Some Fixed Point Theorems for Convex Contraction Mappings and Mappings with Convex Diminishing Diameters. - I (*).

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Summary. – *In this paper we consider several classes of mappings related to the class of contraction mappings by introducing a convexity condition with respect to the iterates of the mappings. Several fixed point theorems are proved for such mappings. Further, in a similar way we consider a related class of mappings satisfying a convexity condition with respect to diameters of bounded sets. In the last part we consider classes of mappings on PM-spaces (probabilistic metric spaces of K. Menger) and some fixed point theorems are given for such classes.*

0. – Introduction.

Let X be a complete metric space with the metric d. In the recent years a great number of papers present generalizations of the well known Banach-Picard contraction principle. Some of these generalizations refer to results containing the Schauder fixed point theorem. The purpose of the present paper is to consider a generalization of the contraction principle by introducing a « convexity » condition concerning the iterates of the mapping. We think that this condition may be adapted for other classes of mappings to obtain some extensions of known fixed point results.

1. – Convex contraction mappings of order 2.

Let X be a complete metric space with the metric d.

Definition 1.1. – A continuous mapping $f\colon X \to X$ is said to be a convex contraction of order 2 if there exist a, b in $(0, 1)$ such that for all x, y in X,

$$d\big(f^2(x), f^2(y)\big) \leqq ad\big(f(x), f(y)\big) + bd(x, y)$$

and where $a + b < 1$.

It is obviously that this class of mappings contains the class of contraction mappings. Concerning the fixed points of mappings which are convex contraction of order 2 we can prove the following.

Theorem 1.2. – *Any convex contraction mapping of order 2 has a fixed point which is unique.*

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PROOF. – Since the uniqueness is trivial we prove only the existence.

Let x_0 be an arbitrary but fixed point in X and consider the orbit of x_0 under f, i.e., the set $(f^n(x_0))_0^\infty$, $f^0(x_0) = x_0$. Set

$$k = \max\Big(d\big(x_0, f(x_0)\big), d\big(f(x_0), f^2(x_0)\big)\Big)$$

and thus for any m,

$$d\big(f^{2m+1}(x_0), f^{2m}(x_0)\big) \leqq ad\big(f^{2m}(x_0), f^{2m-1}(x_0)\big) + bd\big(f^{2m-1}(x_0), f^{2(m-1)}(x_0)\big) .$$

This implies the following relations:

$$d\big(f^3(x_0), f^2(x_0)\big) \leqq ad\big(f^2(x_0), f(x_0)\big) + bd\big(f(x_0), x_0\big) \leqq k(a+b) ,$$

$$d\big(f^4(x_0), f^3(x_0)\big) \leqq ad\big(f^3(x_0), f^2(x_0)\big) + bd\big(f^2(x_0), f(x_0)\big) \leqq \big(ak(a+b) + bk = k(a+b)\big) ,$$

$$d\big(f^5(x_0), f^4(x_0)\big) \leqq ad\big(f^4(x_0), f^3(x_0)\big) + bd\big(f^3(x_0), f^2(x_0)\big) \leqq$$
$$\leqq ak(a+b) + bk(a+b) = k(a+b)^2 .$$

. . . .

An induction argument shows that

$$d\big(f^{2m+1}(x_0), f^{2m}(x_0)\big) \leqq k(a+b)^m .$$

Since

$$d\big(f^{2m-1}(x_0), f^{2m}(x_0)\big) \leqq ad\big(f^{2(m-2)}(x_0), f^{2m-1}(x_0)\big) + bd\big(f^{2m-3}(x_0), f^{2(m-1)}(x_0)\big)$$

we get that

$$d\big(f^{2m-1}(x_0), f^{2m}(x_0)\big) \leqq k(a+b)^m .$$

Now since for $m < n$,

$$d\big(f^m(x_0), f^n(x_0)\big) \leqq d\big(f^m(x_0), f^{m+1}(x_0)\big) + \ldots + d\big(f^{n-1}(x_0), f^n(x_0)\big)$$

from the above estimates we obtain that $\big(f^n(x_0)\big)$ is a Cauchy sequence in X. Indeed, if $m = 21$ then

$$d\big(f^n(x_0), f^n(x_0)\big) \leqq k(a+b)^1 + k(a+b)^1 + k(a+b)^{1+1} + k(a+b)^{1+1} + \ldots \leqq$$
$$\leqq 2k(a+b)^1 \cdot 1/(1-a-b)$$

and a similar estimate holds if $m = 21 + 1$. Since $a + b$ is in $(0, 1)$ we obtain that

$(f^n(x_0))$ is a Cauchy sequence in X. Let $x^* = \lim f^n(x_0)$. Since $f(x^*) = \lim f^{n+1}(x_0)$ we get that $x^* = f(x^*)$.

