Symbolic Dynamics and Nonlinear Semiflows (*) (**).

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Summary. – *For a transverse homoclinic orbit γ of a mapping (not necessarily invertible) on a Banach space, it is shown that the mapping restricted to orbits near γ is equivalent to the shift automorphism on doubly infinite sequences on finitely many symbols. Implications of this result for the Poincaré map of semiflows are given.*

1. – Introduction.

If O is an hyperbolic fixed point of a diffeomorphism $F \in C^k(R^n)$, $k \geq 1$, $n \geq 2$, and W^s, W^u are the stable and unstable manifolds of O, then $q \in W^s \cap W^u$, $q \neq 0$, is said to be transverse homoclinic to O if W^u is transversal to W^s at q, $W^u \pitchfork W^s$. The orbit $\gamma(q) = \{F^n q,\ n \in N;$ set of integers$\}$ through q is called a transverse homoclinic orbit asymptotic to O.

Poincaré was well aware of the fact that the existence of transverse homoclinic orbits implied that the flow defined by F would be very complicated in a neighborhood of q. Birkhoff proved that there must be infinitely many periodic points near q. SMALE [15, 16] showed that there was an integer k and an invariant set I near q of F^k such that F^k restricted to q was equivalent to the shift map σ on the set of doubly infinite sequences on two symbols (see, also, MOSER [11], PALMER [13]). SILNIKOV [14] discussed the set of all orbits of F that remain in a small neighborhood of $\gamma(q)$. He then showed that F on certain subsets of these solutions was equivalent to the shift map σ on the set of doubly infinite sequences on infinitely many symbols.

Our objective in this paper is to generalize these results to the case of $F \in C^k(X)$, where X is a Banach space and F is not necessarily a diffeomorphism. For a hyperbolic fixed point O of F, the local stable set W^s_{loc} and local unstable set W^u_{loc} of O are C^k manifolds (a proof is given below for completeness). However, the behavior of the global stable set W^s and unstable set W^u may not have a nice mani-

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fold structure. Even in the case where W^u is finite dimensional the local dimension may vary with the point on W^u. This necessitates hypotheses on W^u even to define a transverse homoclinic orbit. Under an appropriate hypothesis on W^u (there is an immersion from $W^u_{\mathrm{loc}} \times N$ into W^u which covers $\gamma(q)$), a transverse homoclinic orbit is defined and it is shown that the results of Sil'nikov [14] and Smale [15, 16] are valid. The main theorem is stated and proved in Section 5. The proof is a revised version of the horseshoe argument (see [2], [12]). HOLMES and MARSDEN [6] have also used the properties of horseshoes in the equations of a forced beam. Chaotic motion is discussed in Section 7. The implications for the Poincaré map for flows are given in Section 8. Applications to retarded functional differential equations will appear elsewhere.

2. – Notations and preliminaries.

Let X, Y and Z denote Banach spaces. If U is an open set in X, then $C^k(U, Y)$ is the usual space of functions mapping $\overline{U}$ into Y which are continuous and bounded together with derivatives up through order k. The norm in this space is the supremum of all these derivatives. We also let $C^k(X) = C^k(X, X)$. The symbol $(N^-)(N^+)N$ will denote the (nonpositive integers) (nonnegative integers) integers. By a submanifold of a Banach space Z, we mean a regular submanifold (locally expressed as the graph of a C^1 map from X into Y where $Z = X \oplus Y$ is a splitting of Banach spaces).

If S is a topological space, we let $\Pi_N S$ be the infinite product space with the product topology. An element $\tau \in \Pi_N S$ is a map $\tau\colon N \to S$. Define $\sigma\colon \Pi_N S \to \Pi_N S$ as the shift map, $\tau_1 = \sigma\tau$, $\tau_1(n) = \tau(n+1)$, $n \in N$. If $F \in C^0(S, S)$, a *trajectory of* F is a map $\tau \in \Pi_N S$ such that $\Pi F(\tau) = \sigma(\tau)$, where $\Pi F\colon \Pi_N S \to \Pi_N S$ is defined as $\tau_1 = \Pi F(\tau)$, $\tau_1(n) = F(\tau(n))$, $n \in N$. Obviously ΠF is continuous and the set of all the trajectories of F form a closed subset of $\Pi_N S$, which is a topological subspace with the topology induced from $\Pi_N S$. In a similar way, one defines respectively a positive (negative) trajectory by a map $\tau^+(\tau^-)$. A (positive orbit) (negative orbit) (orbit) will be the range of $(\tau^+)(\tau^-)(\tau)$ and will be denoted by $(O_{\tau^+})(O_{\tau^-})(O_\tau)$. For $\tau \in \Pi_N S$, let $s_n = \tau(n)$, and write $\tau = (\dots, s_{-2}, s_{-1}][s_0, s_1, \dots)$ to indicate that $\tau(0) = s_0$. Thus $\tau_1 = \sigma\tau$ is denoted by $\tau_1 = (\dots, s_{-2}, s_{-1}, s_0][s_1, s_2, \dots)$. And, in this notation, $\Pi F(\tau) = (\dots, Fs_{-2}, Fs_{-1}][Fs_0, Fs_1, \dots)$. We shall use $\tau[i, j]$, $i < j$ integers, to denote the restriction of τ to an interval $[i, j]$.

