A Generalization of Von Neumann's Assignment Problem and K. Fan Optimization Result (*).

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Summary. – *In this paper we generalize the assignment problem in higher dimensions, referring at to another study by the authors. The hide-and-seek game, which is intimately related to the assignment problem, is extended, and an elegant result due to K. Fan about extrema is generalized.*

1. – Introduction.

It is well known that Birkhoff's result on doubly stochastic matrices, which appeared in [1], has produced a great number of appliations in many areas. In particular, VON NEUMANN, in [7], besides giving a very interesting and elementary proof of Birkhoff's theorem applied it to obtain a very elegant solution of the assignment problem via the hide-and-seek two person game. He indicated the wish to extend the study to more than two dimensions. However, he pointed out explicitly the serious difficulties. The authors of this paper, in [5], have attacked and solved the general problem of extremals for « matrices » in any number of dimensions. The aim of this paper is to generalize the assignment problem in higher dimension, referring it to our other study just cited. Therefore, the hide-and-seek game, which is intimately related to the assignment problem, is adequately extended.

Related with it a very elegant result due to K. FAN [4] about extrema is generalized in the same direction and is proved in a much more simpler way.

2. – General hide-and-seek game in k-dimensions.

Let $N_i = N = \{1, 2, ..., n\}$ be a set, for $i = 1, 2, ..., k$. We define the general hide-and-seek game in k-dimension as a zero-sum-two-person game

$$\Gamma = \left\{\bigcup_{i=1}^{k} N_i, N^k; A\right\}.$$

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The hider, the second player, choses a place $(i_1, \ldots, i_k) \in N^k = \bigtimes_{i=1}^{k} N_i$ where he will hide. The seeker choses an « Hyperplane » $\bar{i}_r \in \bigcup_{i=1}^{k} N_i$:

If $i_r = \bar{i}_r$, then the second player pays $A(\bar{i}_r; i_1, \ldots, i_k)$; otherwise, the payoff will be zero. We can thus, write the payoff function as

$$A(\bar{i}_r; i_1, \ldots, i_k) = \alpha(i_1, \ldots, i_k)\, \delta_{i_r \bar{i}_r}$$

where $\delta_{i_r \bar{i}_r}$ is Kronecker's delta.

We assume for reasons of simplicity that the payoff function is strictly positive.

The mixed extension

$$\tilde{\Gamma} = \left\{\widetilde{\bigcup_{i=1}^{k} N_i}, \widetilde{N^k}; E\right\}$$

has the value of the game

$$v = \min_{y \in \widetilde{N^k}} \max_{i_r \in \bigcup_{i=1}^{k} N_i} \left\{\sum_{i_1, \ldots, \hat{i}, \ldots, i_k} \alpha(i_1, \ldots, i_k)\, y(i_1, \ldots, i_k)\right\}$$

where $\hat{i}_r$ indicates that the sum over $i_r \in N_r$ is omitted.

We are now interested in the actual computation of v and in the mixed optimal strategies of the second player. Let us introduce the followings sets, considered already in [5].

$$U_n^k = \left\{x\colon N^k \to \boldsymbol{R}\colon x(i_1, \ldots, i_k) \geqslant 0 \quad \sum_{i_1, \ldots, \hat{i}_r, \ldots, i_k} x(i_1, \ldots, i_k) = 1 \quad \text{for each } i_r \in N_r \text{ and each } r = 1, \ldots, k\right\}$$

where $\boldsymbol{R}$ stands for the reals. This is the set of (k, n) — stochastic matrices.

$$V_n^k = x\colon \left\{N^k \to \boldsymbol{R}\colon x(i_1, \ldots, i_k) \geqslant 0 \quad \sum_{i_1, \ldots, \hat{i}_r, \ldots, i_k} X(i_1, \ldots, i_k) \leqslant 1 \quad \text{for each } i_1 \in N_r \text{ and each } r = 1, \ldots, k\right\}.$$

We have a first result:

LEMMA 1. – $V_n^k = \{x\colon x(i_1, \ldots, i_k) \geqslant 0$ *and there is* $z \in U_n^k\colon z \geqslant x\}$.