We give now an example to show that there exist mappings which are convex contractions of order 2 and are not contractions.

EXAMPLE 1.3. – Let $X = [0, 1]$ with the usual metric and define $f\colon X \to X$ by the relation

$$f(x) = (x^2 + \tfrac{1}{2})/2$$

and it is obviously that this is continuous and not a contraction.

If x, y are arbitrary in X we have,

$$f^2(x) - f^2(y) = (x^2 + y^2)/4(f(x) - f(y)) + 1/8(x + y)(x - y)$$

and thus f is a convex contraction of order 2.

We consider now a more general class of mappings.

DEFINITION 1.4. – Let X be a complete metric space with the metric d. A continuous mapping $f\colon X \to X$ is said to be convex contraction of order n if there exists the positive constants $a_0, \ldots, a_{n-1}$ in $(0, 1)$ such that the following conditions hold:

1) $a_0 + \ldots + a_{n-1} < 1$;

2) for all x, y in X,

$$d(f^n(x), f^n(y)) \leqq a_0 d(x, y) + a_1 d(f(x), f(y)) + \ldots + a_{n-1} d(f^{n-1}(x), f^{n-1}(y)) .$$

The theorem 1.2 may be extended to this class of mappings and since the proof is essentially the same as for theorem 1.2, we mention it only.

THEOREM 1.5. – *If $f\colon X \to X$ is a convex contraction of order n then there exists a fixed point of f and this is unique.*

As is well known some results about contraction mappings were extended to larger classes of continuous mappings, among them we mention the class of contractive mappings. For the reader convenience, we recall the definition of this class of mappings. The mapping $S\colon X \to X$ is said to be contractive if for all x, y in X, $x \neq y$, $d(Sx, Sy) < d(x, y)$.

We consider now the corresponding class of mappings suggested by the class of convex contraction mappings of order 2.

DEFINITION 1.6. – A continuous mapping $f\colon X \to X$ is said to be convex contractive of order 2 if there exists the constants a_0 and a_1 in $(0, 1)$ such that the following conditions hold:

1) $a_0 + a_1 = 1$;

2) $d(f^2(x), f^2(y)) < a_0 d(x, y) + a_1 d(f(x), f(y))$

for all x, y in X.

A well known result of V. V. NEMYTSKI [10] says that if f is a contractive mapping defined on a compact metric space X then f has fixed points in X. The following result is an extension of Nemytski's result.

THEOREM 1.7. – *If $f\colon X \to X$ is a convex contractive mapping of order 2 and X is a compact metric space then there exists a unique point x^* in X such that $f(x^*) = x^*$.*

PROOF. – Let x_0 be an arbitrary point in X and consider the orbit of x_0 under f, i.e., the sequence $(x_n)_0^\infty$, $x_{n+1} = f(x_n)$, $n = 0, 1, 2, \ldots$. Since the space X is compact there exists a subsequence (nj) such that for some point x^* in X,

$$x^* = \lim f^{n_k}(x_0)\,.$$

The continuity of f implies that the following relations hold:

$$f(x^*) = \lim f^{n_k+1}(x_0)\,,$$

$$f^2(x^*) = \lim f^{n_k+2}(x_0)\,,$$

$$f^3(x^*) = \lim f^{n_k+3}(x_0)\,.$$

Let us consider the function defined on X by the formula

$$u(x) = \max\big(d(x, f(x)),\, d(f(x), f^2(x))\big)$$

which is clear a continuous function. Since f is convex contractive of order 2 we obtain that u is nonincreasing with respect to f, i.e.,

$$u(f(x)) \leqq u(x)\,.$$

Now the continuity of u and the above formulas for x^*, $f(x^*)$, $f^2(x^*)$ and $f^3(x^*)$ imply that $u(x^*) = u(f^2(x^*)) = u(f^3(x^*))$. Now if $u(x^*)$ is strictly positive then the property of f to be convex contractive mapping of order 2 implies that

$$u(x^*) = u(f^2(x^*)) < u(x^*)\,.$$

This is a contradiction and thus $u(x^*) = 0$ and x^* is a fixed point for f. It is obviously that x^* is the unique fixed point of f.

In a similar way we can prove the following result which extends a result of EDELSTEIN [2]:

THEOREM 1.8. – *Let $f: X \to X$ be a convex contractive mapping of order 2 and suppose that any orbit $(f^n(x))$, $x \in X$, has a limit point ξ. Then ξ is the unique fixed point of f.*

We consider now a class of mappings suggested by the class of convex contraction mappings and some localization conditions.