Let $\sim$ be an equivalence relation defined in the topological space S. For any $s \in S$, $[s] = \{s_1\colon s_1 \sim s\}$ is said to be the *equivalence class* of s. The *quotient space* $S/\!\sim\, = \{[s]\colon s \in S\}$ is defined with the quotient topology. For a subset $Q \subset S$, define $[Q] = \{[s]\colon s \in Q\}$ as the equivalence class of Q.

Suppose O is a fixed point of $F \in C^k(X)$, $k \geqslant 1$. The fixed point O is hyperbolic if $\sigma(DF(0)) \cap \{|\lambda| = 1\} = \emptyset$, where $\sigma(A)$ denotes the spectrum of a linear operator A. The *unstable set* $W^u(0)$ and the *stable set* $W^s(0)$ of a fixed point O of F are defined by

$$W^u(0) = \bigcup \{\text{negative orbits } O_{\tau^-} \text{ of } F\colon \tau^-(n) \to 0 \text{ as } n \to -\infty\}\,,$$
$$W^s(0) = \bigcup \{\text{positive orbits } O_{\tau^+} \text{ of } F\colon \tau^+(n) \to 0 \text{ as } n \to +\infty\}\,.$$

The *local unstable and stable sets* are defined respectively by

$$W^u(0, U) = \bigcup \{\text{negative orbits } O_{\tau^-} \text{ of } F\colon O_{\tau^-} \subset W^u(0) \cap U\}\,,$$
$$W^s(0, U) = \bigcup \{\text{positive orbits } O_{\tau^+} \text{ of } F\colon O_{\tau^+} \subset W^s(0) \cap U\}\,,$$

where U is an open set containing 0. We use the notation $W^u_{\text{loc}}(0)$, $W^s_{\text{loc}}(0)$, for $W^u(0, U)$, $W^s(0, U)$ if U is not relevant to the problem.

If F is a diffeomorphism, one can always consider complete orbits in the definition of $W^s(0)$. Furthermore, $W^u(0)$, $W^s(0)$ are C^k immersed submanifolds of X [5]. In particular, if the dimension is finite, then the dimension must be the same at every point. The following examples illustrate the differences that can occur with maps.

EXAMPLE 2.1. – $F \in C^k(R^2)$, $k \geqslant 1$, $F(x, y) = (0, 2y)$. For this case, the only fixed point is the origin O and $W^s(O) = \{y = 0\}$, $W^u(O) = \{x = 0\}$. The map F^{-1} is only defined on $W^u(O)$ and is single valued only if the range is restricted to $W^u(O)$.

EXAMPLE 2.2. – We construct a delay differential equation with a hyperbolic equilibrium point having a two-dimensional local unstable manifold. The unstable manifold collapses into a smooth one-dimensional manifold along one of the trajectories, a phenomenon that could not happen in ordinary differential equations. The time one map for this example will have the property that the dimension of the unstable manifold is not the same at every point.