PROOF. – The set at right is obviously included in V_n^k. On the other hand, for a given $x \in V_n^k$, let

$$I_r(x) = \left\{i_r \in N_r\colon \sum_{i_1, \ldots, \hat{i}_r, \ldots, i_k} x(i_1, \ldots, i_k) < 1\right\}.$$

Now consider the sum

$$N(x) = \sum_{r=1}^{k} |I_r(x)|$$

where $|\ |$ indicates the cardinality. If $x \notin U_n^k$, then we have $N(x) > 0$. In other words, there is an $\bar{r} = 1, \dots, n$ such that $I_{\bar{r}}(x) > 0$. In such a case, for each r, we have $I_r(x) > 0$, since

$$\sum_{i_1,\dots,i_k} x(i_1, \dots, i_k) = \sum_{i_r} \sum_{i_1,\dots,\hat{i}_r,\dots,i_k} x(i_1, \dots, i_k) < n .$$

Now define

$$\varepsilon = 1 - \max_{r=1,\dots,k} \max_{i_r \in I_r(x)} \sum_{i_1,\dots,\hat{i}_r,\dots,i_k} x(i_1, \dots, i_k) ,$$

which is strictly positive. With this we construct

$$X_\varepsilon(i_1, \dots, i_k) = \begin{cases} x(i_1, \dots, i_k) + \varepsilon & \text{for only one } (i_1, \dots, i_k) \in \bigtimes_{r=1}^{k} I_r(x) , \\ x(i_1, \dots, i_k) & \text{otherwise} . \end{cases}$$

It follows directly from its definition, that $x_\varepsilon \in U_n^k$, and $x_\varepsilon \geqslant x$ and $N(x) > N(x_\varepsilon)$. Hence, we can repeat the procedure, obtaining a finite sequence

$$x \leqslant x_{\varepsilon_1} \leqslant (x_{\varepsilon_1})_{\varepsilon_2} \leqslant \dots \leqslant \big((x_{\varepsilon_1})\dots\big)_{\varepsilon_s} = \bar{x}$$

$$N(x) > N(x_{\varepsilon_1}) > \dots > N(\bar{x}) = 0 .$$

Therefore $\bar{x} \in U_n^k$ and $x \leqslant \bar{x}$ (q.e.d.).

Using this result, and introducing the set $E(U_n^k)$ of all the extremals of U_n^k, we have

THEOREM 2. – *For the mixed extension $\tilde{\Gamma}$ of the general hide-and-seek game*

$$v = \min_{\bar{y}^e \in E(U_k^n)} v(e) = \min_{\bar{y}^e \in E(U_k^n)} \frac{1}{\displaystyle\sum_{i_1,\dots,i_k} \frac{\bar{y}^e(i_1, \dots, i_k)}{\alpha(i_1, \dots, i_k)}}$$

and y^e is an optimal extremal if $v(e) = v$, where

$$\bar{y}^e(i_1, \dots, i_k) = \frac{\alpha(i_1, \dots, i_k)}{v(e)} y^e(i_1, \dots, i_k) .$$

PROOF. – It is clear that $v > 0$, therefore, if $\tilde{y}$ is an optimal mixed strategy for second player, we have

$$\sum_{i_1,\dots,\hat{i}_r,\dots,i_k} \alpha(i_1, \dots, i_k) \tilde{y}(i_1, \dots, i_k) \leqslant v$$

for each i_r. Then defining

$$\bar{y}(i_1, \dots, i_k) = \frac{\alpha(i_1, \dots, i_k)\tilde{y}(i_1, \dots, i_k)}{v},$$

it holds that $\tilde{y} \in V_n^k$. But, by lemma 1, there exists a $\bar{\bar{y}} \in U_n^k$: $\bar{\bar{y}} \geqslant \bar{y}$. Define

$$\bar{\bar{Z}}(i_1, \dots, i_k) = \frac{v}{\alpha(i_1, \dots, i_k)} \bar{\bar{y}}(i_1, \dots, i_k).$$

Then we have $z \geqslant \tilde{y}$ and we want to prove that indeed this is an equality. Let

$$\theta = \sum_{i_1,\dots,i_k} \tilde{y}(i_1, \dots, i_k) \sum_{i_1,\dots,i_k} \bar{\bar{Z}}(i_1, \dots, i_k) \leqslant 1$$

for $\bar{\bar{Z}} = \theta Z \in N^k$, and

$$\max_{i_r \in \bigcup_{l=t}^{k} N_t} \sum_{i_1,\dots,\hat{i}_r,\dots,i_k} \alpha(i_1, \dots, i_k) z(i_1, \dots, i_k) = \theta v$$

From here, θ must be equal one, because otherwise $\tilde{y}$ would not be optimal. Therefore, the fact that $Z \geqslant \tilde{y}$, implies $\tilde{y} = \bar{\bar{Z}} \in \tilde{N}^k$ and consequently $\bar{\bar{y}} = \bar{y} \in U_n^k$.

