Periodic Solutions of Nonlinear Integral Equations (*).

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Summary. – *The existence of a continuous periodic solution of the system*

$$x(t) = f(t) + \int_{-\infty}^{t} q(t, s, x(s))\, ds ,$$

is studied using Horn's fixed point theorem as the basic tool. First it is assumed that the solutions are bounded in some sense and that they depend continuously on initial functions. Then the required boundedness of solutions are obtained for special cases of q. Also, a few sufficient conditions are provided to ensure the continuous dependence of solutions on initial functions.

1. – Introduction.

The purpose of this paper is to show the existence of a continuous periodic solution of the nonlinear integral system

$$x(t) = f(t) + \int_{-\infty}^{t} q(t, s, x(s))\, ds , \qquad t \in R = (-\infty, \infty) , \tag{1}$$

where x, f, and q are vectors in R^n. Throughout the paper we assume that the following assumptions hold:

($A1$) f is continuous in $t \in R$, $f(t + T) = f(t)$ for some $T > 0$;

($A2$) q is continuous in (t, s, x) for $-\infty < s \leqslant t < \infty$, $x \in R^n$, and $q(t, s, x) = 0$ if $s > t$;

($A3$) $q(t + T, s + T, x) = q(t, s, x)$;

($A4$) q satisfies a local Lipschitz condition in x; i.e., for each $N > 0$ and for each bounded set $\Omega \subset R^n$ there exists an $M > 0$ such that

$$|q(t, s, x_1) - q(t, s, x_2)| \leqslant M|x_1 - x_2|$$

whenever $-N \leqslant s \leqslant t \leqslant N$ and both x_1 and x_2 are in Ω;

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($A5$) There exists a continuous decreasing function $g: (-\infty, 0] \to [1, \infty)$, $g(0) = 1$, $g(r) \to \infty$ as $r \to -\infty$, such that for each continuous *initial function* $\varphi: (-\infty, 0] \to R^n$ satisfying $|\varphi(s)| \leqslant \gamma g(s)$ for some $\gamma > 0$,

$$\int_{-\infty}^{0} q(t, s, \varphi(s))\, ds\,,$$

is continuous for $t \geqslant 0$.

Assumptions ($A1$)-($A5$) ensure that for each continuous initial function φ, there exists a *unique* continuous solution $x(t, 0, \varphi) = x(t, \varphi)$ on an interval $[0, \alpha)$ for some $\alpha > 0$. The function $x(t, \varphi) = \varphi(t)$ on $(-\infty, 0]$. If the solution remains bounded then $\alpha = \infty$. These are well known results on existence, uniqueness and continuation of solutions of nonlinear Volterra integral equations (cf. MILLER [13]).

At first the postulation of g in ($A5$) seems severe. However, if (1) has a fading memory in the sense of BURTON [2, p. 282] then the existence of such a g is assured. Assuming (1) to have a fading memory is realistic because an equation representing a real world situation should remember its past, but the memory should fade with time.

The following lemma gives sufficient conditions for the continuity of

$$\int_{-\infty}^{0} q(t, s, \varphi(s))\, ds$$

stated in ($A5$). A proof of this lemma is available in BURTON [3].

LEMMA 1. – Suppose for each $\gamma > 0$ there exists a continuous function $Q_\gamma: [0, \infty) \to [0, \infty)$ such that $[\varphi: (-\infty, 0] \to R^n$, φ is continuous, $|\varphi(s)| \leqslant \gamma g(s)$, $t \geqslant 0]$ imply

(i) $$|q(t, s, \varphi(s))| \leqslant Q_\gamma(t - s)\,,$$

and

(ii) $$\int_{-\infty}^{t} Q_\gamma(t - s)\, ds < \infty\,.$$

Then for such φ, $\int_{-\infty}^{0} q(t, s, \varphi(s))\, ds$ is continuous for $t \geqslant 0$.

The following is an example which satisfies the conditions of Lemma 1.

