Nonlinear Semigroups and Age-Dependent Population Models (*).

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Summary. – *This paper treats the nonlinear age-dependent population problem* (1) $\varrho(0, a) = \varphi(a)$, $a \in I$; (2) $\varrho(t, 0) = F(\varrho(t, \cdot))$, $t \geq 0$; (3) $\lim_{h\to 0} (\varrho(t+h, a+h) - \varrho(t, a))/h = G(\varrho(t, \cdot))(a)$, $a \in I$, $t \geq 0$, *where* I *is the age range of the population,* $\varrho(t, \cdot)$ *is the unknown age density at time* t, φ *is the known initial age distribution, and the functionals* F *and* G *are nonlinear. The problems of existence, uniqueness, continuous dependence upon initial values, and the positivity of solutions are investigated using the method of nonlinear semigroups.*

1. – Introduction.

Our purpose is prove the existence, uniqueness, and continuous dependence upon initial values of solutions to a general nonlinear age-dependent population model. The theory of nonlinear age-dependent population models initiated with the work of M. GURTIN and R. MACCAMY in [15] and F. HOPPENSTEADT in [11]. Their approach took advantage of the special form of the nonlinearities in the problem to apply the method of characteristics and to convert the equations to a system of nonlinear Volterra integral equations. Other researchers, such as G. DIBLASIO [3], [4], E. SINESTRARI [24], and A. HAIMOVICI [9], [10], have exploited this approach to further develope the theory. The advantages of the Volterra integral equations approach is that the method is direct and the formulas which result provide a useful representation of the solutions. The disadvantage is that the method limits the nature of the nonlinearities which may be treated.

In this paper we adopt a different approach to the problem. Our method views an age-dependent population model as a nonlinear semigroup or dynamical system in a Banach space of initial age distributions. To solve our problem we proceed as follows: (1) in an appropriate Banach space of initial age distributions we define a nonlinear operator A; (2) we show that A generates a nonlinear semigroup $T(t)$, $t \geq 0$ in this Banach space by appealing to the general theory of nonlinear semigroups; (3) we demonstrate that the solutions of our age-dependent population problem are given by the semigroup $T(t)$, $t \geq 0$. The advantages of the semigroup approach are that it removes the technical complexities of the proofs, allows very general

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nonlinearities in the models, and exhibits the dynamical system structure of the solutions. Furthermore, it makes available the theory of operator semigroups in Banach spaces for the investigation of such problems as numerical approximation, asymptotic behavior, and control theory.

The organization of this paper is as follows: In Section 2 we state the results from nonlinear semigroups theory which we will need. In Section 3 we present the nonlinear age-dependent population problem which we will solve. In Section 4 we state our theorems and in Section 5 we give their proofs. In Section 6 we make some concluding remarks.

2. – Nonlinear semigroups.

The facts stated below are taken from [1] and [2].

DEFINITION 2.1. – Let A be a mapping from a subset of a Banach space Y to Y. A is *accretive* provided that there exists $\lambda_0 > 0$ such that if $y_1, y_1 \in D(A)$ and $0 < \lambda < \lambda_0$

$$\|(I + \lambda A)y_1 - (I + \lambda A)y_2\| \geq \|y_1 - y_2\| . \tag{2.1}$$

THEOREM 2.1. – Suppose A is a mapping from a subset of a Banach space Y to Y and there exists $\omega \in R$ such that $A + \omega I$ is accretive. Suppose that there exists $\lambda_1 > 0$ such that if $0 < \lambda < \lambda_1$ then $R(I + \lambda A) = Y$.

If $y \in \overline{D(A)}$ and $t \geq 0$ then

$$\lim_{n\to\infty} (I + 1/nA)^{-([tn]+1)} y \overset{\text{def}}{=} T(t)y \tag{2.2}$$

exists uniformly in bounded intervals of t. Moreover, the family of mappings $T(t)$, $t \geq 0$ so defined is a nonlinear semigroup with generator A in the sense that

(2.3) $T(0) = I$

(2.4) $T(t)(\overline{D(A)}) \subset \overline{D(A)}$ for $t \geq 0$

(2.5) $T(t_1 + t_2) = T(t_1)T(t_2)$ for $t_1, t_2 \geq 0$

(2.6) $T(t)y$ is continuous in t for fixed $y \in \overline{D(A)}$

(2.7) $\|T(t)y_1 - T(t)y_2\| \leq \exp[\omega t]\|y_1 - y_2\|$ for $t \geq 0$ and $y_1, y_2 \in \overline{D(A)}$.