Now, the first transformation gives $\bar{y} \in U_n^k$. But, therefore $\bar{y}$ is a convex combination of extremals:

$$\bar{y} = \sum_{\bar{y}^e \in E(U_n^k)} \lambda_e \bar{y}^e$$

which gives

$$y^e(i_1, \dots, i_k) = \frac{v(e)}{\alpha(i_1, \dots, i_k)} \bar{y}^e(i_1, \dots, i_k),$$

a point $y^e \in \tilde{N}^k$, where

$$v(e) = \frac{1}{\displaystyle\sum_{i_1,\dots,i_k} \frac{\bar{y}^e(i_1, \dots, i_k)}{\alpha(i_1, \dots, i_k)}}.$$

On the other hand, we have

$$\sum_{i_1,\dots,\hat{i}_r,\dots,i_k} \alpha(i_1, \dots, i_k) y^e(i_1, \dots, i_k) = v(e) \geqslant v$$

for i_r, which implies that

$$v = \min_{\bar{y}^e \in E(U_k^n)} v(e) = \min_{\bar{y}^e \in E(U_k^n)} \frac{1}{\displaystyle\sum_{i_1,\dots,i_k} \frac{\bar{y}^e(i_1, \dots, i_k)}{\alpha(i_1, \dots, i_k)}}.$$

Moreover, y^e is optimal and extremal if $v(e) = v$ (q.e.d.).

3. – A simple class of extremals in U_n^k: the diagonals.

With the result just considered in the previous paragraph, we are going to study some general assignment problems in the next section. But for this we need some simple facts about the most simple class of extremals in U_n^k namely: the diagonals. These are the natural generalization in the higher dimensional case of the diagonals of doubly stochastic matrices. For completeness, we repeat here some definitions given by the authors in [5], where the set of extremals is investigated more completely.

An element $x \in U_n^k$ is said to be a *diagonal* if

$$|H(i_r) \cap S(x)| = 1$$

for each $i_r \in N_r$ $r = 1, \ldots, k$. Here

$$H(\bar{i}_r) = \{(i_1, \ldots, i_k) \in N^k\colon i_r = \bar{i}_r\}$$

and $S(x)$ stands for the support of x. A diagonal $x \in D_n^k$ is an extremal vertex of U_n^k. Thus the set of diagonals $D_n^k \subset E(U_n^k)$.

Given a diagonal, it is clear that $x(i_1, \ldots, i_k)$ is either one or zero. Moreover, its projection on the *face* $(1, r)$, that is to say

$$x^r(i_1, i_r) = \sum_{i_2, \ldots, \hat{i}_r, \ldots, i_k} x(i_1, \ldots, i_k)$$

is clearly a diagonal in U_n^2, which determines uniquely a permutation σ^r on N. Therefore, a diagonal in U_n^k determines a set of permutations $\sigma^2, \ldots, \sigma^k$. Defining

$$x^{\sigma^2, \ldots, \sigma^k}(i_1, \ldots, i_k) = \begin{cases} 1 & \text{if } i_2 = \sigma^2(i), \ldots, i_k = \sigma^k(i)\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

we have that $x^{\sigma^2, \ldots, \sigma^k} \in D_n^k$. Therefore the set of diagonals are in one to one correspondence with such a set of permutations.

A further representation of the set D_n^k can be given by the set of all permutations $\sigma^{1,2}, \sigma^{4,3}, \ldots, \sigma^{2k-1,2k}$, where $\sigma^{2r-1,2r}$ indicates a permutation induced by a diagonal in U_n^2 of the face $N_{2r-1,2r}$.

We say that a (k, n)-matrix (that is to says a $x\colon N^k \to \mathbf{R}$), is *P-constant on diagonals* if P is a set of non-overlapping diagonals covering N^k. Furthermore, for each diagonal $z \in P$, $x|S(z)$ is constant.

