) as well as the one relative to the function $(v,E)\to$ $\to m(v,\varepsilon,E)$, except for the continuity of such a function at the points of the set

$$F_\varepsilon=\{(v,E)\in L^2(\Gamma,\mu)\times R_+:\,(v,\varepsilon)\in\mathcal{A},\ \|v\|_{L^2(\Gamma,\mu)}=\varepsilon\,,\ E>\psi(v,\varepsilon)\}\,.$$

Consider now the relations

$$\Phi(0,v,\varepsilon,E)=\varepsilon^{-2}\|v\|^2_{L^2(\Gamma,\mu)}=1\,,\qquad\forall(v,E)\in F_\varepsilon$$
$$\Psi(v,\varepsilon,E)<1\,,\qquad\forall(v,E)\in F_\varepsilon\,.$$

As $\Phi\in C^0(\bar{R}_+\times L^2(\Gamma,\mu)\times R_+\times R_+)$ and $\Psi\in C^1(L^2(\Gamma,\mu)\times R_+\times R_+)$, from the implicit function theorem applied to the equation $\Phi(\lambda,v,\varepsilon,E)=1$ we easily obtain that $\lambda=\varphi(v,\varepsilon,E)\to0$ as $D_\varepsilon\ni(v,E)\to(v_0,E_0)\in F_\varepsilon$. From (4.20) we immediately deduce the wanted relation, i.e.

$$m(v,\varepsilon,E)=\bar{\varrho}(\varphi(v,\varepsilon,E),v,\varepsilon,E)\to0$$
$$\text{in } L^2(\Omega),\ \forall\varepsilon\in R_+\ \text{as}\ D_\varepsilon\in(v,E)\to(v_0,E_0)\in F_\varepsilon\,.\qquad\blacksquare$$

---

(14) D_{ε_0} denotes the intersection of D with the hyperplane $\varepsilon=\varepsilon_0$:

(15) Substitute there $h=(\partial\bar{\varrho}/\partial\lambda)(\lambda,v,\varepsilon,E)$.

A SPECIAL CASE: $\Omega = R^3_-$. Let us consider again the example previously discussed. We can pose problem (P1) in a slightly modified manner: *for every fixed* $(\lambda, v, \varepsilon, E) \in \bar{R}_+ \times L^2(R^2) \times R_+ \times R_+$ *minimize the functional*

$$J(\varrho) = \|\varrho\|^2_{L^2(R^3_-)} + \lambda E^{-2}\|D_{x'}\varrho\|^2_{L^2(R^3_-)} + \lambda\varepsilon^{-2}\|V\varrho - v\|^2_{L^2(R^2)} .$$

Using Fourier transforms with respect to $x' = (x_1, x_2)$ we can write the equation

$$J(\varrho) = \int_{R^3_-}(1 + \lambda E^{-2}|\xi|^2)|\hat{\varrho}(\xi, x_3)|^2\, d\xi\, dx_3 + \\ + \lambda\varepsilon^{-2}\int_{R^2}\left|\int_{-\infty}^{0}|\xi|^{-1}\exp(-|\xi||x_3|)\hat{\varrho}(\xi, x_3)\, dx_3 - \hat{v}(\xi)\right|d\xi .$$

Straightforward calculations give us the explicit formula for the solution to problem (P1), namely:

$$\hat{\bar{\varrho}}(\xi, x_3) = 2\lambda|\xi|^2\exp(-|\xi||x_3|)[\lambda + 2\varepsilon^2(1 + \lambda E^{-2}|\xi|^2)|\xi|^3]^{-1}\hat{v}(\xi) . \tag{4.53}$$

The assertions in theorem 4.1 (and in the remark following it) can be directly verified.

Problem (P2) now consists in minimizing the function

$$\lambda \to \|\bar{\varrho}\|^2_{L^2(R^3_-)} = 2\lambda^2\int_{R^2}|\xi|^3[\lambda + 2\varepsilon^2(1 + \lambda E^{-2}|\xi|^2)|\xi|^3]^{-2}|\hat{v}(\xi)|^2\, d\xi$$

subject to the constraint

$$E^{-2}\int_{R^2}|\xi|^2\, d\xi\int_{-\infty}^{0}|\hat{\bar{\varrho}}(\xi, x_3)|^2\, dx_3 + \varepsilon^{-2}\int_{R^2}|\xi|^{-2}\, d\xi\left|\int_{-\infty}^{2}\hat{\bar{\varrho}}(\xi, x_3)\exp(-|\xi||x_3|)\, dx_3 - \hat{v}(\xi)\right|^0 \leqslant 1 .$$

Taking formula (4.53) into account, the constraint can be written in the form

$$\Phi(\lambda, v, \varepsilon, E) \leqslant 1 ,$$

where

$$\Phi(\lambda, v, \varepsilon, E) = \\ = \varepsilon^{-2}\int_{R^2}\{1 - \lambda[4\varepsilon^2|\xi|^3 + \lambda(1 + 2\varepsilon^2E^{-2}|\xi|^5)][2\varepsilon^2|\xi|^3 + \lambda(1 + 2\varepsilon^2E^{-2}|\xi|^5)]^{-2}\}|\hat{v}(\xi)|^2\, d\xi .$$

Clearly the minimal function $\bar{\varrho}$ will be either 0 or $\bar{\varrho} = \bar{\varrho}(\lambda_0, v, \varepsilon, E)$, λ_0 being a solution to the equation

$$\Phi(\lambda, v, \varepsilon, E) = 1 . \tag{4.54}$$

The properties of the function Φ listed in lemmas 4.1, 4.2, 4.3 can be easily verified. In particular, since Φ is a decreasing function of λ, $\Phi(0, v, \varepsilon, E) = \varepsilon^{-2}\|v\|_{L^2(R^2)}$ and

$$\Psi(v, \varepsilon, E) \underset{\text{def}}{=} \Phi(+\infty, v, \varepsilon, E) = \int_{R^2} |\xi|^5 (E^2/2 + \varepsilon^2 |\xi|^5)^{-1} |\hat{v}(\xi)|^2 \, d\xi \,,$$

we conclude that the necessary and sufficient condition in order a unique (positive) root λ of equation (4.54) to exist is that the triplet (v, ε, E) be a solution to the inequality

$$\Psi(v, \varepsilon, E) \leqslant 1 \,.$$

The situation is pictured as in fig. 1, where now $\bar{\varepsilon}(v) = \|v\|_{L^2(R^2)}$.

Notice that, if v is smooth enough, namely if $v \in W^{5/2,2}(R^2)$, we find a finite value for $E = \lim_{\varepsilon \to 0+} \psi(v, \varepsilon)$.

Moreover, when $\varepsilon = \|v\|_{L^2(R^2)}$, the solution of (4.54) is $\lambda = 0$, which yields $\bar{\varrho} = 0$. In this case the uncertainty on the measurement is so high that the set $\Lambda(v, \varepsilon, E)$ (defined by (4.5)) includes the origin: thus the null function is the solution.

Note added in proofs.

It has been pointed out to us that N. WECK, in Applicable Analysis, **2** (1972), pp. 195-238, considered a problem quite analogous to that studied here; he assumes that the potential v is measured on a large spherical surface containing Ω rather than on the boundary of Ω itself. This paper has some contact points with Weck's.

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