EXAMPLE. – Let $q(t, s, \varphi(s)) = a(t - s) h(s, \varphi(s))$ where $|a(t)| \leqslant \beta(1 + t)^{-4}$ for $t \geqslant 0$ and $|h(t, x)| \leqslant \alpha x^2$ uniformly in t; a and h are continuous. Then for $|\varphi(s)| \leqslant \gamma g(s)$ we have $|q(t, s, \varphi(s))| \leqslant \alpha\beta\gamma^2(1 + t - s)^{-4} g^2(s)$. Let $g(s) = 1 + |s|$. Since $t \geqslant 0$, $s \leqslant 0$, one has $(1 + t - s)^{-2}(1 + |s|)^2 \leqslant 1$. Thus, $|q(t, s, \varphi(s))| \leqslant \alpha\beta\gamma^2(1 + t - s)^{-2}$.

We emphasize that g is selected by examining q of (1). Having selected g, let $(Y, |\cdot|_g) = Y$ be the Banach space of continuous functions $\varphi: (-\infty, 0] \to R^n$ for which

$$|\varphi|_g = \sup\{|\varphi(s)|/g(s): -\infty < s \leqslant 0\}$$

exists and is finite.

DEFINITION 1. – Solutions of (1) are *g-uniform bounded* if for each $B_1 > 0$ there exists a $B_2 > 0$ such that $[\varphi \in Y, |\varphi|_g \leqslant B_1, t \geqslant 0]$ imply $|x(t, \varphi)| < B_2$. Clearly, $B_2 \geqslant B_1$.

DEFINITION 2. – Solutions of (1) are *g-uniform ultimate bounded* for a bound $B > 0$ if for each $B_3 > 0$ there exists a $K > 0$ such that $[\varphi \in Y, |\varphi|_g \leqslant B_3, t \geqslant K]$ imply $|x(t, \varphi)| < B$.

DEFINITION 3. – Solutions of (1) *depend continuously on initial functions* in a set $U \subset Y$ if for each $\varepsilon > 0$ and $J > 0$ there exists a $\delta > 0$ such that $[\varphi, \psi \in U$ with $|\varphi - \psi|_g < \delta]$ imply $|P\varphi - P\psi|_g < \varepsilon$ where $P\varphi(t) = x(t + J, \varphi)$ for $-\infty < t \leqslant 0$.

HORN'S THEOREM [9]. – Let $C_0 \subset C_1 \subset C_2$ be *convex* subsets of a Banach space Z, with C_0 and C_2 *compact* and C_1 *open relative to* C_2. Let $P: C_2 \to Z$ be a continuous mapping such that, for some integer $m > 0$,

$$P^j(C_1) \subset C_2\,, \qquad 1 \leqslant j \leqslant m-1\,,$$

and

$$P^j(C_1) \subset C_0\,, \qquad m \leqslant j \leqslant 2m-1\,,$$

where P^j is the j-th iterate of P. Then P has a fixed point in C_0.

In [2, 4] Burton studied the existence of a periodic solution of a nonlinear *integrodifferential* system by using Horn's theorem as the basic tool. In the present paper we show that the existence of a periodic solution of a nonlinear *integral* system can be shown by using essentially the same techniques that have been used by Burton. The existence of a continuous periodic solution of (1) is shown in Theorem 1 where it is assumed that the solutions are *g-uniform bounded* and *g-uniform ultimate bounded* and that the solutions *depend continuously on initial functions*. We provide a few sufficient conditions to ensure these boundedness for special cases of q in Theorems 2 and 3. In Theorem 4 we show the continuous dependence of solutions on initial functions under suitable conditions.

Horn's theorem is also used by ARINO and HADDOCK [1] for periodic solutions of differential equations with infinite delay. LEITMAN and MIZEL [11] and MILLER and MICHEL [14] are two excellent papers on the periodic solutions of nonlinear integral equations. However, the equations, assumptions, and techniques of these papers are completely different from those of the present paper.

2. – Periodic solution.

LEMMA 2. – If $x(t)$ is a solution of (1) on $[0, \infty)$ then $x(t+T)$ is also a solution of (1) on $[0, \infty)$.