DEFINITION 1.9. – A continuous mapping $f: X \to X$ is called locally convex contraction of infinite order if there exists a sequence of positive numbers $(a_i)_0^\infty$ in $(0, 1)$, $a_0 + a_1 + a_2 + \ldots < 1$, and for each $x \in X$ there exists an integer $n = n(x)$ such that for all $y \in X$,

$$d(f^n(x), f^n(y)) \leqq a_0 d(x, y) + a_1 d(f(x), f(y)) + \ldots + a_n d(f^{n-1}(x), f^{n-1}(y)) .$$

REMARK 1.10. – If $a_1 = a_{i+1} = 0$, $i = 1, 2, 3, \ldots$, then this class coincides with the class considered by SEHGAL [12].

An easy modification of the proof in the SEHGAL paper [12] gives us the following result:

THEOREM 1.11. – *If $f: X \to X$ is a locally convex contraction of infinite order then there exists a unique point x^* in X such that $f(x^*) = x^*$.*

2. – Two-sided convex contraction mappings.

We consider now another class of mappings suggested by the class of mappings satisfying the following condition: the mapping S is defined on a complete metric space X and for some a, b in $(0, 1)$ the following conditions are satisfied:

1) $a + b < 1$;

2) for all x, y in X,

$$d(Sx, Sy) \leqq ad(x, Sx) + bd(y, Sy)$$

holds. We note that there exist a number of papers in which results about this class of mappings are presented.

Our class is considered in the following definition.

DEFINITION 2.1. – A continuous mapping $f: X \to X$, defined on a complete metric space X is said to be a two-sided convex contraction mapping if there exist a_1, a_2, b_1, b_2 in $(0, 1)$ such that the following inequalities hold:

1) $a_1 + a_2 + b_1 + b_2 < 1$;

2) $d(f^2(x), f^2(y)) \leqq a_1 d(x, f(x)) + a_2 d(f(x), f^2(x)) + b_1 d(y, f(y)) + b_2 d(f(y), f^2(y))$

holds for all $x \neq y$ in X.

A related class of mappings is considered in the following definition.

DEFINITION 2.2. – A continuous mapping $f: X \to X$ is said to be of convex type 2 if there exist the positive numbers $c_0, c_1, a_1, a_2, b_1, b_2$ such that the following inequality holds:

$$d(f^2(x), f^2(y)) = c_0 d(x, y) + c_1 d(f(x), f(y)) + a_1 d(x, f(x)) + a_2 d(f(x), f^2(y)) + \\ + b_1 d(y, f(y)) + b_2 d(f(y), f^2(y))$$

for all $x \neq y$ in X $(c_0 + c_1 + a_1 + a_2 + b_1 + b_2 < 1)$.

It is clear that the class of mappings considered in the Definition 2.2 is larger than the classes of mappings considered in the Definitions 1.1 and 2.1.

Concerning the problem of fixed points for mappings considered in the Definition 2.1 we have the following result.

THEOREM 2.3. – *For any two-sided convex contraction mapping there exists a unique fixed point.*

PROOF. – Let x_0 be an arbitrary point in X and consider the orbit of x_0 under f. Set

$$k = \max \big(d(x_0, f(x_0)), d(f(x_0), f^2(x_0))\big)$$

and then we get

$$d(f^2(x_0), f^3(x_0)) \leqq a_1 d(x_0, f(x_0)) + a_2 d(f(x_0), f^2(x_0)) + b_1 d(f^2(x_0), f(x_0)) + b_2 d(f^2(x_0), f^3(x_0))$$

and thus

$$d(f^2(x_0), f^3(x_0)) \leqq (a_1 + b_1 + a_2)/(1 - b_2) k .$$

Similarly we obtain the following relation,

$$d(f^3(x_0), f^4(x_0)) \leqq (a_1 + a_2 + b_1)/(1 - b_2) k$$

and an induction argument shows that for all m,

$$d(f^m(x_0), f^{m+1}(x_0)) \leqq \big((a_1 + a_2 + b_1)/(1 - b_2)\big)^{m-2} k .$$

From these estimates we obtain that $(f^n(x_0))$ is a Cauchy sequence.

Then it is clear that $x^* = \lim f^n(x_0)$ is a fixed point for f and clear is the unique point with this property.

In a similar way we can prove the following result about the fixed points for the mappings considered in the Definition 2.2.