Now, we have the following result concerning the convex hull $K(D_n^k)$ of diagonals:

THEOREM 3. – *A (k, n)-stochastic matrix $x \in K(D_n^k)$ if and only if $x = \sum_{j \in J} x_j$, where each x_j is P_j-constant on diagonals for some P_j.*

Proof. – We will firstly show that there exists a partition of the set of diagonals $P_1, P_2, \ldots, P_t$ such that for each $j = 1, \ldots, t$.

$$\bigcup_{x \in P_j} S(x) = N^k \quad \text{and} \quad S(x_1) \cap S(x_2) = \emptyset$$

for each $x_1, x_2 \in P_j$. Here, $\emptyset$ indicates the empty set.

Indeed, consider the set of permutations on N, $\sigma(1), \sigma(2), \ldots, \sigma(n)$ defined by

$$\sigma(i)(j) = j + i - 1 \bmod n$$

for $j \in N$. Therefore let $x(\sigma(i))$ be a diagonal on $N \times N$ with values one on $(k, \sigma(i)(k))$ and zero otherwise. From here $S\big(x(\sigma(i))\big) \cap S\big(x(\sigma(j))\big) = \emptyset$ when $i \neq j$, and

$$\bigcup_{i=1}^{n} S\big(x(\sigma(i))\big) = N \times N$$

which gives us a partition $P_c = \{\sigma(1), \ldots, \sigma(n)\}$ for the doubly stochastic case. At this point, we loosely write « permutations » or « diagonals », indifferently, since there is a one-to-one correspondence between them.

Now we construct all the other partitions, just by giving a permutation τ. Ineded, define $\tau(i)$ as the permutation

$$\tau(i)(i) = \sigma(i)\big(\tau(i)\big) ,$$

where $\sigma(i) \in P_c$. It is easy to see that $P_\tau = \{\tau(1), \ldots, \tau(n)\}$ is another suitable partition. Now if $\tau = \sigma(j) \in P_c$ for a given j, then

$$\tau(i)(k) = \sigma(i)\big(\sigma(j)(k)\big) = \sigma(i)\big(\text{mod}_n(j + k - 1)\big) = \text{mod}_n\,(i + j + k - 2) ,$$

and, taking $l = \text{mod}_n\,(i + j - 1)$, we get $\sigma(l) = \tau(i)$, which implies that for any cyclic permutation τ, P_c is invariant. The same holds for any P_τ. Therefore we have $(n-1)!$ suitable partition P_τ in the case $k = 2$.

Now, in the general case when $k \geq 2$, consider the faces $(1, r)$. Consider in each face the set of cyclic permutations $\sigma^r(1), \ldots, \sigma^r(n)$ given already above, then by picking $\sigma^2(i_2), \sigma^3(i_3), \ldots, \sigma^k(i_k)$ we construct a diagonal $x^{\sigma^2(i_2), \ldots, \sigma^k(i_k)}$. We have

$$S(x^{\sigma^2(i_2), \ldots, \sigma^k(i_k)}) \cap S(x^{\sigma^2(\bar{i}_2), \ldots, \sigma^k(\bar{i}_k)}) = \emptyset$$

if $i_r \neq \bar{i}_r$ for some r. Furthermore,

$$\bigcup_{(i_2, \ldots, i_k) \in N_2 \times \ldots \times N_k} S(x^{\sigma^2(i_2), \ldots, \sigma^k(i_k)}) = N^k ,$$

From here, we have

$$P^k_c = \{x^{\sigma^2(i_2),\ldots,\sigma^k(i_k)}\,(i_2, \ldots, i_k) \in N_2 \times \ldots \times N_k\}$$

and $|P^k_c| = n^{k-1}$. As in the case $k = 2$, it is easy to see that any other partition can be given by permutations $\tau^2, \ldots, \tau^k$

$$P^k_{\tau^2,\ldots,\tau^k} = \{x^{\tau^2(i_2),\ldots,\tau^k(i_k)}\colon (i_2, \ldots, i_k) \in N_2 \times \ldots \times N_k\}\ .$$

Again, if $\tau^2 = \sigma^2(i_2), \ldots, \tau^k = \tau^k(i_k)$ then

$$P^k_{\sigma^2(i_2),\ldots,\sigma^k(i_k)} = P^k_c\ .$$

Therefore, since are exactly $(n!)^{k-1}$ diagonals in U^k_n we get $[(n-1)!]^{k-1}$ different permutations $P^k_{\tau^2,\ldots,\tau^k}$.