The proof of Lemma 2 is a simple calculation.

LEMMA 3. – Suppose the function g of ($A5$) also satisfies the following property: for each $H > 0$ there exists a $\mu > 0$ such that $g(s-H) \leqslant \mu g(s)$ on $(-\infty, 0]$. Then for $M > 0$, $L > 0$, and $H > 0$, the set

$$C = \{\varphi \in Y\colon |\varphi(s)| \leqslant M\sqrt{g(s-H)},\ |\varphi(u) - \varphi(v)| \leqslant L|u-v|\}$$

is compact and convex.

A proof of Lemma 3 is available in BURTON [2, page 170].

The proof of Theorem 1 is similar to the proof of Theorem 4.3.5 of BURTON [2] but there are some significant differences in assumptions. Therefore, we provide the proof of Theorem 1.

THEOREM 1. – Suppose ($A1$)-($A5$) hold and the function g of ($A5$) satisfies the condition of Lemma 3. Suppose also the following assumptions hold:

(a) Solutions of (1) are g-uniform bounded and g-uniform ultimate bounded;

(b) If U is any bounded subset of Y then solutions of (1) depend continuously on initial functions in U;

(c) For each $\alpha > 0$ and each $H > 0$ there exists an $L_1(\alpha, H) > 0$ such that if $\varphi\colon (-\infty, 0] \to R^n$, φ is continuous and $|\varphi(s)| \leqslant \alpha\sqrt{g(s-H)}$, then

$$\left|\int_{-\infty}^{0} q(u, s, \varphi(s))\, ds - \int_{-\infty}^{0} q(v, s, \varphi(s))\, ds\right| \leqslant L_1|u-v|$$

for $u, v \geqslant 0$;

(d) For each $\beta > 0$ there exists an $L_2(\beta) > 0$ such that if $y\colon [0, \infty) \to R^n$, y is continuous and $|y(s)| \leqslant \beta$, then

$$\left|\int_{0}^{u} q(u, s, y(s))\, ds - \int_{0}^{v} q(v, s, y(s))\, ds\right| \leqslant L_2|u-v|$$

for $u, v \geqslant 0$;

(e) There exists an $L_3 > 0$ such that

$$|f(u) - f(v)| \leqslant L_3|u-v|$$

for u, v in R.

Then (1) has a continuous T-periodic solution on R.

PROOF. – For $B>0$ of Def. 2 find $B_2>B$ and $K_1>0$ such that $|\varphi|_g\leqslant B$ implies $[|x(t,\varphi)|<B_2$ for $t\geqslant 0$ and $|x(t,\varphi)|<B$ for $t\geqslant K_1]$. For B_2 find $B_3>B_2$ and $K>0$ such that $|\varphi|_g\leqslant B_2$ implies $[|x(t,\varphi)|<B_3$ for $t\geqslant 0$ and $|x(t,\varphi)|<B$ for $t\geqslant K]$. Find $H>0$ with $B\sqrt{g(-H)}=B_3$. Consider $L_1(B,H)$ of (c), $L_2(B_3)$ of (d), and L_3 of (e). Let $L=L_1+L_2+L_3$. Let

$$C_0=\{\varphi\in Y\colon |\varphi(s)|\leqslant B\sqrt{g(s)},\ |\varphi(u)-\varphi(v)|\leqslant L|u-v|\},$$

$$C_2=\{\varphi\in Y\colon |\varphi(s)|\leqslant B\sqrt{g(s-H)},\ |\varphi(u)-\varphi(v)|\leqslant L|u-v|\},$$

