Structural Assignment of Neumann Boundary Feedback Parabolic Equations: the unbounded Case in the Feedback Loop (*) (**).

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Summary. – *A parabolic equation defined on a bounded domain is considered, with input acting in* the Neumann (or mixed) boundary conditions, *and expressed as a* specified feedback *of the solution x of the form*: $\langle \gamma x, w\rangle g$, *where* $w \in L_2(\Omega)$, $g \in L_2(\Gamma)$ *and* γ *is a continuous operator for* $\sigma < \frac{3}{4}$: $H^{2\sigma}(\Omega) \to L_2(\Omega)$. *The* free *system is assumed unstable. In this case, the boundary feedback stabilization problem (in space dimension larger or equal to two) follows from an essentially more general result recently established by the authors in* [L8]: *under algebraic (full rank), verifiable conditions at the unstable eigenvalues, one can select* boundary *vectors, so that the corresponding feedback solutions decay* in the uniform operator norm *exponentially at* $t \to \infty$. *Here, this stabilization peoblem is pushed further and made more precise, under the additional assumption that the original* free *system be self-adjoint*: *we show, in fact, that one can further restrict the* boundary *vectors, so that the corresponding feedback solutions have the following more precise desirable* structural property (*the same enjoyed by* free stable *systems*): *they can be expressed as an infinite linear combination of decaying exponentials. A semigroup approach is employed. Since* structure *of feedback solutions is sought, the analysis here is much more technical and vastly different from* [L8], *where only* norm upper bound *was the goal.*

1. – Introduction and statement of main result.

Let Ω be a bounded open domain in R^ν with boundary Γ, assumed to be an $(\nu-1)$-dimensional variety with Ω locally on one side of Γ (¹). Let $A(\xi, \partial)$ be a uniformly strongly elliptic operator of order two in Ω of the form $A(\xi, \partial) = \sum_{|\alpha|\leqslant 2} a_\alpha(\xi)\,\partial^\alpha$, with smooth real coefficients a_α, where the symbol ∂ denotes differentiation. We begin by considering a diffusion open-loop system based on Ω with input applied on Γ through mixed (elastic) boundary conditions; that is

$$(1.1)\qquad \frac{\partial x}{\partial t}(t, \xi) = -A(\xi, \partial)\, x(t, \xi) \qquad \text{in } (0, T]\times\Omega$$

$$(1.2)\qquad x(0, \xi) = x_0(\xi)\,, \qquad \xi \in \Omega$$

$$(1.3)\qquad \frac{\partial x(t, \zeta)}{\partial \eta} + b(\zeta)\, x(t, \zeta) = f(t, \zeta) \qquad \text{in } (0, T]\times\Gamma \ (\textit{Mixed B.C.})\,.$$

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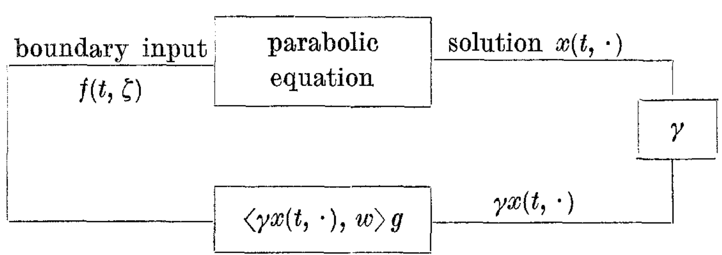
(¹) Assumptions on Γ will be imposed as needed; see the statement of Theorem 1.2.

Here, $f(t, \zeta)$ is the input function or control function (or forcing term), defined on $(0, T]\times\Gamma$, which influences the solution $x(t, \xi)$. In (1.3), $b(\cdot)$ is also a real function defined on Γ and $\partial/\partial\eta$ is the (outward) normal derivative. The Neumann case is obtained when $b \equiv 0$. It is known [F1] that: the operator A, consisting of $-A(\xi, \partial)$ with zero boundary conditions, *generates an analytic* semigroup on $L_2(\Omega)$, which we shall denote by the convenient notation $\exp[At]$, $t\geqslant 0$.

The Boundary Feedback Closed-Loop System. We now demand that the input function $f(t, \zeta)$ be expressed in a feedback form as a linear operator (of finite dimensional range) acting, in particular, as a gradient operator of the solution vector $x(t, \xi)$; that is, if γ denotes *any* continuous operator: $H^{2\sigma}(\Omega) \to L_2(\Omega)$, for *any* fixed $2\sigma < \frac{3}{2}$, we demand in this paper that the feedback operator be a continuous operator from $H^{2\sigma}(\Omega)$ into a J-dimensional subspace of $L_2(\Omega)$ of the form:

$$f(t, \zeta) = \sum_{j=1}^{J} \langle(\gamma x)(t, \cdot), w_j(\cdot)\rangle g_j(\zeta) \quad \text{on } (0, T]\times\Gamma. \tag{1.4}$$

Here, w_j and g_j are fixed vectors in $L_2(\Omega)$ and $L_2(\Gamma)$ respectively, and the symbol $\langle\cdot, \cdot\rangle$ denotes the inner product in $L_2(\Omega)$. The vectors $\{g_j\}_{j=1}^{J}$ are assumed to be linearly independent. For $J = 1$, we write w and g instead of w_1 and g_1. The special situation when γ is the gradient operator, i.e. $\gamma h = \nabla h$ is covered when $2\sigma = \frac{1}{2}$.



The Feedback System.

The following result may be proved as in [L3]; here Γ may have finitely many conical points [K3].

THEOREM 1.1 [L3]. – The feedback closed-loop solutions $x(t, x_0)$ of (1.1)-(1.4), can be expressed simply as $x(t, x_0) = S_F(t)x_0$, $x_0 \in L_2(\Omega)$, $t\geqslant 0$, where $S_F(t)$ defines a (feedback) C_0-semigroup which is analytic and compact on $L_2(\Omega)$ for $t > 0$, and whose generator A_F has compact resolvent on $L_2(\Omega)$. □

Theorem 1.1 states that the (well known) properties of the *open loop* (*free*) system, i.e., with $f(t, \zeta) \equiv 0$ are preserved by the *closed-loop* system.

REMARK 1.1. – The proof of [L3] actually shows that the feedback semigroup $S_F(t)$ has the same properties listed above on all spaces $H^{\frac{3}{2}-\varepsilon}(\Omega)$, $0 < \varepsilon \leqslant \frac{3}{2}$. The structural assignment result of the present paper is topologically consistent with the described regularity of feedback solutions. We also refer to [T3] for regularity results, obtained by different techniques, that complement, and neither fully imply, nor are fully implied by, Theorem 1.1. □

Since Ω is a bounded domain, the resolvent operator $R(\lambda, A)$ is compact [D2, p. 1740]. Hence the spectrum $\sigma(A)$ of A is only point spectrum and consists of a sequence of isolated distinct eigenvalues $\{\lambda_k\}$, $k = 1, 2, \ldots$, $|\lambda_k| \to \infty$, with corresponding normalized linearly independent eigenvectors $\{\Phi_{km}\}$, $k = 1, 2, \ldots, \mathscr{M}_k$ ($\mathscr{M}_k$ being the geometric multiplicity of λ_k). As is well known, since $\exp[At]$ is analytic, the $\{\lambda_k\}$ are contained in a triangular sector delimited by the rays: $a + \varrho \exp[\pm i\theta]$, $0 \leqslant \varrho < \infty$, $\pi/2 < \theta < \pi$, a real, with no finite accumulation point. Thus, at the right of any vertical line in the complex plane there are at most finitely many of them. Our standing *assumption*—for the problem considered in this paper to be significant—is that: there are $(K-1)$ eigenvalues $\lambda_1, \ldots, \lambda_{K-1}$ at the right of the imaginary axis ordered, say, by decreasing real parts

$$\ldots \leqslant \operatorname{Re} \lambda_K < 0 \leqslant \operatorname{Re} \lambda_{K-1} \leqslant \ldots \operatorname{Re} \lambda_2 \leqslant \operatorname{Re} \lambda_1 . \tag{1.5}$$

Thus, the generator A is unstable, in the sense that there are free solutions (corresponding to $f(t, \zeta) \equiv 0$), say the eigensolutions with $1 \leqslant k \leqslant K-1$, that blow up in time, in fact exponentially. Under this preliminary assumption, it is of interest in boundary control theory to pose the following general *boundary feedback stabilization problem*: identify large classes of vectors w_j, g_j, $j = 1, \ldots, J$, for the least possible J, such that all solutions of the corresponding closed loop feedback system (1.1)-(1.4) decay to zero as $t \to \infty$ in the strongest possible uniform operator norm. A solution to this problem is provided by Theorem 1.2. To formulate it, we need to introduce the following number.

DEFINITION 1.1. – Let the integer l_T ($1 \leqslant l_T \leqslant \dim X_u$) denote the number of *linearly independent* Dirichlet traces $\{\Phi_{km}|_\Gamma\}$, $k = 1, \ldots, K-1$; $m = 1, \ldots, \mathscr{M}_k$ *corresponding to* the normalized eigenfunctions associated with the *unstable* eigenvalues in (1.5). □

We next introduce the $J \times \mathscr{M}_k$ matrix

$$W_k \equiv \begin{vmatrix} \langle w_1, \gamma\Phi_{k1}\rangle, & \langle w_1, \gamma\Phi_{k2}\rangle, & \ldots, & \langle w_1, \gamma\Phi_{k\mathscr{M}_k}\rangle \\ \langle w_2, \gamma\Phi_{k1}\rangle, & \langle w_2, \gamma\Phi_{k2}\rangle, & \ldots, & \langle w_2, \gamma\Phi_{k\mathscr{M}_k}\rangle \\ \vdots & \vdots & & \vdots \\ \langle w_J, \gamma\Phi_{k1}\rangle, & \langle w_J, \gamma\Phi_{k2}\rangle, & \ldots, & \langle w_J, \gamma\Phi_{k\mathscr{M}_k}\rangle \end{vmatrix}$$

associated with each unstable eigenvalue λ_k of A, and the $J \times (\dim X_u)$ matrix $W = [W_1, W_2, \ldots, W_{K-1}]$; here X_u is the (unstable) subspace of $L_2(\Omega)$, generated by the eigenvectors of $\{\lambda_k\}_{k=1}^{K-1}$. With X_s the (stable) subspace which is the orthogonal complement of X_u in $L_2(\Omega)$, we let P and Q be the orthogonal projections: $L_2(\Omega)$ onto X_u or X_s, respectively. For $x \in L_2(\Omega)$, we write $x_u = Px$ and $x_s = Qx$.

THEOREM 1.2 (Stabilization). – Let $\nu = \dim \Omega \geqslant 2$, and let Ω either have (²) a C^∞-boundary Γ, or else be a parallelepiped. Let the (necessarily point (³)) spectrum of the generator A satisfy the instability condition (1.5). Let the restriction A_u of A on the unstable subspace X_u be diagonalizable (⁴) on X_u. Let the given vectors w_j, $j = 1, \dots, J$ satisfy the following full rank conditions

$$\text{rank } W_k = \mathscr{M}_k\,, \qquad k = 1, \dots, K-1 \tag{1.6)(a}$$

at the $K-1$ unstable eigenvalues in (1.5) and, moreover,

$$\dim X_u \leqslant l_T + l_w - 1 \tag{1.6)(b}$$

where l_w is defined by

$$\text{rank } W = l_w (\max\{\mathscr{M}_k,\, k = 1, \dots, K-1\} \leqslant l_w)\,.$$

Then, there exist boundary vectors $g_j \in L_2(\Gamma)$, whose minimal number is discussed in Remark 2.1 below, such that, for all $0 \leqslant s \leqslant \sigma < \frac{3}{4}$, we have that the corresponding feedback solutions $x(t, t_0) = S_F(t) x_0$ satisfy

$$|S_F(t)|_{\mathscr{L}(H^{2s}(\Omega))} \leqslant M_{\delta s} \exp[-\delta t]\,, \qquad t \geqslant 0 \tag{1.7}$$

for some positive constants δ and $M_{\delta s}$, provided the vectors $\overline{w}_{js} = (-A_s)^\sigma Q \gamma^* w_j \in X_s$, defined as in the sobsequent Eqs (2.16)-(2.17), are in a sufficiently small sphere of X_s, depending on δ and the g_j's. Here $|\ |_{\mathscr{L}(H^{2s}(\Omega))}$ is the uniform operator norm of $H^{2s}(\Omega)$. Moreover, we may require that $-\delta = \text{Re}\,\lambda_K + \varepsilon$, for any preassigned $\varepsilon > 0$. Eq. (1.7) implies the expected conclusion on the spectrum location of the feedback generator A_F:

$$\sup \text{Re}\,\sigma(A_F) \leqslant \lim_{t\to\infty} \frac{\ln |S_F(t)|}{t} \leqslant -\delta < 0\,. \quad \square$$

(²) This assumption on Γ is needed *only* to invoke Corollary 2.2 in [S1] to guarantee (3A.11) in Appendix 3A of [L8]. Otherwise, Γ may have finitely many conical points [K3].

(³) Since A has compact resolvent on $L_2(\Omega)$.

(⁴) The assumption that A_u be diagonalizable is only for the convenience to have « clean », easy-to-check tests such as (1.6) (a)-(b), expressed in terms of (not necessarily orthogonal) normalized eigenvectors $\{\Phi_{km}\}$. Otherwise, resort to the Jordan canonical form is necessary.

PROOF. – The proof is obtained as in Theorem 1.2 of [L8], except that now the content of the footnote to Eq. (3.25) must be used to obtain invertibility of the term in the brackets by means of $|\overline{w}_s|$ sufficiently small □

REMARK 1.1. – As substantiated in [L8], the *stabilized* feedback semigroup in (1.7) is generally *not a contraction*; i.e. $M_\delta > 1$ in (1.7). □

Theorem 1.2 gives only a desired norm-upperbound for the feedback solutions corresponding to the suitable vectors w_j and g_j claimed there; it provides however no information regarding the *structure* of such feedback solutions, nor does it give any specific description regarding the *spectral properties* of the corresponding feedback generator A_F beyong the indispensable spectrum location, say: $\operatorname{Re} \sigma(A_F) \leqslant -\delta < 0$.

With this preliminary background, we can finally introduce the problem investigated in this paper. The purpose of the present paper is to push further the above stabilization result and, in fact, to pose and solve a *more precise* problem—which we call *structural assignment problem.* This regards the *exact structure* of the feedback solutions (not merely their norm upper bound), as an infinite linear combination of decaying exponentials (cf. footnote, (6)). To achieve this, we will restricte our attention to the natural case where the original generator A with zero mixed boundary conditions is *self-adjoint* (whereby, if A is stable, the desired structural property is automatically satisfied with vectors $g_j \equiv 0$). With A selfadjoint, the eigenvalues $\{\lambda_k\}$ in (1.5) are real and the corresponding eigenvectors $\{\Phi_{km}\}$ form an orthonormal system in $L_2(\Omega)$. We now state the main results of the present paper in the technically simpler situation where all eigenvalues have geometric multiplicity equal to one. We then write Φ_k instead of Φ_{k1}.

THEOREM 1.3. – In addition to the hypotheses through (1.6*b*) of Theorem 1.2, we assume that the original generator A be self-adjoint and have (real) eigenvalues as in (1.5) with geometric multiplicities $\mathcal{M}_k \equiv 1$. We assume further that $w_j \in L_2(\Omega)$ satisfy

$$0 \neq \langle w_j, \gamma\Phi_m\rangle \leqslant \text{const}/m^{1+2\sigma/\nu}, \quad m = 1, 2, \ldots, \; j = 1, \ldots, J \tag{1.8a}$$

so that the vectors $\overline{w}_{js} = (-A_s)^\sigma Q\gamma^* w_j$ defined as in the subsequent Eqs. (2.16)-(2.17), satisfy

$$0 \neq \langle \overline{w}_{js}, \Phi_m\rangle \leqslant \text{const}/m, \quad m = 1, 2, \ldots; \; j = 1, \ldots, J \tag{1.8b}$$

(as one sees via (3.21)). Then

$$\left\{ \begin{array}{l} \text{for all such vectors } \overline{w}_{js} \text{ with sufficiently} \\ \text{small } l_\infty\text{-norm} \end{array} \right. \tag{1.8c}$$

there exist boundary vectors $g_j \in L_2(\Gamma)$, whose minimal number is discussed in Remark 2.1 below, such that the corresponding feedback solutions $x(t, x_0) = S_F(t)x_0$ of the *feedback system* (1.1)-(1.4), with initial datum

$$x_0 \in \mathscr{D}\big((cI - A)^{\frac{3}{4}-\varrho}\big) = H^{\frac{3}{2}-2\varrho}(\Omega) \qquad \big(\text{see } (2.6)\big) \tag{1.9) (5}$$

can be written for $t \geqslant 0$ as:

$$\langle (cI - A)^{\frac{3}{4}-\varrho} x(t, x_0), y\rangle = \sum_{r=1}^{K-1} b_i \exp[c_i t] + \sum_{r=1}^{\infty} \chi_r \exp[\alpha_r t] \tag{1.10) (6}$$

for any $y \in L_2(\Omega)$. (Notice that the equality is in the weak-topology of $H^{\frac{3}{2}-2\varrho}(\Omega)$). In (1.10), the $\{c_i\}_{i=1}^{K-1}$ are negative constants, which can be preassigned in any chosen interval (in particular, at the left of λ_K), that replace the unstable eigenvalues $\lambda_1, \ldots, \lambda_{K-1}$, while the $\{\alpha_r\}$ are a suitable sequence of negative constants having the same asymptotic behavior as the $\{\lambda_k\}$: $[\alpha_k - \lambda_k] \to 0$ as $k \to \infty$. (See (3.38).) Moreover, the coefficients $\{\chi_r\}$ are in l_1 and, along with the coefficients $\{b_i\}$, are exhibited in the proof as dependent on y, the initial datum in $H^{\frac{3}{2}-2\varrho}(\Omega)$, and on the system parameters, including the sought-after vectors $g_j \in L_2(\Gamma)$: (see equation (3.58*b*) and Eq. (3.59) which depend on the sequence $\{d_r\}$. The sequence $\{d_r\}$ is related to the sequence $\{n_r\}$ by (3.46)-(3.47) which, in turn, is related to the initial point and the system's parameters via (3.16)).