3. – Nonlinear age-dependent population models.

Let X be a Banach space with norm $\| \ \|$. Let $0 \leq r \leq +\infty$ and let $I = [0, r]$ if $r < +\infty$ and $I = [0, \infty)$ if $r = +\infty$. Denote by L^1 the Banach space $L^1(I; X)$ with norm $\| \ \|_{L^1}$. Let $\sigma \geq 0$ and denote by C_σ the Banach space of continuous functions φ from I to X such that $\|\exp[\sigma a]\varphi(a)\|$ is bounded for $a \in I$, and the norm of C_σ is given by

$$\|\varphi\|_\sigma = \sup_{a\in I} \|\exp[\sigma a]\varphi(a)\|\,, \qquad \varphi \in C_\sigma\,.$$

We define nonlinear operators F and G as follows:

(3.1) $F\colon C_0 \cap L^1 \to X$ and there exists a constant $|F|$ such that for all $\varphi_1, \varphi_2 \in C_0 \cap L^1$, $\|F(\varphi_1) - F(\varphi_2)\| \leq |F| \|\varphi_1 - \varphi_2\|_{L^1}$

(3.2) $G\colon C_\sigma \to C_\sigma$ for some constant $\sigma > |F|$ and there exists a constant $|G|$ such that for all $\varphi_1 \varphi_2 \in C_\sigma$, $\|G(\varphi_1) - G(\varphi_2)\|_\sigma \leq |G| \|\varphi_1 - \varphi_2\|_\sigma$.

Our problem is to find a function ϱ from $[0, \infty) \times I$ to X satisfying

(3.3) $\varrho(0, a) = \varphi(a)\,, \quad a \in I$

(3.4) $\varrho(t, 0) = F(\varrho(t, \cdot))\,, \quad t \geq 0$

(3.5) $D\varrho(t, a) = G(\varrho(t, \cdot))(a)$, $t \geq 0$, $a \in I$, where φ is given in C_σ, $\varrho(t, \cdot) \in C_\sigma$ for $t \geq 0$, and D is defined by

$$D\varrho(t, a) = \lim_{h\to 0^+} \big(\varrho(t+h, a+h) - \varrho(t, a)\big)/h\,.$$

4. – Statement of the theorems.

We define

(4.1) $M = \{\varphi \in C_\sigma\colon \varphi(0) = F(\varphi)\}$

(4.2) $A\colon C_\sigma \to C_\sigma\,, \quad A\varphi = \varphi' - G(\varphi)\,, \quad D(A) = \{\varphi \in M\colon \varphi' \in C_\sigma\}\,.$

THEOREM 4.1. – Let (3.1), (3.2), (4.1), (4.2) hold, and let $\omega = \sigma + |G|$. The following are true:

(4.3) $R(I + \lambda A) = C_\sigma \quad$ for $0 < \lambda < 1/2\omega$

(4.4) $A + \omega I$ is accretive

(4.5) $\overline{D(A)} = M\,.$

Let X_+ be a closed cone X (see [19], p. 14) and define

(4.6) $C_{\sigma,+} = \{\varphi \in C_\sigma\colon \varphi(a) \in X_+ \text{ for all } a \in I\}$

(4.7) $M_+ = C_{\sigma,+} \cap M\,.$

THEOREM 4.2. – Suppose the hypothesis of Theorem 4.1 and in addition suppose

(4.8) $F(C_{\sigma,+}) \subset X_+$

(4.9) there exists a positive constant α such that $(G + \alpha I)(C_{\sigma,+}) \subset C_{\sigma,+}$.

Then for $0 < \lambda < 1/(2\omega + \alpha)$, $(I + \lambda A)^{-1}(C_{\sigma,+}) \subset C_{\sigma,+}$.

THEOREM 4.3. – Let the hypothesis of Theorem 4.1 hold. Then A is the generator of a nonlinear semigroup $T(t)$, $t \geqslant 0$ in M as in Theorem 2.1. If, in addition (4.8) and (4.9) hold, then $T(t)(M_+) \subset M_+$ for all $t \geqslant 0$.

THEOREM 4.4. – Let the hypothesis of Theorem 4.1 hold. Let $\varphi \in M$ and define the function ϱ from $[0, \infty) \times I$ to X by $\varrho(t, a) = \big(T(t)\varphi\big)(a)$, $t \geqslant 0$, $a \in I$. Then ϱ satisfies (3.3), (3.4), (3.5). If (4.8) and (4.9) hold and $\varphi \in M_+$, then $\varrho(t, a) \in X_+$ for all $t \geqslant 0$ and $a \in I$. If the mapping G satisfies

(4.10) $G\colon L^1 \to L^1$ and there exists a constant K such that for all $\varphi_1, \varphi_2 \in L^1$, $\|G(\varphi_1) - G(\varphi_2)\|_{L^1} \leqslant K \|\varphi_1 - \varphi_2\|_{L^1}$

then ϱ is the unique continuous function from $[0, \infty) \times I$ to X which satisfies (3.3), (3.4), and (3.5).