THEOREM 2.4. – *If $f: X \to X$ is a mapping of convex type 2 then there exists a unique fixed point of f.*

We note that we can consider a class of mappings related to the class of generalized contractions. For the reader convenience, we recall that a mapping $f: X \to X$ defined on a metric space X is said to be generalized contraction if there exists a function α on X with walues in $[0, 1)$ such that for all x, y in X the following inequality holds:

$$d(f(x), f(y)) \leqq \alpha(x)\, d(x, y) .$$

The class we consider is introduced as follows.

DEFINITION 2.5. – A continuous mapping $f: X \to X$ is called of generalized convex type 2 if there exist the positive functions on X, $c_0(,)$, $c_1(,)$, $a_1(,)$, $a_2(,)$, $b_1(,)$, $b_2(,)$ such that the following inequalities hold:

1) $c_0(x) + c_1(x) + a_1(x) + a_2(x) + b_1(x) + b_2(x) < 1$;

2) $d(f^2(x), f^2(y)) \leqq c_0(x)\, d(x, y) + c_1(x)\, d(f(x), f(y)) + a_1(x)\, d(x, f(x)) +$
$+ a_2(x)\, d(f(x), f^2(x)) + b_1(x)\, d(y, f(y)) + b_2(x)\, d(f(y), f^2(y))$.

It is clear that for some appropriate functions c_i, a_i, b_i, the class of mappings considered in the Definition 2.5 reduces to the mappings considered above.

We close this section with some remarks about a result in [14] about the fixed points of certain self maps of an interval.

First we note that the arguments in the proof of Theorem 1 in [14] gives also the following result.

THEOREM 2.6. – *Let $f: [a, b] \to [a, b]$ with the property that a, b are in $f([a, b])$. Suppose that for some positive r, s, $r + s = 1$ the following inequality hold:*

$$|f(x) - f(y)| \leqq r|x - f(x)| + s|y - f(y)| .$$

Then the midpoint of $[a, b]$ is a fixed point of f.

Also, the same arguments (essentially the use of the fact that $[a, b]$ is a convex, closed and bounded subset in an uniformly convex space) permits us to give the following result.

THEOREM 2.7. – *Let C be a closed, convex and bounded subset in an uniformly convex Banach space X. Suppose that $f\colon C \to C$ is a continuous mapping satisfying the following conditions:*

1) *for some r, s in $[0, 1)$, $r + s = 1$, $\|f(x) - f(y)\| \leqq r\|x - f(x)\| + s\|y - f(y)\|$, x, y in C;*

2) $\partial C \subseteq f(C)$.

Then f has a fixed point in C.

We note that it is not difficult to see that any point of C which is the midpoint of a diametral segment of C (we call a segment $[x, y]$ x, y in C diametral if $\|x - y\| = = \operatorname{diam}(C)$) is a fixed point of f.

3. – Fixed point theorems for mapping with convex diminishing diameters.

Let X be a complete metric space with the metric d. For any subset M of X we consider the diameter $d(M)$ defined as $\sup\big(d(x, y), x, y \in M\big)$ and we say that a set in X is bounded if its diameter is finite.

If $f\colon X \to X$ is a contraction mapping then it is obvious that for any bounded set M in X, $d\big(f(M)\big) \leqq kd(M)$ where k is the contraction constant of f $\big($i.e., $d\big(f(x)$ $f(y)\big) \leqq kd(x, y)\big)$.

We consider now a class of mappings suggested by the class of convex contraction mappings of order n as well as of the class of mappings satisfying the above inequality with respect to diameters.

DEFINITION 3.1. – A mapping $f\colon X \to X$ is said to be with locally convex diminishing diameters of infinite order if there exists a sequence of positive numbers $(a_i)_0^\infty$, $\sum a_i < 1$, and for each bounded set M in X there exists an integer $n = n(M)$ such that the following inequality holds:

$$d\big(f^n(M)\big) \leqq a_0 d(M) + a_1 d\big(f(M)\big) + \ldots + a_{n-1} d\big(f^{n-1}(M)\big) .$$

If $a_1 = a_{i+1} = 0$, $i = 0, 1, 2, \ldots$, then we say that f is with locally power diminishing diameters.

LEMMA 3.2. – If g, $T\colon X \to X$ is a continuous mapping satisfying the property that there exists $k \in [0, 1)$ and for each $x \in X$ there exists an integer $n = n(x)$ with the property that for all $y \in X$, $d\big(f^n(x), f^n(y)\big) \leqq kd(x, y)$ then f is with locally power diminishing diameters.

We remark that the mappings with the property stated in the Lemma 3.2 were considered in Sehgal's paper [12].