The vectors g_j are given by: $g_j = \bar{g}_j + g_j^*$, where the $\bar{g}_j$'s are the solution of the finite moment problem (3A.7) in Appendix 3A of [L8] as applied to the present case, unique in the space

$$\mathscr{F} \equiv \operatorname{span}\{\Phi_{km}|_\Gamma,\ k = 1, \ldots, K-1;\ m = 1, \ldots, \mathscr{M}_k\}$$

and the g_i^*'s are any vectors orthogonal to $\mathscr{F}$.

An expansion similar to (1.10) holds, this time in the weak topology of $L_2(\Omega)$, if the initial datum is only assumed in $L_2(\Omega)$. □

In the self-adjoint case, the stabilization result is then recaptured as a consequence of Thm. 1.3, via a double application of the Uniform Boundedness Principle.

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(5) Here c is a positive constant greater than the largest unstable eigenvalue, so that the fractional powers are well-defined.

(6) A (scalar or vector valued) function $z(t)$ of the form

$$z(t) = \sum_{k=1}^{\infty} a_k \exp[\alpha_k t], \qquad t \in R^+$$

with α_k *negative* real numbers, where $\alpha_k \to -\infty$ as $k \to \infty$, and where $\{a_k\} \in l_1$ will be called in this paper a *function of the class IDE* (*Infinite linear combination of Decaying Exponentials*).

A reformulation of expansion (1.10) in terms of *spectral properties* of the feedback generator is given next. To appreciate it, one should note that the feedback generator A_F, corresponding to those special vectors w_j and g_j as in Theorem 1.3, that produce feedback solutions of class IDE *cannot in general be a self-adjoint operator* (7), so that an orthonormal basis in $L_2(\Omega)$ of eigenvectors of A_F is out of question.

COROLLARY 1.4. – The following spectral properties hold for the feedback generator A_F corresponding to the vectors w_j and g_j of Theorem 1.3:

(i) the distinct constants $\{c_i\}_{i=1}^{K-1}$, and $\{\alpha_r\}_{r=1}^{\infty}$ are eigenvalues of such feedback generator A_F;

(ii) the corresponding (normalized) eigenvectors $\{e_{F,i}\}_{i=1}^{K-1}$, and $\{e'_{F,r}\}_{r=1}^{\infty}$ form a (Schauder) basis in $L_2(\Omega)$ (non-orthogonal, when the g_j's or the w_j's are not all zero) so that the following expansions apply:

$$x = \sum_{i=1}^{K-1} \eta\,(x)\, e_{F,i} + \sum_{r=1}^{\infty} \eta'_r(x)\, e_{F,r}\,, \qquad x \in L_2(\Omega) \tag{1.11}$$

$$A_F x = \sum_{i=1}^{K-1} c_i\, \eta_i(x)\, e_{F,i} + \sum_{r=1}^{\infty} \alpha_r \eta'_r(x)\, e_{F,r}\,, \qquad x \in \mathscr{D}(A_F) \tag{1.12}$$

where the bounded linear functionals $\{\eta_i\}$, $\{\eta'_r\}$ and the eigenvectors $\{e_{F,i}\}$, $\{e'_{F,r}\}$, are biorthogonal sequences, say:

$$\eta_n(e_{F,m}) = \begin{matrix} 1 & n = m \\ 0 & n \neq m \end{matrix}\,.$$

Similarly for $(\eta'_n\}$ and $\{e'_{F,m}\}$. Thus

$$S_F(t)\,x_0 = \sum_{i=1}^{K-1} \eta_i(x) \exp\,[c_i t]\, e_{F,i} + \sum_{r=1}^{\infty} \eta'_r(x) \exp\,[\alpha_r t]\, e'_{F,r}\,. \qquad \square$$

The general case in which $\mathscr{M}_k \not\equiv 1$ can presumably be handled similarly. However, its detailed treatment would have considerably overloaded the presentation, particularly at the notational level. It is therefore analyzed only in the first part of the proof (section 2), while the more technical part of the proof (section 3) is restricted to the case $\mathscr{M}_k \equiv 1$.

Our present results on the structural assignment problem for (1.1)-(1.4) represent a step forward over the same problem for a parabolic system with Dirichlet B.C. with «interior observation» treated in [L2], where γ was the identity operator in

(7) Since, in this case, A_F would be dissipative, contrary to Remark 1.1.

$L_2(\Omega)$. This extension is achieved here by refining in a number of places our original proof in [L2], and by further sharpening some crucial estimates in our analysis. However, because of assumptions (1.8) (*b-c*), the present results do not cover yet the fully boundary feedback case, where γ would be a continuous operator $H^{2\sigma}(\Omega) \to \to L_2(\Gamma)$, $2\sigma < \frac{3}{2}$, in particular the Dirichlet trace $\gamma x = x|_\Gamma$ (« boundary observation »).

In this very important case, however, a solution to the boundary stabilization problem was provided in [L8] in the non necessarily self-adjoint case, thereby disproving a belief to its impossibility, expressed in [F3] on a basis of a one-dimensional ($\nu = 1$) negative example.

2. – Proof of Theorem 1.3.

Preliminaries and finite dimensional part for general $\mathcal{M}_k$.

As in our previous work [L1-L3, L8, T3-T4], our approach to treating nonhomogenous boundary problems consists of replacing the feedback system described by (1.1)-(1.3) and (1.4) with a corresponding semigroup-rooted abstract version. This was developed only very recently with the aim to model, through a variation of parameter type formula, nonsmooth *boundary* input parabolic equations. See the original references [B1, B2, W1] for the full development, at least in the Dirichlet case. The mixed case can be treated similarly (see comments in [T3] following (2.4*M*)). See also the very general and unifying treatment given in [L1]. In the case of the boundary feedback given by (1.4), these abstract semigroup versions give rise to the following integral model:

$$x(t) = \exp[At]x_0 - \int_0^t A \exp[A(t-\tau)]\, M \sum_{j=1}^{J} \langle \gamma x(\tau), w_j \rangle g_j \, d\tau \,. \tag{2.1}$$

Here, $-A$ is the elliptic operator with zero mixed B.C.; the « mixed map » is the continuous linear map $L_2(\Gamma) \to H^{\frac{3}{2}}(\Omega)$ [L5], [N1] defined by $y = Mg$, where

$$-A(\xi, \partial)y = 0 \quad \text{in } \Omega; \qquad \left(\frac{\partial y}{\partial \eta} + by\right)_\Gamma = g \,. \tag{2.2}$$

Since Ω is a bounded domain, the resolvent operator $R(\beta, A)$ is compact [D2; p. 1740]. Hence, the spectrum $\sigma(A)$ of A is only point spectrum and consists of a sequence of real isolated eigenvalues $\{\lambda_k\}$ with no finite accumulation point: $\lambda_k \to -\infty$, and with a corresponding orthonormal basis of eigenvectors $\{\Phi_{km}\}$, $k = 1, 2, \ldots$; $m = 1, \ldots, \mathcal{M}_k$, $\mathcal{M}_k$ being the geometric multiplicity of λ_k. Following now a procedure introduced in [T2], we let $X = L_2(\Omega)$ be decomposed into two orthogonal subspaces X_u and X_s corresponding, respectively, to the subsets $\{\lambda_1, \ldots, \ldots, \lambda_{K-1}\}$ and $\{\lambda_k, k \geq K\}$ of the spectrum of A that satisfies assumption (1.5). (The subscripts u and s stand for «stable» and «unstable», respectively.) Here, we appeal to the standard decomposition theorem as in [K1, Thm. 6.17, p. 178]. With P denoting the orthogonal projection of $L_2(\Omega)$ onto X_u and $Q = I - P$, then $Q\mathscr{D}(A) \subset \subset \mathscr{D}(A)$, X_u and X_s are invariant under A and hence under the semigroup $\exp[At]$. As for the spectra, we have $\sigma(A_u) = \{\lambda_1, \ldots, \lambda_{K-1}\}$, $\sigma(A_s) = \{\lambda_k, k \geqslant K\}$, where $A_\bullet$ is the restriction of A on $X_\bullet$. A_u is bounded, in fact, finite-dimensional.

Finally, P and Q commute with A, hence with the semigroup $\exp[At]$. We shall henceforth use the notation $x_u = Px$ and $x_s = Qx$. Notice that the fractional powers of $(-A_s)$ are well defined. Notice also that the definition of M in (2.2) always implies $Mg \notin \mathscr{D}(A)$, unless $g = 0$. Thus, for $g \neq 0$, we always have $QMg \notin \mathscr{D}(A_s)$. However, the following relations, which we later apply crucially, hold:

$$\mathscr{D}\big((-A_s)^{\frac{3}{4}-\varrho}\big) = QH^{\frac{3}{2}-2\varrho}(\Omega), \quad 0 < \varrho \leqslant \tfrac{3}{4} \tag{2.3}$$

with norm

$$|x|_{H^{\frac{3}{2}-2\varrho}(\Omega)} = |(-A_s)^{\frac{3}{4}-\varrho} x|_{L_2(\Omega)} . \tag{2.4}$$

Relations (2.3) are contained in the literature of fractional powers [F2], [L5], [L1; App. B], [M2; p. 187]. Now, elliptic theory [L4; pp. 187-188], [N1] shows that

$$\text{range of } M \subset H^{\frac{3}{2}}(\Omega) \tag{2.5}$$

and from (2.3) we then obtain

$$Q\,[\text{range of } M] \subset QH^{\frac{3}{2}-2\varrho}(\Omega) = \mathscr{D}\big((-A_s)^{\frac{3}{4}-\varrho}\big), \quad 0 < \varrho \leqslant \tfrac{3}{4} . \tag{2.6}$$

Having introduced the relevant machinery, we are now in a position to begin the proof. We project (2.1) onto X_u and X_s. By virtue of (2.6), the projections of the solution $x(t)$ in (2.1) onto X_u and X_s are, respectively

$$x_u(t) = \exp[A_u t]x_{0u} - \int_0^t \exp[A_u(t-\tau)] \sum_{j=1}^{J} A_u PMg_j[\langle \gamma x_u(\tau), w_j\rangle + \langle \gamma x_s(\tau), w_j\rangle]\, d\tau . \tag{2.7}$$

$$x_s(t) = \exp[A_s t]x_{0s} - \int_0^t (-A_s)^{\frac{1}{4}+\varrho} \exp[A_s(t-\tau)] \sum_{j=1}^{J} (-A_s)^{\frac{3}{4}-\varrho} \cdot QMg_j[\langle \gamma x_s(\tau), w_j\rangle + \langle \gamma x_s(\tau), w_j\rangle]\, d\tau . \tag{2.8}$$

These projections are *coupled*. Considering the unperturbed part of Eq. (2.7) we are led to study the equation

$$\dot z = A_u z + \sum_{j=1}^{J} A_u PMg_j \langle \gamma z, w_j\rangle , \quad z \in X_u,\ z(0) = x_{0u} . \tag{2.9}$$

We then observe that this can be more conveniently rewritten as

$$\dot{z} = \bar{A}_u z , \tag{2.10}$$

where $\bar{A}_u$ is a square matrix of size equal to $\dim X_u$, depending on the vectors $A_u PMg_j$ and w_j (besides A_u and γ). This can be seen by using in X_u the basis of orthonormal eigenvectors Φ_{km}, $k = 1, \ldots, K-1$, which make the matrix corresponding to the operator A_u diagonal. As we are seeking suitable boundary vectors. $g_j \in L_2(\Gamma)$, which produce the desired stabilized feedback semigroup, we find it convenient to consider the projections (2.7) and (2.8) after setting

$$p_j \overset{\text{df}}{=} A_u PMg_j; \quad q_j \overset{\text{df}}{=} (-A_s)^{\frac{3}{4}-\varrho} QMg_j , \quad j = 1, \ldots, J , \tag{2.11}$$

and to think of the vectors p_j and q_j as, for the time being, just vectors in X_u and X_s, respectively, without any connection with the vectors g_j which generate them. The question of synthesizing p_j and q_j through an appropriate g_j will be taken up later on. (Appendices 4 A-B.)

With (2.11) in mind, we obtain a suitable structural assignment of the solution to (2.10), for a suitable choice of vectors p_j, through the following lemma.

LEMMA 2.1. – Suppose that the vectors $w_j \in L_2(\Omega)$ are chosen as to satisfy the full rank conditions (1.6) (*a*) at the unstable eigenvalues and, moreover, condition (1.6) (*b*).

Then

(i) there exist vectors p_j, $j = 1, \ldots, J$ in X_u such that the corresponding matrix $\bar{A}_u$ in (2.10) has a set of eigenvalues arbitrarily close to any preassigned set of ($\dim X_u$)—complex numbers (appearing in complex conjugate pairs, if A_u and p_j are all real).

In particular, these eigenvalues of $\bar{A}_u$ may be preassigned to be all distinct, equal to negative constants c_i, $i = 1, \ldots, \dim X_u$ and, for instance

$$\operatorname{Re} \lambda_{k+1} < c_{\dim X_u} < \ldots < c_1 < \operatorname{Re} \lambda_K < 0 , \tag{2.12}$$

in which case the solution to (2.9), or equivalently to (2.10), is an X_u-function of the form:

$$z(t) = \exp[\bar{A}_u t] x_{0u} = \sum_{i=1}^{\dim X_u} \exp[c_i t] \langle x_{0u}, \psi_i \rangle \psi_i . \tag{2.13}$$

Here, ψ_i is the normalized eigenvector of $\bar{A}_u$ corresponding to the simple eigenvalue c_i and the $\{\psi_i\}$, $i = 1, \ldots, \dim X_u$ form a basis on X_u.

(ii) Moreover, when $\dim \Omega = \nu \geq 2$, each vector p_j, $j = 1, \ldots, J$ can indeed be synthesized, as required by the left equality of (2.11), by any of the infinitely many vectors $g_j \in L_2(\Gamma)$, satisfying the moment problem (3 A.7) of the proof in Appendix 3 A in [L8]. The minimal number J required is discussed in Remark 2.1 below. The case $\dim \Omega = \nu = 1$ is also included, provided $\dim X_u \leqslant 3$.

PROOF OF LEMMA 2.1. – See e.g. Appendix 3A of [L8], where the proof is given in the more general case of γ being a continous operator: $H^{2\sigma}(\Omega) \to L_2(\Gamma)$. □

REMARK 2.1. – Conditions (1.6) (*a*) and the definition of l_w in particular imply

$$J \geqslant \max\{\mathcal{M}_k,\ k = 1, \dots, K-1\} \quad \text{and} \quad J \geqslant l_w . \tag{2.14}$$

Moreover, the proof given in Appendix 3 A of [L8] shows that the largest multiplicity of the unstable eigenvalues is indeed the minimum number of boundary vectors g_i required for the conclusion of Lemma 2.1, provided that the Dirichlet traces

$$\{\Phi_{km}|_\Gamma\}, \quad k = 1, \dots, K-1;\ m = 1, \dots, \mathcal{M}_k$$

of the eigenfunctions are linearly independent (8).
Otherwise, more vectors $g_j \in L_2(\Gamma)$ are needed. For instance, if $\mathcal{M}_k \equiv 1$, $1 \leqslant k \leqslant K-1$ and (9) $l_T < \dim X_u = K-1$, then J suitable vectors $g_j \in L_2(\Gamma)$ that satisfy the moment problem (3A.7) in Appendix 3A of [L8], where

$$J \geqslant \dim X_u - l_T + 1 ,$$

will suffice (10). A full analysis of the situation, amounting to a certain *output stabilizability* problem in X_u, is contained in the proof in Appendix 3A of [L8]. □

Application of Lemma 2.1 to the unperturbed part of the projection (2.7) allows us to rewrite (2.7) more conveniently as

$$x_u(t) = \exp[\bar{A}_u t]x_{0u} - \int_0^t \exp[\bar{A}_u(t-\tau)] \sum_{j=1}^{J} p_j \langle \gamma x_s(\tau), w_j \rangle \, d\tau . \tag{2.15}$$

To handle the coupled projections (2.7) and (2.8), we need the following consideration. Fix $2\sigma < \frac{3}{2}$ in the definition of γ, given above (1.4).

Therefore, if we take $H^{2\sigma}(\Omega)$ and $L_2(\Omega)$ as pivot spaces [A1, p. 48], it follows that the adjoint operator γ^* is a continuous operator: $L_2(\Omega) \to H^{2\sigma}(\Omega)$.

(8) This is the case when Ω is a parallelepiped.

(9) This is the case when Ω is a sphere.

(10) In the case of one dimensional Ω, where J and l_T are at most equal to two, the unstable eigenspace cannot be of dimensions larger than three.

By (2.3), we can then write

$$\langle \gamma x(t), w_j\rangle_\Gamma = \langle \gamma x_u(t), w_j\rangle + \langle \gamma x_s(t), w_j\rangle = = (x_u(t), \gamma^* w_j)_{H^{2\sigma}(\Omega)} + (x_s(t), \gamma^* w_j)_{H^{2\sigma}(\Omega)} = = \langle A_u^\sigma x_u(t), \tilde{w}_{ju}\rangle + \langle (-A_s)^\sigma x_s(t), \overline{w}_{js}\rangle \tag{2.16}$$

where $\langle\, ,\rangle$ is the inner product in $L_2(\Omega)$ and where we have set

$$\tilde{w}_{ju} \overset{\text{df}}{=} A_u^\sigma P\gamma^* w_j\,, \qquad \overline{w}_{js} = (-A_s)^\sigma Q\gamma^* w_j\,. \tag{2.17}$$

Notice that in (2.17), $Q\gamma^* w_j$ is written, by (2.3), as belonging to the largest possible fractional power of $(-A_s)$, and hence a further « transfer » of an additional fractional power of $(-A_s)$ from the left to the right of the second inner product in (2.16) is not allowed.