5. – Proofs of the theorems.

We first give

PROOF OF (4.3). – Let $0 < \lambda < 1/\omega$ and let $\psi \in C_\sigma$. We observe that $\varphi \in D(A)$ and $(I + \lambda A)\varphi = \psi$ if and only if

$$\varphi(a) = \exp[-a/\lambda]\Big[F(\varphi) + \int_0^a \exp[b/\lambda]\big(G(\varphi)(b) + \psi(b)/\lambda\big)\, db\Big]. \tag{5.1}$$

Fix a point $x \in X$ and define the mapping K_x from C_σ to C_σ by

$$(K_x\varphi)(a) = \exp[-a/\lambda]\Big[x + \int_0^a \exp[b/\lambda]\big(G(\varphi)(b) + \psi(b)/\lambda\big)\, db\Big]. \tag{5.2}$$

If $\varphi_1, \varphi_2 \in C_\sigma$, then

$$\exp[\sigma a]\|(K_x\varphi_1)(a) - (K_x\varphi_2)(a)\| \leqslant \exp[-a(1-\sigma\lambda)/\lambda]\int_0^a \exp[b(1-\sigma\lambda)/\lambda]|G|\|\varphi_1 - \varphi_2\|_\sigma\, db \leqslant |G|\|\varphi_1 - \varphi_2\|_\sigma \lambda/(1-\sigma\lambda)$$

Thus,

$$\|K_x\varphi_1 - K_x\varphi_2\|_\sigma \leqslant \|\varphi_1 - \varphi_2\|_\sigma(\lambda\omega - \sigma\lambda)/(1-\sigma\lambda)\,.$$

Since $\lambda\omega < 1$, K_x is a strict contraction from C_σ to C_σ: By the Contraction Mapping Theorem there is a unique point $\varphi_x \in C_\sigma$ such that $K_x\varphi_x = \varphi_x$. Observe that for $x_1, x_2 \in X$

$$(5.3)\qquad \begin{aligned} &\exp[\sigma a]\|\varphi_{x_1}(a) - \varphi_{x_2}(a)\| \\ &\leqslant \exp[-a(1-\sigma\lambda)/\lambda\Big[\|x_1 - x_2\| + \int_0^a \exp[b(1-\sigma\lambda)/\lambda]|G|\|\varphi_{x_1} - \varphi_{x_2}\|_\sigma\, db\Big] \\ &\leqslant \|x_i - x_2\| + \lambda|G|\|\varphi_{x_1} - \varphi_{x_2}\|_\sigma/(1-\sigma\lambda) \end{aligned}$$

and hence

$$(5.4)\qquad \|\varphi_{x_1} - \varphi_{x_2}\| \leqslant \|x_1 - x_2\|(1-\lambda\sigma)/(1-\lambda\omega)\,.$$

Define the mapping k from X to X by $k(x) = F(\varphi_x)$, $x \in X$. If $x_1, x_2 \in X$, then (3.1) and (5.4) yield

$$\begin{aligned} &\|k(x_1) - k(x_2)\| \leqslant |F|\int_0^r \|\varphi_{x_1}(a) - \varphi_{x_2}(a)\|\, da \\ &\leqslant |F|\int_0^r \exp[-a/\lambda]\Big[\|x_1 - x_2\| + \int_0^a \exp[b(1-\sigma\lambda)/\lambda]\exp[\sigma b]\|G(\varphi_{x_1})(b) - G(\varphi_{x_2})(b)\|\, db\Big]\, da \\ &\leqslant |F|[\lambda\|x_1 - x_2\| + \lambda|G|\|\varphi_{x_1} - \varphi_{x_2}\|_\sigma/(1-\sigma\lambda)\,\sigma] \\ &\leqslant \lambda|F|[\|x_1 - x_2\| + |G|\|x_1 - x_2\|/(1-\lambda\omega)/\sigma] \\ &= \lambda|F|\|x_1 - x_2\|\big((1-\lambda\sigma)\,\omega\big)/(1-\lambda\omega)\,\sigma\,. \end{aligned}$$

Since $\lambda\omega < \frac{1}{2}$ and $\sigma > |F|$, k is a strict contraction from X to X, and thus there exists a unique point $x_0 \in X$ such that $k(x_0) = x_0 = F(\varphi_{x_0})$. Hence, φ_{x_0} is the unique solution of (4.10) and so $(I + \lambda A)\varphi_{x_0} = \psi$.