$$C_1=\{\varphi\in Y\colon |\varphi|_g<B_2\}\cap C_2\,.$$

Using Lemma 3 one sees that C_0, C_1, and C_2 satisfy the necessary properties for Horn's theorem. For $\varphi\in C_2$ let $P\varphi=x(t+T,\varphi)$ for $-\infty<t\leqslant 0$. The map P is continuous by (b). It follows from Lemma 2 that $P^j\varphi=x(t+jT,\varphi)$. Indeed, by Lemma 2, both $x(t+T,\varphi)$ and $x(t,P\varphi)$ are solutions of (1) on $[0,\infty)$. Notice that both have the same initial function $P\varphi$. Thus, by the uniqueness of solutions we have: $x(t+T,\varphi)=x(t,P\varphi)$. Now, $P^2\varphi=P(P\varphi)=x(t+T,P\varphi)=x(t+2T,\varphi)$. Continuing this process one obtains $P^j\varphi=x(t+jT,\varphi)$. Let $m>0$ with $mT>K+H$. In Lemma 4 we show that $P^j(C_1)\subset C_2$ for all j, and $P^j(C_1)\subset C_0$ for $j\geqslant m$. Therefore by Horn's theorem there exists a φ in C_0 such that $P\varphi=\varphi$ which implies $x(t+T,\varphi)=x(t,\varphi)$ as required.

LEMMA 4. – Consider the sets C_0, C_1, C_2, the map P^j, and the number m defined in the proof of Theorem 1. Then $P^j(C_1)\subset C_2$ for all j, and $P^j(C_1)\subset C_0$ for $j\geqslant m$.

PROOF. – First we show that for $\varphi\in C_1$

$$|P^j\varphi(u)-P^j\varphi(v)|\leqslant L|u-v|$$

for $u,v\in(-\infty,0]$.

Since $P^j\varphi(t)=x(t+jT,\varphi)$,

$$|P^j\varphi(u)-P^j\varphi(v)|=|x(u+jT,\varphi)-x(v+jT,\varphi)|\,.$$

Suppose $u+jT\leqslant 0$ and $v+jT\leqslant 0$. Then

$$|x(u+jT,\varphi)-x(v+jT,\varphi)|=|\varphi(u+jT)-\varphi(v+jT)|\leqslant L|u-v|$$

by definition.

Suppose $u + jT > 0$ and $v + jT > 0$. Then

$$\begin{aligned}|x(u + jT, \varphi) - x(v + jT, \varphi)| &\leqslant |f(u + jT) - f(v + jT)| \\ &\quad + \left|\int_{-\infty}^{0} q(u + jT, s, \varphi(s))\, ds - \int_{-\infty}^{0} q(v + jT, s, \varphi(s))\, ds\right| \\ &\quad + \left|\int_{0}^{u+jT} q(u + jT, s, x(s, \varphi))\, ds - \int_{0}^{v+jT} q(v + jT, s, x(s, \varphi))\, ds\right. \\ &\leqslant L_3|u - v| + L_1|u - v| + L_2|u - v| = L|u - v| .\end{aligned}$$

Suppose $u + jT > 0$ and $v + jT < 0$. Then

$$\begin{aligned}|x(u + jT, \varphi) - x(v + jT, \varphi)| &= |x(u + jT, \varphi) - \varphi(v + jT)| \\ &\leqslant |x(u + jT, \varphi) - x(0, \varphi)| + |\varphi(0) - \varphi(v + jT)| \\ &\leqslant L(u + jT) + L(- v - jT) = Lu - Lv = L|u - v| .\end{aligned}$$

This proves that

$$|P^j\varphi(u) - P^j\varphi(v)| \leqslant L|u - v|$$

for all $u, v \in (-\infty, 0]$.

To show that $P^j(C_1) \subset C_2$ for all j, it now remains to show that

$$|P^j\varphi(s)| = |x(s + jT, \varphi)| \leqslant B\sqrt{g(s - H)}$$

for all j, and $-jT \leqslant s \leqslant 0$, i.e.,

$|x(t, \varphi)| \leqslant B\sqrt{g(t - jT - H)}$ for all j, and $0 \leqslant t \leqslant jT$. We know that $|x(t, \varphi)| < B_3$ for $0 \leqslant t \leqslant jT$ for any j. Also, for $0 \leqslant t \leqslant jT$ we have $-jT - H \leqslant t - jT - H \leqslant -H$. So, $B\sqrt{g(t - jT - H)} \geqslant B\sqrt{g(-H)} = B_3$. Thus, we have

$$|x(t, \varphi)| < B\sqrt{g(t - jT - H)} \quad \text{for all } j\,, \quad \text{and} \quad 0 \leqslant t \leqslant jT\,.$$

