An Existence Theorem for Compressible Viscous Fluids (*).

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Sunto. – *Si dimostra un teorema di esistenza (locale nel tempo) per il sistema di equazioni che descrive il moto di un fluido viscoso comprimibile.*

Compressible viscous fluids have been studied by several authors in the last thirty years. The first result was a uniqueness theorem proved by GRAFFI [4] in 1953 for barotropic fluids, i.e. fluids for which the pressure p depends only on the density ϱ. Then in 1959 SERRIN [14] proved a uniqueness theorem for general fluids in a bounded domain.

With regard to the Cauchy problem in $\boldsymbol{R}^3$, an existence theorem was proved in 1962 by NASH [12]; independently ITAYA [5], [6] and VOL'PERT - HUDJAEV [20] obtained existence theorems in other classes of functions (the latter work, however, studies a set of equations that is physically correct only for barotropic fluids).

These existence results are all local in time. In 1980 MATSUMURA - NISHIDA [9], [10] proved that the solution exists for all time for small initial data.

With regard to the initial-boundary value problem, an existence theorem was proved in 1976 by SOLONNIKOV [16], for barotropic fluids with constant viscosities. The solution is found in the class $W_q^{2,1}$, $q > 3$ (see for instance LADYZENSKAJA - SOLONNIKOV - URAL'CEVA [7] for the definition of this space). In the general case TANI [17] obtained an existence theorem for bounded or unbounded domains, the solution belonging to Hölder spaces. Finally, BÖHM [1] has extended the result of SOLONNIKOV to more general cases and TON [18] has proved the existence of a weak solution (however both these authors modify a little the correct physical case).

Also these results are local in time. No global result is known for the initial-boundary value problem in dimension greater than one.

In this paper we prove an existence theorem (local in time) for some initial-boundary value problems which are physically reasonable. The equations are written in the general form, and the solution is found in Sobolev spaces of Hilbert type. The proof is based on the method of successive approximation, and is rather simple in concept. One must however make some straightforward but tedious calculations, due to the unavoidable complexity of the system of equations. The basic estimates are obtained by using some well-known theorems of LIONS - MAGENES [8]

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and in this way our result turns out to be strictly related to the general theory of parabolic equations.

1. – Statement of the problem and main results.

Let Ω be a bounded connected open subset of R^3, locally on one side of its boundary Γ. Set $Q_T \equiv]0, T[\times \Omega$ and $\Sigma_T \equiv]0, T[\times \Gamma$. The equations that we want to study are

$$
(1.1)\quad \begin{cases}
\varrho\left[\dfrac{\partial v}{\partial t} + (v\cdot\nabla)v - b\right] = -\nabla p + \sum\limits_k [D_k(\mu D_k v) + D_k(\mu\nabla v^k)] + \\
\qquad\qquad + \nabla\left[\left(\zeta - \dfrac{2}{3}\mu\right)\operatorname{div} v\right] & \text{in } Q_T\,, \\
\dfrac{\partial\varrho}{\partial t} + \operatorname{div}(\varrho v) = 0 & \text{in } Q_T\,, \\
c_V\varrho\left[\dfrac{\partial\theta}{\partial t} + v\cdot\nabla\theta\right] = -\theta\dfrac{\partial p}{\partial\theta}\operatorname{div} v + \sum\limits_k D_k(\chi D_k\theta) + \varrho r + \\
\qquad\qquad + \dfrac{\mu}{2}\sum\limits_{i,k}(D_k v^i + D_i v^k)^2 + \left(\zeta - \dfrac{2}{3}\mu\right)(\operatorname{div} v)^2 & \text{in } Q_T\,, \\
v_{|\Gamma} = \bar{v} & \text{on } \Sigma_T\,, \\
\theta_{|\Gamma} = \bar{\theta} & \text{on } \Sigma_T\,, \\
v_{|t=0} = v_0 & \text{in } \Omega\,, \\
\varrho_{|t=0} = \varrho_0 & \text{in } \Omega\,, \\
\theta_{|t=0} = \theta_0 & \text{in } \Omega\,,
\end{cases}
$$

where the velocity $v = v(t, x)$, the density $\varrho = \varrho(t, x)$, and the absolute temperature $\theta = \theta(t, x)$ are the unknowns; $b = b(t, x)$ is the external force field per unit mass and $r = r(t, x)$ is the heat supply per unit mass per unit time; $v_0 = v_0(x)$, $\varrho_0 = \varrho_0(x) > 0$, and $\theta_0 = \theta_0(x)$ are the initial data; and $\bar{v} = \bar{v}(t, y)$, $\bar{\theta} = \bar{\theta}(t, y)$ are the boundary data, defined for $y \in \Gamma$.

Moreover we consider the following constitutive equations

$$
(1.2)\qquad p = \bar{p}(\varrho, \theta)\,, \qquad c_V = \bar{c}_V(\varrho, \theta)\,,
$$

$$
\mu = \bar{\mu}(\varrho, \theta, v) > 0\,, \qquad \zeta = \bar{\zeta}(\varrho, \theta, v) \geqslant 0\,, \qquad \chi = \bar{\chi}(\varrho, \theta, v) > 0\,,
$$

which express how the pressure p, the specific heat at constant volume c_V, the coefficients of viscosity μ and ζ, and the coefficient of heat conductivity χ depend on the unknowns ϱ, θ and v.

Let $n = n(y)$ be the unit outward normal vector to Γ. We prove the following result

THEOREM A. – *Let* $\Gamma \in C^4$; $b \in H^{2,1}(Q_{T_0})$, $r \in H^{2,1}(Q_{T_0})$; $\bar{v} \in H^{7/2,7/4}(\Sigma_{T_0})$ *with* $\bar{v}\cdot n = 0$ *on* Σ_{T_0}, $\bar{\theta} \in H^{7/2,7/4}(\Sigma_{T_0})$; $v_0 \in H^3(\Omega)$, $\varrho_0 \in H^3(\Omega)$, $\theta_0 \in H^3(\Omega)$; $\bar{p} \in C^3$, $\bar{c}_V \in C^3$ *with* $0 < \bar{c}_1 \leqslant \bar{c}_V \leqslant \bar{c}_2$; $\bar{\mu} \in C^3$ *with* $\bar{\mu} \geqslant \bar{\mu}_1 > 0$, $\bar{\zeta} \in C^3$, $\bar{\chi} \in C^3$ *with* $\bar{\chi} \geqslant \bar{\chi}_1 > 0$. *Suppose that the data satisfy the compatibility conditions (see Remark* 3.1). *Then there exist* $T_1 \in]0, T_0]$, $v \in H^{4,2}(Q_{T_1})$, $\varrho \in C^0([0, T_1]; H^3(\Omega))$ *with* $\varrho > 0$ *in* $\bar{Q}_{T_1}$, $\theta \in H^{4,2}(Q_{T_1})$ *such that* (v, ϱ, θ) *is a solution of* (1.1) *in* Q_{T_1}.

For the definition of the spaces $H^{r,s}(Q_T)$, see for instance LIONS - MAGENES [8], chap. 4.

Other boundary value problems are considered in Remark 3.3.

Uniqueness of the solution is proved in [19] (see also GRAFFI [4] and SERRIN [14]).

2. – Proof of Theorem A for constant coefficients.