Continuing with the proof of Theorem 1.3, we see that, by (2.17), the projections (2.15) and (2.8) can then be rewritten in the equivalent form

$$x_u(t) = \exp[\overline{A}_u t]x_{0u} - \int_0^t \exp[\overline{A}_u(t-\tau)] \sum_{j=1}^J p_j \langle (-A_s)^\sigma x_s(\tau), \overline{w}_{js}\rangle\, d\tau\,. \tag{2.18}$$

$$x_s(t) = \exp[A_s t]x_{0s} - \int_0^t (-A_s)^{\frac{1}{4}+\varrho} \exp[A_s(t-\tau)] \sum_{j=1}^J q_j[\langle (-A_s)^\sigma x_s(\tau), \overline{w}_{js}\rangle + \langle A_u^\sigma x_u(\tau), \tilde{w}_{ju}\rangle]\, d\tau \tag{2.19}$$

which we shall find more useful.

REMARK 2.2. – With reference to Lemma 2.1, a vector in X_u will always be referred to the basis $\{\psi_i\}_{i=1}^{\dim X_u}$. On the other hand, a vector in X_s will always be referred to the basis $\{\Phi_{km}\}_{k=K}^{\infty}$. We shall also adopt, henceforth, the following short notation:

if $v \in X_u$, we set $(v)_i = \langle v, \psi_i\rangle$, $i = 1, \dots, \dim X_u$;

if $v \in X_s$, we set $[v]_{km} = \langle v, \Phi_{km}\rangle$, $k = K, K+1, \dots$. □

REMARK 2.3. – For handy reference below, we collect here the following results and observations. Let $\lambda_k \neq c_i$. Then,

$$\int_0^t \exp[\lambda_k(t-\tau)]\exp[c_i\tau]\, d\tau = \frac{\exp[c_i t] - \exp[\lambda_k t]}{c_i - \lambda_k}\,, \tag{2.20a}$$

and

$$\int_\sigma^t \exp[\lambda_k(t-\tau)]\exp[c_i(\tau-\sigma)]\, d\tau = \frac{\exp[c_i(t-\sigma)] - \exp[\lambda_k(t-\sigma)]}{c_i - \lambda_k}\,. \tag{2.20b}$$

In other words, convolving two different exponentials preserves the exponential character. By contrast, we have

$$\int_0^t \exp[c(t-\tau)]\exp[c\tau]\,d\tau = t\exp[ct]\,.$$

Thus, convolving the *same* exponentials destroys its exponential character. These remarks will play a crucial role in the development given below in proving the desired structural properties of the feedback solutions. □

3. – Continuation of Proof of Theorem 1.3.

Infinite dimensional part, when $\mathcal{M}_k \equiv 1$.

For simplicity of notation we assume henceforth that the conclusion of Lemma 2.1 holds with just one vector; i.e. with $J = 1$, in which case we write w, g, p, q instead of w_1, g_1, p_1, q_1.

The proof now preceeds through a lengthy sequence of intermediate results. As the geometric multiplicity $\mathcal{M}_k$ of all eigenvalues is assumed identically one: $\mathcal{M}_k \equiv 1$, we then consistently write Φ_k for Φ_{k1} throughout. With the constant ϱ in (2.3) fixed once and for all, we also set

$$(-A_s)^{\frac{3}{4}-\varrho}QMg = q = \{q_k\}_{k=K}^{\infty} \in X_s = QL_2(\Omega) \tag{3.1}$$

for the sought-after vector in X_s, and

$$A_u PMg = p = \{p_i\}_{i=1}^{K-1} \in X_u = PL_2(\Omega) \tag{3.2}$$

for the corresponding vector in X_u, provided by Lemma 2.1. Here, according to the convention of Remark 2.2, we mean explicitly:

$$q_k = \langle q, \Phi_k\rangle\,, \quad k \geqslant K \quad \text{but} \quad p_i = \langle p, \psi_i\rangle\,, \quad i = 1, \dots, K-1\,. \tag{3.3}$$

We first consider the projections (2.18), (2.19) for $J = 1$, thinking at first of p and q in (3.1), (3.2) as, for the time being, just vectors in X_u and X_s, respectively, without any connection with the vector $g \in L_2(\Gamma)$ which generates them. The question of synthesizing p *and* q through an appropriate g will be taken up toward the end of the proof, in Appendices 4A-4B. (The question of synthesizing *only* p was solved in the proof of Lemma 2.1.)

Finally, throughout this section, the initial point x_{0s} is assumed to lie in $\mathcal{D}((-A_s)^{\frac{3}{4}-\varrho})$, with no further explicit mention made (see (1.9)).

3.1. *Reduction to a Volterra Integral Equation in* $\mathscr{d}(t) = \langle(-A_s)^\sigma x_s(t), \overline{w}_s\rangle$.

Our starting point is the pair of projections (2.18), (2.19) rewritten now for $J = 1$

$$x_u(t) = \exp[\overline{A}_u t]x_{0u} - \int_0^t \exp[\overline{A}_u(t-\tau)]p\,\langle(-A_s)^\sigma x_s(\tau), \overline{w}_s\rangle\,d\tau \tag{3.4}$$

$$x_s(t) = \exp[A_s t]x_{0s} - \int_0^t (-A_s)^{\frac{1}{4}+\varrho}\exp[A_s(t-\tau)]q[\langle(-A_s)^\sigma x_s(\tau), \overline{w}_s\rangle + \langle x_u(\tau), \tilde{w}_u\rangle]\,d\tau\,, \tag{3.5}$$

where we have defined the vector $\tilde{w}_u \in X_u$ by

$$\tilde{w}_u = A_u^\sigma \overline{w}_u, \quad \text{i.e., by} \quad \langle A_u^\sigma x_u(\tau), \overline{w}_u\rangle = \langle x_u(\tau), \tilde{w}_u\rangle\,. \tag{3.6}$$

Furthermore, set

$$\sigma + \tfrac{1}{4} + \varrho \overset{\text{df}}{=} \delta < 1\,, \quad \text{for suitable } \varrho, \text{ once } \sigma \text{ is assigned} \tag{3.7}$$

and introduce the unknown function $\mathscr{d}(t)$:

$$\mathscr{d}(t) = \langle(-A_s)^\sigma x_s(t), \overline{w}_s\rangle\,. \tag{3.8}$$

Applying $(-A_s)^\sigma$ to (3.5) and taking the inner product with $\overline{w}_s$ yields, by virtue of (3.8),

$$\mathscr{d}(t) = \langle\exp[A_s t](-A_s)^\sigma x_{0s}, \overline{w}_s\rangle - \int_0^t \langle(-A_s)^\delta \exp[A_s(t-\tau)]q, \overline{w}_s\rangle[\mathscr{d}(\tau) + \langle x_u(\tau), \tilde{w}_u\rangle]\,d\tau\,. \tag{3.9}$$

We next compute, by means of (3.4) and a change in the order of integration,

$$\int_0^t (-A_s)^\delta \exp[A_s(t-\tau)]q\,\langle x_u(\tau), \tilde{w}_u\rangle\,d\tau = \int_0^t (-A_s)^\delta \exp[A_s(t-\tau)]q\,\langle\exp[\overline{A}_u\tau]x_{0u}, \tilde{w}_u\rangle\,d\tau - \int_0^t\int_\sigma^t (-A_s)^\delta \exp[A_s(t-\tau)]q\,\langle\exp[\overline{A}_u(\tau-\sigma)]p, \tilde{w}_u\rangle\,\mathscr{d}(\sigma)\,d\tau\,d\sigma\,.$$

By (2.3), (2.12) and the notational convention in Remark 2.2,

$$\int_0^t (-A_s)^\delta \exp[A_s(t-\tau)] q \langle x_u(\tau), \tilde{w}_u\rangle \, d\tau =$$

$$= \sum_{k=K}^{\infty} \sum_{i=1}^{K-1} \Bigg\{ \int_0^t \lambda_k^\delta \exp[\lambda_k(t-\tau)] \exp[c_i\tau](x_{0u})_i(\tilde{w}_u)_i \, d\tau -$$

$$- \int_0^t \int_\sigma^t \lambda_k^\delta \exp[\lambda_k(t-\tau) \exp[c_i(\tau-\sigma)](\tilde{w}_u)_i p_i d(\sigma) \, d\tau \, d\sigma \Bigg\} q_k \Phi_k =$$

$$= \sum_{k=K}^{\infty} \sum_{i=1}^{K-1} \Bigg\{ \frac{\exp[c_i t] - \exp[\lambda_k t]}{c_i - \lambda_k} (x_{0u})_i (\tilde{w}_u)_i \lambda_k^\delta -$$

$$- \int_0^t \frac{\exp[c_i(t-\sigma)] - \exp[\lambda_k(t-\sigma)]}{c_i - \lambda_k} d(\sigma)\, d\sigma (\tilde{w}_u)_i p_i \lambda_k^\delta \Bigg\} q_k \Phi_k . \qquad \text{(by (2.20))}$$

Hence, (3.9) becomes

$$d(t) = n(t) + \int_0^t h(t-\tau)\, d(\tau)\, d\tau\,, \qquad 0 \leqslant t < \infty\,, \tag{3.10}$$

where, in the notational convention of Remark 2.2

$$\begin{aligned} n(t) &= \langle \exp[A_s t](-A_s)^\sigma x_{0s}, \overline{w}_s\rangle - \\ &\quad - \sum_{k=K}^{\infty}\sum_{i=1}^{K-1} \frac{\exp[c_i t] - \exp[\lambda_k t]}{c_i - \lambda_k} (x_{0u})_i (\tilde{w}_u)_i [\overline{w}_s]_k q_k \lambda_k^\delta = \\ &= \sum_{i=1}^{K-1} \exp[c_i t](x_{0u})_i(\tilde{w}_u)_i \sum_{k=K}^{\infty} \frac{[\overline{w}_s]_k q_k \lambda_k^\delta}{\lambda_k - c_i} + \\ &\quad + \sum_{k=K}^{\infty} \exp[\lambda_k t][\overline{w}_s]_k \left\{ [(-A_s)^\sigma x_{0s}]_k - q_k \lambda_k^\delta \sum_{i=1}^{K-1} \frac{(x_{0u})_i(\tilde{w}_u)_i}{\lambda_k - c_i} \right\}, \end{aligned} \tag{3.11}$$

$$\begin{aligned} h(t) &= -\langle (-A_s)^\delta \exp[A_s t] q, \overline{w}_s\rangle + \\ &+ \sum_{k=K}^{\infty}\sum_{i=1}^{K-1} \frac{\exp[c_i t] - \exp[\lambda_k t]}{c_i - \lambda_k} p_i (\tilde{w}_u)_i q_k [\overline{w}_s]_k \lambda_k^\delta = - \sum_{i=1}^{K-1} \exp[c_i t] p_i (\tilde{w}_u)_i \sum_{k=K}^{\infty} \frac{q_k [\overline{w}_s]_k \lambda_k^\delta}{\lambda_k - c_i} - \\ &\quad - \sum_{k=K}^{\infty} \exp[\lambda_k t] q_k [\overline{w}_s]_k \lambda_k^\delta \left\{ 1 - \sum_{i=1}^{K-1} \frac{p_i(\tilde{w}_u)_i}{\lambda_k - c_i} \right\}. \end{aligned} \tag{3.12}$$

We can rewrite (3.11) and (3.12) in a simplified manner as

$$n(t) = \sum_{r=1}^{\infty} n_r \exp[\beta_r t]\,, \qquad 0 \leqslant t < \infty\,, \tag{3.13}$$

$$h(t) = \sum_{r=1}^{\infty} h_r \exp[\beta_r t]\,, \qquad 0 < t < \infty \tag{3.14}$$

where

$$\beta_r = c_r\,, \quad r = 1, \dots, K-1; \qquad \text{and} \qquad \beta_r = \lambda_r\,, \quad r = K, K+1, \dots \tag{3.15}$$

$$n_r = (x_{0u})_r (\tilde{w}_u)_r \sum_{k=K}^{\infty} \frac{q_k [\overline{w}_s]_k \lambda_k^\delta}{\lambda_k - c_r}\,, \qquad r = 1, \dots, K-1\,. \tag{3.16a}$$

$$n_r = \frac{[(-A_s)^{\frac{3}{4}-\varrho} x_{0s}]_r [\overline{w}_s]_r}{\lambda_r^{1-\delta}} - q_r [\overline{w}_s]_r \lambda_r^\delta \sum_{i=1}^{K-1} \frac{(x_{0u})_i (\tilde{w}_u)_i}{\lambda_r - c_i}\,, \qquad r = K, K+1, \dots\,. \tag{3.16b}$$

(Here, x_{0s} is written in the largest fractional power of $(-A_s)$ compatible with the assumption (1.9)).

$$h_r = -p_r (\tilde{w}_u)_r \sum_{k=K}^{\infty} \frac{q_k [\overline{w}_s]_k \lambda_k^\delta}{\lambda_k - c_r}\,, \qquad r = 1, \dots, K-1\,. \tag{3.17a}$$

$$h_r = -q_r [\overline{w}_s]_r \lambda_r^\delta \left\{ 1 - \sum_{i=1}^{K-1} \frac{p_i (\tilde{w}_u)_i}{\lambda_r - c_i} \right\}, \qquad r = K, K+1, \dots\,. \tag{3.17b}$$

Notice that the existence of the solution $d(t)$ in (3.10), as an analytic f-nction for $t > 0$, is already known through the existence result Theorem 1.1. One may alternatively invoke the standard theory of linear Volterra integral equations [M1]. It should be kept in mind that the functions $n(t)$, $h(t)$, and hence $d(t)$, depend upon q.

The above expressions will play a crucial role in the analysis, given below, of the Volterra equation (3.10). Notice that each coefficient n_r and h_r, for $r \geqslant K$, depends only on the corresponding coordinate q_r; while for $1 \leqslant r \leqslant K-1$ it depends in a cumulative way on all $\{q_k\}_{k=K}^{\infty}$. This fact will be a source of difficulties. We also remark that we shall henceforth borrow freely from Eq. (3.15) both the notation $\{\beta_r\}$ in place of $\{c_r\}$ and $\{\lambda_r\}$, or the other way around.

3.2. *Existence of admissible vectors q generating Volterra solutions $d(t)$ of class IDE with negative exponents $\{a_r\}_{r=1}^{\infty}$ all distinct from all $\{\beta_r\}_{r=1}^{\infty}$.*

In order to establish that the solution $d(t)$ of (3.10) is of class IDE for a suitable vector q, we find it convenient to associate with Eq. (3.10) the following sequence of auxiliary Volterra equations:

$$d_N(t) = n_N(t) + \int_0^t h_N(t-\tau)\, d_N(\tau)\, d\tau\,, \tag{3.18}$$

where $N = 1, 2, \dots$, and

$$n_N(t) = \sum_{r=1}^{N} n_r \exp[\beta_r t]\,, \tag{3.19a}$$

$$0 \leqslant t < \infty\,.$$

$$h_N(t) = \sum_{r=1}^{N} h_r \exp[\beta_r t] \tag{3.19b}$$

LEMMA 3.1. – Let the initial point x_{0s} be in $\mathscr{D}((-A_s)^{\frac{3}{4}-\varrho})$ and also let $q \in QL_2(\Omega)$. Then,

(i) the corresponding sequences

$$(-A_s)^{1-\delta}\{n_r\}_{r=K}^{\infty} \overset{\text{df}}{=} \{\lambda_r^{1-\delta} n_r\}_{r=K}^{\infty} \quad \textit{and} \quad (-A_s)^{-\delta}\{h_r\}_{r=K}^{\infty} = \{\lambda_r^{-\delta} h_r\}_{r=K}^{\infty}$$

defined by (3.16), (3.17), all belong to the space l_1; moreover

$$\lambda_r^{1-\delta} n_r \leqslant \frac{\text{const}}{r}; \qquad \lambda_r^{-\delta} h_r \leqslant \frac{\text{const}}{r};$$

(ii) the corresponding functions $\mathscr{n}(t)$ and $\mathscr{h}(t)$ are functions of class IDE on $t \geqslant 0$ and $t > 0$, respectively, and they are both absolutely Laplace transformable;

(iii) the corresponding functions $\mathscr{n}_N(t)$ and $\mathscr{h}_N(t)$ in (3.19) converge uniformly over $\overline{\boldsymbol{R}^+}$ and $[a, \infty)$, $a > 0$, respectively to the functions $\mathscr{n}(t)$ and $\mathscr{h}(t)$ in (3.13) and (3.14).

PROOF. – Conclusion (i) is immediate from the explicit expressions (3.16) and (3.17) of the coefficients via (1.8*b*). As a consequence, $\mathscr{n}(t)$ and $\mathscr{h}(t)$ are the uniform limits over $\overline{\boldsymbol{R}^+}$ and $[a, \infty)$ of the decaying exponentials, thus establishing conclusion (ii), as required by definition of class IDE. Conclusion (i) also clearly implies (iii). □

We start with a general result which will be refined and complemented below in Theorem 3.9.

PROPOSITION 3.2. – For any vector $q \in QL_2(\Omega)$, the corresponding solutions $\mathscr{d}_N(t)$ to the Volterra equation (3.18) converge uniformly over $\overline{\boldsymbol{R}^+}$ to the corresponding solution $\mathscr{d}(t)$ of the Volterra equation (3.10).