To show that $P^j(C_1) \subset C_0$ for $j \geqslant m$, we need to show that

$$|P^j\varphi(s)| = |x(s + jT, \varphi)| \leqslant B\sqrt{g(s)} \quad \text{for } -\infty < s \leqslant 0\,,$$

i.e.,

$$|x(t, \varphi)| \leqslant B\sqrt{g(t - jT)} \quad \text{for } -\infty < t \leqslant jT\,.$$

For $t \leqslant 0$ we know that $|x(t, \varphi)| = |\varphi(t)| \leqslant B\sqrt{g(t - H)}$. For $j \geqslant m$ we have $jT > H$ So, $t - jT < t - H$. This implies that $B\sqrt{g(t - jT)} > B\sqrt{g(t - H)}$. Thus,

$$|x(t, \varphi)| < B\sqrt{g(t - jT)} \quad \text{for } t \leqslant 0\,.$$

For $0 < t < K$ we know that $|x(t, \varphi)| < B_3$. Also, for $0 < t < K$ we have $-jT < t - jT < K - jT < -H$ if $j \geqslant m$. So, $B\sqrt{g(t - jT)} > B\sqrt{g(-H)} = B_3$. Thus,

$$|x(t, \varphi)| < B\sqrt{g(t - jT)} \quad \text{for } 0 < t < K .$$

Finally, for $K \leqslant t \leqslant jT$ we know that $|x(t, \varphi)| < B$. Also, for $K \leqslant t \leqslant jT$ we have $K - jT \leqslant t - jT \leqslant 0$. So, $B\sqrt{g(t - jT)} \geqslant B\sqrt{g(0)} = B$. Thus,

$$|x(t, \varphi)| < B\sqrt{g(t - jT)} \quad \text{for } K \leqslant t \leqslant jT .$$

This completes the proof of Lemma 4.

3. – g-uniform boundedness and g-uniform ultimate boundedness of solutions. Continuous dependence of solutions on initial functions.

LEMMA 5. – Let $z(t)$ and $\alpha(t)$ be continuous in $t \in R^+ = [0, \infty)$. Suppose $z(t) \in L^1(R^+)$ and $\alpha(t) \to 0$ as $t \to \infty$. Then

$$m(t) = \int_0^t z(t - s)\alpha(s)\, ds \tag{2}$$

tends to zero as $t \to \infty$.

PROOF. – From (2) we have for $t \geqslant \tau > 0$

$$|m(t)| \leqslant \int_0^\tau |z(s)||\alpha(t - s)|\, ds + \int_\tau^t |z(s)||\alpha(t - s)|\, ds .$$

If $A(\sigma) = \sup\{|\alpha(s)|: s \geqslant \sigma\}$ then

$$|m(t)| \leqslant A(t - \tau)\int_0^\infty |z(s)|\, ds + A(0)\int_\tau^\infty |z(s)|\, ds .$$

Since $A(\sigma) \to 0$ as $\sigma \to \infty$ we get

$$\limsup_{t \to \infty} |m(t)| \leqslant A(0)\int_\tau^\infty |z(s)|\, ds .$$

This is true for all $\tau > 0$ and the right hand side tends to zero as $\tau \to \infty$. This completes the proof of Lemma 5.

LEMMA 6. – Let $a(t) \in C(R^+) \cap L^1(R^+)$. Suppose there exists a continuous decreasing function $p\colon (-\infty, 0] \to [1, \infty)$, $p(0) = 1$, $p(r) \to \infty$ as $r \to -\infty$ with

$$\int_0^\infty |a(s)| p(-s)\, ds \leqslant M$$

for some $M > 0$. Then $w(t) = \int_{-\infty}^0 |a(t-s)| p(s)\, ds \leqslant M$ for $t \in R^+$, and $w(t) \to 0$ as $t \to \infty$.