In this section we suppose that the coefficients $\bar{\mu}$, $\bar{\zeta}$, $\bar{\chi}$ are constant. Consider the operators

$$L_1 \equiv \frac{\partial}{\partial t} - \frac{1}{\varrho_0(x)}\left[\bar{\mu}\Delta + \left(\bar{\zeta} + \frac{1}{3}\bar{\mu}\right)\nabla \operatorname{div}\right] \equiv \frac{\partial}{\partial t} + A(x, D)\,, \tag{2.1}$$

$$L_{2,v} \equiv \frac{\partial}{\partial t} + v\cdot\nabla\,, \tag{2.2}$$

$$L_3 \equiv \frac{\partial}{\partial t} - \frac{1}{\varrho_0(x)c_0(x)}\bar{\chi}\Delta \equiv \frac{\partial}{\partial t} + B(x, D)\,, \tag{2.3}$$

where $c_0(x) \equiv \bar{c}_V(\varrho_0(x), \theta_0(x))$.

One starts from $(v_0, \theta_0, \varrho_0)$ and obtains $(v_n, \theta_n, \varrho_n)$ from $(v_{n-1}, \theta_{n-1}, \varrho_{n-1})$ by solving these problems:

$$\begin{cases} L_1(v_n) = \left(\dfrac{1}{\varrho_{n-1}} - \dfrac{1}{\varrho_0}\right)\left[\bar{\mu}\Delta v_{n-1} + \left(\bar{\zeta} + \dfrac{1}{3}\bar{\mu}\right)\nabla \operatorname{div} v_{n-1}\right] + \\ \qquad + b - (v_{n-1}\cdot\nabla)v_{n-1} - \dfrac{1}{\varrho_{n-1}}\nabla p_{n-1} \equiv F_{n-1} + b & \text{in } Q_T\,, \\ v_{n|\Gamma} = \bar{v} & \text{on } \Sigma_T\,, \\ v_{n|t=0} = v_0 & \text{in } \Omega\,, \end{cases} \tag{2.4}$$

$$\begin{cases} L_3(\theta_n) = \left(\dfrac{1}{\varrho_{n-1}c_{n-1}} - \dfrac{1}{\varrho_0 c_0}\right)\bar{\chi}\Delta\theta_{n-1} + \dfrac{\bar{\mu}}{2c_{n-1}\varrho_{n-1}}\sum\limits_{i,k}(D_k v^i_{n-1} + D_i v^k_{n-1})^2 + \\ \qquad + \dfrac{1}{\varrho_{n-1}c_{n-1}}\left(\bar{\zeta} - \dfrac{2}{3}\bar{\mu}\right)(\operatorname{div} v_{n-1})^2 - v_{n-1}\cdot\nabla\theta_{n-1} - \\ \qquad - \dfrac{1}{\varrho_{n-1}c_{n-1}}\theta_{n-1}\dfrac{\partial p_{n-1}}{\partial\theta}\operatorname{div} v_{n-1} + \dfrac{r}{c_{n-1}} \equiv G_{n-1} & \text{in } Q_T\,, \\ \theta_{n|\Gamma} = \bar{\theta} & \text{on } \Sigma_T\,, \\ \theta_{n|t=0} = \theta_0 & \text{in } \Omega\,, \end{cases} \tag{2.5}$$

$$(2.6)\qquad \begin{cases} L_{2,v_n}(\varrho_n) = -\varrho_{n-1}\operatorname{div} v_n & \text{in } Q_T\,, \\ \varrho_{n|t=0} \quad = \varrho_0 & \text{in } \Omega\,, \end{cases}$$

where

$$p_j(t,x) \equiv \bar{p}(\varrho_j(t,x), \theta_j(t,x))\,, \qquad c_j(t,x) \equiv \bar{c}_V(\varrho_j(t,x), \theta_j(t,x))\,,$$
$$\frac{\partial p_j}{\partial\theta}(t,x) \equiv \frac{\partial\bar{p}}{\partial\theta}(\varrho_j(t,x), \theta_j(t,x))\,, \qquad \text{for each } j\geq 0\,.$$

Set

$$(2.7)\qquad \|b\|_{2,1,T_0} + \|\bar{v}\|_{7/2,7/4,T_0} + \|\bar{\theta}\|_{7/2,7/4,T_0} + \|r\|_{2,1,T_0} + \\ + \|\varrho_0\|_3 + \|\theta_0\|_3 + \|v_0\|_3 \equiv \frac{F}{4} < +\infty\,,$$

$$(2.8)\qquad \bar{\varrho}_1 \equiv \tfrac{1}{2}\inf_{\bar{\Omega}} \varrho_0(x) > 0\,, \qquad \bar{\varrho}_2 \equiv 2\sup_{\bar{\Omega}} \varrho_0(x)\,,$$

where $\|\cdot\|_k$ and $\|\cdot\|_{r,s,T}$ denote the usual norms in $H^k(\Omega)$ and $H^{r,s}(Q_T)$ (or $H^{r,s}(\Sigma_T)$).

From now on each constant c, C_j, C'_j, $\bar{C}_j$, $\bar{C}'_j$, K_j and each instant T_{j+1}, $j\geq 0$, will depend at most on the data of the problem Ω, T_0, r, $\bar{p}$, $\bar{c}_V$, $\bar{\mu}$, $\bar{\zeta}$, $\bar{\chi}$, ϱ_0, θ_0, v_0. Other possible dependences will be explicitly pointed out.

We want to prove that there exist $T_1 = T_1(F) \in]0, T_0]$ and a constant C_0 such that

$$(2.9)\qquad \sup_{[0,T_1]} \|\varrho_n(t)\|_3 + \|v_n\|_{4,2,T_1} + \|\theta_n\|_{4,2,T_1} \leq C_0 F\,, \qquad n\geq 1\,,$$

$$(2.10)\qquad 0 < \bar{\varrho}_1 \leq \varrho_n(t,x) \leq \bar{\varrho}_2 \quad \text{in } \bar{Q}_{T_1},\ n\geq 0\,.$$

From (2.9) one obtains at once (see chap. 1, Theor. 3.1 and Theor. 3.2 of [8]) that v_n and θ_n belong to $C^0([0,T_1]; H^3(\Omega))$ and

$$(2.11)\qquad \sup_{[0,T_1]} \|v_n(t)\|_3 \leq c\{\|v_n\|_{4,2,T_1} + \|v_n(0)\|_3 + \|\dot{v}_n(0)\|_1\}\,,$$

$$(2.12)\qquad \sup_{[0,T_1]} \|\theta_n(t)\|_3 \leq c\{\|\theta_n\|_{4,2,T_1} + \|\theta_n(0)\|_3 + \|\dot{\theta}_n(0)\|_1\}\,,$$

where $\dot{f} \equiv \partial f/\partial t$. Note that the constant c in (2.11) and (2.12) doesn't depend on the length of the time interval T_1.

One proves (2.9) and (2.10) by induction. First of all one has to find the existence theorems for (2.4), (2.5) and (2.6).

LEMMA 2.1. – *$A(x, D)$ is a uniformly strongly elliptic system, with elliptic constant $2\bar{\mu}/\bar{\varrho}_2$.*

PROOF. – One has only to observe that for each $\xi \in \mathbf{R}^3$, $\eta = \alpha + i\beta \in \mathbf{C}^3$

$$\operatorname{Re}\sum_{i,j} \xi_i\xi_j\eta_i\bar{\eta}_j = \sum_{i,j} \xi_i\xi_j(\alpha_i\alpha_j + \beta_i\beta_j) = \Big(\sum_i \xi_i\alpha_i\Big)^2 + \Big(\sum_i \xi_i\beta_i\Big)^2 \geq 0\,,$$

and that $-(1/\varrho_0)\bar{\mu}\Delta$ is uniformly strongly elliptic with constant $2\bar{\mu}/\bar{\varrho}_2$ (see for instance MORREY [11], § 6.5 for the definition and other properties of strongly elliptic systems).