PROOF OF (4.4). – Let $0 < \lambda < (\sigma - |F|)/|F|\omega$ and let $\psi_1, \psi_2 \in C_\sigma$, $\varphi_1, \varphi_2 \in D(A)$ such that $(I + \lambda A)\varphi_i = \psi_i$, $i = 1, 2$. It suffices to show that

$$(5.5)\qquad \|\psi_1 - \psi_2\|_\sigma \geqslant \|\varphi_1 - \varphi_2\|_\sigma(1-\lambda\omega)\,.$$

First observe from (5.1) and (3.1) that

$$\begin{aligned} \|F(\varphi_1) - F(\varphi_2)\| &\leqslant |F|\int_0^r \|\varphi_1(a) - \varphi_2(a)\|\, da \\ &\leqslant |F|\int_0^r \exp[-a/\lambda]\{\|F(\varphi_1) - F(\varphi_2)\| \end{aligned}$$

$$+\int_0^a \exp[b/\lambda][|G|\exp[-\sigma b]\|\varphi_1-\varphi_2\|_\sigma+\exp[-\sigma b]\|\psi_1-\psi_2\|_\sigma/\lambda]\,db\}\,da$$
$$\leqslant |F|\{\lambda\|F(\varphi_1)-F(\varphi_2)\|$$
$$+\lambda[|G|\|\varphi_1-\varphi_2\|_\sigma+\|\psi_1-\psi_2\|_\sigma/\lambda]/(1-\lambda\sigma)\sigma\}\,.$$

Now solve this inequality to obtain

$$\|F(\varphi_1)-F(\varphi_2)\|\leqslant|F|(\lambda|G|\|\varphi_1-\varphi_2\|_\sigma+\|\psi_1-\psi_2\|_\sigma)/(1-|F|\lambda)(1-\lambda\sigma)\sigma\,. \tag{5.6}$$

From (5.1) and (5.6) we obtain

$$\exp[\sigma a]\|(\varphi_1(a)-\varphi_2(a))\|\leqslant\exp[-a/\lambda]\{\|F(\varphi_1)-F(\varphi_2)\|$$
$$+\int_0^a\exp[b/\lambda-\sigma b][|G|\|\varphi_1-\varphi_2\|_\sigma+\|\psi_1-\psi_2\|_\sigma/\lambda]\,db\}$$
$$\leqslant\exp[-a/\lambda][|F|/(1-|F|\lambda)\sigma+(\exp[a(1-\sigma\lambda)/\lambda]-1)]$$
$$\times[\lambda|G|\|\varphi_1-\varphi_2\|_\sigma+\|\psi_1-\psi_2\|_\sigma]/(1-\lambda\sigma)\,.$$

Since $\lambda<(\sigma-|F|)/|F|\omega<(\sigma-|F|)/|F|\,\sigma$, we have $|F|/(1-|F|\lambda)\,\sigma<1$, and thus

$$\exp[\sigma a]\|\varphi_1(a)-\varphi_2(a)\|\leqslant[\lambda|G|\|\varphi_1-\varphi_2\|_\sigma+\|\psi_1-\psi_2\|]/(1-\lambda\sigma)\,.$$

Hence,

$$\|\varphi_1-\varphi_2\|_\sigma\leqslant[\lambda|G|\|\varphi_1-\varphi_2\|_\sigma+\|\psi_1-\psi_2\|_\sigma]/(1-\lambda\sigma)\,.$$

Now solve this inequality to obtain (5.5).

PROOF OF (4.5). – Define the subspace $C_{\sigma,0}$ of C_σ by $C_{\sigma,0}=\{\varphi\in C_\sigma\colon\varphi(0)=0\}$ and define $A_0\colon C_{\sigma,0}\to C_{\sigma,0}$ by

$$A_0\varphi=\varphi'\,,\qquad D(A_0)=\{\varphi\in C_{\sigma,0}\colon\varphi'\in C_{\sigma,0}\}\,.$$