Now, if $x \in K(D^k_n)$, then it is a convex combination of diagonals

$$x = \sum_{j\in J} \lambda_j x_j = \sum_{\tau^2,\ldots,\tau^k}\ \sum_{j\in P^k_{\tau^2,\ldots,\tau^k}} \lambda_j x_j = \sum_{\tau^2,\ldots,\tau^k} x_{\tau^2,\ldots,\tau^k}$$

where $x_{\tau^1,\ldots,\tau^k}$ is clearly $P^k_{\tau^2,\ldots,\tau^k}$-constant on diagonals.

Vice-versa, if $x \in U^k_n$ and

$$x = \sum_{\tau^2,\ldots,\tau^k} x_{\tau^2,\ldots,\tau^k}$$

where $x_{\tau^2,\ldots,\tau^k}$ is $P^k_{\tau^2,\ldots,\tau^k}$-constant on diagonals, then

$$x_{\tau^2,\ldots,\tau^k} = \sum_{d\in P^k_{\tau^2,\ldots,\tau^k}} x_{\tau^2,\ldots,\tau^k}(d)\, X^{\tau^2,\ldots,\tau^k}$$

where

$$X_{\tau^2,\ldots,\tau^k}(d) = X_{\tau^2,\ldots,\tau^k}(i_1, \ldots, i_k) \quad \text{in } (i_1, \ldots, i_k) \in d.$$

From here

$$x = \sum_{\tau^2,\ldots,\tau^k} \Big[\sum_{d\in P^k_{\tau^2,\ldots,\tau^k}} x_{\tau^2,\ldots,\tau^k}(d)\Big] X^{\tau^2,\ldots,\tau^k} = \sum_{\tau^2,\ldots,\tau^k} \lambda_{\tau^2,\ldots,\tau^k}\, X^{\tau^2,\ldots,\tau^k}$$

where in the last sum the coefficients are convex because $x \in U^n_k$ (q.e.d.).

4. – A variant of the game.

Here we are going to consider a slight modification of the hide-and-seek game presented in the first paragraph. We are concerned with a subclass of extremals, the diagonals.

These will permit us to study the generalized assignment problems given in the next section.

Using the result given above, we can modify Lemma 1, obtaining the result given in the following Lemma.

Let p^k be the set of k-matrices $x \geqslant 0$ such that

$$x = \sum_{\tau^2, \dots, \tau^k} x_{\tau^2, \dots, \tau^k}$$

where $x_{\tau^2, \dots, \tau^k}$ is $P^k_{\tau^2, \dots, \tau^k}$-constant on diagonals.

LEMMA 4. – $V_{D^k_n} = P^k \cap V^k_n = \{x \in P^k\colon$ there is a $z \in K(D^k_n)\colon z \geqslant x\}$.

PROOF. – Let X be an element of $V_{D^k_n}$, then

$$x = \sum_{\tau^2, \dots, \tau^k} x_{\tau^2, \dots, \tau^k}$$

where $x_{\tau^2, \dots, \tau^k}$ is $P^k_{\tau^2, \dots, \tau^k}$-constant on diagonals.

Let $I_r(x)$ be the same set introduced in the proof of Lemma 1. Again, if a $I_r(x) \neq \emptyset$, all of them are non-empty. Therefore by taking the same ε, define

$$x_\varepsilon(i_1, \dots, x_k) = \begin{cases} x(i_1, \dots, i_k) + \varepsilon & \text{if } (i_1, \dots, i_k) \in S(x^{\bar\tau^2, \dots, \bar\tau^k}) \\ & \text{for only one } \bar\tau^2, \dots, \bar\tau^k, \\ x(i_1, \dots, i_k) & \text{otherwise}. \end{cases}$$

It is immediate that $x_\varepsilon \in K(D^k_n)$ and $x_\varepsilon \geqslant x$. The reason that here the process terminates with only one step is due to the fact that all the $x_{\tau^2, \dots, \tau^k}$ are $P^k_{\tau^2, \dots, \tau^k}$-constant on diagonals. The remaining inclusion is trivial (q.e.d.).