Observe also that the operator

$$A_1(D) \equiv \varrho_0(x)A(x, D) = -\bar{\mu}\Delta - (\bar{\zeta} + \tfrac{1}{3}\bar{\mu})\nabla \operatorname{div}$$

is the «elasticity operator» (see for instance NEČAS [13], chap. 3, Prop. 7.2). □

Let $T \in]0, T_0]$. Then one has

LEMMA 2.2. – *If $F_{n-1} \in H^{2,1}(Q_T)$ and the (necessary) compatibility conditions are satisfied (see Remark 3.1), then (2.4) has a unique solution $v_n \in H^{4,2}(Q_T)$ and*

$$(2.13)\qquad \|v_n\|_{4,2,T} \leqslant C_1\{\|F_{n-1}\|_{2,1,T} + \|b\|_{2,1,T_0} + \|\bar{v}\|_{7/2,7/4,T_0} + \|v_0\|_3 + \|\dot{v}_n(0)\|_1\} \leqslant$$
$$\leqslant C_1'\left\{\frac{F}{2} + \|F_{n-1}\|_{2,1,T}\right\}.$$

PROOF. – One applies Theor. 5.2, chap. 4 of LIONS - MAGENES [8], where $H = H^2(\Omega)$, $\mathcal{H} = L^2(\Omega)$, and A is defined by

$$Aw = A(x, D)w$$
$$D_H(A) = \{w \in H | Aw \in H,\ w_{|\Gamma} = 0\}.$$

$A(x, D)$ is a unbounded closed operator in H and has the following properties (see § 4, (4.8) and (4.11)):

(i) $D_H(A) = H^4(\Omega) \cap H_0^1(\Omega)$;

(ii) for each $w \in D_H(A)$, for each $\lambda \in \boldsymbol{C}$ with $\operatorname{Re}\lambda > \lambda_0 + 1$ (see (4.6) for the definition of λ_0) one has

$$\|w\|_2 \leqslant C_2\|(A + \lambda)w\|_0,$$

where C_2 doesn't depend on λ.

Moreover $A + \lambda$ is an isomorphism from $D_H(A)$ (endowed with the graph norm) in H for each $\lambda \in \boldsymbol{C}$ with $\operatorname{Re}\lambda > \lambda_0 + 1$.

Finally one obtains

$$\begin{aligned}\|w\|_{D_H(A)} + |\lambda|^2\|w\|_0 &= \|w\|^2 + \|Aw\|_2 + |\lambda|^2\|w\|_0 \leqslant \\ &\leqslant \|w\|_2 + \|(A+\lambda)w\|_2 + |\lambda|\|w\|_2 + |\lambda|\|\lambda w\|_0 \leqslant \\ &\leqslant (1 + |\lambda|)\|w\|_2 + \|(A+\lambda)w\|_2 + |\lambda|\|(A+\lambda)w\|_0 + |\lambda|\|Aw\|_0 \leqslant \\ &\leqslant (1 + |\lambda|)\|w\|_2 + \|(A+\lambda)w\|_2 + |\lambda|\|(A+\lambda)w\|_0 + \left(\frac{4}{3}\bar{\mu} + \bar{\zeta}\right)\frac{1}{\bar{\varrho}_1}|\lambda|\|w\|_2 \leqslant \\ &\leqslant C_3(1 + |\lambda|)\|(A+\lambda)w\|_0 + \|(A+\lambda)w\|_2,\end{aligned}$$

where C_3 doesn't depend on λ.

Hence the result follows as in Theor. 5.3, chap. 4 of [8]. Note also that the constant C_1 doesn't depend on T. Moreover, since $\|\dot{v}_n(0)\|_1 \leqslant K_1(F/4)$, the second estimate in (2.13) is obtained by taking $C_1' \equiv \max(C_1, C_1 K_1)$. □

From this lemma one obtains at once that v_1 (that is the solution of (2.4) for $n = 1$) satisfies

$$\|v_1\|_{4,2,T} \leqslant C_1' \left\{\frac{F}{2} + C_4 T^{1/2}\left[\|v_0\|_3^2 + (1 + \|\varrho_0\|_2^2)(1 + \|\theta_0\|_2^2 + \|\varrho_0\|_2^2)(\|\varrho_0\|_3 + \|\theta_0\|_3)\right]\right\} \leqslant \leqslant C_1' \left\{\frac{F}{2} + C_5 T^{1/2} F(1 + F^4)\right\}.$$

Then one can choose $T \leqslant T_2 = T_2(F)$ and obtain

$$\|v_1\|_{4,2,T} \leqslant C_1' F . \tag{2.14}$$

LEMMA 2.3. – *If $G_{n-1} \in H^{2,1}(Q_T)$ and the (necessary) compatibility conditions are satisfied (see Remark 3.1), then (2.5) has a unique solution $\theta_n \in H^{4,2}(Q_T)$ and*

$$\|\theta_n\|_{4,2,T} \leqslant C_6 \{\|G_{n-1}\|_{2,1,T} + \|\bar{\theta}\|_{7/2,7/4,T_0} + \|\theta_0\|_3 + \|\dot{\theta}_n(0)\|_1\} \leqslant C_6' \{F/2 + \|G_{n-1}\|_{2,1,T}\}. \tag{2.15}$$

PROOF. – One has only to observe that $B(x, D)$ is uniformly strongly elliptic, with elliptic constant $2\tilde{\chi}/\bar{c}_2\bar{\varrho}_2$, and apply Theor. 5.3, chap. 4 of [8]. Moreover, one has $\|\dot{\theta}_n(0)\|_1 \leqslant K_2(F/4)$, and then the second estimate in (2.15) is obtained by staking $C_6' \equiv \max(C_6, C_6 K_2)$. □

Hence, as before, one obtains that θ_1 satisfies

$$\|\theta_1\|_{4,2,T} \leqslant C_6' \left\{\frac{F}{2} + C_7 T^{1/2} \left[\left\|\frac{1}{\varrho_0 c_0}\right\|_2 \|v_0\|_3^2 + \|v_0\|_2 \|\theta_0\|_3 + + \left\|\frac{1}{\varrho_0 c_0}\right\|_2 \|\theta_0\|_2 (1 + \|\varrho_0\|_2^2 + \|\theta_0\|_2^2)\|v_0\|_3\right] + C_7' \|r\|_{2,1,T}\right\} \leqslant \leqslant C_6' \left\{\frac{F}{2} + C_8 T^{1/2} F^2 (1 + F^8) + C_7' \|r\|_{2,1,T}\right\},$$

and by choosing $T \leqslant T_3 = T_3(F)$ one has

$$\|\theta_1\|_{4,2,T} \leqslant C_6' F , \tag{2.16}$$

since $\|r\|_{2,1,T} \to 0$ as $T \to 0^+$.