The proof of Lemma 6 is trivial.

In the next theorem we assume that the resolvent of the kernel a of the linear Volterra integral equation

$$x(t) = f(t) + \int_0^t a(t-s) x(s)\, ds$$

is integrable. Results on the integrability of the resolvents can be found in GRIPENBERG [5, 6, 7], GROSSMAN [8], MILLER [12], and PALEY and WIENER [15].

THEOREM 2. – Let $q(t, s, x) = a(t-s)x(s)$ where $a(t) \in C(R^+) \cap L^1(R^+)$. Suppose there exists an $M > 0$ such that $\int_0^\infty |a(s)| g(-s)\, ds \leqslant M$. Suppose also that the following conditions hold:

(i) $b(t)$, the resolvent of $a(t)$, is of class $L^1(R^+)$;

(ii) For each $\gamma > 0$ and for each continuous initial function φ with $|\varphi(s)| \leqslant \gamma g(s)$, $\int_{-\infty}^0 a(t-s)\varphi(s)\, ds$ is continuous for $t \geqslant 0$.

Then solutions of

$$x(t) = f(t) + \int_{-\infty}^t a(t-s) x(s)\, ds \tag{3}$$

are g-uniform bounded and g-uniform ultimate bounded.

PROOF. – Consider a continuous $\varphi\colon (-\infty, 0] \to R^n$ with $|\varphi(s)| \leqslant \gamma g(s)$ for some $\gamma > 0$. By (ii), (3) has a unique continuous solution $x(t, \varphi)$ for $t \geqslant 0$. Since $b(t)$ is the resolvent of $a(t)$, the solution $x(t, \varphi)$ can be obtained by

$$x(t, \varphi) = f(t) + \mu(t) - \int_0^t b(t-s)\{f(s) + \mu(s)\}\, ds \tag{4}$$

where

$$\mu(t) = \int_{-\infty}^0 a(t-s)\varphi(s)\, ds\,.$$

Let $|f(t)|\leqslant N$ for $t\in R$, and $\int_0^\infty |b(t)|\,dt\leqslant L$. Then using Lemma 6 in (4) we get

$$|x(t,\varphi)|\leqslant N + |\mu(t)| + NL + \Big|\int_0^t b(t-s)\mu(s)\,ds\Big|\leqslant N + \gamma M + NL + \gamma ML$$

for $t\geqslant 0$. This proves that $x(t,\varphi)$ is g-uniform bounded.

The function $\mu(t)$ is continuous by (ii), and $\mu(t)\to 0$ as $t\to\infty$ by Lemma 6. Since $a(t)$ is continuous, $b(t)$ is continuous for $t\geqslant 0$ (cf. MILLER [13, p. 202]). Also, $b(t)\in L^1(R^+)$ by (i). So, by Lemma 5, $\int_0^t b(t-s)\mu(s)\,ds\to 0$ as $t\to\infty$. Thus, we can choose the ultimate bound $B = N(1+L)+1$.

Equation (3) is linear and of *convolution* type. Results on the existence and uniqueness of periodic solutions of linear integral equations of *nonconvolution* type are available in ISLAM [10].

THEOREM 3. – Let $q(t,s,x) = a(t-s)h(s,x)$. Let the functions a and g satisfy the properties stated in Theorem 2. Suppose (ii) of Theorem 2 and the following conditions hold:

(i) h is continuous in (t,x) for $t\in R$, $x\in R^n$; $h(t,x)$ satisfies a local Lipschitz condition in x, and $|h(t,x)|\leqslant\alpha|x|$ uniformly in t for some $\alpha>0$;

(ii) There exists a $\beta>0$ such that

$$\int_0^\infty |a(t)|\,dt\leqslant\beta\,, \qquad \text{and} \qquad \alpha\beta<1\,.$$

Then solutions of

$$x(t) = f(t) + \int_{-\infty}^t a(t-s)h\big(s,x(s)\big)\,ds \tag{5}$$

are g-uniform bounded and g-uniform ultimate bounded.