We now deduce the mixed extension of the general hide-and-seek game. Let

$$\bar\Gamma_{D^k_n} = \left\{\bigcup_{i=r}^{k} N_i, \Sigma_\alpha; E\right\}$$

where $\Sigma_\alpha \subset \widetilde{N}^k$ is the set of probabilities

$$X = \sum_{\tau^2, \dots, \tau^k} X(\tau^2, \dots, \tau^k)$$

where

$$X(\tau^2, \dots, \tau^k)(i_1, \dots, i_k) = c(d)\frac{1}{\alpha(i_1, \dots, i_k)} \qquad \text{for } (i_1, \dots, i_k) \in d \in P^k_{\tau^2,\dots,\tau^k}\,.$$

Obviously Σ_α is compact and convex.

We have the following result which can be proved following the lines of Theorem 2, using Lemma 4

THEOREM 5. – *For the extension* $\bar{\Gamma}_{D^k_n}$

$$v(D^k_n) = \min_{y\in\Sigma_\alpha} \max_{i_r \in \bigcup_{i=r} N_i} E(y, i_r) = \min_{\bar{y}^e \in D^k_n} v(\bar{y}^e) = \min_{\tau^2,\dots,\tau^k} \frac{1}{\sum\limits_{(i_1,\dots,i_k)\in S(x^{\tau_2,\dots,\tau_k})} \big(1/\alpha(i_1, \dots, i_k)\big)}\,.$$

On the optimal mixed strategies for the second player, the transformation $\alpha(\cdot)\bar{y}(\cdot) = v(D^k_n)\bar{y}(\cdot)$ sends Σ_α to $v(D^k_n)$. Moreover, the coefficients $c(d)$ satisfy

$$\sum_{\tau^2,\dots,\tau^k} \sum_{d\in P^k_{\tau^2,\dots,\tau^k}} c(d) \sum_{(i_1,\dots,i_k)\in d} \frac{1}{\alpha(i_1, \dots, i_k)} = 1$$

because $x \in N^k$.

In the case that $k = 2$, both Theorems 2 and 5 give von Neumann's result.

5. – Applications to some assignment problems.

In this section we are going to discuss two generalized assignment problems and we will relate them with the results already obtained in the previous paragraphs.

PROBLEM 1. – We have k groups of n-persons each and k groups of n-jobs each. Any given group of persons corresponds uniquely to a definite a group of jobs. Say the i-th group corresponds with the i-th.

The value of the i_1-person in the j_1-job, ..., the i_k-person in the j_k-job is

$$a(i_1, j_1, \dots, i_k, j_k)\,.$$

The problem is to find assynments $\tau^1, \dots, \tau^k$ for all the groups such as to maximize the total value

$$\sum_{i_1,\dots,i_k} a\big(i_1, \tau^1(i_1), \dots, i_k, \tau^k(i_k)\big)\,.$$

If we introduce a transformation given by

$$a(i_i, j_1, \dots, i_k, j_k) + \sum_{r=1}^{k} b_{i_r} + \sum_{r=1}^{k} c_{i_r}$$

where the b's and the c's are constants, then the maximization problem is invariant. Thus we can assume that the payoff function a be positive.

Taking

$$\alpha(i_1, j_1, .., i_k, j_k) = \frac{1}{a(i_1, j_1, ..., i_k, j_k)}$$

the maximum value of the assignment problem is $1/v(D_n^k)$ and it is reached on a diagonal by Theorem 5.

PROBLEM 2. – Now, we have a group of n persons and $k-1$ different groups of exactly n types of jobs. A person i must perform a job j_r in each group $r = 2, ..., k$ such that there is exactly one person performing the job j_r in each group r. Here the value of the person i performing the jobs $j_1, ..., j_{k-1}$ is

$$a(i, j_2, ... j_k) .$$

An assignment in this problem is a set of permutations $\tau^2, ..., \tau^k$ corresponding to the faces $(1, r)$, $r = 2, ..., k$, as introduced before Theorem 3. The total value for this assignment is

$$\sum_i a\big(i, \tau^2(i), ..., \tau^k(i)\big) .$$

By a procedure analogous to the one given in problem I, Theorem 5 gives the right answer also in this case.

6. – A generalization of K. Fan's optimization result.