LEMMA 2.4. – *Let v_n be the solution of (2.4) determined in Lemma 2.2. Then (2.6) has a unique solution $\varrho_n \in L^\infty(0, T; H^3(\Omega))$ and*

$$\sup_{[0,T]} \|\varrho_n(t)\|_3 \leqslant \exp\left(c\int_0^T \|v_n(\tau)\|_3 \, d\tau\right) \left\{\|\varrho_0\|_3 + c\int_0^T \|\varrho_{n-1}(\tau) \operatorname{div} v_n(\tau)\|_3 \, d\tau\right\}. \tag{2.17}$$

PROOF. – Since $v_n \in H^{4,2}(Q_T)$, then (see [8], chap. 1, Theor. 3.1)

$$v_n \in C^0([0, T]; H^3(\Omega)) \subset C^0([0, T]; C^1(\bar{\Omega})) .$$

Hence the existence of a unique solution ϱ_n follows from the method of characteristics (observe that $v_n \cdot n_{|\Gamma} = \bar{v} \cdot n = 0$ on Σ_T).

Let now α be a multi-index with $|\alpha| \leqslant 3$. Then one has

$$\int_\Omega \left(\frac{\partial}{\partial t} D^\alpha \varrho_n\right) D^\alpha \varrho_n \, dx = -\int_\Omega D^\alpha(v_n \cdot \nabla \varrho_n) D^\alpha \varrho_n \, dx - \int_\Omega D^\alpha(\varrho_{n-1} \operatorname{div} v_n) D^\alpha \varrho_n \, dx ;$$

by adding in α and integrating by parts the term

$$-\int_\Omega v_n \cdot \nabla (D^\alpha \varrho_n)^2 \, dx ,$$

it follows

$$\frac{1}{2} \frac{d}{dt} \|\varrho_n(t)\|_3^2 \leqslant c \left[\|v_n(t)\|_3 \|\varrho_n(t)\|_3^2 + \|\varrho_{n-1}(t) \operatorname{div} v_n(t)\|_3 \|\varrho_n(t)\|_3 \right] .$$

By Gronwall's lemma one obtains easily (2.17). □

From (2.17) one sees that if $T \leqslant T_2$

$$\sup_{[0,T]} \|\varrho_1(t)\|_3 \leqslant \exp\left(c \int_0^T \|v_1(\tau)\|_3 \, d\tau\right) \|\varrho_0\|_3 \left\{1 + c \int_0^T \|v_1(\tau)\|_4 \, d\tau\right\} \leqslant$$
$$\leqslant \exp\,(c C_1' F T^{1/2}) \|\varrho_0\|_3 \{1 + c C_1' F T^{1/2}\} .$$

Hence there exists $T_4 = T_4(F)$ such that if $T \leqslant \min\,(T_2, T_4)$ one obtains

$$\sup_{[0,T]} \|\varrho_1(t)\|_3 \leqslant F . \tag{2.18}$$

Moreover, let $U_1(s, t, x)$ be the solution of

$$\frac{d U_1(s, t, x)}{ds} = v_1(s, U_1(s, t, x)) , \qquad U_1(t, t, x) = x ;$$

then from the explicit formula for ϱ_1

$$\varrho_1(t, x) = \varrho_0(U_1(0, t, x)) - \int_0^t \varrho_0(U_1(s, t, x)) \operatorname{div} v_1(s, U_1(s, t, x)) \, ds$$

it follows that in Q_T one has

$$\varrho_1(t,x) \leqslant \frac{\bar{\varrho}_2}{2}\left\{1 + \int_0^t \|\operatorname{div} v_1(s)\|_{L^\infty(\Omega)}\, ds\right\} \leqslant \frac{\bar{\varrho}_2}{2}\{1 + T^{1/2} c C_1' F\},$$

$$\varrho_1(t,x) \geqslant 2\bar{\varrho}_1 - \frac{\bar{\varrho}_2}{2} T^{1/2} c C_1' F.$$

Then there exists $T_5 = T_5(F)$ such that if $T \leqslant \min(T_2, T_5)$ one obtains

$$\bar{\varrho}_1 \leqslant \varrho_1(t,x) \leqslant \bar{\varrho}_2 \quad \text{in } Q_T. \tag{2.19}$$

Hence one chooses $T_6 \leqslant \min(T_2, T_3, T_4, T_5)$ and $C_0 \geqslant 1 + C_1' + C_6'$, and obtains (2.9), (2.10) for $n = 1$.

Suppose now that (2.9), (2.10) are satisfied for each $j \leqslant n-1$. We have to obtain them for the index n. One has only to estimate the norms of F_{n-1} and G_{n-1} in $H^{2,1}(Q_T)$. The first estimate is obtained by a straightforward calculation. Set $\|\cdot\|_{q;k} \equiv \|\cdot\|_{L^q(0,T;H^k(\Omega))}$, $1 \leqslant q \leqslant +\infty$, $k \geqslant 0$; one has

$$\begin{aligned}
\|F_{n-1}\|_{2,1,T} &\leqslant 2\|F_{n-1}\|_{2;2} + \|\dot{F}_{n-1}\|_{2;0} \leqslant \\
&\leqslant C_9 \Bigg\{ \left\|\frac{\varrho_0 - \varrho_{n-1}}{\varrho_0 \varrho_{n-1}}\right\|_{\infty;2} \|v_{n-1}\|_{2;4} + \left\|\frac{1}{\varrho_{n-1}^2}\dot{\varrho}_{n-1}\right\|_{2;1} \|v_{n-1}\|_{\infty;3} + \\
&+ \left\|\frac{\varrho_0 - \varrho_{n-1}}{\varrho_0 \varrho_{n-1}}\right\|_{\infty;2} \|\dot{v}_{n-1}\|_{2;2} + \|v_{n-1}\|_{\infty;2}\|v_{n-1}\|_{2;3} + \\
&+ \|v_{n-1}\|_{\infty;2}\|\dot{v}_{n-1}\|_{2;1} + \left\|\frac{1}{\varrho_{n-1}}\right\|_{\infty;2} (1 + \|\nabla\varrho_{n-1}\|^2_{\infty;1} + \|\nabla\theta_{n-1}\|^2_{\infty;1})\cdot \\
&\cdot(\|\varrho_{n-1}\|_{2;3} + \|\theta_{n-1}\|_{2;3}) + \left\|\frac{1}{\varrho_{n-1}^2}\dot{\varrho}_{n-1}\right\|_{2;1} (\|\varrho_{n-1}\|_{\infty;2} + \|\theta_{n-1}\|_{\infty;2}) + \\
&+ (1 + \|\varrho_{n-1}\|_{\infty;2} + \|\theta_{n-1}\|_{\infty;2})(\|\dot{\varrho}_{n-1}\|_{2;1} + \|\dot{\theta}_{n-1}\|_{2;1})\Bigg\}.
\end{aligned} \tag{2.20}$$