PROOF. – Let us consider an integer m such that $2\cdot k^m < 1$. If M is a bounded set in X then pick a point x in M and consider the following sequence of integers:

$$n_1 = n(x),\ x_1 = f^n(x),\ n_2 = n(x_1),\ x_2 = f^{n_2}(x_1),\ \dots,$$
$$n_m = n(x_{m-1}),\ x_{m+1} = f^{n_{m+1}}(x_m),\ \dots .$$

Now if y, z are arbitrary points in M we have,

$$d\big(f^{n_1+\dots+n_m}(y), f^{n_1+\dots+n_m}(z)\big) \leqq d\big(f^{n_1+\dots+n_m}(x), f^{n_1+\dots+n_m}(y)\big) + $$
$$+ d\big(f^{n_1+\dots+n_m}(x), f^{n_1+\dots+n_m}(z)\big) \leqq kd\big(f^{n_1+\dots+n_{m-1}}(x), f^{n_1+\dots+n_{m-1}}(y)\big) +$$
$$+ kd\big(f^{n_1+\dots+n_{m-1}}(x), f^{n_1+\dots+n_{m-1}}(z)\big) \leqq \dots \leqq k^m d(x, y) + k^m d(x, z) \leqq 2\cdot k^m d(M)$$

and the assertion of the Lemma is proved.

For the fixed point theorem which follows we suppose that the metric space X is bounded.

THEOREM 3.3. – *Let $f\colon X \to X$ be a mapping with locally power diminishing diameters. Then there exists a unique point x^* in X such that $f(x^*) = x^*$.*

PROOF. – Since the uniqueness is obvious we prove only the existence. To this end we consider the following sequence of sets:

$$\bar{X} \supseteq \bar{f}(X) \supseteq \bar{f}^2(X) \supseteq \dots \supseteq \bar{f}^n(X) \supseteq \bar{f}^{n+1}(X) \supseteq \dots$$

and we remark that $\lim d\big(f^n(X)\big) = 0$. This clear implies that $\lim d\big(\bar{f}^n(X)\big) = 0$.

Indeed, we consider the sequence of sets:

$$X_1 = \bar{f}^{n_1}(X),\ n_1 = n(X),\ n_2 = n(X_1),\ X_2 = \bar{f}(^{n_2}X_1),\ \dots$$

and we get immediately that

$$d(X_p) \leqq k^p d(X)\ .$$

Thus $\lim d(X_p) = 0$ and this implies obviously that $\lim d\big(f^n(X)\big) = 0$.

In this case by the Cantor's theorem we get that $\cap\, \bar{f}^n(X) = (x^*)$.

We note also that from this relation we have that for any $x \in X$, $\lim f^n(x)$ exists and is x^*. Of course if f is supposed to be continuous then x^* is a fixed point of f. In the general case we consider the set

$$G = \big(x^*, f(x^*), f^2(x^*), \dots\big)$$

and from the above result G is closed. Obviously G is invariant for f and the image of G through f is exactly G. Now if G contains more than one point we obtain a contradiction since f is with locally power diminishing diameters.

Thus $G = (x^*)$ and this is the unique fixed point for f.

We prove now a fixed point theorem related to Nemytski's result.

THEOREM 3.4. – *Let X be a compact metric space with the metric d. Suppose that $f: X \to X$ is a continuous mapping such that for any invariant set M in X, M with the property that $d(M) \neq 0$, $d(f(M)) < d(M)$. Then f has a fixed point in X.*

PROOF. – We consider the family $\mathcal{F}$ of all closed subset of X which are invariant under f. We define a partial order on this family by the inclusion. It is easy to see that with this order we can apply the Zorn's Lemma. Thus we can consider a set F which is a minimal element and we show that F reduces to a point. Suppose that this is not so and we consider that sequence of closed sets $(\bar{f}^n(F))$ and we remark that it is decreasing. Set $M_F = \cap \bar{f}^n(F)$. Then M_F is nonempty, invariant under f and of course $M_F \subseteq F$. If F is not equal to M_F then this contradicts the definition of F. Thus we must have $F = M_F$.

If $d(F) \neq 0$ then $d(f(F)) < d(F)$ and then $\bar{f}(F)$ is a closed subset of X which is invariant under f and smaller than F. This is a contradiction with the definition of F. Thus $d(F) = 0$ and the theorem is proved.

COROLLARY 3.5. – *Let X be as above and $f: X \to X$ be a continuous mapping such that for some integer m, and for each subset of X with the diameter nonzero, $d(f^m(M)) <$ $< d(M)$. Then f has a unique fixed point in X.*

4. – Convex contraction mappings in generalized metric spaces.