In connection with the set V_n^2 there appear many interesting problems of maximization with many applications.

One is the known extremal problem presented by K. FAN in [4], which is concerned with the maximazation of a payoff function

$$f^2(x) = \sum_{i=1}^{n} \sum_{i=1}^{n} a_{ij} X_{ij}$$

on the set V_n^2.

Here we are going to generalize the problem and give suitable caracterization of the extremum.

Let

$$f^k(x) = \sum_{i_1,...,i_k} a_{i_1,...,i_k}, \vartheta(i_1, ..., i_k)$$

be a linear a function defined on V_n^k. Then, we have the following result which considerably generalizes the corresponding one due to K. FAN in [4], which is proved in an elegant way, based upon a strong result of convex sets. Here, we give a more simple proof.

THEOREM 6. – *For any* (k, n)*-matrix* $a_{i_1,\dots,i_k}$*, we have*

$$\max_{x\in V_n^k} f^k(x) = \min_{(y_{i_1},\dots,y_{i_k})\in A(a)} \sum_{r=1}^{k} \sum_{i_r=1}^{n} y_{i_r}$$

where $A(a)$ *is the set of* $(y_{i_1}, \dots, y_{i_k}) \geqslant 0$ *such that* $y_{i_1} + \dots + y_i \geqslant a_{i_1,\dots,i_k}$ *for each* $(i_1, \dots, i_k) \in N^k$.

PROOF. – The problem of maximization can be expressed in a linear programing as follows

$$\max_{\substack{C\cdot x\leqslant b\\ x\geqslant 0}} f^k(x)$$

where C is $kn\times n^k$ matrix defined on $\left(\bigcup_{i=r}^{k} N_i\right)\times N^k$ with values

$$C(\bar{i}_r;\, i_1, \dots, i_k) = \delta_{\bar{i}_r i_r}$$

where δ is Kronecker's delta. The vector b of dimension $nk\times 1$ has all its values equal one. As in [5], the set V_u^k is uniquely determined by both the inequalities given above. $C\cdot x$ indicates the matrix product of C with the vector x. Now consider its dual problem

$$\min_{\substack{v'\cdot C\leqslant a\\ v\geqslant 0}} \sum_{r=1}^{n} \sum_{i_r=1}^{x} b_{i_r} v_{i_r}$$

where v' is the transpose of v and a is a (k, n)-matrix. Clearly there always exist a vector satisfying the previous inequalities. But such a set is nothing else that $A(a)$. By the duality in linear programming (see for example NIKAIDO [6] pag. 133), both maximum and minimum values coincide (q.e.d.).

At this point we would like to emphasize the fact that K. Fan's original proof is valid for the most general case considered here. Indeed, it is almost a literally translation of the case with $k = 2$.

With a slight modification in the above proof, using now a variant of the dual theorem in linear programming (as for example BURGER [2] pag. 117) we obtain the following general result, which generalizes that due to EGERVÁRY [3] in the case of $k = 2$.

THEOREM 7. – *For any* (k, n)*-matrix* a*, we have*

$$m^k(a) = \max_{x\in E(U_n^k)} f^n(x) = \min_{(y_{i_1},\dots,y_{i_k})\in B(a)} \sum_{r=1}^{n} \sum_{i_r=1}^{n} y_{i_r}$$

where now $B(a)$ *is the set of* $(y_{i_1}, \ldots, y_{i_k})$ *such that* $(y_{i_1} + \ldots + y_{i_k}) \geqslant a_{i_1, \ldots, i_k}$ *for each* $(i_1, \ldots, i_k) \in N^k$.

We would like to indicate that one can write down the previous results in the form of dynamic programming. For the last one, let

$$F(y^1, \ldots, y^k; a) = \min_{(y_{i_1}, \ldots, y_{i_k}) \in B(a)} \sum_{r=1}^{k} \sum_{i_r=1}^{n} y_{i_r}$$

where $y^r = \{y_{i_r}\}_{i_r \in N}$, then

$$m^k(a) = F(y^1, \ldots, y^k; a) =$$

$$= \min_{y'} \left[\sum_{i_1=1}^{n} y_{i_1} + F(y^2, \ldots, y^k; a - y^1) \right] = \min_{y'} \left[\sum_{i_1=1}^{n} y_{i_1} + m^{k-1}(a - y') \right].$$

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