If one observes that $\varrho_{n-1} \in \operatorname{Lip}([0,T]; H^2(\Omega))$ and that

$$\|f\|_{2;k} \leqslant T^{1/2}\|f\|_{\infty;k},$$

from (2.9) and (2.10) it follows that

$$\|F_{n-1}\|_{2,1,T} \leqslant \bar{C}_1(F) T^{1/2}.$$

Hence from (2.13)

$$\|v_n\|_{4,2,T} \leqslant C_1'\left\{\frac{F}{2} + \bar{C}_1(F) T^{1/2}\right\},$$

and when $T \leqslant T_7 = T_7(F)$ one has

$$\|v_n\|_{4,2,T} \leqslant C_1' F.$$

On the other hand one obtains

(2.21) $$\|G_{n-1}\|_{2,1,T} \leqslant 2\|G_{n-1}\|_{2;2} + \|\dot G_{n-1}\|_{2;0} \leqslant$$

$$\leqslant C_{10}\Bigg\{\left\|\frac{\varrho_0 c_0 - \varrho_{n-1}c_{n-1}}{\varrho_0 c_0 \varrho_{n-1}c_{n-1}}\right\|_{\infty;2}[\|\theta_{n-1}\|_{2;4} + \|\dot\theta_{n-1}\|_{2;2}] +$$

$$+ \left\|\frac{1}{\varrho_{n-1}c_{n-1}}\right\|_{\infty;2}[\|v_{n-1}\|_{2;3}\|v_{n-1}\|_{\infty;3} + \|v_{n-1}\|_{2;3}\|\theta_{n-1}\|_{\infty;3}(1 + \|\nabla\varrho_{n-1}\|^2_{\infty;1} + \|\nabla\theta_{n-1}\|^2_{\infty;1})] +$$

$$+ \|v_{n-1}\|_{\infty;3}(\|\dot v_{n-1}\|_{2;1} + \|\dot\theta_{n-1}\|_{2;1}) + \|\theta_{n-1}\|_{\infty;2}\|\dot v_{n-1}\|_{2;1} +$$

$$+ \|v_{n-1}\|_{\infty;3}\|\theta_{n-1}\|_{\infty;2}(\|\dot\varrho_{n-1}\|_{2;0} + \|\dot\theta_{n-1}\|_{2;0}) + \|v_{n-1}\|_{\infty;2}\|\dot\theta_{n-1}\|_{2;3} +$$

$$+ \left\|\frac{1}{\varrho^2_{n-1}c^2_{n-1}}(\dot\varrho_{n-1}c_{n-1} + \varrho_{n-1}\dot c_{n-1})\right\|_{2;1}[\|\theta_{n-1}\|_{\infty;3} + \|v_{n-1}\|^2_{\infty;3} + \|\theta_{n-1}\|_{\infty;2}\|v_{n-1}\|_{\infty;2}] +$$

$$+ \left(\left\|\frac{1}{c_{n-1}}\right\|_{\infty;2} + \|\dot c_{n-1}\|_{\infty;0}\right)\|r\|_{2;2} + \|\dot r\|_{2;0}\Bigg\}.$$

If one observes that $c_{n-1}\in L^\infty(0, T; H^2(\Omega))$, $\dot\theta_{n-1}\in C^0([0, T]; H^1(\Omega))$ (from Theor. 3.1, chap. 1 of [8]), and that by interpolation

$$\dot\theta_{n-1}\in L^2(0, T; H^2(\Omega)),$$

i.e. $\theta_{n-1}\in C^{1/2}([0, T]; H^2(\Omega))$, it follows by some calculation that

$$\|G_{n-1}\|_{2,1,T} \leqslant \bar C_2(F)T^{1/2} + \bar C_2'(F)\|r\|_{2,1,T}.$$

Hence from (2.15) one obtains that

$$\|\theta_n\|_{4,2,T} \leqslant C_6'\left\{\frac{F}{2} + \bar C_2(F)T^{1/2} + \bar C_2'(F)\|r\|_{2,1,T}\right\},$$

and when $T \leqslant T_8 = T_8(F)$ one has

$$\|\theta_n\|_{4,2,T} \leqslant C_6' F.$$

Now we can obtain also the estimate for ϱ_n: from (2.17) it follows

$$\sup_{[0,T]}\|\varrho_n(t)\|_3 \leqslant \exp\left(c\int_0^T\|v_n(\tau)\|_3\,d\tau\right)\left\{\|\varrho_0\|_3 + c\int_0^T\|\varrho_{n-1}(\tau)\operatorname{div} v_n(\tau)\|_3\,d\tau\right\} \leqslant$$

$$\leqslant \exp(cC_1'FT^{1/2})\left\{\frac{F}{2} + cC_1'F^2T^{1/2}\right\}.$$

Hence there exists $T_9 = T_9(F)$ such that if $T \leqslant \min(T_7, T_9)$, then

$$\sup_{[0,T]}\|\varrho_n(t)\|_3 \leqslant F.$$

Moreover from the explicit formula for ϱ_n, it follows that in Q_T

$$\varrho_n(t, x) \leqslant \frac{\bar{\varrho}_2}{2}\{1 + 2T^{1/2}cC_1'F\}$$

and

$$\varrho_n(t, x) \geqslant 2\bar{\varrho}_1 - \bar{\varrho}_2 T^{1/2}cC_1'F .$$

Hence one can find $T_{10} = T_{10}(F)$ such that if $T \leqslant \min(T_7, T_{10})$ it follows that

$$\bar{\varrho}_1 \leqslant \varrho_n(t, x) \leqslant \bar{\varrho}_2 \quad \text{in } Q_T .$$

Finally, one chooses

$$T_1 = \min\{T_6, T_7, T_8, T_9, T_{10}\}, \qquad C_0 = 1 + C_1' + C_6'; \tag{2.22}$$

consequently, (2.9) and (2.10) are proved in Q_{T_1}.

We can prove now the convergence of the successive approximations. From Theor. 4.1, chap. 1 in [8] it follows that $H^{4,2}(Q_{T_1})$ is continuously embedded into $H^{3/4}(0, T_1; H^{5/2}(\Omega))$, which is compactly embedded in $C^0([0, T_1]; H^2(\Omega))$ from Ascoli - Arzelà's and Rellich's theorems.

Hence there exist subsequences v_{n_k} and θ_{n_k} which converge in $C^0([0, T_1]; H^2(\Omega))$. Analogously, one obtains that $\dot{v}_{n_k}$ and $\dot{\theta}_{n_k}$ converge in $C^0([0, T_1]; L^2(\Omega))$.

Moreover, from (2.6) we obtain that

$$\sup_{[0,T_1]} \|\dot{\varrho}_n(t)\|_2 \leqslant cC_1'F^2 ,$$

hence ϱ_n is bounded in $H^{3/4}(0, T_1; H^{9/4}(\Omega))$ (see chap. 1, Theor. 4.1 and Theor. 9.6 of [8]), which is compactly embedded in $C^0([0, T_1]; H^2(\Omega))$. Analogously, we prove that $\dot{\varrho}_n$ is bounded in $H^{3/4}(0, T_1; H^{5/4}(\Omega))$, which is compactly embedded in $C^0([0, T_1]; H^1(\Omega))$.

Hence we can pass to the limit in the space $C^0([0, T_1]; L^2(\Omega))$ in (2.4), (2.5) and (2.6), and we find a solution (v, ϱ, θ) of problem (1.1), such that $v \in H^{4,2}(Q_{T_1})$, $\theta \in H^{4,2}(Q_{T_1})$, $\varrho \in L^\infty(0, T_1; H^3(\Omega))$, and moreover

$$\|v\|_{4,2,T_1} \leqslant C_1'F, \quad \|\theta\|_{4,2,T_1} \leqslant C_6'F, \quad \sup_{[0,T_1]} \|\varrho(t)\|_3 \leqslant F, \quad 0 < \bar{\varrho}_1 \leqslant \varrho(t, x) \leqslant \bar{\varrho}_2 \quad \text{in } Q_{T_1}.$$

Observe also that, from the uniqueness theorem (see [**19**]), the whole sequence $(v_n, \varrho_n, \theta_n)$ converges to the solution (v, ϱ, θ) of problem (1.1).