First we recall the notion of generalized metric space. We consider for this the extended real line which consists of all points of the real line and two points denoted by $-\infty$ and 00 with the usual order relation $-\infty \leqq x \leqq \infty$.

DEFINITION 4.1. – A function $d: X \times X \to R_e$ (the extended real line) where X is an abstract set, is called a generalized metric if the following properties hold:

1) $d(x, y) = d(y, x)$;

2) $d(x, y) = 0$ if and only if $x = y$;

3) $d(x, z) \leqq d(x, y) + d(y, z)$ (if $d(x, y) = \infty$ or $d(y, z) = \infty$ then we consider $d(x, y) + d(y, z) = \infty$),

hold for all x, y, z in X.

A set X with a generalized metric is called, after Luxemburg, a generalized metric space. Of course we can define in this setting all notions known in the theory of metric spaces.

The following result presents a fixed point theorem for a class of mappings on generalized complete metric spaces.

THEOREM 4.2. – *Let (X, d) be a complete generalized metric space and $f\colon X \to X$ be a mapping satisfying the following properties:*

1) $d(f^2(x), f^2(y)) \leqq ad(x, y) + bd(f(x), f(y))$, $a + b < 1$, x, y in X;

2) *for any point $x \in X$ there exists an integer n such that $d(f^n(x), f^{n+1}(x))$ and $d(f^{n+1}(x), f^{n+2}(x))$ are less that ∞;*

3) *if $f(x) = x$, $f(y) = y$ then $d(x, y) < \infty$.*

Then there exists a unique point $x^* \in X$ such that $x^* = f(x^*)$.

PROOF. – Let x_0 be an arbitrary point in X and consider the orbit of x_0 under f. Then the property 2 assures the existence of an integer n such that $d(f^n(x_0), f^{n+1}(x_0)) < \infty$, $d(f^{n+1}(x_0), f^{n+2}(x_0)) < \infty$. Then according to the property 1 we get that

$$d(f^{n+2}(x_0), f^{n+3}(x_0)) = ad(f^n(x_0), f^{n+1}(x_0)) + bd(f^{n+1}(x_0), f^{n+2}(x_0)) \leqq k(a + b)$$

where

$$k = \max\big(d(f^n(x_0), f^{n+1}(x_0)), d(f^{n+1}(x_0), f^{n+2}(x_0))\big) .$$

Further we obtain that

$$d(f^{n+3}(x_0), f^{n+4}(x_0)) \leqq k(a + b)$$
$$d(f^{n+4}(x_0), f^{n+5}(x_0)) \leqq k(a + b)^2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$d(f^{n+2m}(x_0), f^{n+2m+1}) \leqq k(a + b)^m$$
$$d(f^{n+2m-1}(x_0), f^{n+2m}(x_0)) \leqq k(a + b)^m .$$

An induction argument shows that these are true for all integers and from these estimates we obtain that the sequence $(f^n(x_0))$ is in fact a Cauchy sequence. Then $x^* = \lim f^n(x_0)$ is a fixed point for f and the theorem is proved.

5. – Sequences of convex contraction mappings.

If $f: X \to X$ is a convex contraction mapping of order 2, i.e., a continuous mapping satisfying the relation

$$d(f^2(x), f^2(y)) \leqq ad(f(x), f(y)) + bd(x, y)$$

then we say that f is an (a, b)-convex contraction mapping.

THEOREM 5.1. – *Let $(f_n)_1^\infty$ be a sequence of (a, b)-convex contraction mappings defined on a complete metric space X with the metric d. If $(x_n)_1^\infty$ is the sequence of fixed point of (f_n), i.e., $x_n = f_n(x_n)$ and the sequence (f_n) converges pointwise to a mapping f which is continuous then* $\lim x_n$ *exists and is a fixed point for f.*

PROOF. – Since the sequence (f_n) converges pointwise to f we get that f is an (a, b)-convex contraction mapping. Let z be the fixed point of f. Then we have,

$$d(z, x_n) - d(z, f_n^2(z)) \leqq d(x_n, f_n^2(z))$$

and f_n being an (a, b)-convex contraction mapping,

$$d(x_n, f_n^2(z)) \leqq d(f_n^2(x_n), f_n^2(z)) = ad(x_n, f_n(z)) + bd(x_n, z)$$

and thus

$$d(x_n, z) = \big(d(z, f_n^2(z)) + ad(x_n, f_n(z))\big)/(1 - b) .$$

Further we get,

$$d(x_n, z) \leqq 1/(1 - b)\cdot\Big(d(z, f_n^2(z)) + a\big(d(x_n, z)\big) + d(z, f_n^2(z))\Big)$$

and then

$$d(x_n, z)\big(1 - a/(1 - b)\big) \leqq 1/(1 - b)\, d(z, f_n^2(z)) + a/(1 - b)\, d(z, f_n(z)) .$$

The pointwise converegence of the sequence (f_n) implies now that $\lim d(x_n, z) = 0$. This proves our assertion.