Then, $-A_0$ is the infinitesimal generator of the strongly continuous linear contraction semigroup $T_0(t)$, $t\geqslant0$ in $C_{\sigma,0}$ given by $T_0(t)\varphi(a)=\varphi(a-t)$ for $a\geqslant t$ and $T_0(t)\varphi(a)=0$ for $0\leqslant a\leqslant t$. Further, for $\lambda>0$ and $\psi\in C_{\sigma,0}$,

$$(I+\lambda A_0)^{-1}\psi(a)=\int_0^a\exp[(b-a)/\lambda]\psi(b)/\lambda\,db\,. \tag{5.7}$$

It is well known (e.g. [31], p. 241) that

$$\lim_{\lambda\to0}(I+\lambda A_0)^{-1}\psi=\psi\qquad\text{for all }\psi\in C_{\sigma,0}\,. \tag{5.8}$$

Let $\psi \in M$, let $0 < \lambda < 1/2\,\omega$, and let $\varphi_\lambda \in D(A)$ such that $(I + \lambda A)\varphi_\lambda = \psi$. From (5.1) and (5.7) we have that

$$\varphi_\lambda(a) = \exp[-a/\lambda]F(\varphi_\lambda) + \int_0^a \exp[(b-a)/\lambda]\,G(\varphi_\lambda)(b)\,db + (I + \lambda A_0)^{-1}(\psi - \psi(0))(a) + (1 - \exp[-a/\lambda])\,\psi(0)\,.$$

Thus,

$$\exp[\sigma a](\varphi_\lambda(a) - \psi(a)) = \exp[\sigma a]\{\exp[-a/\lambda][F(\varphi_\lambda) - F(\psi)] + \int_0^a \exp[(b-a)/\lambda]\,G(\varphi_\lambda)(b)\,db + (I + \lambda A_0)^{-1}(\psi - \psi(0))(a) - (\psi - \psi(0))(a)\}$$

and so

$$\|\varphi_\lambda - \psi\|_\sigma \leqslant |F|\,\|\varphi_\lambda - \psi\|_{L^1} + \lambda\|G(\varphi)\|_\sigma/(1-\sigma\lambda) + \|(I + \lambda A_0)^{-1}(\psi - \psi(0)) - (\psi - \psi(0))\|_\sigma\,.$$

Now solve this inequality to obtain

$$\|\varphi_\lambda - \psi\|_\sigma \leqslant \big(\sigma/(\sigma - |F|)\big)\big(\lambda[|G|\,\|\varphi_\lambda\|_\sigma + \|G(0)\|_\sigma]/(1-\lambda\sigma) + \|(I + \lambda A_0)^{-1}(\psi - \psi(0)) - (\psi - \psi(0)\|_\sigma\big)\,. \tag{5.9}$$

Observe that (5.9) implies that $\|\varphi_\lambda\|_\sigma$ is bounded independent of λ in $(0, 1/2\omega)$. Now use (5.8) and (5.9) to argue that $\lim_{\lambda\to 0^+} \varphi_\lambda = \psi$, and hence conclude that $\psi \in \overline{D(A)}$.

Let $\psi \in \overline{D(A)}$ so that there exists a sequence $\{\varphi_n\}$ such that $\varphi_n(0) = F(\varphi_n)$ and $\{\varphi_n\}$ converges to ψ in C_σ. By the continuity of F we have that $\psi(0) = F(\psi)$ and hence $\psi \in M$.

PROOF OF THEOREM 4.2. – The proof of Theorem 4.2 is very similar to the proof of (4.3). Let $0 < \lambda < 1/(2\omega + \alpha)$ and let $\psi \in C_{\sigma,+}$. Then $\varphi \in D(A)$ such that $(I + \lambda A)\varphi = \psi$ if and only if

$$\varphi(a) = \exp[-a(1+\lambda\alpha)/\lambda]\Big\{F(\varphi) + \int_0^a \exp[b(1+\lambda\alpha)/\lambda][(G + \alpha I)(\varphi)(b) + \psi(b)/\lambda]\,db\Big\}\,. \tag{5.10}$$

Let $x \in X_+$ and by (4.9) we may define a mapping J_x from $C_{\sigma,+}$ to $C_{\sigma,+}$ by

$$(J_x\chi)(a) = \exp[-a(1+\lambda\alpha)/\lambda]\Big\{x + \int_0^a \exp[b(1+\lambda\alpha)/\lambda][(G + \alpha I)(\chi)(b) + \psi(b)/\lambda]\,db\Big\}\,.$$

For $\chi_1, \chi_2 \in C_{\sigma,+}$

$$\exp[\sigma a]\|(J_x\chi_1)(a) - (J_x\chi_2)(a)\| \leqslant (|G| + \alpha)\,\|\chi_1 - \chi_2\|_\sigma\,\lambda/(1 + \lambda(\alpha - \sigma))\,.$$