Finally, one can see that $\varrho \in C^0([0, T_1]; H^3(\Omega))$. In fact, the representation formula for $\varrho(t, x)$ can be written as

$$\varrho(t, x) = \varrho_0(U(0, t, x)) - \int_0^t \varrho(s, U(s, t, x)) \operatorname{div} v(s, U(s, t, x))\, ds \tag{2.23}$$

or

$$\varrho(t, x) = \varrho_0(U(0, t, x)) \exp\left[-\int_0^t \operatorname{div} v(s, U(s, t, x))\, ds\right] \tag{2.24}$$

(see for instance ITAYA [5], or SOLONNIKOV [16]).

Since $U(s, t, x) \in C^0([0, T_1]\times[0, T_1]; H^3(\Omega))$ (as in BOURGUIGNON - BREZIS [2], Lemma A.6), and for each $s, t \in [0, T_1]$, $U(s, t, x)$ is a C^1 diffeomorphism from $\bar{\Omega}$ on $\bar{\Omega}$, with $U(\partial\Omega) \subset \partial\Omega$, one obtains easily that $\varrho \in C^0([0, T_1]; H^3(\Omega))$ (see also BOURGUIGNON - BREZIS [2], Lemmas A.3, A.4, A.5).

Observe also that is $\inf_{\bar{\Omega}} \varrho_0(x) > 0$, then from (2.24) it follows that every solution of $(1.1)_2$ in Q_T is positive in Q_T, for each $T > 0$.

3. – Proof of Theorem A in the general case.

Suppose now that $\bar{\mu}, \bar{\zeta}, \bar{\chi}$ belong to C^3, with $\bar{\mu} \geqslant \bar{\mu}_1 > 0$, $\bar{\zeta} \geqslant 0$, $\bar{\chi} \geqslant \bar{\chi}_1 > 0$.

Consider the operators

$$\begin{aligned}\tilde{L}_1 w \equiv \frac{\partial w}{\partial t} - \frac{1}{\varrho_0(x)} \left\{\sum_k [D_k(\mu_0 D_k w) + D_k(\mu_0 \nabla w^k)] + \nabla\left[\left(\zeta_0 - \frac{2}{3}\mu_0\right)\operatorname{div} w\right]\right\} \equiv \\ \equiv \frac{\partial w}{\partial t} + \tilde{A}(x, D) w,\end{aligned} \tag{3.1}$$

$$L_{2,v} \equiv \frac{\partial}{\partial t} + v\cdot\nabla, \tag{3.2}$$

$$\tilde{L}_3 \equiv \frac{\partial}{\partial t} - \frac{1}{\varrho_0(x) c_0(x)} \sum_k D_k(\chi_0 D_k) \equiv \frac{\partial}{\partial t} + \tilde{B}(x, D), \tag{3.3}$$

where $\mu_0(x) \equiv \bar{\mu}(\varrho_0(x), \theta_0(x), v_0(x))$, $\zeta_0(x) \equiv \bar{\zeta}(\varrho_0(x), \theta_0(x), v_0(x))$, $\chi_0(x) \equiv \bar{\chi}(\varrho_0(x), \theta_0(x), v_0(x))$.

The terms F^j_{n-1} and G_{n-1} become respectively

$$\begin{aligned}F^j_{n-1} \equiv \frac{1}{\varrho_0}\Big\{\sum_k D_k[(\mu_{n-1} - \mu_0) D_k v^j_{n-1} + (\mu_{n-1} - \mu_0) D_j v^k_{n-1}] + \\ + D_j\left[\left(\zeta_{n-1} - \zeta_0 - \frac{2}{3}\mu_{n-1} + \frac{2}{3}\mu_0\right)\operatorname{div} v_{n-1}\right]\Big\} + \\ + \left(\frac{1}{\varrho_{n-1}} - \frac{1}{\varrho_0}\right)\Big\{D_k[\mu_{n-1} D_k v^j_{n-1}] + D_k[\mu_{n-1} D_j v^k_{n-1}] + D_j\left[\left(\zeta_{n-1} - \frac{2}{3}\mu_{n-1}\right)\operatorname{div} v_{n-1}\right]\Big\} - \\ - (v_{n-1}\cdot\nabla) v_{n-1} - \frac{1}{\varrho_{n-1}} \nabla p_{n-1},\end{aligned} \tag{3.4}$$

$$(3.5)\qquad G_{n-1} \equiv \frac{1}{\varrho_0 c_0}\sum_k D_k(\chi_{n-1}-\chi_0)D_k\theta_{n-1}] + \left(\frac{1}{\varrho_{n-1}c_{n-1}}-\frac{1}{\varrho_0 c_0}\right)D_k(\chi_{n-1}D_k\theta_{n-1}) + \\ + \frac{\mu_{n-1}}{2c_{n-1}\varrho_{n-1}}\sum_{i,k}(D_k v^i_{n-1}+D_i v^k_{n-1})^2 + \frac{\zeta_{n-1}-\frac{2}{3}\mu_{n-1}}{c_{n-1}\varrho_{n-1}}(\operatorname{div} v_{n-1})^2 - \\ - v_{n-1}\cdot\nabla\theta_{n-1} - \frac{1}{\varrho_{n-1}c_{n-1}}\theta_{n-1}\frac{\partial p_{n-1}}{\partial\theta}\operatorname{div} v_{n-1} + \frac{r}{c_{n-1}},$$

where $\mu_j(t,x) \equiv \bar{\mu}(\varrho_j(t,x), \theta_j(t,x), v_j(t,x))$, and analogously for ζ_j and χ_j, $j\geq 1$.

One easily verifies that

$$\|\mu_{n-1}-\mu_0\|_{\infty;2} \leq \bar{C}_3(F)\,T^{1/2},$$
$$\|\mu_{n-1}\|_{\infty;3},\qquad \|\mu_0\|_3 \leq \bar{C}_4(F),$$
$$\|\dot{\mu}_{n-1}\|_{\infty;1} \leq \bar{C}_5(F),$$

and analogously for ζ_{n-1} and χ_{n-1}. These estimates are sufficient to obtain the estimates for F_{n-1} and G_{n-1}, by proceeding as in § 2.

On the other hand, $\tilde{A}(x, D)$ is a uniformly strongly elliptic system, with elliptic constant $2\bar{\mu}_1/\bar{\varrho}_2$, and $\tilde{B}(x, D)$ is uniformly strongly elliptic, with elliptic constant $2\bar{\chi}_1/\bar{c}_2\bar{\varrho}_2$.

Lemma 2.2 and Lemma 2.3 are proved in the same way (see § 4 for the properties of $\tilde{A}(x, D)$), and then the existence of a solution of (1.1) is obtained by successive approximation without any other change.

REMARK 3.1. – The compatibility conditions (see Lemma 2.2 and Lemma 2.3) are (see [8], chap. 4, Prop. 2.2)

$$(3.6)\qquad v_{0|\Gamma} = \bar{v}(0, y)$$

$$\frac{\partial\bar{v}}{\partial t}(0,y) = \frac{1}{\varrho_0}\left\{\sum_k [D_k(\mu_0 D_k v_0) + D_k(\mu_0\nabla v_0^k)] + \\ + \nabla\left[\left(\zeta_0-\frac{2}{3}\mu_0\right)\operatorname{div} v_0\right]\right\} + b(0) + F_{n-1}(0,y) \quad \text{on } \Gamma,$$

that is

$$(3.7)\qquad \frac{\partial\bar{v}}{\partial t}(0,y) = \frac{1}{\varrho_0}\left\{\sum_k [D_k(\mu_0 D_k v_0) + D_k(\mu_0\nabla v_0^k)] + \\ + \nabla\left[\left(\zeta_0-\frac{2}{3}\mu_0\right)\operatorname{div} v_0\right]\right\} + b(0) - (v_0\cdot\nabla)v_0 - \frac{1}{\varrho_0}\nabla p_0 \quad \text{on } \Gamma,$$

where $b(0) = b(0, x)$.