THEOREM 5.2. – *Let $(f_n)_1^\infty$ be a sequence of mappings converging uniformly to an (a, b)-convex contraction mapping f. If $(x_n)_1^\infty$ is a sequence of fixed points of the mappings in the sequence then* $\lim x_n$ *exists and is a fixed point of f.*

PROOF. – Let z be the fixed point of f. If x_n is a fixed point of f_n then we have

$$d(x_n, z) \leqq d(x_n, f^2(x_n)) + d(f^2(x_n), f^2(z)) \leqq d(x_n, f^2(x_n)) + ad(f(x_n), z) + bd(x_n, z)$$

and thus

$$d(x_n, z) \leqq 1/(1-b)\cdot\big(d(x_n, f^2(x_n))\big) + ad(f(x_n), z) \leqq 1/(1-b)\big(d(f\text{-}(x_n), f^2(x_n))\big) + \\ + a\big(d(f(x_n), f_n(x_n)) + bd(f_n(x_n), z)\big)$$

and from the uniform convergence and $f_n(z) \to z$, $f_n^2(z) \to z$, $\lim d(x_n, z) = 0$ and the assertion is proved.

6. – Mapping with diminishing probabilistic diameters on PM-spaces.

In 1942 K. MENGER initiated the study of probabilistic metric spaces [9]. A probabilistic metric space (briefly a PM-space) is a space in which the distance between two points is described by a probabilistic distribution.

Let I be the closed unit interval and Δ denote the set of all nondecreasing, left-continuous functions on R such that $F(0) = 0$ and the range is a subset of I. H will denote the function defined via

$$H(x) = \begin{cases} 0 & x \leqq 0\,, \\ 1 & x > 0\,. \end{cases}$$

A triangular norm (briefly a t-norm) is a function T mapping $I \times I$ into I which is associative, nondecreasing in each place and satisfies $T(a, 1) = a$ for each $a \in I$. A t-norm will be called l.c. t-norm if it is left continuous. Some t-norm of importance to us are:

$$T_m(a, b) = \text{maximum}\,(a + b - 1, 0)$$

$$\text{Prod}\,(a, b) = ab$$

$$\text{Min}\,(a, b) = \text{minimum}\,(a, b)\,.$$

In Δ we consider an order relation $F \leqq G$ if $F(x) \leqq G(x)$ for all $x \in R$ and $F < G$ if $F \leqq G$ and $F \neq G$. A triangle function (briefly a t-function) is a function mapping $\Delta \times \Delta$ into Δ which is associative, commutative, non-decreasing in each place and satisfies $\tau(F, H) = F$ for each $F \in \Delta$.

In what follows we shall assume that the t-functions satisfy the condition

$$(*) \qquad \sup\big(\tau(F, F),\, F < H\big) = H\,.$$

If T is an l.c. t-norm then the mapping τ defined via

$$\tau(F, G)(x) = \sup\big(T(F(ax), g(bx)), a + b = 1\big)$$

is an l.c. t-function.

A probabilistic metric space is an ordered pair $(S, \mathcal{F})$ where S is an abstract set and $\mathcal{F}$ is a mapping from $S \times S$ into R whose value $\mathcal{F}(p, q)$ at any pair (p, q) s idenoted by F_{pq} and the following conditions are satisfied:

1) $\lim\limits_{x\to\infty} F_{pq}(x) = 1$ for all p, q in S;

2) for all $p, q \in S$, $F_{pq} = H$ if and only if $p = q$;

3) $F_{pq} = F_{qp}$ for all $p, q \in S$,

and either for some t-norm T;

4_m) $F_{pr}(x + y) \geqq T\big(F_{pq}(x), F_{qr}(y)\big)$ for all $p \cdot q, r \in S$ and all $x, y \geqq 0$, or for some t-function τ;

4_s) $F_{pr} \geqq \tau(F_{pq}, F_{qr})$ for all $p \cdot q, r \in S$.

Suppose that $(S, \mathcal{F})$ satisfies the axiom 4_m) with the t-norm T be a continuous function. If A is a nonempty subset of S then the function

$$D_A(x) = \sup_{t<x}\Big(\inf_{p,q\in A} F_{pq}(t)\Big)$$

is called the probabilistic diameter of A. (It is easy to see that D_A is in Δ.)