PROOF. – Let $\hat{\mathscr{h}}(s) = \sum_{r=1}^{\infty} h_r/(s-\beta_r)$ be the Laplace transform of $\mathscr{h}(t)$ of (3.14) for $\operatorname{Re} s > 0$. By Lemma 3.1 (i) on $\{h_r\}$, we can achieve $|1-\hat{\mathscr{h}}(s)| > \varrho_u > 0$ for $\operatorname{Re} s > u$, for a suitably large u. Then the (onesided) Laplace transform $\hat{\mathscr{d}}_N(s)$ of $\mathscr{d}_N(t)$ exists here by

$$\hat{\mathscr{d}}_N(s) = \frac{\hat{\mathscr{n}}_N(s)}{1-\hat{\mathscr{h}}_N(s)}, \qquad N = K, K+1, \ldots$$

and the uniqueness of the solution $\mathscr{d}_N(t)$ to (3.18). Then, Lemma 3.1 (iii) and the definition of the (one-sided) Laplace transform imply that, as $N \to \infty$, the functions $\hat{\mathscr{n}}_N(s)$ and $\hat{\mathscr{h}}_N(s)$ are uniformly convergent to $\hat{\mathscr{n}}(s)$ and $\hat{\mathscr{h}}(s)$, respectively, over $\operatorname{Re} s$ suitably large. Again, by the uniqueness property of the solution $\mathscr{d}(t)$ to

(3.10), the functions $\hat{d}_N(s)$ also converge uniformly over Re s suitably large to the function $\hat{d}(s)$. But then, the inverse Laplace integral (see [D1; Thm. 24.4, p. 157]) implies that, as $N \to \infty$, $d_N(t)$ converges uniformly to $d(t)$ over $[0, \infty)$. □

We next establish some properties enjoyed by the Laplace transform of solutions $d(t)$.

PROPOSITION 3.3. – For any vector $q \in QL_2(\Omega)$, the Laplace transform $\hat{d}(s)$ of the corresponding solution $d(t)$, extended over the entire complex plane $\boldsymbol{C}$ in a natural way by the right-hand side of (3.20) below, is a meromorphic function over $\boldsymbol{C}$.

Moreover, if, for a suitable q, the corresponding (continuous over $\overline{\boldsymbol{R}^+}$) solution $d(t)$ of (3.10) is of class IDE, then $\hat{d}(s)$ has countably many simple poles $\{\alpha_r\}$, α_r real and negative, which are simple zeros for $[1 - \hat{h}(s)]$: $\hat{h}(\alpha_r) \equiv 1$. Such poles are either finitely many or else, if infinitely many, have moduli tending to infinity:

$$|\alpha_r| \to \infty \quad \text{as } r \to \infty .$$

PROOF. – As in the preceding proof, we have explicitly, from (3.13), (3.14),

$$\hat{d}(s) = \frac{\hat{n}(s)}{1 - \hat{h}(s)} = \frac{\sum_{r=1}^{\infty} \frac{n_r}{s - \beta_r}}{1 - \sum_{r=1}^{\infty} \frac{h_r}{s - \beta_r}}, \tag{3.20}$$

which is the Laplace transform of $d(t)$ for Re $s > 0$ and is extended to $\boldsymbol{C}$ by the expression on the right-hand side. As the ratio of two meromorphic functions over $\boldsymbol{C}$ (with common poles $\{\beta_r\}_{r=1}^{\infty}$, in fact), $\hat{d}(s)$ is meromorphic and hence its poles are either finitely many or else their moduli tend to infinity [K2]. In addition, $\hat{d}(s)$ admits an expansion as the sum of its principal parts plus an entire function (MITTAG-LEFFLER, Theorem [K2]; [L4]). The poles of $\hat{d}(s)$ are zeros of the denominator $[1 - \hat{h}(s)]$. If the real s_0 is such a zero with multiplicity m, then the term $t^{m-1} \exp[s_0 t]$ occurs in the antitransform of the Mittag-Leffler expansion of $\hat{d}(s)$. Hence, the statement on the $\{\alpha_r\}$ is a consequence of the assumed IDE character of $d(t)$. □

We proceed now to characterize the admissible vectors q (recall Eq. (3.1)) whose corresponding solution $d(t)$ are function of the class IDE with the additional requirement that the exponents be *all* different from the set $\{\beta_r\}_{r=1}^{\infty}$ in (3.15) (recall Remark 2.3).

REMARK 3.1. – We refer here to a basic known result on the asymptotic behavior of the eigenvalues of second-order self-adjoint elliptic differential operators, which will play a crucial role below. If ν denotes, as in the Introduction, the dimension of the euclidean space containing the domain Ω, then (see [T. 1; pp. 392-395], [Cl; Ch. VI, §§ 3.3-3.4]) the estimate

$$\beta_k = \lambda_k \sim k^{2/\nu}, \quad k = K, K+1, \dots \tag{3.21}$$

holds. Here, and hereafter, the symbol $\sim$ means that the left-hand side can be estimated by the right-hand side from below and from above with the aid of constants independent of the variable in question (k, in this case) going to infinity. □

We now let m be the smallest (non-negative) integer stricly greater than $((\nu/2)-1)$. Then, (3.21) implies

$$\sum_{k=1}^{\infty} \frac{1}{\beta_k^{m+1}} \sim \sum_{k=1}^{\infty} \frac{1}{k^{2(m+1)/\nu}} < \infty .$$

Therefore, by virtue of the Weistrass factorization theorem [L4; p. 390], the function $\mathfrak{B}(s)$, defined by

$$\mathfrak{B}(s) = \prod_{k=1}^{\infty} \left(1 - \frac{s}{\beta_k}\right) E(s, \beta_k, m) \tag{3.23}$$

where

$$E(s, \beta_k, m) \overset{\text{df}}{=} \exp\left[\frac{s}{\beta_k} + \frac{1}{2}\left(\frac{s}{\beta_k}\right)^2 + \dots + \frac{1}{m}\left(\frac{s}{\beta_k}\right)^m\right], \tag{3.24}$$

is an entire function with zeros of multiplicity one precisely at the points $\{\beta_k\}_{k=1}^{\infty}$ and no other zero (the integer m is the genus of $\mathfrak{B}(s)$). Then, the meromorphic function $\hat{d}(s)$ in (3.20) can be rewritten as the ratio of two entire functions, in fact

$$\hat{d}(s) = \frac{\hat{n}(s)\,\mathfrak{B}(s)}{\mathfrak{D}(s)}$$

where

$$\mathfrak{D}(s) = (1 - \hat{h}(s))\,\mathfrak{B}(s), \tag{3.25}$$

while $\mathfrak{D}(s)$ and $(1-\hat{h}(s))$ have precisely the same zeros. To motivate our further analysis, let us now assume, in the light of Proposition 3.3, that there exists a vector $q \in QL_2(\Omega)$ (this assumption will be shown later to be non-void) such that the corresponding function $(1-\hat{h}(s))$, obtained through (3.17), has countably many negative zeroes, all simple, of the form $\{\alpha_k\}_{k=1}^{\infty}$, any α_k being different from all the $\{\beta_j\}_{j=1}^{\infty}$, but with a similar asymptotic behavior: $\alpha_k \sim \beta_k$. Then, the function

$$\prod_{k=1}^{\infty} \left(1 - \frac{s}{\alpha_k}\right) E(s, \beta_k, m)$$

is well defined and vanishes precisely at $\{\alpha_k\}_{k=1}^{\infty}$. By standard complex analysis theory [K3; p. 6], such a function differs from $\mathfrak{D}(s)$ at most by a factor $\exp[z(s)]$, where $z(s)$ is an entire function; that is,

$$\mathfrak{D}(s) = \exp[z(s)] \prod_{k=1}^{\infty} \left(1 - \frac{s}{\alpha_k}\right) E(s, \beta_k, m) .$$

As a matter of fact, $\exp[z(s)]$ must be a constant and, in fact, equal to $A_\infty = \prod_{k=1}^{\infty} (\alpha_k/\beta_k)$, provided this infinite product is well defined, i.e. provided $(\alpha_k - \beta_k)/\beta_k \in l_1$. To see this, one writes

$$\exp[z(s)] = \left(1 - \sum_{r=1}^{\infty} \frac{h_r}{s - \beta_r}\right)\left(\prod_{k=1}^{\infty} \frac{\alpha_k}{\beta_k}\right) \prod_{k=1}^{\infty} \frac{(1 - \beta_k/s)}{(1 - \alpha_k/s)}$$

from which one obtains the limit value A_∞ by letting s go to infinity in any way except along the negative real axis; this, leads to $\exp[z(s)]$ as being $\mathcal{O}(1)$, and hence, by Liouville's Theorem, as being the constant A_∞. We have thus proved the first part of the following claim, whose assumption, as already remarked, will be shown later to be non-void.

PROPOSITION 3.4. – Let there exist a vector $q = \{q_k\}_{k=K}^{\infty} \in QL_2(\Omega)$ whose corresponding function $\hat{h}(t)$ in (3.14), obtained through the constants $\{h_k\}_{k=1}^{\infty}$ of (3.17), satisfies

$$\hat{h}(\alpha_k) = 1 , \quad k = 1, 2, \dots \quad \text{with multiplicity one} ,$$

for a negative sequence $\{\alpha_k\}_{k=1}^{\infty}$ with

(3.26) $\qquad \alpha_k \neq \beta_j , \quad \alpha_k \sim \beta_k , \quad$ and $\quad (\alpha_k - \beta_k)/\beta_k \in l_1 , \quad k = j = 1, 2, \dots .$

Then, the following identity over $\boldsymbol{C}$ holds:

$$(1 - \hat{h}(s))\,\mathfrak{B}(s) = \mathfrak{D}(s) , \tag{3.27}$$

where $\mathfrak{B}s)$ is defined by (3.23) and

$$\mathfrak{D}(s) = A_\infty \prod_{k=1}^{\infty} \left(1 - \frac{s}{\alpha_k}\right) E(s, \beta_k, m) , \tag{3.28}$$

where

$$A_\infty = \prod_{k=1}^{\infty} \left(\frac{\alpha_k}{\beta_k}\right).$$

Moreover, the correspending sequence h_r is expressed by

$$h_r = \frac{A_\infty \beta_r \prod_{k=1}^{\infty}\left(1-\frac{\beta_r}{\alpha_k}\right)E(\beta_r, \beta_k, m)}{\prod_{\substack{k=1\\k\neq r}}^{\infty}\left(1-\frac{\beta_r}{\beta_k}\right)E(\beta_r, \beta_k, m)}, \qquad r = 1, 2, \ldots . \tag{3.29}$$

PROOF. – The entire proposition was proved above, following (3.25), except for the expressions (3.29), which we now derive as a consequence of (3.27). For the assumed q, rewrite (3.27) explicitly as

$$\left(1-\frac{h_r}{s-\beta_r}-\sum_{\substack{j=1\\j\neq r}}^{\infty}\frac{h_j}{s-\beta_j}\right)\mathfrak{B}(s) = \mathfrak{D}(s) .$$

In other words, by (3.23), for $r = 1, 2, \ldots,$

$$\left(1-\sum_{\substack{j=1\\j\neq r}}^{\infty}\frac{h_j}{s-\beta_j}\right)\mathfrak{B}(s) + \frac{h_r}{\beta_r}\prod_{\substack{k=1\\k\neq r}}^{\infty}\left(1-\frac{s}{\beta_k}\right)E(s, \beta_k, m) = \mathfrak{D}(s) .$$

We now set $s = \beta_r$ in the above expression. Using $\mathfrak{B}(\beta_r) = 0$ and (3.28), we obtain the desired formulas (3.29). □

The following Lemma will be needed.

LEMMA 3.5. – For $\alpha_k \sim \beta_k$, we have

(i) $$\frac{\prod_{\substack{k=1\\k\neq r}}^{\infty}\left(1-\frac{\beta_r}{\alpha_k}\right)E(\beta_r, \beta_k, m)}{\prod_{\substack{k=1\\k\neq r}}^{\infty}\left(1-\frac{\beta_r}{\beta_k}\right)E(\beta_r, \beta_k, m)} \sim 1;$$

(ii) $$\frac{\prod_{\substack{k=1\\k\neq r}}^{\infty}\left(1-\frac{\alpha_r}{\beta_k}\right)E(\alpha_r, \beta_k, m)}{\prod_{\substack{k=1\\k\neq r}}^{\infty}\left(1-\frac{\alpha_r}{\alpha_k}\right)E(\alpha_r, \beta_k, m)} \sim 1 .$$

PROOF OF LEMMA 3.5. – We have, for $r, k = 1, 2, \ldots,$

$$\frac{\beta_r}{\alpha_k} \sim \frac{\beta_r}{\beta_k} \quad \text{and also} \quad \frac{\alpha_r}{\beta_k} \sim \frac{\alpha_r}{\alpha_k}, \tag{3.30}$$

and the conclusion follows. □

COROLLARY 3.6. – Under the assumptions of Proposition 3.4, the following asymptotic estimate holds for the sequence h_r in (3.29) generated by the assumed vector q, as $r \to \infty$,

$$h_r \sim \alpha_r - \beta_r \sim q_r[\overline{w}_s]_r \beta_r^\delta \to 0 \quad \text{as } r \to \infty . \tag{3.31}$$

PROOF. – The first $\sim$ on the left stems from (3.29) via Lemma 3.5 (i). The second $\sim$ on the right then follows from (3.17*b*), as the term in $\{\ \}$ in that equation is obviously ~ 1. By $q_r[\overline{w}_s]_r \leqslant \text{const}/r$ (see (1.8*b*)) and by (3.21), we get

$$[q_r \overline{w}_s]_r \beta_r^\delta \leqslant \text{const} \frac{r^{2\delta/\nu}}{r} \to 0 \quad \text{as } r \to \infty$$

for $\nu = \dim \Omega \geqslant 2$, as assumed, cf. (3.7). □

REMARK 3.2. – The proof of Proposition 3.4 and Corollary 3.6 actually refines the initial assumption of (3.26) by leading to the more refined conclusion (3.31). In fact, (3.31) yields by virtue also of (3.21) as $k \to \infty$, and of (1.8′)

$$\frac{\alpha_k - \beta_k}{\beta_k} \sim \frac{1}{k^{1+2(1-\delta)/\nu}} \in l_1 \tag{3.32}$$

and Eq. (3.26) follows. Note that estimate (3.31) relates the assumed q and $\{\alpha_k\}_{k=1}^\infty$. Reference to Fig. 3.1 below will greatly help in following the rest of the proof. □

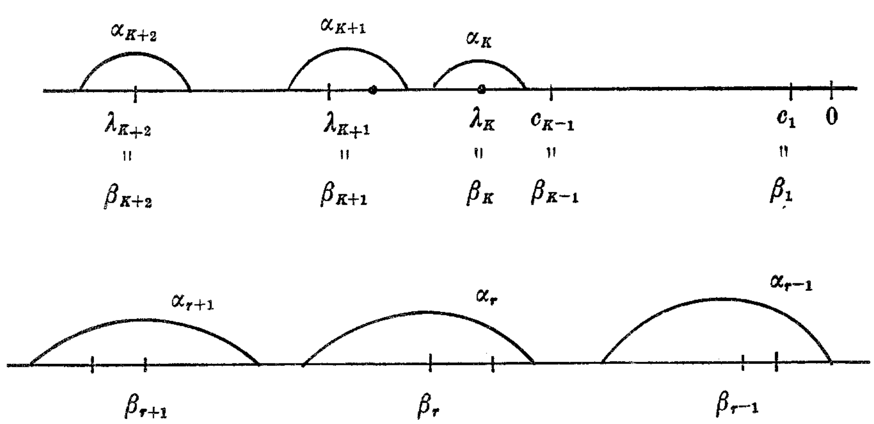

Figure 3.1. Asymptotic behavior of the constants $\{\alpha_r\}$ with respect to the constants $\{\beta_r\}$ (see (3.31) and (3.39)).

We now tackle the problem of the existence of vectors q, as postulated in Proposition 3.4 and Corollary 3.6. To this end, it is convenient to introduce the following definition which is motivated by (3.17) and the paragraph below (3.17).

DEFINITION 3.1. – An l_1-sequence $\{\tilde{h}_k\}_{k=1}^{\infty}$ will be said to *satisfy* the *Realizability Conditions* for the problem under study if it satisfies the conditions, for $r = 1, \dots, K-1$,

$$\tilde{h}_r = p_r(\tilde{w}_u)_r \sum_{k=K}^{\infty} \frac{\tilde{h}_k}{(\lambda_k - c_r)\left\{1 - \sum_{i=1}^{K-1} \frac{p_i(w_u)_i}{\lambda_k - c_i}\right\}}, \tag{3.33}$$

which are crucial for the realizability of such $\{\tilde{h}_k\}_{k=1}^{\infty}$ through a vector q as demanded by (3.17). □

Reversing the procedure of Proposition 3.4 and Corollary 3.6, we now first assign a sequence $\{\alpha_k\}_{k=1}^{\infty}$, with appropriate asymptotic behavior as suggested by (3.31), (see Fig. 3.1), and then solve (3.27) for a suitable sequence $\{\tilde{h}_k\}_{k=1}^{\infty}$ (see Proposition 3.7). We then study in Theorem 3.8 how to force such a solution $\{\tilde{h}_k\}_{k=1}^{\infty}$ to satisfy the Realizability Conditions as well. Actually, even more is accomplished by the following two results.

PROPOSITION 3.7. – Fix an arbitrary vector $v = \{v_k\}_{k=1}^{\infty} \in l_\infty$ with $v_k \neq 0$.
Next, assign a negative sequence $\{\alpha_k\}_{k=1}^{\infty}$, which satisfies

$$\alpha_k \neq \beta_j, \quad k = j = 1, 2, \dots, \tag{3.34}$$

and the asymptotic estimate

$$\alpha_k - \beta_k \sim [\bar{w}_s]_k v_k \beta_k^{\delta}, \tag{3.35}$$

which implies $(\alpha_k - \beta_k)/\beta_k \in l_1$ (see Remark 3.2 and (1.8*b*)).
Then, there exists a sequence $\{\tilde{h}_j\}_{j=1}^{\infty}$ such that

$$\left(1 - \sum_{j=1}^{\infty} \frac{\tilde{h}_j}{s - \beta_j}\right) \mathcal{B}(s) = \mathcal{D}(s), \tag{3.36}$$

with $\mathcal{B}(s)$ and $\mathcal{D}(s)$ as in (3.23) and (3.28), respectively. Identity (3.36) then implies

$$\hat{\tilde{h}}(\alpha_k) \stackrel{\text{df}}{=} \sum_{j=1}^{\infty} \frac{\tilde{h}_j}{\alpha_k - \beta_j} \equiv 1, \quad k = 1, 2, \dots$$

and, moreover, uniquely determines the $\{\tilde{h}_j\}_{j=1}^{\infty}$, according to formula (3.29), written for $\tilde{h}_j$.