Since $\lambda\omega < 1$, $(|G| + \alpha)\,\lambda/(1 + \lambda(\alpha - \sigma)) < 1$, and hence J_x is a strict contraction from $C_{\sigma,+}$ to $C_{\sigma,+}$. Let χ_x be the unique point of $C_{\sigma,+}$ such that $J_x\chi_x = \chi_x$. Observe that for $x_1, x_2 \in X_+$

$$((\exp[\sigma a]\|\chi_{x_1}(a) - \chi_{x_2}(a)\| \leqslant \|x_1 - x_2\| + \lambda(|G| + \alpha)\|\chi_{x_1} - \chi_{x_2}\|_\sigma/(1 + \lambda(\alpha - \sigma))$$

and hence

$$\|\chi_{x_1} - \chi_{x_2}\|_\sigma \leqslant (1 + \lambda(\alpha - \sigma))\|x_1 - x_2\|/(1 - \lambda\omega)\,. \tag{5.11}$$

By (4.8) we may define a mapping j from X_+ to X_+ by $j(x) = F(\chi_x)$.

If $x_1, x_2 \in X_+$, then (3.1) and (5.11) yield

$$\begin{aligned}\|j(x_1) - j(x_2)\| &\leqslant |F|\int_0^r \|\chi_{x_1}(a) - \chi_{x_2}(a)\|\,da\\ &\leqslant \lambda|F|[\|x_1 - x_2\| + (|G| + \alpha)\|\chi_{x_1} - \chi_{x_2}\|_\sigma/(1 + \lambda(\alpha - \sigma))\,\sigma]\\ &\leqslant \lambda|F|\|x_1 - x_2\|(\omega(1 - \lambda\sigma) + \alpha)/(1 - \lambda\omega)\,\sigma\,.\end{aligned}$$

Since $|F|/\sigma < 1$ and $\lambda < 1/(2\omega + \alpha)$, we see that j is a strict contraction from X_+ to X_+. Thus, there exists a unique point $x_0 \in X_+$ such that $x_0 = j(x_0) = F(\chi_{x_0})$. Hence, $\varphi = \chi_{x_0}$ is the unique solution of (5.10) and $\chi_{x_0} \in C_{\sigma,+}$.

PROOF OF THEOREM 4.3. – The proof follows immediately from Theorem 2.1, 4.1, and 4.2.

Before giving the proof of Theorem 4.4 we establish two lemmas. The first lemma is a slight modification of a result of A. PLANT in [20] and the second lemma is a slight modification of a result of H. SCHWARZ in [25], p. 222.

LEMMA 5.1. – Let the hypothesis of Theorem 4.4 hold and let $0 \leqslant t_1 < t_2$ and $0 \leqslant a_1 < a_2 < r$. Then

$$\int_{t_1}^{t_2}[\varrho(t, a_2) - \varrho(t, a_1)]\,dt + \int_{a_1}^{a_2}[\varrho(t_2, a) - \varrho(t_1, a)]\,da = \int_{a_1}^{a_2}\int_{t_1}^{t_2} G(\varrho(t, \cdot))(a)\,dt\,da\,. \tag{5.12}$$

PROOF. – Define for $n = 1, 2, \ldots, t \geqslant 0$, and $a \in I$

$$\varrho_n(t, a) = (I + 1/nA)^{-([tn]+1)}\varphi(a)\,.$$

From Theorem 2.1 we have that $\lim_{n\to\infty}\varrho_n(t, a) = \varrho(t, a)$ uniformly in a and bounded intervals of t. For $n, m = 1, 2, \ldots,$

$$\begin{aligned}n[\varrho_n(m/n, a) - \varrho_n((m - 1)/n, a)] &= -A\varrho_n(m/n, a)\\ &= -d/da\varrho_n(m/n, a) + G(\varrho_n(m/n, \cdot))(a)\,.\end{aligned}$$

Then, for $a_1, a_2 \in I$, $a_1 < a_2$, $\varrho_n(m/n, a_2) - \varrho_n(m/n, a_1)$

$$= \int_{a_1}^{a_2} d/da\,\varrho_n(m/n, a)\, da$$

$$= \int_{a_1}^{a_2} \{-n[\varrho_n(m/n, a) - \varrho_n((m-1)/n, a)] + G(\varrho_n(m/n, \cdot))(a)\}\, da\,.$$

Divide by n and add for $m = p+1, \ldots, q$:

$$\sum_{m=p+1}^{q} [\varrho_n(m/n, a_2) - \varrho_n(m/n, a_1)]/n =$$

$$= \int_{a_1}^{a_2} [\varrho_n(p/n, a) - \varrho_n(q/n, a)]da + \int_{a_1}^{a_2} \sum_{m=p+1}^{q} G(\varrho_n(m/n, \cdot))(a)/n\, da\,.$$

Let $n \to \infty$ such that $p/n \to t_1$ and $q/n \to t_2$, and (5.12) follows.