A topology on a PM-space is defined as follows: if $p \in S$ then an (ε, λ)-neighbourhood is the set $U_p(\varepsilon, \lambda)$ defined via

$$U_p(\varepsilon, \lambda) = \big(q, q \in S, F_{pq}(\varepsilon) > 1 - \lambda\big)$$

and a sequence of points $(p_n) \subset S$ is a Cauchy sequence if $F_{p_n p_m} \to H$ (pointwise). A PM-space is said to be complete if every Cauchy sequence is convergent.

Now we consider on a PM-space in which 4_m) is satisfied a class of mappings containing as a special case the contraction mappings of SEHGAL [11]. We recall that a mapping $f\colon S \to S$ on a PM-space is said to be a contraction mapping if there exists $k \in (0, 1)$ such that for all p, q in S, $F_{f(p)f(q)}(x) \geqq F_{pq}(x/k)$.

DEFINITION 5.1. – Let $(S, \mathcal{F})$ be a PM-space satisfying 4_m) and T be a continuous mapping. A map $f\colon S \to S$ is said to be with diminishing probabilistic diameters if for each bounded set A in S,

$$D_{f(A)}(x) \geqq D_A(x/k)$$

for all $x \in R$ and where $k \in (0, 1)$.

We recall that a set M in a PM-space is said to be bounded if

$$\sup_x D_A(x) = 1$$

and semibounded if the above sup is in $(0, 1)$, unbounded if $D_A = 0$.

From just the definition of mappings in the Definition 5.1 it is clear that any contraction is in this class.

Concerning the fixed points for such mappings we have the following results.

THEOREM 5.2. – *If $f\colon S \to S$ is a mapping with diminishing diameters then it has at most one fixed point.*

PROOF. – Let $\operatorname{Fix} f = (p, p \in S, f(p) = p)$ and suppose that this set contains more than one point. Take p, q in $\operatorname{Fix} f$ and $p \neq q$. The set

$$M = (p, q)$$

is bounded in S and consider the diameter D_M. From the property of the mapping f and the fact that p, q are in $\operatorname{Fix} f$ we get that

$$D_{f(M)}(x) = D_M(x) \geqq D_M(x/k)$$

for some $k \in (0, 1)$. Since for each integer n we have $D_M(x) \geqq D_M(x/k^n)$ and $D_M = F_{pq}$ we get that $F_{pq} = H$. This implies that $p = q$ and thus $\operatorname{Fix} f$ contains at most one point.

To formulate the following result first we give the following definition.

DEFINITION 5.3. – Let $f\colon S \to S$ be a mapping with diminishing diameters and M_0 be any bounded invariant subset for f in S. Define the sequence of sets (M_n) via $M_n = f(M_{n-1})$, $n = 1, 2, 3, \ldots$ and set

$$G_{M_0}(x) = \inf_n \big(D_{M_n}(x)\big) .$$

THEOREM 5.4. – *If $(S, \mathcal{F})$ is a complete* PM-*space satisfying* $4_s)$ *under the triangle function τ_T satisfying* $(*)$ *then, either*

1) *f has a unique fixed point in S;*

2) *for every bounded invariant set M_0, $\sup_x G_{M_0}(x) \in (0, 1)$.*

PROOF. – Suppose that for some bounded invariant set $M_0 \subset S$, $\sup_x G_{M_0}(x) \in (0, 1)$ and consider the sequence of sets $M_n = f(M_{n-1})$ and then for every $x \in R$ we have

$$D_{M_n}(x) = D_{f(M_{n-1})}(x) \geqq D_{M_{n-1}}(x/k)$$

and then

$$D_{M_n}(x) \geqq D_{M_0}(x/k^{n-1}) .$$

Since D_A is a non-decreasing function, the above relation implies that

$$\lim D_{M_n}(x) = 1$$

for all $x \in R$. Now we remark that $M_0 \supseteq M_1 \supseteq M_2 \supseteq \ldots$ and this holds also in the closure with respect to the (ε, λ)-topology. Since for any bounded set $L \subset S$, the probabilistic diameter of L and Cl L are the same, we obtain that $\cap \bar{M}_n = (p)$. Now we show that p is a fixed point for f. Since $\cap \bar{M}_n = (p)$ it follows that for every $p_0 \in M_0$, $(f^n(p_0))$ converges in the (ε, λ)-topology to p. To prove that $p = f(p)$ we remark that the set $(p \cdot f(p), f^2(p), \ldots)$ is closed in the (ε, λ)-topology and invariant for f. Since f is with diminishing probabilistic diameters, this set reduces to p.

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