Analogously, for the absolute temperature θ one must require

$$(3.8)\qquad \theta_{0|\Gamma} = \bar{\theta}(0, y)$$

$$(3.9)\qquad \frac{\partial\bar{\theta}}{\partial t}(0,y) = \frac{1}{\varrho_0 c_0}\sum_k D_k(\chi_0 D_k v_0) + \frac{\mu_0}{2c_0\varrho_0}\sum_{i,k}(D_k v_0^i + D_i v_0^k)^2 + \\ + \frac{\zeta_0-\frac{2}{3}\mu_0}{c_0\varrho_0}(\operatorname{div} v_0)^2 - v_0\cdot\nabla\theta_0 - \frac{1}{\varrho_0 c_0}\theta_0\frac{\partial p_0}{\partial\theta}\operatorname{div} v_0 + \frac{1}{c_0}r(0) \quad \text{on } \Gamma,$$

where $r(0) = r(0, x)$.

REMARK 3.2. – From thermodynamics one expects that (see fot instance SERRIN [15])

$$\overline{p} > 0\,, \quad \frac{\partial \overline{p}}{\partial \varrho} > 0\,, \quad \theta > 0\,. \tag{3.10}$$

One can require the first two for the assigned function $\overline{p}$, but the third assertion must be proved.

From the maximum principle, if $\bar{\theta} \geqslant 0$ on Σ_T and $\theta_0 \geqslant 0$ in $\bar{\Omega}$, and if $r \geqslant 0$ in $\bar{Q}_T$, one obtains that $\theta(t, x) \geqslant 0$ in $\bar{Q}_T$, for each $T > 0$.

Moreover, if $\theta_0(x) \geqslant 2\bar{\theta}_1 > 0$ and $\bar{\theta}(t, y) \geqslant 2\bar{\theta}_1 > 0$, and if $r \geqslant 0$ in $\bar{Q}_{T_0}$, from the maximum principle one obtains that

$$\theta(t, x) \geqslant 2\bar{\theta}_1 - eT \max_{\bar{Q}_T} \left| \frac{1}{c_V \varrho} \theta \frac{\partial p}{\partial \theta} \operatorname{div} v \right| \qquad \text{in } \bar{Q}_T\,.$$

Then for T small enough, depending on the data of the problem, one has

$$\theta(t, x) \geqslant \bar{\theta}_1 > 0 \qquad \text{in } \bar{Q}_T\,.$$

REMARK 3.3. – In case $\bar{\chi}$ is constant, $\bar{\chi} \equiv \bar{\chi}_1 > 0$, one can solve problem (1.1) also for the different boundary conditions

$$\bar{\chi}_1 \frac{\partial \theta}{\partial n} = \bar{\theta} \qquad \text{or} \qquad \bar{\chi}_1 \frac{\partial \theta}{\partial n} + k\theta = k\bar{\theta}\,,$$

where k is a positive constant and $\bar{\theta} \in H^{5/2, 5/4}(\Sigma_{T_0})$.

One proves Lemma 2.3 in the same way, since these boundary value problems for the operator $\tilde{B}(x, D)$ are still «regular elliptic» (see [8], chap. 4, § 4).

Obviously, one has to rewrite the compatibility conditions, which in these cases must be satisfied to order $i < \frac{3}{4}$ (see [8], chap. 4, Prop. 2.2); namely, one must suppose that

$$\bar{\chi}_1 \frac{\partial \theta_0}{\partial n}\bigg|_\Gamma = \bar{\theta}(0, y) \qquad \text{or} \qquad \bar{\chi}_1 \frac{\partial \theta_0}{\partial n}\bigg|_\Gamma + k\theta_{0|\Gamma} = k\bar{\theta}(0, y)\,.$$

Finally, the maximum principle for θ can be applied also in these cases with minor changes.

4. – Appendix.

Consider the operator

$$\tilde{A}(x, D)u \equiv -\frac{1}{\varrho_0(x)} \left\{ \sum_k [D_k(\mu_0 D_k u) + D_k(\mu_0 \nabla u^k)] + \nabla \left[\left(\zeta_0 - \frac{2}{3}\mu_0 \right) \operatorname{div} u \right] \right\}, \tag{4.1}$$

(μ_0 and ζ_0 defined in § 3), which becomes

$$A(x, D)u = -\frac{1}{\varrho_0(x)}\left[\bar{\mu}\Delta u + \left(\bar{\zeta} + \frac{1}{3}\bar{\mu}\right)\nabla \operatorname{div} u\right] \tag{4.2}$$

when μ and ζ are constant.

One can write $\tilde{A}$ in divergence form

$$(\tilde{A}u)^r = \sum_s \tilde{A}_{rs}u^s = -\sum_{i,j,s} D_i(a_{ij}^{rs}D_j u^s) + \sum_{j,s} b_j^{rs}D_j u^s ,$$

where

$$a_{ij}^{rs} \equiv \frac{1}{\varrho_0}\mu_0\delta_{rs}\delta_{ij} + \frac{1}{\varrho_0}\mu_0\delta_{is}\delta_{rj} + \frac{1}{\varrho_0}\left(\zeta_0 - \frac{2}{3}\mu_0\right)\delta_{ri}\delta_{js} ,$$

$$b_j^{rs} \equiv D_j\left(\frac{1}{\varrho_0}\right)\mu_0\delta_{rs} + D_s\left(\frac{1}{\varrho_0}\right)\mu_0\delta_{rj} + D_r\left(\frac{1}{\varrho_0}\right)\left(\zeta_0 - \frac{2}{3}\mu_0\right)\delta_{js} .$$

Consider the bilinear form associated to $\tilde{A}$

$$a(u, v) \equiv \sum_{i,j,r,s}\int_\Omega a_{ij}^{rs}D_j u^s D_i\bar{v}^r\,dx + \sum_{j,s,r}\int_\Omega b_j^{rs}D_j u^s\bar{v}^r\,dx \tag{4.3}$$

and define

$$a_\lambda(u, v) \equiv a(u, v) + \lambda(u, v)_{L^2(\Omega)} . \tag{4.4}$$

The form $a_\lambda(u, v)$ is coercive in $H_0^1(\Omega)$, for $\lambda \in \boldsymbol{C}$ such that $\operatorname{Re}\lambda \geqslant \lambda_0$ large enough, and moreover $a_\lambda(u, v)$ is bounded in $H_0^1(\Omega)$ for each $\lambda \in \boldsymbol{C}$.