PROOF. – See Appendix 4B of [L9] for a similar situation. □

What follows is a main result that affirms the existence of admissible vectors q (cf. (3.1)) as postulated in Propostion 3.4 and Corollary 3.6, provided that it has a suitable small l_∞-norm and all its coordinates $\langle q, \Phi_k \rangle = q_k$, of q, $k = K, K+1, \dots$ are nonzero.

THEOREM 3.8. – Let a vector $q \in X_s$ as in (3.1) be given; i.e. $q = (-A_s)^{\frac{3}{4}-\varrho} QMg = \{q_k\}_{k=K}^{\infty}$ for some $g \in L_2(\Gamma)$, where now the coordinates (11) $q_k = (-\lambda_k)^{-\frac{1}{4}-\varrho}(g, \Phi_k|_\Gamma)_\Gamma$ will be required (12) to be different from zero. Then, for all vectors $\overline{w}_s$ which satisfy condition (1.8*b*) and, which, in addition, possess a norm $|\overline{w}_s|_{l_\infty}$ sufficiently small, one can construct:

(i) a vector $\overline{v} = \{\overline{v}_k\}_{k=K}^{\infty} \in l_2$, with

$$0 \neq |\overline{v}_k| \leqslant C_v, \qquad k = K, K+1, \dots \tag{3.37}$$

where C_v is sufficiently small so that the corresponding sequence $\{\tilde{\alpha}_k\}_{k=K}^{\infty}$ defined by

$$\tilde{\alpha}_k - \beta_k = [\overline{w}_s]_k \beta_k^{\delta} \overline{v}_k, \qquad k = K, K+1, \dots \tag{3.38}$$

has its terms $\tilde{\alpha}_k$ negative, distinct, and satisfying

$$\tilde{\alpha}_k \neq \beta_j, \qquad k = K, K+1, \dots; \; j = 1, 2, \dots; \tag{3.39}$$

(ii) a sequence $\{\tilde{\alpha}_i\}_{i=1}^{K-1}$ of negative, distinct constants with

$$\tilde{\alpha}_i \neq \tilde{\alpha}_k, \quad \text{and} \quad \tilde{\alpha}_i \neq \beta_j, \qquad i = 1, \dots, K-1; \; k = K, K+1, \dots; \; j = 1, 2, \dots \tag{3.40}$$

such that the corresponding sequence $\{\overline{h}_r\}_{r=1}^{\infty}$ with $\{\beta_r^{-\delta}\overline{h}_r\}_{r=1}^{\infty} \in l_1$ defined, according to Proposition 3.7, by

$$\overline{h}_r = \frac{\beta_r A_\infty \prod_{k=1}^{\infty}\left(1 - \dfrac{\beta_r}{\tilde{\alpha}_k}\right) E(\beta_r, \beta_k, m)}{\prod_{\substack{k=1 \\ k \neq r}}^{\infty}\left(1 - \dfrac{\beta_r}{\beta_k}\right) E(\beta_r, \beta_k, m)}, \qquad r = 1, 2, \dots, \tag{3.41}$$

where the exponential function $E(\beta_r, \beta_k, m)$ is defined in (3.24) and A_∞ is specified following (3.28)), satisfies the Realizability Condition (3.33).

---

(11) See Appendix 4A, below (4A.4).

(12) This is possible since $\Phi_k|_\Gamma \neq 0$ for C^∞-boundaries Γ [S1, Cor. 2.2] and for parallelepipeds.

Moreover, the function $\bar{h}(s) \stackrel{\mathrm{df}}{=} \sum_{r=1}^{\infty} \bar{h}_r/(s-\beta_r)$, corresponding to such a sequence $\{\bar{h}_r\}_{r=1}^{\infty}$ satisfies $1-\hat{h}(\bar{\alpha}_k)=0$, $k=1,2,\ldots$ with multiplicity one, while $\hat{h}(s)\neq 1$ for $s\neq\bar{\alpha}_k$.

Therefore, according to Proposition 3.4, the function $\hat{h}(s)$ defined above satisfies (3.27). □

PROOF. – Conclusions (i) and (ii) are proved in Appendices 4A and 4B. These appendices construct, ultimately by means of a fixed point technique, the sequence $\{\bar{\alpha}_i\}_{i=1}^{K-1}$ for which the Realizability Conditions hold, and, in addition, the claimed vector $\bar{v}$ and the sequence $\{\bar{\alpha}_k\}_{k=K}^{\infty}$.

Moreover, the continuity of the map $\{v_k\}_{k=K}^{\infty}\to\{\alpha_i\}_{i=1}^{K-1}: l_\infty\to \boldsymbol{R}^{K-1}$, needed in Appendix 4A, is proved in Appendix 4B.

To show that such $\{\beta_r^{-\delta}\bar{h}_r\}_{r=1}^{\infty}$ is in l_1, we need only invoke part (i) of Lemma 3.5 to obtain the $\sim$ on

$$\bar{h}_r \sim \bar{\alpha}_r - \beta_r = [\overline{w}_s]_r \bar{v}_r \beta_r^{\delta}\,, \tag{3.42}$$

while the equality on the right stems from (3.38). The Schwarz inequality applied to (3.42) ensures then that $\{\beta_r^{-\delta} h_r\}_{r=1}^{\infty}\in l_1$.

As to the claim for the corresponding function $\hat{h}(s)$, this stems from (3.27), which holds by virtue of Proposition 3.7. □

The next result establishes that any admissible vector q that fulfills (3.27) also furnishes the desired solution to (3.10): $d(t)$, of the special class IDE, with exexponents α_r all different from the constants $\{\beta_r\}$.

THEOREM 3.9. – Let q be an admissible vector with corresponding function $\hat{h}(s)$, obtained through (3.17), that satisfies (3.27) for negative constants $\{\alpha_k\}$ obeying the estimate (3.31). (Such vectors q are provided by the proof of Theorem 3.8.) Then, the following properties hold for the corresponding solution $d(t)$ to the Volterra equation (3.10):

(i) the solution $d(t)$ has the form

$$d(t) = \sum_{r=1}^{\infty} d_r \exp[\alpha_r t]\,, \qquad t\in\overline{\boldsymbol{R}^+}; \tag{3.43}$$

that is,

$$\hat{d}(s) = \sum_{r=1}^{\infty} \frac{d_r}{s-\alpha_r}\,. \tag{3.44}$$

(For Re $s>0$, $\hat{d}(s)$ is the one-sided Laplace transform of $d(t)$ and is extended over $\boldsymbol{C}$ through the right-hand side of (3.44).)

(ii) The coefficients $\{d_r\}$ satisfy the conditions

$$(3.45) \qquad \sum_{r=1}^{\infty} |d_r \alpha_r^{1-\delta}| < \infty\,, \quad \text{and} \quad d_r \alpha_r^{1-\delta} \leqslant \frac{\text{const}}{r}\,, \quad r = 1, 2, \ldots$$

analogous to the properties of $\{n_r \beta_r^{1-\delta}\}$ of Lemma 3.1 (i).

(iii) The coefficients d_r are the residues, res $\hat{d}(\alpha_r)$, of $\hat{d}(s)$ at $\{\alpha_r\}$:

$$(3.46) \qquad d_r = \operatorname{res} \hat{d}(\alpha_r)\,.$$

Consequently, $d(t)$ is a function of the special class IDE.

PROOF. – To prove (3.44), we apply the Mittag-Leffler expansion to $\hat{d}(s)$. To this end, the following proposition is crucial.

PROPOSITION 3.10. – Under the hypothesis of Theorem 3.9, the residues res $\hat{d}(\alpha_r)$ of the function $\hat{d}(s)$ at the points α_r satisfy the estimate

$$\operatorname{res} \hat{d}(\alpha_r) = \mathcal{O}\left(\left(\frac{r^{2\delta/\nu}}{r^{1+\delta/\nu}}\right) + |n_r|\right) \qquad \text{as } r \to \infty\,.$$

PROOF OF PROPOSITION 3.10. – With $\mathcal{B}(s)$ and $\mathcal{D}(s)$ the entire functions given by (3.23) and (3.28), we have from (3.20) and (3.27)

$$(3.47) \qquad \operatorname{res} \hat{d}(\alpha_r) = \lim_{s \to \alpha_r} (s-\alpha)\frac{\hat{n}(s)\,\mathcal{B}(s)}{\mathcal{D}(s)} = -\frac{\hat{n}(\alpha_r)\alpha_r(\beta_r-\alpha_r)\prod\limits_{\substack{k=1\\k\neq r}}^{\infty}\left(1-\dfrac{\alpha_r}{\beta_k}\right)E(\alpha_r, \beta_k, m)}{A_\infty \beta_r \prod\limits_{\substack{k=1\\k\neq r}}^{\infty}\left(1-\dfrac{\alpha_r}{\alpha_k}\right)E(\alpha_r, \beta_k, m)}$$

$$(3.48) \qquad \sim \hat{n}(\alpha_r)(\alpha_r - \beta_r)\,,$$

by Lemma 3.5 (ii), and $\alpha_r \sim \beta_r$, where

$$(3.49) \qquad \hat{n}(\alpha_r) = \sum_{i=1}^{\infty} \frac{n_i}{\alpha_r - \beta_i}\,.$$

As $\{n_j \beta_j^{1-\delta}\}_{j=1}^{\infty}$ belongs to l_1 and $|n_i \beta_i^{1-\delta}| \leqslant c/i$, according to Lemma 3.1 (i), we now need to invoke part (i) of the following Lemma. Part (ii) will be needed later on in Lemma 3.12 and Theorem 3.14.

LEMMA 3.11. – For any vector $b = \{b_j\}_{j=1}^{\infty}$ such that $b_j \leqslant \text{const}/j$, the following estimates hold for $\alpha_i \sim \beta_i \overset{\text{def}}{=} \lambda_i$:

(i):

1) $\displaystyle\sum_{i=r+1}^{\infty} \frac{b_i \lambda_i^\delta}{\lambda_i(\lambda_i - \alpha_r)} = \mathcal{O}\left(\frac{1}{\alpha_r^{2-\varepsilon'-\delta}}\right)$;

2) $\displaystyle\sum_{i=1}^{r-1} \frac{b_i \lambda_i^\delta}{\lambda_i(\lambda_i - \alpha_r)} = \mathcal{O}\left(\frac{1}{\alpha_r}\right)$;

where ε' is an arbitrary positive number. In addition, by symmetry with the above,

(ii):

1) $\displaystyle\sum_{i=k+1}^{\infty} \frac{b_i \alpha_i^\delta}{\alpha_i(\lambda_k - \alpha_i)} = \mathcal{O}\left(\frac{1}{\lambda_k^{2-\delta-\varepsilon'}}\right)$;

2) $\displaystyle\sum_{i=1}^{k-1} \frac{b_i \alpha_i^\delta}{\alpha_i(\lambda_k - \alpha_i)} = \mathcal{O}\left(\frac{1}{\lambda_k}\right)$.

PROOF. – The proof is similar to the one given in Appendix 4E of [L9]. □

Continuing with the proof of Proposition 3.10, we apply Lemma 3.11 (i) to the sum in (3.49), after multiplying the numerator and denominator by $\lambda_i^{1-\delta}$ as allowed by Lemma 3.1 (i), to obtain

$$\hat{n}(\alpha_r) = \mathcal{O}\left(\frac{1}{\alpha_r}\right) + \frac{n_r}{\alpha_r - \beta_r}. \tag{3.50}$$

Inserting (3.50) into (3.48) yields

$$\text{res}\ \hat{d}(\alpha_r) = \mathcal{O}\left(\frac{\alpha_r - \beta_r}{\alpha_r}\right) + \mathcal{O}(n_r). \tag{3.51}$$

The desired conclusion of Proposition 3.10 then follows from (3.51) via

$$\frac{\alpha_r - \beta_r}{\alpha_r} = \mathcal{O}\left(\frac{r^{2\delta/\nu}}{r} \cdot \frac{1}{r^{2/\nu}}\right), \tag{3.52}$$

which is the result of (3.37)-(3.38), of $q_r[\overline{w}_s]_r \leqslant \text{const}/r$ (see (1.8*b*)) and of $\alpha_r \sim \beta_r \sim r^{2/\nu}$, from (3.21). □

Returning to the proof of Theorem 3.9, we see from Proposition 3.10 and Lemma 3.1 (i) that we can apply the Mittag-Leffler Theorem [L3; p. 394], [K3; p. 37ff] to obtain

$$\hat{d}(s) = \sum_{r=1}^{\infty} \frac{\text{res}\ \hat{d}(\alpha_r)}{s - \alpha_r} + e(s), \tag{3.53}$$

where $e(s)$ is an entire function. But from (3.20), we see that $\hat{d}(s)$ goes to zero for $s \to \infty$ in any way except along the negative real axis and this leads to $e(s) = \mathcal{O}(1)$ and hence, by Liouville's Theorem, to $e(s) \equiv 0$.

We can then rewrite (3.53) as in (3.44), from which relations (3.46) follow immediately. We now prove (3.45). To this end, we invoke (3.46), and (3.51), to obtain

$$d_r \alpha_r^{1-\delta} \sim \frac{\alpha_r - \beta_r}{\alpha_r^\delta} + n_r \alpha_r^{1-\delta} \sim h_r \alpha_r^{-\delta} + n_r \alpha_r^{1-\delta} \tag{3.54}$$

where the right-hand side estimate makes use of (3.42). Then (3.45) follows via $\alpha_r \sim \beta_r$ from Theorem 3.8 (ii) and Lemma 3.1 (i). The proof of Theorem 3.9 is thus complete. □

3.3. *For $\mathscr{d}(t)$ as in § 3.2, the projections $x_u(t)$ and $x_s(t)$ are of the special class IDE.*

With the existence of representation (3.43) for some admissible vector q guaranteed by Theorems 3.8 and 3.9, the following Lemma will be useful in later arguments. The proof of this lemma, however, takes place in the t-domain.

LEMMA 3.12. – The solution $\mathscr{d}(t)$ of the integral equation (3.10) has the form (3.43) as a function of class IDE, where

$$\alpha_r \neq \beta_j\,, \qquad r, j = 1, 2, \ldots,$$

if and only if the following conditions are satisfied.

$$\sum_{i=1}^{\infty} \frac{h_i}{\alpha_r - \beta_i} \equiv 1\,, \qquad r = 1, 2, \ldots, \tag{3.55}$$

$$n_r = -h_r \sum_{i=1}^{\infty} \frac{d_i}{\alpha_i - \lambda_r}\,, \qquad r = K, K+1, \ldots, \tag{3.56a}$$

$$\frac{(x_{0u})_r}{p_r} = \sum_{i=1}^{\infty} \frac{d_i}{\alpha_i - c_r}\,, \qquad r = 1, \ldots, K-1\,. \tag{3.56b}$$

PROOF. – Let $\mathscr{d}(t)$ be given by (3.43). Then, inserting (3.43), (3.13) and (3.14) into (3.10) and equating to zero, by linear independence argument, all the coefficients of exponentials results, after straightforward computations, in (3.55), as well as in

$$n_r \equiv -h_r \sum_{i=1}^{\infty} \frac{d_i}{\alpha_i - \beta_r}\,, \qquad r = 1, 2, \ldots. \tag{3.57}$$

Relation (3.55) means, of course, that the $\{\alpha_r\}_{r=1}^{\infty}$ are zeros of the denominator $(1 - \hat{\mathscr{k}}(s))$ of $\hat{\mathscr{d}}(s)$. For $r = K, K+1, \ldots,$ (3.57) leads to (3.56a) via (3.15). For $r = 1, \ldots, K-1$, however, the ratios n_r/h_r in (3.57) are computed via (3.16a) and (3.17a), thus leading to (3.56b).

Reversing the steps of the above procedure proves the opposite direction. □

REMARK 3.3. – Summing up (3.57) in r from one through ∞ and using (3.55) yields $\sum_{r=1}^{\infty} n_r = -\sum_{r=1}^{\infty} d_r$ $\square$

A final Lemma is needed.

LEMMA 3.13. – For any admissible vector q provided by Theorems 3.8 and 3.9 the corresponding solution $\mathscr{d}(t)$ of the form (3.43) to equation (3.10) satisfies the following asymptotic estimates:

$$\frac{d_r}{\lambda_r - \alpha_r} = \mathcal{O}\left(\frac{[(-A_s)^{\frac{3}{4}-\varrho} x_{0s}]_r}{q_r \lambda_r}\right) + \mathcal{O}\left(\frac{1}{\lambda_r}\right).$$

PROOF. – By (3.56a), we can write for $r = K, K+1, \ldots,$

$$\frac{d_r}{\lambda_r - \alpha_r} = \frac{n_r}{h_r} - \sum_{\substack{j=1 \\ j \neq r}}^{\infty} \frac{d_j}{\lambda_r - \alpha_j}.$$

Next, by virtue of (3.45) and Lemma 3.11 part (ii), we have

$$\sum_{\substack{j=1 \\ j \neq r}}^{\infty} \frac{d_j}{\lambda_r - \alpha_j} = \mathcal{O}\left(\frac{1}{\lambda_r}\right).$$

Finally, (3.16b) and (3.17b) provide

$$\frac{n_r}{h_r} = \mathcal{O}\left(\frac{[(-A_s)^{\frac{3}{4}-\varrho} x_{0s}]_r}{q_r \lambda_r}\right) + \mathcal{O}\left(\frac{1}{\lambda_r}\right)$$

and the Lemma is proved. $\square$

We are finally ready now to draw the desired conclusions to the solution $x(t)$.