LEMMA 5.2. – Let Y be a Banach space and let f be a function from $R\times R\times R$ to Y such that f_{123} exists and is continuous on $R\times R\times R$ and f_{23} exists on $R\times R\times R$. Then f_{231} exists and equals f_{123} on $R\times R\times R$.

PROOF. – Let $(r, s, t) \in R\times R\times R$ and let $\varepsilon > 0$. Since f_{123} is continuous at (r, s, t) there exists $\delta > 0$ such that if $0 < |h|, |k|, |j| < \delta$, then

$$\|f_{123}(r+h, s+k, t+j) - f_{123}(r, s, t)\| < \varepsilon\,.$$

Let $z \in Y^*$ and define the function f^z from $R\times R\times R$ to R by $f^z = z(f)$. Choose h, k, j such that $0 < |h|, |k|, |j| < \delta$ and define

$$\begin{aligned}\Delta^z = f^z(r+h, s+k, t+j) &- f^z(r+h, s+k, t)\\ &- f^z(r+h, s, t+j) + f^z(r+h, s, t) - f^z(r, s+k, t+j)\\ &+ f^z(r, s+k, t) + f^z(r, s, t+j) - f^z(r, s, t)\,,\end{aligned}$$

$$\begin{aligned}u(h) = f^z(r+h, s+k, t+j) &- f^z(r+h, s+k, t)\\ &- f^z(r+h, s, t+j) + f^z(r+h, s, t)\,.\end{aligned}$$

By the Mean Value Theorem there exists $\theta \in (0,1)$ such that

$$\begin{aligned}\Delta^z &= u(h) - u(0) = hu'(\theta_1 h)\\ &= h[f_1^z(r+\theta_1 h, s+k, t+j) - f_1^z(r+\theta_1 h, s+k, t)\\ &\quad - f_1^z(r+\theta_1 h, s, t+j) + f_1^z(r+\theta_1 h, s, t)]\,.\end{aligned}$$

Define

$$v(k) = f_1^z(r + \theta_1 h, s + k, t + j) - f_1^z(r + \theta_1 h, s + k, t)$$

There exists $\theta_2, \theta_3 \in (0, 1)$ such that

$$\begin{aligned} \Delta^z = h[v(k) - v(0)] = hkv'(\theta_2 k) \\ = hk[f_{12}^z(r + \theta_1 h, s + \theta_2 k, t + j) - f_{12}^z(r + \theta_1 h, s + \theta_2 k, t)] \\ = hkj[f_{123}^z(r + \theta_1 h, s + \theta_2 k, t + \theta_3 j)]. \end{aligned}$$

Thus

$$|\Delta^z/hkj - f_{123}^z(r, s, t)| < \varepsilon |z|$$

But

$$\lim_{j\to 0} \lim_{k\to 0} \Delta^z/hkj = [f_{23}^z(r + h, s, t) - f_{23}^z(r, s, t)]/h.$$

Thus, for $0 < |h| < \delta$

$$|[f_{23}^z(r + h, s, t) - f_{23}^z(r, s, t)] - f_{123}^z(r, s, t)| \leqslant \varepsilon |z|.$$

The conclusion now follows by the Hahn-Banach Theorem. (e.g. [23], p. 190.)

PROOF OF THEOREM 4.4. – The proof of (3.3) follows from (2.3). The proof of (3.4) follows from (2.4) and (4.5). To prove (3.5) define for $t \geqslant 0$, $0 \leqslant a < r$, $0 \leqslant h < < r - a$

$$f(h, a, t) = \int_h^{a+h} \int_h^{t+h} \varrho(s, b)\, ds\, db.$$

By Lemma 5.1

$$f_1(h, a, t) = \int_h^{t+h} [\varrho(s, a + h) - \varrho(s, h)]\, ds + \int_h^{a+h} [\varrho(t + h, b) - \varrho(h, b)]\, db = \int_h^{a+h} \int_h^{t+h} G(\varrho(s, \cdot))(b)\, ds\, db.$$

Then, $f_{123}(h, a, t) = G(\varrho(t + h, \cdot))(a + h)$ and $f_{23}(h, a, t) = \varrho(t + h, a + h)$. By Lemma 5.2 $f_{231}(h, a, t)$ exists and is $f_{123}(h, a, t)$. Hence, $f_{123}(0, a, t) = D\varrho(t, a) = G(\varrho(t, \cdot))(a)$ and (3.5) is established.