PROOF. – Consider an initial function φ with $|\varphi(s)|\leqslant\gamma g(s)$. Since h is continuous, it follows from (ii) of Theorem 2 that $\int_{-\infty}^0 a(t-s)h\big(s,\varphi(s)\big)\,ds$ is continuous for $t\geqslant 0$. Therefore, (5) has a unique continuous solution $x(t,\varphi)$ on $[0,\tau)$ for some $\tau>0$. We show that $x(t,\varphi)$ remains bounded on $[0,\tau)$, and the bound is independent of τ. This will essentially imply that the solution $x(t,\varphi)$ is defined on R^+ and $x(t,\varphi)$ is g-uniform bounded.

Let $|f(t)|\leqslant N$ for $t\geqslant 0$. From (5) we have for $0\leqslant t<\tau$

$$|x(t,\varphi)|\leqslant N + \alpha\gamma w(t) + \alpha\int_0^t |a(t-s)||x(s,\varphi)|\,ds \tag{6}$$

where

$$w(t) = \int_{-\infty}^{0} |a(t-s)| g(s)\, ds .$$

Let $v(t) = \{\sup |x(s, \varphi)| \colon 0 \leqslant s \leqslant t,\ t < \tau\}$. Now, from (6) and Lemma 6 we obtain

$$|x(t, \varphi)| \leqslant N + \alpha\gamma M + \alpha\beta v(t) , \qquad 0 \leqslant t < \tau .$$

Since $v(t)$ is nondecreasing, one readily sees that

$$|x(t, \varphi)| \leqslant v(t) \leqslant \frac{N + \alpha\gamma M}{1 - \alpha\beta} .$$

This proves that $x(t, \varphi)$ is g-uniform bounded.

Let $V(t) = \{\sup |x(s, \varphi)| \colon s \geqslant t,\ t \in R^+\}$, and $V_0 = \lim_{t\to\infty} V(t)$. Note that $V(t)$ is non-increasing and $|x(t, \varphi)| \leqslant V(t)$. By Lemma 5 and (ii),

$$\lim_{t\to\infty} \int_0^t |a(t-s)| V(s)\, ds = V_0 \lim_{t\to\infty} \int_0^t |a(t-s)|\, ds \leqslant \beta V_0 .$$

Now, since $w(t) \to 0$ as $t \to \infty$, a few calculations on (6) yield

$$\limsup_{t\to\infty} |x(t, \varphi)| \leqslant \frac{N}{1 - \alpha\beta} .$$

Thus, we can choose the ultimate bound $B = N/(1 - \alpha\beta) + 1$.

THEOREM 4. – Let solutions of (1) be g-uniform bounded. Let U be any bounded subset of Y. In addition to $(A4)$, suppose q also satisfies the following: for a given $\varepsilon > 0$ and $L > 0$ there exists a $D > 0$ such that $[\varphi \in U,\ t \in [0, L]]$ imply

$$\int_{-\infty}^{-D} |q(t, s, \varphi(s))|\, ds < \varepsilon .$$

Then solutions of (1) depend continuously on initial functions in U.

PROOF. – Let $J > 0$ and $\varepsilon > 0$ be given and let $\varphi \in U$. It is sufficient to find a $\delta > 0$ such that $[\psi \in U,\ |\varphi - \psi|_g < \delta]$ imply $|x(t, \varphi) - x(t, \psi)| < \varepsilon$ for $0 \leqslant t \leqslant J$.

Let $\beta > 0$ be such that $[\varphi \in U,\ t \geqslant 0]$ imply $|x(t, \varphi)| < \beta$. For $-J \leqslant s \leqslant t \leqslant J$ and for Ω bounded by β, find $M = M_1$ of $(A4)$. Choose $\varepsilon_1 > 0$ such that $2\varepsilon_1 \exp[M_1 J] < \varepsilon/2$. For this ε_1 and for J, find $D \geqslant J$ of the hypothesis. Let $\alpha = \max\{|\varphi(s)| \colon -D \leqslant s \leqslant 0\}$. For $-D \leqslant s \leqslant t \leqslant D$ and for Ω bounded by α, find $M = M_2$ of $(A4)$.