In fact, let $u \in H^2(\Omega) \cap H_0^1(\Omega)$; one obtains

$$\begin{aligned} a_\lambda(u, u) = \int_\Omega \left[\frac{\mu_0}{\varrho_0}|Du|^2 + \frac{\mu_0}{\varrho_0}\sum_{j,i}D_j u^i D_i\bar{u}^j + \frac{\zeta_0 - \frac{2}{3}\mu_0}{\varrho_0}|\operatorname{div} u|^2\right]dx + \\ + \sum_{j,s}\int_\Omega\left[D_j\left(\frac{1}{\varrho_0}\right)\mu_0 D_j u^s\bar{u}^s + D_s\left(\frac{1}{\varrho_0}\right)\mu_0 D_j u^s\bar{u}^j + \right. \\ \left. + D_s\left(\frac{1}{\varrho_0}\right)\left(\zeta_0 - \frac{2}{3}\mu_0\right)D_j u^j\bar{u}^s\right]dx + \lambda\int_\Omega |u|^2\,dx . \end{aligned}$$

Integrating by parts one has

$$\sum_{i,j}\int_\Omega \frac{\mu_0}{\varrho_0}D_j u^i D_i\bar{u}^j\,dx = \int_\Omega\left\{\frac{\mu_0}{\varrho_0}|\operatorname{div} u|^2 - \sum_{i,j}\left[D_j\left(\frac{\mu_0}{\varrho_0}\right)u^i D_i\bar{u}^j - D_i\left(\frac{\mu_0}{\varrho_0}\right)u^i D_j\bar{u}^j\right]\right\}dx ;$$

hence

$$\begin{aligned} |a_\lambda(u, u)| \geqslant \operatorname{Re} a_\lambda(u, u) \geqslant \frac{2\bar{\mu}_1}{\bar{\varrho}_2}\int_\Omega |Du|^2\,dx + \operatorname{Re}\lambda\int_\Omega |u|^2\,dx - \\ - c_*\int_\Omega\left[\left|D\left(\frac{\mu_0}{\varrho_0}\right)\right| + \left|D\left(\frac{1}{\varrho_0}\right)\right|\right]|Du|\,|u|\,dx \geqslant \frac{\bar{\mu}_1}{\bar{\varrho}_2}\int_\Omega |Du|^2\,dx + \\ + \left\{\operatorname{Re}\lambda - \frac{2\bar{\varrho}_2 c_*^2}{\bar{\mu}_1}\left[\left\|D\left(\frac{\mu_0}{\varrho_0}\right)\right\|^2_{L^\infty(\Omega)} + \left\|D\left(\frac{1}{\varrho_0}\right)\right\|^2_{L^\infty(\Omega)}\right]\right\}\int_\Omega |u|^2\,dx \geqslant \frac{\bar{\mu}_1}{\bar{\varrho}_2}\int_\Omega |Du|^2\,dx \end{aligned} \tag{4.5}$$

when λ is such that

$$\operatorname{Re}\lambda \geqslant \frac{2\bar{\varrho}_2 c_*^2}{\bar{\mu}_1}\left[\left\|D\left(\frac{\mu_0}{\varrho_0}\right)\right\|^2_{L^\infty(\Omega)} + \left\|D\left(\frac{1}{\varrho_0}\right)\right\|^2_{L^\infty(\Omega)}\right] \equiv \lambda_0 . \tag{4.6}$$

Observe that $\mu_0 \in H^3(\Omega)$ and $\varrho_0 \in H^3(\Omega)$, hence λ_0 is finite.

If $u \in H^1_0(\Omega)$, one obtains the same result by approximation.

Moreover, one easily verifies that

$$|a_\lambda(u, v)| \leqslant c_\lambda \|u\|_{H^1_0(\Omega)} \|v\|_{H^1_0(\Omega)} , \tag{4.7}$$

for each $u, v \in H^1_0(\Omega)$.

From Lax-Milgram's lemma, given $f \in L^2(\Omega)$, one finds a unique solution $u \in H^1_0(\Omega)$ of

$$a_\lambda(u, v) = (f, v)_{L^2(\Omega)} , \quad \forall v \in H^1_0(\Omega) .$$

Moreover, $\tilde{A}$ is a uniformly strongly elliptic system, and consequently by regularization one has that $u \in H^2(\Omega) \cap H^1_0(\Omega)$, and that $u \in H^4(\Omega) \cap H^1_0(\Omega)$ when $f \in H^2(\Omega)$. One has only to observe that the regularity $a^{rs}_{ij} \in H^3(\Omega)$, $b^{rs}_j \in H^2(\Omega)$ is sufficient to apply the standard methods (see for instance GIUSTI [3], chap. 3, § 3, 4, 5).

Setting

$$\mathcal{H} = L^2(\Omega) , \quad H = H^2(\Omega) ,$$

one has obtained in this way that

$$D_{\mathcal{H}}(\tilde{A}) = H^2(\Omega) \cap H^1_0(\Omega) , \quad D_H(\tilde{A}) = H^4(\Omega) \cap H^1_0(\Omega) . \tag{4.8}$$

Suppose now that $u \in D_{\mathcal{H}}(\tilde{A})$ is a solution of $(\tilde{A} + \lambda)u = f$; then

$$(\tilde{A} + \lambda)u = (\tilde{A} + \lambda_0)u + (\lambda - \lambda_0)u = f$$

i.e.

$$(\lambda - \lambda_0)u = f - (\tilde{A} + \lambda_0)u . \tag{4.9}$$

Taking the scalar product in $L^2(\Omega)$ of equation (4.9) with u, from (4.7) and (4.5) one has

$$\begin{aligned} |\lambda - \lambda_0| \|u\|_0^2 = |(f, u)_{L^2(\Omega)} - a_{\lambda_0}(u, u)| &\leqslant \|f\|_0 \|u\|_0 + |a_{\lambda_0}(u, u)| \leqslant \\ &\leqslant \|f\|_0 \|u\|_0 + c_{\lambda_0} \|u\|^2_{H^1_0(\Omega)} \leqslant \|f\|_0 \|u\|_0 + c_{\lambda_0} \frac{\bar{\varrho}_2}{\bar{\mu}_1} |a_\lambda(u, u)| \leqslant \\ &\leqslant \left(1 + c_{\lambda_0} \frac{\bar{\varrho}_2}{\bar{\mu}_1}\right) \|f\|_0 \|u\|_0 \quad \text{for each } \lambda \in \mathbf{C},\ \operatorname{Re}\lambda \geqslant \lambda_0 . \end{aligned}$$

Hence, by choosing the constant large enough, one finds

$$\|u\|_0 \leqslant \frac{c}{1+|\lambda|}\|(\tilde{A}+\lambda)u\|_0 \quad \forall \lambda \in \boldsymbol{C},\ \operatorname{Re}\lambda > \lambda_0 + 1\,. \tag{4.10}$$

Moreover, from the well-known elliptic a priori estimates and from (4.5), (4.10) one obtains

$$\begin{aligned}\|u\|_2 \leqslant c(\|\tilde{A}u\|_0 + \|u\|_{H_0^1(\Omega)}) \leqslant c[\|(\tilde{A}+\lambda)u\|_0 + (1+|\lambda|)\|u\|_0] \leqslant \\ \leqslant C_2\|(\tilde{A}+\lambda)u\|_0 \quad \forall \lambda \in \boldsymbol{C},\ \operatorname{Re}\lambda > \lambda_0 + 1\,,\end{aligned} \tag{4.11}$$

where C_2 doesn't depend on λ.

Finally, one observes that $\tilde{A}$ is a closed (unbounded) operator in $\mathcal{H}$ and in H, and that $\tilde{A} + \lambda$ is an isomorphism from $D_{\mathcal{H}}(\tilde{A})$ into $\mathcal{H}$ and from $D_H(\tilde{A})$ into H, for each $\lambda \in \boldsymbol{C}$ such that $\operatorname{Re}\lambda > \lambda_0 + 1$.

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