If (4.8) and (4.9) hold and $\varphi \in M_+$, then $\varrho(t, a) \in X_+$ for all $t \geqslant 0$ and $a \in I$ by virtue of Theorem 4.3.

Finally, let (4.10) hold and suppose that η is a continuous function from $[0, \infty)\times I$ to X satisfying (3.3), (3.4), and (3.5). Let $\xi = \|\varrho - \eta\|$ and for $0\leqslant t$, $0\leqslant a < r$, $0 < h < r - a$

$$\int_0^t\int_0^a [\xi(s+h, b+h) - \xi(s, b)]\, db\, ds = \int_0^t \int_a^{a+h} \xi(s, b)\, db\, ds - \int_0^t\int_0^h \xi(s, b)\, db\, ds + \int_t^{t+h}\int_h^{a+h} \xi(s, b)\, db\, ds - \int_0^h\int_h^{a+h} \xi(s, b)\, db\, ds\,.$$

Divide by h and let $h \to 0$ to obtain

$$\int_0^t \xi(s, a)\, ds - \int_0^t \xi(s, 0)\, ds + \int_0^a \xi(t, b)\, db - \int_0^a \xi(0, b)\, db \leqslant \int_0^t\int_0^a \|G(\varrho(s, \cdot))(b) - G(\eta(s, \cdot))(b)\|\, db\, ds\,. \tag{5.13}$$

From (5.13), (3.1), and (4.10) we obtain

$$\begin{aligned}\int_0^r \xi(t, b)\, db &\leqslant \int_0^t \xi(s, 0)\, ds + \int_0^r \xi(0, b)\, db + \int_0^t\int_0^r \|G(\varrho(s, \cdot))(b) - G(\eta(s, \cdot))(b)\|\, db\, ds\\ &= \int_0^t \|F(\varrho(s, \cdot)) - F(\eta(s, \cdot))\|\, ds + \int_0^t\int_0^r \|G(\varrho(s, \cdot))(b) - G(\eta(s, \cdot))(b)\|\, db\, ds\\ &\leqslant (|F| + K)\int_0^t\int_0^r \xi(s, b)\, db\, ds\,.\end{aligned}$$

By Gronwall's Lemma $\int_0^r \xi(t, b)\, db = 0$ for all $t \geqslant 0$ and hence $\xi(t, b) = 0$ for all $t\geqslant 0$ and $b \in I$.

6. – Concluding remarks.

The semigroup approach to age-dependent population models has recently been used by J. PRUSS in [22]. In the problem treated in [22] the birth process of equation (3.4) involves a linear functional F. Thus, the abstract equation associated with the problem can be expressed in semilinear form. That is, the operator A in (4.2) can be decomposed into a linear strongly continuous semigroup generator and a locally Lipschitz nonlinear operator. In [22] this semilinear form is exploited to develop an existence theory and a theory of equilibrium solutions and their local stability.

There are close similarities between age-dependent population models and functional and functional differential equations. The papers of J. DYSON and R. VILLELLA-BRESSAN [7], [8] and of the author [26], [27], and [28], as well as many others, treat nonlinear functional and functional differential equations, from a semigroup point of view. For such equations the generators have the form $A\varphi = \varphi'$ with the boundary conditions $\varphi(0) = F(\varphi)$ (functional equations) or $\varphi'(0) = F(\varphi)$ (functional differential equations). The idea of renorming the Banach spaces of initial functions has been exploited for these equations in [8], [27], [28], and [21]. For our age-dependent population problem the renorming device is recessitated by the fact that our generator A may not satisfy an accretiveness condition in C_0, but does in C_σ.

The form of the nonlinearities in our model is very general, but does require a global Lipschitz condition. The age-dependent model of Gurtin and MacCamy and its extensions by other authors have a special form which is only locally Lipschitz. Another point of difference of our theory is that in our model condition (3.4) must be satisfied for $t = 0$, and thus conditions (3.4) and (3.5) are necessarily compatible. The methods of Gurtin and MacCamy do not require this compatibility condition, but rather the more general condition that (3.4) holds for $t > 0$. In a forthcoming paper we will study general age-dependent population models with locally Lipschitz nonlinearities and without the compatibility condition. We will also generalize the model to allow for a diffusion process to occur in the dynamics of the population.

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