Let $\psi \in U$ and $|\varphi - \psi|_g < \delta$ so that $k\delta M_2 D \exp[M_1 J] < \varepsilon/2$ where $k = \{\max g(s) \colon -D \leqslant s \leqslant 0\}$. Let $x_1(t) = x(t, \varphi)$ and $x_2(t) = x(t, \psi)$. Then from (1) we obtain for

$0 \leqslant t \leqslant J$

$$|x_1(t) - x_2(t)| \leqslant 2\varepsilon_1 + \int_{-D}^{0} M_2 |\varphi(s) - \psi(s)|\, ds + \int_{0}^{t} M_1 |x_1(s) - x_2(s)|\, ds \leqslant 2\varepsilon_1 + M_2 Dk\delta + \int_{0}^{t} M_1 |x_1(s) - x_2(s)|\, ds .$$

By Gronwall's inequality, we get

$$|x_1(t) - x_2(t)| \leqslant (2\varepsilon_1 + M_2 Dk\delta) \exp [M_1 J] < \varepsilon .$$

This completes the proof of Theorem 4.

REFERENCES

[1] O. ARINO - J. HADDOCK, *Estimates for periodic solutions of differential equations with infinite delay*, Notices Amer. Math. Soc., (29) **5** (1984), p. 57.

[2] T. A. BURTON, *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Academic Press, New York, 1985.

[3] T. A. BURTON, *Periodic solutions of integrodifferential equations*, J. London Math. Soc., (2) **31** (1985), pp. 537-548.

[4] T. A. BURTON, *Toward unification of periodic theory*, in *Differential Equations: Qualitative Theory*, pp. 127-141 (B. SZ.-NAGY and L. HATVANI (eds)), North-Holland (1987).

[5] G. GRIPENBERG, *On positive, nonincreasing resolvents of Volterra equations*, J. Differential Equations, **30** (1978), pp. 380-390.

[6] G. GRIPENBERG, *On the resolvents of Volterra equations with nonincreasing kernels*, J. Math. Anal. Appl., **76** (1980), pp. 134-145.

[7] G. GRIPENBERG, *On the asymptotic behavior of resolvents of Volterra equations*, SIAM J. Math. Anal., **11** (1980), pp. 654-662.

[8] S. I. GROSSMAN, *Integrability of resolvents of certain Volterra integral equations*, J. Math. Anal. Appl., **48** (1974), pp. 785-793.

[9] W. A. HORN, *Some fixed point theorems for compact maps and flows in Banach spaces*, Trans. Amer. Math. Soc., **149** (1970), pp. 391-404.

[10] M. N. ISLAM, *Periodic solutions of Volterra integral equations*, Internat. J. Math. Math. Sci., to appear.

[11] M. J. LEITMAN - V. J. MIZEL, *Asymptotic stability and the periodic solutions of*

$$x(t) + \int_{-\infty}^{t} a(t-s)\, g(s, x(s))\, ds = f(t) ,$$

J. Math. Anal. Appl., **66** (1978), pp. 606-625.

[12] R. K. MILLER, *On Volterra integral equations with non-negative integrable resolvents*, J. Math. Anal. Appl., **22** (1968), pp. 319-340.

[13] R. K. MILLER, *Nonlinear Volterra Integral Equations*, Benjamin, Melno Park, California, 1971.

[14] R. K. MILLER - A. N. MICHEL, *On the response of nonlinear multivariable interconnected feedback systems to periodic input signals*, IEEE Trans. Circuits. Syst., CAS-**27** (1980), pp. 1088-1097.

[15] R. E. A. C. PALEY - N. WIENER, *Fourier transforms in the complex domain*, Amer. Math. Soc., Colloquium Publications, 1934.