THEOREM 3.14. – For any admissible vector q provided by Theorems 3.8 and 3.9 the projection $x_u(t)$ of the solution $x(t)$ is an X_u-function of the special class IDE ([13]).

PROOF. – In view of (2.13), it is enough to show the desired conclusion for the integral term of (3.4). By (2.14), (3.8) and (3.43), this term can be rewritten for

([13]) Since X_u is finite dimensional and A_u an operator on it, then $A_u^r x_u(t)$ for any power r is also of the special class IDE.

$t \geqslant 0$ as

$$(3.58)\qquad \bar{x}_u(t) \overset{\text{df}}{=} \int_0^t \exp[\bar{A}_u(t-\tau)]p\, d(\tau)\, d\tau$$

$$= \sum_{i=1}^{K-1}\left\{\int_0^t \exp[c_i(t-\tau)]p_i \sum_{=1}^{\infty} d_r \exp[\alpha_r \tau]\, d\tau\right\}\psi_i$$

$$(by\ (2.20)) \quad = \sum_{i=1}^{K-1}\left\{p_i \sum_{r=1}^{\infty} d_r \frac{\exp[\alpha_r t] - \exp[c_i t]}{\alpha_r - c_i}\right\}\psi_i$$

$$= \sum_{i=1}^{K-1}\left\{-p_i\left(\sum_{r=1}^{\infty}\frac{d_r}{\alpha_r - c_i}\right)\exp[c_i t] + p_i \sum_{r=1}^{\infty}\frac{d_r}{\alpha_r - c_i}\exp[\alpha_r t]\right\}\psi_i$$

$$(by\ (3.56b)) \quad = \sum_{i=1}^{K-1}\left\{-(x_{0u})_i \exp[c_i t] + p_i \sum_{r=1}^{\infty}\frac{d_r}{\alpha_r - c_i}\exp[\alpha_r t]\right\}\psi_i\,, \qquad t > 0\,.$$

$$(3.58b)\qquad (by\ (2.13)) \quad = -\exp[\bar{A}_u t]x_0 + \sum_{i=1}^{K-1} p_i \sum_{r=1}^{\infty}\frac{d_r}{\alpha_r - c_i}\exp[\alpha_r t]\,.$$

But, since the $\{c_i\}$ and the $\{\alpha_r\}$ were chosen (see Fig. 3.1)

$$\inf_{\substack{r=1,2,\ldots\\ i=1,\ldots,K-1}} |\alpha_r - c_i| = \gamma > 0\,,$$

relations (3.45) in Theorem 3.9 imply *a fortiori* that the infinite sum in r in (3.58*b*) is a function of the special class IDE. The desired conclusion is then contained in (3.58*b*). □

The proof for the relevant result for $x_s(t)$ on X_s passes through the following theorem.

THEOREM 3.15. – For any admissible vector q provided by Theorems 3.8 and 3.9, the function $(-A_s)^{\frac{3}{4}-\varrho}x_s(t)$ is of the special class IDE in the X_s-weak topology. More precisely, we have for $y \in X_s = QL_2(\Omega)$

$$(3.59)\qquad \langle(-A1)^{\frac{3}{4}-\varrho}x_s(t), y\rangle = \sum_{r=1}^{\infty}\exp[\alpha_r t]d_r\left[1 + \sum_{i=1}^{K-1}\frac{(\tilde{w}_u)_i p_i}{\alpha_r - c_i}\right]\left[\sum_{k=K}^{\infty}\frac{\lambda_k q_k y_k}{\alpha_r - \lambda_k}\right].$$

PROOF. – We have from (3.5), (3.4):

$$(3.60)\qquad \langle(-A_s)^{\frac{3}{4}-\varrho}x_s(t), y\rangle = \langle\exp[A_s t](-A_s)^{\frac{3}{4}-\varrho}x_{0s}, y\rangle +$$

$$+\left\langle\int_0^t A_s \exp[A_s(t-\tau)]q\langle\exp[\bar{A}_u\tau]x_{0u}, \tilde{w}_u\rangle\, d\tau, y\right\rangle +$$

$$+\left\langle\int_0^t A_s \exp[A_s(t-\tau)]q[\langle\bar{x}_u(\tau), w_u\rangle + d(\tau)]\, d\tau, y\right\rangle,$$

with $d(\cdot)$ and $\bar{x}_u(\cdot)$ defined by (3.8) and (3.58), respectively.

Since by (3.58*b*) and (3.43),

$$(3.61)\quad \langle \bar{x}_u(\tau), \tilde{w}_u\rangle + d(\tau) = -\langle \exp[\bar{A}_u\tau]x_{0u}, \tilde{w}_u\rangle + \\ + \sum_{r=1}^{\infty} d_r\left\{1+\sum_{i=1}^{K-1}\frac{(\tilde{w}_u)_i p_i}{\alpha_r - c_i}\right\}\exp[\alpha_r\tau]\,.$$

we see that the first term in (3.61), once inserted in (3.60), cancels the second term in (3.60). Hence, substituting (3.61) into (3.60) and using the convolution relations (2.20) yields

$$(3.62)\quad \langle(-A_s)^{\frac{3}{4}-\varrho}x_s(t), y\rangle = \langle \exp[A_s t](-A_s)^{\frac{3}{4}-\varrho}x_{0s}, y\rangle \\ + \sum_{k=K}^{\infty}\sum_{r=1}^{\infty}\lambda_k q_k y_k d_r\left\{1+\sum_{i=1}^{K-1}\frac{(\tilde{w}_u)_i p_i}{\alpha_r - c_i}\right\}\frac{\exp[\alpha_r t]-\exp[\lambda_k t]}{\alpha_r - \lambda_k} \\ = \sum_{k=K}^{\infty}\exp[\lambda_k t]\left\{\lambda_k q_k y_k\left(\sum_{r=1}^{\infty}\frac{d_r\left(1+\sum_{i=1}^{K-1}\frac{(\tilde{w}_u)_i p_i}{\alpha_r - c_i}\right)}{\lambda_k - \alpha_r}\right) + \lambda_k^{\frac{3}{4}-\varrho}[x_{0s}]_k y_k\right\} \\ + \sum_{r=1}^{\infty}\left\{\exp[\alpha_r t]d_r\left(1+\sum_{i=1}^{K-1}\frac{(\tilde{w}_u)_i p_i}{\alpha_r - c_i}\right)\left(\sum_{k=K}^{\infty}\frac{\lambda_k q_k y_k}{\alpha_r - \lambda_k}\right)\right\}.$$

To ascertain that all the infinite sums in (3.62) are well defined, we observe preliminarily that

$$(3.63a)\quad \sum_{r=1}^{\infty}\left|\frac{d_r}{\alpha_r - \lambda_k}\right| = \sum_{r=1}^{k-1}\left|\frac{d_r\alpha_r^{1-\delta}}{\alpha_r^{1-\delta}(\alpha_r - \lambda_k)}\right| + \sum_{r=k+1}^{\infty}\left|\frac{d_r\alpha_r^{1-\delta}}{\alpha_r^{1-\delta}(\alpha_r - \lambda_k)}\right| + \left|\frac{d_k}{\alpha_k - \lambda_k}\right|$$

$$(3.63b)\quad = \mathcal{O}\left(\frac{1}{\lambda_k}\right) + \mathcal{O}\left(\frac{[(-A_s)^{\frac{3}{4}-\varrho}x_{0s}]_k}{\lambda_k q_k}\right),$$

which follows when Lemma 3.11 (ii) is applied to the first two sums in (3.63*a*) (a valid procedure when (3.45) is invoked) and Lemma 3.13 is applied to the last term in (3.63*a*). Hence, (3.63*b*) gives

$$(3.64)\quad \sum_{k=K}^{\infty}|\lambda_k q_k y_k|\left(\sum_{r=1}^{\infty}\left|\frac{d_r}{\alpha_r - \lambda_k}\right|\right) \leqslant \text{const}\sum_{k=K}^{\infty}|q_k y_k| + \left|((-A_s)^{\frac{3}{4}-\varrho}x_{0s})_k y_k\right| \\ \leqslant \text{const}\,(|q| + |(-A_s)^{\frac{3}{4}-\varrho}x_{0s}|)|y|\,.$$

Therefore, the following interchange of order of summation is allowed

$$(3.65)\quad \sum_{k=K}^{\infty}|\lambda_k q_k y_k|\sum_{r=1}^{\infty}\left|\frac{d_r}{\alpha_r - \lambda_k}\right| = \sum_{r=1}^{\infty}|d_r|\sum_{k=K}^{\infty}\left|\frac{\lambda_k q_k y_k}{\alpha_r - \lambda_k}\right|,$$

showing by (3.64) that, as $|\alpha_r| \to \infty$, the following sequence in r is in l_1:

$$\left\{ d_r \left(1 + \sum_{i=1}^{K-1} \frac{(\tilde{w}_u)_i p_i}{\alpha_r - c_i} \right) \left(\sum_{k=K}^{\infty} \frac{\lambda_k q_k y_k}{\alpha_r - \lambda_k} \right) \right\}_{r=1}^{\infty} \in l_1 . \tag{3.66}$$

Since the term $\sum_{i=1}^{K-1}$ goes to zero as $r \to \infty$, we conclude from (3.65) and (3.66) that Eq. (3.62) is well defined as a function of class IDE. To complete the proof of Theorem 3.15, it remains to show that the first sum $\sum_{k=K}^{\infty}$ in Eq. (3.62) is, in fact, identically zero. To this end, we recall the definition of $\mathscr{d}(t)$ (Eq. (3.8)) and its expansion (3.43) for q admissible as assumed, along with $\sigma < \frac{3}{4} - \varrho$ (Eq. (1.11)). We then deduce that for $y = \bar{y}$ with (see (2.17)):

$$\bar{y} = (-A_s)^{\frac{3}{4}+\varrho+\sigma} \overline{w}_s \in X_s$$

the first sum $\sum_{k=K}^{\infty}$ in (3.62) vanishes identically. This implies (14) that all its coefficients are identically zero,

$$\bar{y}_k \left[\lambda_k q_k \left(\sum_{r=1}^{\infty} \frac{d_r \left(1 + \sum_{i=1}^{K-1} \frac{(\tilde{w}_u)_i p_i}{\alpha_r - c_i} \right)}{\lambda_k - \alpha_r} \right) + \lambda_k^{\frac{3}{4}-\varrho} [x_{0s}] \right] \equiv 0 , \qquad k = K, K+1 ,$$

We then divide by the non-zero coefficient $\bar{y}_k$ [see (1.8'): $[\overline{w}_s]_k \neq 0$] and obtain the desired conclusion. Theorem 3.15 is fully proved. □

To finish off the proof of Theorem 1.3, we need to tackle the *synthesis* problem of the vectors $p \in X_u$ and $q \in X_s$ by a suitable boundary vector $g \in L_2(\Gamma)$, as dictated by (3.1) and (3.2). More precisely, we seek a vector g in $L_2(\Gamma)$ such that:

(i) it satisfies the finite moment problem (3A.7) of Appendix A of [L8], involving only the coordinates $(g, \Phi_k|_\Gamma)_\Gamma$, $k = 1, \dots, K-1$;

(ii) the vector $q = (-A_s)^{\frac{3}{4}-\varrho} QMg \in X_s$ has all coordinates (15)

$$\langle q, \Phi_r \rangle = -\lambda_r^{-\frac{1}{4}-\varrho} (g, \Phi_r|_\Gamma)_\Gamma , \qquad r = K, K+1, \dots$$

different from zero, so that the procedure of Appendices 4A-B applies.

---

(14) If $\sum_{k=K}^{\infty} z_k \exp[\lambda_k t] \equiv 0$, $t \geq 0$, with $\{z_k\} \in l_1$, then term-by-term Laplace transforming gives

$$\frac{z_k}{\lambda - \lambda_k} + \sum_{\substack{j=1 \\ j \neq k}}^{\infty} \frac{z_j}{\lambda - \lambda_j} \equiv 0 \qquad \text{for } \lambda \not\equiv \{\lambda_k\}_{k=K}^{\infty}$$

by analytic continuation. Integrating along a small circle centered in λ_k yields by Cauchy's theorem $z_k \equiv 0$ as desired.

(15) We are using here $M^* \Phi_k = -(1/\lambda_k)(g, \Phi_k|_\Gamma)_\Gamma$ (see [L7, Lemma 4.1]).

This can be accomplished by means of boundary vectors g of the type $g = \bar{g} + g^*$, where $\bar{g}$ is the solution, unique in the space $\mathscr{F} = \text{span}\,\{\Phi_k|_\Gamma\}_{k=1}^{K-1}$, of the moment problem (3A.7) in [L8], and g^* is any vector orthogonal to $\mathscr{F}$. In fact, we compute

$$A_u PMg^* = \sum_{k=1}^{K-1} \lambda_k \langle Mg^*, \Phi_k\rangle \Phi_k = -\sum_{k=1}^{K-1} (g^*, \Phi_k|_\Gamma)_\Gamma \Phi_k = 0$$

so that for the stabilizing vector p in X_u of Lemma 2.1 we get $A_u PMg = A_u PM\bar{g}$. The proof of Theorem 1.3 is thus complete. □

PROOF OF COROLLARY 1.4. – We write more conveniently $\{\gamma_n\}_{n=1}^{\infty}$ for the sequence $\{c_i\}_{i=1}^{K-1}$, and $\{\alpha_r\}_{r=1}^{\infty}$. As noted at the end of Theorem 1.3, the expansion (1.10) holds in the weak topology of $L_2(\Omega)$, when $x_0 \in L_2(\Omega)$ in which case, we can write

$$\langle S_F(t)x_0, y\rangle = \sum_{n=1}^{\infty} u_n(x_0, y) \exp[\gamma_n t], \qquad t \geq 0 \tag{3.67}$$

for the desired feedback semigroup $S_F(t)$ on $L_2(\Omega)$; where the $u_n(x_0, y)$'s are constants depending on x_0 and y, which form an l_1-sequence; moreover, u_n is a bounded linear functional on x_0 for y fixed, and similarly on y for x_0 fixed. Thus, $u_n(x_0, y) = \langle B_n x_0, y\rangle$ for bounded operators B_n on $L_2(\Omega)$. Application of the Laplace transform to (3.67) (term by term application is legal) yields for the resolvent of A_F:

$$\langle R(\mu, A_F)x_0, y\rangle = \sum_{n=1}^{\infty} \frac{\langle B_n x_0, y\rangle}{\mu - \gamma_n}, \qquad \mu \neq \{\gamma_n\} \tag{3.68}$$

after extension by analytic continuation. The constants $\{\gamma_n\}$ are then simple poles of the resolvent and thus simple eigenvalues of A_F [T5, Thm. 5.8-A, p. 306] with corresponding eigenvectors $e_{F,n}$.

Next compute around a small circle Γ_n, centered at a fixed $\gamma_{\bar{n}}$ and containing no other point of the sequence $\{\gamma_n\}$

$$\int_{\Gamma_{\bar{n}}} R(\mu, A_F)x_0\, d\mu = \int_{\Gamma_{\bar{n}}} \frac{B_n x_0}{\mu - \gamma_{\bar{n}}}\, d\mu = 2\pi i B_{\bar{n}} x_0$$

by Cauchy Theorem. Thus, B_n is the projection from $L_2(\Omega)$ onto the one dimensional eigenspace of A_F spanned by the normalized eigenvector $e_{F,n}$, along $(I - B_n)L_2(\Omega)$: $B_n x = \eta_n(x) e_{F,n}$, $\eta_n(x) = \text{scalar}$. Then, $\eta_n(e_{F,n}) = 1$ and $\eta_n(e_{F,m}) = 0$, $n \neq m$. From (3.67) with $t = 0$

$$x = \sum_{n=1}^{\infty} B_n x = \sum_{n=1}^{\infty} \eta_n(x)\, e_{F,n}, \qquad x \in L_2(\Omega) \tag{3.69}$$

so that $\{e_{F,n}\}_{n=1}^{\infty}$ is a basis on $L_2(\Omega)$. Since B_n commutes with A_F, we also obtain

$$A_F x = \sum_{n=1}^{\infty} B_n A_F x = \sum_{n=1}^{\infty} A_F B_n x = \sum_{n=1}^{\infty} \gamma_n \eta_n(x) e_{F,n}, \qquad x \in \mathscr{D}(A_F) \tag{3.70}$$

as desired. Expansions (3.69)-(3.70) can be written out explicitly as in (1.11)-(1.12) respectively. □

Appendix 4A.

Proof of Theorem 3.8 (i)-(ii).

(1) If $v = \{v_k\}_{k=K}^{\infty}$ is any vector satisfying

$$0 \neq |v_k| \leqslant C_v, \qquad k = K, K+1, \ldots \tag{4A.1}$$

we define a corresponding sequence $\{\alpha_k\}_{k=K}^{\infty}$ of scalars by setting

$$\alpha_k - \beta_k = [\overline{w}_s]_k \beta_k^{\delta} v_k, \qquad k = K, K+1, \ldots . \tag{4A.2}$$

Then by (1.8′), (3.21), and (4A.1)

$$[\overline{w}_s]_k \beta_k^{\delta} v_k \leqslant \text{const} \frac{k^{2\delta/\nu}}{k} \to 0 \qquad \text{as } k \to \infty$$

for $\dim \Omega = \nu \geqslant 2$ as assumed, and so $\alpha_k \sim \beta_k$ and $(\alpha_k - \beta_k)/\beta_k \in l_1$, by Remark 3.2. Thus, we conclude that: if C_v in (4A.1) is sufficiently small, then the constants α_k, $k = K, K+1, \ldots$ in (4A.2) are all *real* and *negative*, like the corresponding β_k's, as desired.

(2) If $\{c_i\}_{i=1}^{K-1}$ are the distinct negative constants obtained through Lemma 2.1 and e.g. required to be

$$\begin{array}{ccccccccc} \lambda_{K+1} < & \lambda_K < & c_{K-1} < & \ldots & < c_1 < 0 , \\ \| & \| & \| & & \| \\ \beta_{K+1} & \beta_K & \beta_{K-1} & & \beta_1 \end{array}$$

we consider vectors $\{a_i\}_{i=1}^{K-1}$ in the $\boldsymbol{R}^{K-1}$ sphere $\mathscr{S}_c$

$$\mathscr{S}_c = \{\{a_i\}_{i=1}^{K-1} : |a_i - c_i| \leqslant \varrho_c\}, \qquad i = 1, \ldots, K-1 \tag{4A.3}$$

with ϱ_c sufficiently small, so that all coordinates $a_1, \ldots, a_{K-1}$ are negative and distinct.

(3) We now let q be a vector of the form

$$q = (-A_s)^{\frac{3}{4}-\varrho} QMg \in QL_2(\Omega) \quad \text{for some } g \in L_2(\Gamma)\,. \tag{4A.4}$$

Since

$$q = \sum_{r=K}^{\infty} (-\lambda_r)^{\frac{3}{4}-\varrho} \langle Mg, \Phi_r \rangle \Phi_r = \sum_{r=K}^{\infty} (-\lambda_r)^{\frac{3}{4}-\varrho} (g, M^*\Phi_r)_\Gamma \Phi_r$$

and $M^*\Phi_r = ((-1)/(\lambda_r))\Phi_r|_\Gamma$ [L7; Lem. 4.1], we deduce that the coordinates q_r of q are

$$q = \{q_r\}_{r=K}^{\infty} = \{(-\lambda_r)^{-\frac{1}{4}-\varrho}(g, \Phi_r|_\Gamma)_\Gamma\}_{r=K}^{\infty}$$

and therefore, by Corollary 2.2 in [S1] for C^∞-boundary Γ and for parallelepipeds, they can and will all be required to be different from zero: $q_r \neq 0$, $r = K, K+1, \ldots$:

(4) Next, with q as in (4A.4) *fixed*, and for each $\{a_i\}_{i=1}^{K-1}$ in the sphere $\mathscr{S}_c$ we define a non linear operator T_y, depending on the vector $y = \{q_r[\overline{w}_s]_r\}_{r=K}^{\infty}$ from $\{a_i\}_{i=1}^{K-1} \to \{q'_r\}_{r=K}^{\infty}$, by setting: $\{q'_r\}_{r=K}^{\infty} = T_y\{a_i\}_{r=1}^{K-1}$ with

$$q'_r \equiv q_r[\overline{w}_s]_r \frac{\prod_{k=1}^{K-1}\left(1-\dfrac{\beta_r}{\beta_k}\right)E(\beta_r, \beta_k, m)}{\prod_{k=1}^{K-1}\left(1-\dfrac{\beta_r}{a_k}\right)E(\beta_r, \beta_k, m)}, \quad (\beta_k \equiv c_k);\ r = K, K+1, \ldots \tag{4A.5}$$

which we shall consider as acting from the (*closed*) *sphere* $\mathscr{S}_c$ *in* R^{K-1} *into* l_∞.

For the operator T_y the following claim is easily verified: *with the radius* ϱ_c *fixed in advance, and for a given vector* q *as in* (4A.4) (so that such q is in l_2, hence in l_∞), *one can select a sufficiently small sphere for the vectors* $\overline{w}_s \in X_s$ in (1.8*b*)—*as assumed. in* (1.8*c*)— *such that all the corresponding operators* T_y *which are defined through* (4A.5) *map the sphere* $\mathscr{S}_c$ *into an arbitrarily small neighborhood of the origin in* l_∞. This assertion follows from the definition of T_y in (4A.5) and the fact that for points $\{a_i\}_{i=1}^{K-1}$ in $\mathscr{S}_c$ we have: $\inf_{r,i} \{|\beta_r - a_i|: \{a_i\} \in \mathscr{S}_c\} > 0$, where the inf is taken over all $r = K, K+1, \ldots$ and $i = 1, \ldots, K-1$.

(5) Next, motivated by (3.17*b*) and (3.29), we define a *non-linear operator* F: $\{v_k\}_{k=K}^{\infty} \to \{(Fv)_r\}_{r=K}^{\infty} = \{f_r\}_{r=K}^{\infty}$ by

$$\text{(i)} \quad f_r = (Fv)_r \equiv \gamma_r \prod_{k=K}^{\infty}\left(1-\frac{\beta_r}{\alpha_k}\right)E(\beta_r, \beta_k, m)\,, \qquad r = K, K+1, \ldots \tag{4A.6}$$

where the constants

$$\text{(ii)} \quad \gamma_r = \frac{-\beta_r^{1-\delta}A_\infty}{\left\{1-\displaystyle\sum_{i=1}^{K-1}\frac{p_i(\overline{w}_u)_i}{\beta_r - c_i}\right\}\displaystyle\prod_{\substack{k=K\\k\neq r}}^{\infty}\left(1-\frac{\beta_r}{\beta_k}\right)E(\beta_r, \beta_k, m)} \tag{4A.6}$$

depend only on data $\lambda_k = \beta_k$ and on parameters obtained through Lemma 2.1: $(w_u)_i$, p_i and c_i. By virtue of (4A.2), Eq. (4A.6) (i) gives $\{(Fv)_r\}_{r=K}^{\infty}$ explicitly in terms of $\{v_k\}_{k=K}^{\infty}$:

$$\text{(4A.6)} \qquad \text{(iii)} \quad f_r = (Fv)_r = \gamma_r \prod_{k=K}^{\infty} \left(1 - \frac{\beta_r}{\beta_k + [\overline{w}_s]_k v_k \beta_k^{\delta}}\right) E(\beta_r, \beta_k, m)\,, \qquad r = K, K+1, \ldots .$$

We shall consider F *as acting from a neighborhood of the origin in* l_∞, *see* (4A.1), *into* l_∞. Notice that F maps $\{v_k \equiv 0\}_{k=K}^{\infty}$ into $\{(Fv)_r \equiv 0\}_{r=K}^{\infty}$, more generally, if one coordinate $v_{\bar{k}} = 0$, then by (4A.6) (iii) the corresponding coordinate $(Fv)_{\bar{k}} = 0$ as well. Notice that, because of Lemma 3.5, it follows from (4A.6) that

$$\text{(4A.7)} \qquad (Fv)_r \sim \frac{\beta_r^{1-\delta}}{\alpha_r}(\alpha_r - \beta_r) \sim \frac{\alpha_r - \beta_r}{\beta_r^{\delta}} = v_r[\overline{w}_s]_r \qquad \text{as } r \to \infty$$

where, in the last step, we have used (4A.2).

The following proposition, to be proved at the end of the present Appendix, will be paramount in our treatment.

PROPOSITION 4A.1. – The inverse mapping theorem applies to the operator F defined above; i.e. there is a neighborhood $\mathcal{N}_v$ of $v = 0$ in l_∞ such that F is one—to—one in $\mathcal{N}_v$ with F^{-1} continuous in the $l_\infty \to l_\infty$ topology. □

An important consequence of both the claim regarding T_y *and* Proposition 4A.1 is: *with the radius* ϱ_c *of* $\mathcal{S}_c$ *fixed in advance, and for a given vector* q *as in* (4A.4), *one can select a sufficiently small sphere for the vectors* $\overline{w}_s \in X_s$ *in* (1.8*b*)—*as assumed in* (1.8*c*) *such that the corresponding composite map* $F^{-1}T_y$ *of* T_y *followed by* F^{-1} *is well defined and maps the sphere* $\mathcal{S}_c$ *into an arbitrarily small neighborhood of the origin in* l_∞.

(7) It will be shown in the subsequent Appendix 4B that the *map* $G: \{v_k\}_{k=K}^{\infty} \to \{\alpha_i\}_{i=1}^{K-1}$ from *a neighborhood of the origin of* l_∞ *into* R^{K-1}, *which produces the constants* $\alpha_1, \ldots, \alpha_{K-1}$ *for which the Realizability Conditions* (3.33) *hold, is in fact continuous.*

$$\begin{array}{ccc} \{a_i\}_{i=1}^{K-1} \in \mathcal{S}_c & & \Big\{\{\alpha_i\}_{i=1}^{K-1} \text{ for which the R.C. holds}\Big\} \in \mathcal{S}_c \\ T_y \Big\downarrow & & \Big\uparrow G \\ l_\infty \ni \{q'_r\}_{r=K}^{\infty} & \xrightarrow{\;F^{-1}\;} & \{v_k\}_{k=K}^{\infty} \in l_\infty . \end{array}$$

Notice that, if we apply F^{-1} on the vector $\{q'_r\}$ given by (4A.5) for a preassigned q, we get a vector $\{v_k\}$, whose corresponding $\{\alpha_k\}_{k=K}^{\infty}$ via (4A.2) are such that

$$q'_r = \gamma_r \prod_{k=K}^{\infty} \left(1 - \frac{\beta_r}{\alpha_r}\right) E(\beta_r, \beta_k, m)\,, \qquad r = K, K+1, \ldots$$

so that, by (4A.7),

$$q'_r \sim v_r[\overline{w}_s]_r \tag{4A.7'}$$

while (4A.5) gives $q'_r \sim q_r[\overline{w}_s]_r$. Thus: the preassigned vector q and the obtained vector $v = F^{-1}q'$, with $q' = T_y a = T_y\{a_i\}_{i=1}^{K-1}$ satisfy

$$q_r \sim v_r\,. \tag{4A.7''}$$

Thus, if q is only in l_2, so is v.

(8) As a consequence of the last two statements, we obtain a conclusive result, which we state formally:

PROPOSITION 4A.2. – With a radius ϱ_c of $\mathscr{S}_c$ fixed in advance, *and a* given vector q as in (4A.4), one can select a suitably small sphere for the vectors $\overline{w}_s \in X_s$ in (1.8*b*)—as assumed in (1.8*c*)—such that the corresponding composite map $GF^{-1}T_y$ of T_y followed by F^{-1} and by G is well defined and maps the (closed) sphere $\mathscr{S}_c$ into itself. □

Since $GF^{-1}T_y$ is continuous, being the composition of continuous maps, Brower's fixed point theorem applies and produces (at least) a fixed point $\{\overline{a}_i\}_{i=1}^{K-1} \in \mathscr{S}_c$, with all coordinates distinct and negative. The corresponding vector $\{q'_r\}_{r=K}^{\infty} = T_y\{\overline{a}_i\}_{i=1}^{K-1}$ has all its coordinates different from zero, by (1.8*b*) and also since all coordinates q_r were taken $\neq 0$; hence (by the observation above Prop. 4A.1), the corresponding vector $\{\overline{v}_k\}_{k=K}^{\infty} = F^{-1}T_y\{\overline{a}_i\}_{i=1}^{K-1}$ is $\sim \overline{q}_r$ and has also all its coordinates different from zero.

As to the sequence $\{\tilde{\alpha}_i\}_{i=1}^{K-1}$ in the conclusion (ii) of Theorem 3.8, we then take a fixed point $\tilde{\alpha}_i = \overline{a}_i$, $i = 1, \ldots, K-1$. As to the sequence $\{\tilde{\alpha}_k\}_{k=K}^{\infty}$ in the conclusion (i) of Theorem 4.8, we take instead

$$\tilde{\alpha}_k = \beta_k + [\overline{w}_s]_k \beta_k^{\delta} \overline{v}_k\,, \qquad k = K, K+1, \ldots\,.$$

With this choice

$$\tilde{\alpha}_k \neq \beta_j\,, \quad \begin{matrix} k = K, K+1, \ldots \\ j = 1, 2, \ldots \end{matrix} \quad \text{and} \quad \begin{matrix} k = 1, \ldots, K-1 \\ \text{but } k \neq j \end{matrix}$$

from Appendix 4B. It remains to show

$$\tilde{\alpha}_i \neq c_i \equiv \beta_i\,, \qquad i = 1, \ldots, K-1\,.$$

In fact, if—say—$\tilde{\alpha}_1 = c_1$, then by (4B.1) in Appendix 4B with $a_i = \tilde{\alpha}_i$, $i = 1, \ldots, K-1$ we would have that the corresponding $\tilde{h}_1(a) = 0$. Since $\{\tilde{\alpha}_i\}_{i=1}^{K-1}$

makes the Realizability Conditions (3.33) hold, it follows that Eqs. (3.17*a*-*b*) apply to the corresponding sequence $\{h_r(a) = h_r\}_{r=1}^{\infty}$ with p and $\{c_i\}_{i=1}^{K-1}$ coming from Lemma 2.1 and with q the vector as in (4A.4) for which Brower's fixed point theorem holds. But then from (3.17*a*-*b*), we see that the condition $h_r = 0$, $r = 1, \ldots, K-1$ can always be avoided by slightly changing, if necessary, say just one c_i.

(9) To conclude the proof of *Theorem* 4.8 it remains to establish Proposition 4A.1

PROOF OF PROPOSITION 4A.1. – We need to verify that the operator F defined in (4A.6) satisfies the following properties [L10, p. 266; M3, p. 116]:

(*a*) F admits a well defined Frechet derivative $F'(v)$ in a neighborhood $\mathcal{N}_v$ of the origin in l_∞, and, moreover, the map $v \to F'(v)$ is continuous in $\mathcal{N}_v$ in the topology of $l_\infty \to l_\infty$;

(*b*) the operator $F'(v = 0)$ is invertible; i.e. $[F'(v = 0)]^{-1}$ exists in l_∞. The validity of (*a*) will follow a fortiori once we show the following assertion: *that the second Frechet derivative $F''(v)$ is well defined as a continuous operator $l_\infty \to l_\infty$.*

In fact, $F''(v)$ is an infinite matrix with the following structure:

$$F''(v) = \begin{vmatrix} E_K \\ E_{K+1} \\ E_{K+2} \\ \vdots \end{vmatrix}$$

where

$$E_r = \begin{vmatrix} \dfrac{\partial}{\partial v_1}\dfrac{\partial f_r}{\partial v_1}, & \dfrac{\partial}{\partial v_2}\dfrac{\partial f_r}{\partial v_1}, & \dfrac{\partial}{\partial v_3}\dfrac{\partial f_r}{\partial v_1}, \ldots \\ \dfrac{\partial}{\partial v_1}\dfrac{\partial f_r}{\partial v_2}, & \dfrac{\partial}{\partial v_2}\dfrac{\partial f_r}{\partial v_2}, & \dfrac{\partial}{\partial v_3}\dfrac{\partial f_r}{\partial v_2}, \ldots \\ \dfrac{\partial}{\partial v_1}\dfrac{\partial f_r}{\partial v_3}, & \dfrac{\partial}{\partial v_2}\dfrac{\partial f_r}{\partial v_3}, & \dfrac{\partial}{\partial v_3}\dfrac{\partial f_r}{\partial v_3}, \ldots \\ \cdots & \cdots & \cdots \end{vmatrix}$$

for $r = K, K+1, \ldots$. According to well known results [T5, p. 220], $F''(v)$ defines a continuous operator: $l_\infty \to l_\infty$, provided

$$\text{(4A.8)} \qquad \begin{matrix}\text{sup}\\ \text{over all}\\ \text{rows}\end{matrix} \{l_1\text{-norm of a row}\} = \sup_r \left\{ \sup_l \sum_{j=1}^{\infty} \left| \frac{\partial^2 f_r}{\partial v_j\, \partial v_l} \right| \right\} < \infty .$$

From (4A.6) we compute after setting $Z_k = [\overline{w}_s]_k \beta_k^\delta$:

$$\text{(4A.9)}\quad \text{for } j \neq l: \frac{\partial^2 f_r}{\partial v_j\, \partial v_l} = \frac{\partial}{\partial v_j}\left[\gamma_r \prod_{\substack{k=K\\ k=l}}^{\infty}\left(1 - \frac{\beta_r}{\beta_k + Z_k v_k}\right)\frac{E(\beta_r, \beta_k, m)\beta_r Z_l E(\beta_r, \beta_l, m)}{(\beta_l + Z_l v_l)^2} = \right.$$

$$= \gamma_r \left\{\prod_{\substack{k=K\\ k\neq l\\ k\neq j}}^{\infty}\left(1 - \frac{\beta_r}{\beta_k + Z_k v_k}\right)E(\beta_r, \beta_k, m)\right\}\frac{\beta_r Z_j E(\beta_r, \beta_j, m)\beta_r Z_l E(\beta_r, \beta_l, m)}{(\beta_j + Z_j v_j)^2(\beta_l + Z_l v_l)^2} =$$

$$= \frac{f_r \beta_r Z_j \beta_r Z_l}{(1 - \beta_r/(\beta_j + Z_j v_j))(1 - \beta_r/(\beta_l + Z_l v_l))(\beta_j + Z_j v_j)^2(\beta_l + Z_l v_l)^2} =$$

$$\left. = \frac{f_r \beta_r^2 Z_j Z_l}{(\beta_j - \beta_r + Z_j v_j)(\beta_l - \beta_r + Z_l v_l)(\beta_j + Z_j v_j)(\beta_l + Z_l v_l)}\right].$$

Similarly from (4A.6) we obtain for $j = l$:

$$\text{(4A.10)}\quad \frac{\partial^2 f_r}{\partial v_l^2} = \gamma_r \left\{\prod_{\substack{k=K\\ k\neq l}}^{\infty}\left(1 - \frac{\beta_r}{\beta_k + Z_k v_k}\right)E(\beta_r, \beta_k, m)\right\}\frac{-2\beta_r Z_l^2 E(\beta_r, \beta_l, m)}{(\beta_l + Z_l v_l)^3} =$$

$$= \frac{-2 f_r \beta_r Z_l^2}{(\beta_l - \beta_r + Z_l v_l)(\beta_l + Z_l v_l)^2}.$$

To verify (4A.8), we need, according to (4A.9) and (4A.10), to check that the following two quantities Σ_1 and Σ_2 be finite:

$$\text{(4A.11)}\quad \Sigma_1 = \sup_r\left\{\sup_l \sum_{\substack{j=1\\ j\neq l}}^{\infty}\left|\frac{\partial^2 f_r}{\partial v_j\, \partial v_l}\right|\right\} =$$

$$= \sup_r\left\{|f_r \beta_r^2| \sup_l \frac{|Z_l|}{|\beta_l - \beta_r + Z_l v_l||\beta_l + Z_l v_l|}\left[\sum_{\substack{j=1\\ j\neq l}}^{\infty}\frac{|Z_j|}{|\beta_j - \beta_r + Z_j v_j||\beta_j + Z_j v_j|}\right]\right\}.$$

$$\text{(4A.12)}\quad \Sigma_2 = \sup_r\left\{\sup_l\left|\frac{\partial^2 f_r}{\partial v_l^2}\right|\right\} = \sup_r\left\{2|f_r \beta_r| \sup_l \frac{Z_l^2}{|(\beta_l - \beta_r + Z_l v_l)(\beta_l + Z_l v_l^2)|}\right\}.$$

We first handle Σ_1. In the sequel we shall use with no further mention that $Z_k = [\overline{w}_s]_k \beta_k^\delta \leqslant (\text{const}/k) k^{2\delta/\nu} \to 0$ as $k \to \infty$. (From (1.8*b*) and (3.21)). In order to show that Σ_1 is finite, it will suffice to establish that the following quantities Σ_1' and Σ_1'' be finite, where Σ_1' refers to the case $l \neq r$ and Σ_1'' refers to the case $l = r$:

$$\text{(4A.13)}\quad \Sigma_1' = \sup_r\left\{|f_r \beta_r^2| \sup_{l\neq r}\frac{|Z_l|}{|(\beta_l - \beta_r + Z_l v_l)\beta_l|}\left[\sum_{\substack{j=1\\ j\neq l}}^{\infty}\frac{|Z_j|}{|(\beta_j - \beta_r + Z_j v_j)\beta_j|}\right]\right\} \leqslant$$

$$\leqslant \sup_r\left\{|f_r \beta_r^2| \sup_{l\neq r}\frac{|Z_l|}{|(\beta_l - \beta_r)\beta_l|}\left[\sum_{\substack{j=1\\ j\neq l,r}}^{\infty}\frac{|Z_j|}{|(\beta_j - \beta_r)\beta_j|} + \frac{|Z_r|}{|Z_r v_r \beta_r|}\right]\right\}.$$

$$(4A.14)\qquad \Sigma_1'' = \sup_r \left\{ |f_r\beta_r^2| \frac{|Z_r|}{|Z_r v_r \beta_r|} \sum_{\substack{j=1\\ j\neq r}}^{\infty} \frac{|Z_j|}{|(\beta_j - \beta_r + Z_j v_j)\beta_j|} \right\}$$

$$\text{by (4A.7)}\qquad \leqslant \text{const} \sup_r \left\{ |\beta_r| \sum_{\substack{j=1\\ j\neq r}}^{\infty} \frac{|Z_j|}{|(\beta_j - \beta_r)\beta_j|} \right\}.$$

To conclude that Σ_1' and Σ_1'', and hence Σ_1, are finite, we now need to invoke the following two estimates:

$$(4A.15)\qquad \sum_{\substack{j=1\\ j\neq r}}^{\infty} \frac{|Z_j|}{|(\beta_j - \beta_r)\beta_j|} = \mathcal{O}\left(\frac{1}{\beta_r}\right) \qquad \text{(see the independent Lemma 3.11)},$$

$$(4A.16)\qquad \sup_{\substack{r,l\\ r\neq l}} \frac{|\beta_r Z_l|}{|\beta_l(\beta_l - \beta_r)|} \leqslant \text{const} \qquad \text{(see the final Remark 4A.1)}.$$

We now use (4A.15) at the level of the infinite sum term in (4A.13). We then obtain from (4A.13), since $f_r \leqslant \text{const}$ and $(f_r/v_r) \leqslant \text{const}$ (cf. (4A.7$'$)):

$$\Sigma_1' \leqslant \text{const} \sup_r \left\{ \sup_{r\neq l} \frac{|\beta_r Z_l|}{|(\beta_l - \beta_r)\beta_l|} \right\} < \infty$$

from which the finiteness of Σ_1' follows via (4A.16). The finiteness of Σ_1'' follows directly from (4A.14) via (4A.15). The proof that $\Sigma_1 < \infty$ is complete. The proof that $\Sigma_2 < \infty$ is simpler. From (4B.12), we compute

$$(4A.17)\qquad \Sigma_2 \leqslant \text{const} \sup_r \left\{ |f_r\beta_r| \sup_{l\neq r} \frac{|Z_l^2|}{|(\beta_l - \beta_r)\beta_l^2|} \right\} + \text{const} \sup_r \left\{ \frac{|f_r \beta_r Z_r^2|}{|Z_r v_r \beta_r^2|} \right\}.$$

Since $f_r \leqslant \text{const}$, $(f_r/v_r) \leqslant \text{const}$ and $(Z_l/\beta_l) \to 0$, and (4A.16), we easily conclude from (4A.17) that $\Sigma_2 < \infty$ as desired. The proof that the second Frechet derivative $F''(v)$ is a bounded operator $l_\infty \to l_\infty$ is thus complete.

To finish the proof of Proposition 4A.1, it remains to show statement (*b*) on the invertibility of $F'(v = 0)$. This is quickly done as follows. The Frechet derivative $F'(v)$ is an infinite matrix with entries

$$F'(v) = \left|\left(\frac{\partial f_r}{\partial v_j}\right)\right|, \qquad j = r = K, K+1, \ldots .$$

Starting from (4A.5) (ii), we compute directly, again with $Z_k = [\overline{w}_s]_k \beta_k^\delta$

$$(4A.18)\qquad \frac{\partial f_r}{\partial v_j} = \gamma_r \prod_{\substack{k=K\\ k\neq j}}^{\infty} \left(1 - \frac{\beta_r}{\alpha_k}\right) E(\beta_r, \beta_k, m) \frac{\partial}{\partial v_j} \left(1 - \frac{\beta_r}{\beta_r + Z_j v_j}\right) E(\beta_r, \beta_j, m) =$$

$$= \frac{f_r}{(1 - \beta_r/(\beta_j + Z_j v_j))} \frac{-\beta_r Z_j}{(\beta_r + Z_j v_j)^2} = \frac{-f_r \beta_r Z_j}{(\beta_j - \beta_r + Z_j v_j)(\beta_r + Z_j v_j)}.$$

Setting $v = 0$, i.e. $v_j \equiv 0$, $j = K, K+1, \dots$ implies, as we know (see below (4A.7)) $f_r \equiv 0$, $r = K, K+1, \dots$ Thus from (4A.18) we get

$$\left.\frac{\partial f_r}{\partial v_j}\right|_{v=0} = 0 \qquad \text{for } r \neq j\,,$$

$$\left.\frac{\partial f_r}{\partial v_r}\right|_{v=0} = \frac{-f_r}{v_r}\,.$$

Thus, $F'(v = 0)$ is an infinite matrix whose off-diagonal terms all vanish, and whose main diagonal terms are $-f_r/v_r$, $r = K, K+1, \dots$. Since $f_r/v_r \sim 1$ by (4A.7), we can conclude that $F'(v = 0)$ is invertible as an operator: $l_\infty \to l_\infty$.

REMARK 4A.1. – To prove estimate (4A.16), rewrite

$$\sup_{\substack{r,l\\ r\neq l}} \frac{|\beta_r Z_l|}{|\beta_l(\beta_l - \beta_r)|} = \sup_{\substack{r,l\\ r\neq l}} \frac{|Z_l|}{|\beta_l(\beta_l/\beta_r - 1)|} \tag{4A.19}$$

and since $Z_l/\beta_l \to 0$ as $l \to \infty$, we only have to worry if $[(\beta_l/\beta_r) - 1]$ becomes unbounded. Thus, taking $r = l - 1$ we estimate (4A.19) by using $Z_l = [\overline{w}_s]_l \beta_l^\delta$ where $[\overline{w}_s]_l \leqslant \text{const}/l$ (cf. (1.8*b*)) and $\beta_l \sim l^{2/\nu}$ (cf. (3.21)). It is left to the reader to check that the sup is bounded.

Proposition 4A.1 is thus fully proved. □

Appendix 4B.

Determination of $\{\alpha_i\}_{i=1}^{K-1}$ *from which the Realizability Condition* (3.33) *hold. Continuity of* G: $\{v_k\}_{k=K}^{\infty} \to \{\alpha_i\}_{i=1}^{K-1}$ *from* $l_\infty \to \boldsymbol{R}^{K-1}$.

Let $a = \{a_1, \dots, a_{K-1}\}$ be a set of distinct negative numbers in the sphere $\mathscr{S}_c$ defined by (4A.3) each different from all β_j. Motivated by (3.29), we define a sequence $\{\tilde{h}_r(a)\}_{r=1}^{\infty}$, depending on a, by

$$\tilde{h}_r(a) = \frac{\beta_r A_\infty \prod_{k=1}^{K-1}\left(1 - \frac{\beta_r}{a_k}\right) E(\beta_r, \beta_k, m) \prod_{k=K}^{\infty}\left(1 - \frac{\beta_r}{\alpha_k}\right) E(\beta_r, \beta_k, m)}{\prod_{\substack{k=1\\ k\neq r}}^{\infty}\left(1 - \frac{\beta_r}{\beta_k}\right) E(\beta_r, \beta_k, m)}\,. \tag{4B.0}$$

We then try to determine the parameters $\{a_1, \dots, a_{K-1}\}$ in such a way that the sequence $\{\tilde{h}_r(a)\}_{r=1}^{\infty}$ satisfies the Realizability Condition (3.33).

The sequence $\tilde{h}_r(a)$ in (4B.0) can be more conveniently rewritten as

$$\tilde{h}_r(a) = e_r \prod_{k=1}^{K-1}\left(1 - \frac{\beta_r}{a_k}\right), \qquad r = 1, 2, \dots \tag{4B.1}$$

where the coefficients e_r are defined by

$$(4B.2)\qquad e_r = \frac{\beta_r A_\infty \prod_{k=K}^{\infty}\left(1-\frac{\beta_r}{\alpha_k}\right)E(\beta_r,\beta_k,m)\prod_{k=1}^{K-1}E(\beta_r,\beta_k,m)}{\prod_{\substack{k=1\\k\neq r}}^{\infty}\left(1-\frac{\beta_r}{\beta_k}\right)E(\beta_r,\beta_k,m)}, \qquad r=1,2,\dots$$

where $\alpha_k = \beta_k + [\overline{w}_s]_k \beta_k^\delta v_k$ by (3.38).

We then determine the negative parameters $a_1, \dots, a_{K-1}$ in such a way that the Realizability Conditions (3.33), rewritten now as

$$(4B.3)\qquad e_i \prod_{k=1}^{K-1}\left(1-\frac{\beta_i}{a_k}\right) = p_i(w_u)_i \sum_{\substack{r=K\\i\neq j}}^{\infty} \frac{e_r \prod_{k=1}^{K-1}\left(1-\frac{\beta_r}{a_k}\right)}{(\lambda_r - c_i)\left\{1-\sum_{j=1}^{K-1}\frac{p_j(w_u)_j}{\lambda_r - c_j}\right\}}, \qquad i=1,\dots,K-1$$

are satisfied. To this end, we use the identity

$$\prod_{k=1}^{K-1}(\beta_r - a_k) = \beta_r^{K-1} - \left(\sum_{k=1}^{K-1} a_k\right)\beta_r^{K-2} + \left(\sum_{\substack{i,j=1\\i\neq j}}^{K-1} a_i a_j\right)\beta_r^{K-3} - \\ -\left(\underbrace{\sum_{i,j,k=1}^{K-1}}_{\text{distinct}} a_i a_j a_k\right)\beta_r^{K-4} + \dots + (-1)^{K-1}\prod_{j=1}^{K-1} a_j,$$

on the right and, for $r = i$, on the left of (4B.3) and apply the sum $\sum_{r=K}^{\infty}$ on each power of β_r in the previous identity separately. Then, by setting

$$(4B.4)\qquad \overline{A}_{i,K-l} = p_i(w_u)_i \sum_{r=K}^{\infty} \frac{e_r \beta_r^{K-l}}{(\lambda_r - c_i)\left\{1-\sum_{j=1}^{K-1}\frac{p_j(w_u)_j}{\lambda_r - c_j}\right\}},$$

$$l = 1, \dots, K;\ i = 1, \dots, K-1$$

and

$$A_{i,K-} = (-1)^{l-1}\overline{A}_{i,K-l} + (-1)^l e_i \beta_i^{K-l}, \qquad l=1,\dots,K;\ i=1,\dots,K-1,$$

the Realizability Conditions (4B.3) can be rewritten as a multilinear algebraic system (see also Eq. (3.15)),

$$(4B.5)\qquad A_{i,K\ 1} + A_{i,K-2}\left(\sum_{k=1}^{K-1} a_k\right) + A_{i,K-3}\left(\sum_{\substack{i,j=1\\i\neq j}}^{K-1} a_i a_j\right) + \\ + A_{i,K-4}\left(\underbrace{\sum_{i,j,k=1}^{K-1}}_{\text{distinct}} a_i a_j a_k\right) + \dots + \dots A_{i,0}\left(\prod_{j=1}^{K-1} a_j\right) = 0,$$

of $(K-1)$ equations in $(K-1)$ unknowns, for which we seek a *negative* solution: $a_1, \ldots, a_{K-1}$ (that is, all a_i negative).

Notice that the infinite series defining each coefficient $\bar{A}_{i,K-l}$ through (4B.4) is $\sim \sum_{r=K}^{\infty} e_r \beta_r^{K-l-1}$. Therefore, the following claim is relevant.

Through Eq. (3B.2) one can define a non-linear map: $\{v_k\}_{k=K}^{\infty} \to \{e_r \beta_r^{K-l-1}\}_{r=K}^{\infty}$ that we view from $l_\infty \to l_1$. *We claim that this map is continuous.*

In fact, by simply comparing Eq. (4B.2) and Eq. (4A.5) (ii) in Appendix 4A, with γ_r defined by (4A.5) (iii), we see that

$$e_r \sim \frac{\beta_r}{\prod_{k=1}^{K-1}\left(1-\dfrac{\beta_r}{\beta_k}\right)} \cdot \frac{q'_r \beta_r^\delta [\bar{w}_s]_r}{\beta_r}.$$

It suffices to consider the case $l = 1$. Here we get

$$\beta_r^{K-2} e_r \sim q'_r [\bar{w}_s]_r \beta_r^{\delta-1} = \mathcal{O}\left(\frac{q'_r}{r^{1+2(1-\delta)/\nu}}\right) \tag{4B.6}$$

where $\delta < 1$ and $\nu \geqslant 2$: in the last step we have made use of assumption (1.8′) on $\bar{w}$ and of Eq. (3.21) for β_r. Appendix 4A shows a fortiori that the map: $\{v_k\}_{k=K}^{\infty} \to \{q'_r\}_{r=K}^{\infty}$ from $l_\infty \to l_\infty$ is continuous. Thus, the desired claim follows from (4B.6). We conclude that: *the map* $v = \{v_k\}_{k=K}^{\infty} \to A_{i,K-l}$ *is continuous from* $l_\infty \to R^1$. We next want to show that when the l_∞-norm of v is sufficiently small, the system (4B.6) does admit a negative solution. To establish this, we make use of an observation plus a continuity argument.

The observation is that, when

$$\alpha_k \equiv \beta_k, \quad k = K, K+1, \ldots, \tag{4B.7}$$

then a solution of distinct roots for the system (4B.3) is given by

$$a_i = \beta_i \overset{\text{df}}{=} c_i < 0, \quad i = 1, \ldots, K-1 \tag{4B.8}$$

with the $c_i < 0$ coming from Lemma 2.1. In fact, under assumption (4B.7), it follows from (4B.2) that

$$e_r \equiv 0, \quad r = K, K+1, \ldots$$

and hence from the right-hand side of (4B.3) we deduce that system (4B.6) reduces to

$$e_i \prod_{k=1}^{K-1}\left(1-\frac{\beta_i}{a_k}\right) = 0, \quad i = 1, \ldots, K-1. \tag{4B.9}$$

In other words, $\bar{A}_{i,K-l} \equiv 0$ in this case (see Eq. (4B.5)).

Referring to Eq. (3.14), however, since

$$c_i \stackrel{\mathrm{df}}{=} \beta_i \neq \alpha_k (\equiv \beta_k)\,, \quad i = 1, \ldots, K-1;\ k = K, K+1, \ldots,$$

it follows from (4B.2) that $e_i \neq 0$, $i = 1, \ldots, K-1$. Hence, the desired conclusion (4B.8) is a consequence of (4B.9). This proves the observation.

For convenience of language, we shall call the situation under assumption (4B.7) the *original situation*. We now use a continuity argument. First, we argue that the roots of a multilinear system like (4B.6) depend continuously on the real coefficients of the system. Second, we argue that these coefficients, as shown above depend continuously on the sequence $\{v_k\}_{k=K}^{\infty} \in l_\infty$. Therefore, if the vector v is sufficiently small in the l_∞-norm, the new coefficients $A_{i,K-l}$ are a slight perturbation of the original ones. Since the roots of the original situation (4B.9) are distinct and negative, as described in (4B.8), so will the new roots $\{\alpha_i\}_{i=1}^{K-1}$ be.

Thus the map G, needed in Appendix 4A: $\{v_k\}_{k=K}^{\infty} \to \{\alpha_i\}_{i=1}^{K-1}$ defined from $l_\infty \to \to \boldsymbol{R}^{K-1}$ is continuous. □

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