Consider the delay equation

$$\dot{x}(t) = \alpha(x(t))\, x(t) + \beta(x(t))\, x(t-1)\,,$$

where $x \in R$, $\alpha(x)$ and $\beta(x)$ are defined as

$$\big(\alpha(x), \beta(x)\big) = \begin{cases} \left(\dfrac{2e-1}{e-1}, -\dfrac{e^2}{e-1}\right), & |x| \leqslant 1\,; \\ (1, 0)\,, & |x| \geqslant 2\,; \\ 1 = \alpha(x) + \beta(x)\, e^{-1} \text{ when } 1 \leqslant |x| \leqslant 2\,. \\ \text{Also, } \alpha(x) \text{ and } \beta(x) \in C^\infty(R)\,. \end{cases}$$

The origin 0 is an equilibrium point of (2.1). Equation (2.1) is linear in a neighborhood of 0 and has $\lambda_1 = 1$ and $\lambda_2 = 2$ as the positive characteristic values. All the other characteristic values have negative real parts. Thus (see [3]), there is a neighborhood U of 0 such that $\dim W^u(0, U) = 2$. Let $x(t) = \varepsilon e^t$ be a solution issuing from $W^u(0, U)$. For some large $\bar{t} > 0$ we have $\inf_{-1\leq\theta\leq 0}|X_{\bar{t}}(\theta)| > 2$, and in a neighborhood of $X_{\bar{t}}$, (2.1) becomes $\dot{x}(t) = x(t)$. Let $\psi \in C[-1, 0]$ be in a amall neighborhood of $X_{\bar{t}}$ and suppose that there is a solution passing through ψ in the negative direction. It is easy to see that $\psi(\theta) = \eta e^{\bar{\eta}+\theta}$ with-near ε. Therefore, the unstable set in this neighborhood of $X_{\bar{t}}$ is a smooth manifold but of dimension 1.

Take the time one map $F = T(1)$ of the solution map $T(t)$ of (2.1). We have an example with the property that the hyperbolic fixed point 0 of F has a local two dimensional unstable manifold which collapses into a one dimensional manifold.

Suppose $F \in C^k(X)$, $k \geq 1$ and 0 is an hyperbolic fixed point of F. We shall prove that $W^u_{\text{loc}}(0)$ and $W^s_{\text{loc}}(0)$ are submanifolds in § 3. An orbit O_τ is an *homoclinic orbit asymptotic to a fixed point* 0 *of* F if $O_\tau \subset W^u(0) \cap W^s(0)$ and $O_\tau \not\equiv \{0\}$. An homoclinic orbit O_τ asymptotic to a fixed point 0 of F is said to be a *transverse homoclinic orbit* if

1) 0 is an hyperbolic fixed point;

2) for any sufficiently large pair of integers $i, j > 0$, such that $\tau(-i) \in W^u_{\text{loc}}(0)$ and $\tau(j) \in W^s_{\text{loc}}(0)$, F^{i+j} sends a disc in $W^u_{\text{loc}}(0)$ containing $\tau(-i)$ diffeomorphically onto its image which is transverse to $W^s_{\text{loc}}(0)$ at $\tau(j)$.

Notice that $W^u(0)$, $W^s(0)$ may not have a manifold structure even in a small neighborhood of O_τ. However, condition 2) implies that we can attach to each $\tau(k) \in O_\tau$, $k \in \mathbf{N}$, small pieces of submanifolds $W^u_{\text{loc}}(\tau(k)) \subset W^u(0)$ and $W^s_{\text{loc}}(\tau(k)) \subset W^s(0)$ diffeomorphic to $W^u_{\text{loc}}(0)$ and $W^s_{\text{loc}}(0)$, respectively, and such that

$$W^u_{\text{loc}}(\tau(k)) \pitchfork W^s_{\text{loc}}(\tau(k)) \quad \text{at } \tau(k) \in O_\tau . \tag{2.2}$$

Furthermore, $FW^u_{\text{loc}}(\tau(k-1)) \supset W^u_{\text{loc}}(\tau(k))$ and $FW^s_{\text{loc}}(\tau(k)) \subset W^s_{\text{loc}}(\tau(k+1))$. This can be done as follows. If i, j are given as in condition 2), then $W^u_{\text{loc}}(\tau(k)) = W^u_{\text{loc}}(0)$, $k \leq -i$ and $W^s_{\text{loc}}(\tau(k)) = W^s_{\text{loc}}(0)$, $k \geq j$. $W^u_{\text{loc}}(\tau(k))$, $k \geq j$ is defined as a disc in $F^{k+i}W^u_{\text{loc}}(\tau(-i))$, diffeomorphic to $W^u_{\text{loc}}(0)$ by 2). For $-i < k < j$, $F^{k+i}W^u_{\text{loc}}(\tau(-i))$ still contains a disc covering $\tau(k)$, and shall be defined as $W^u_{\text{loc}}(\tau(k))$, since $(F^{i+j})^{-1}F^{j-k}$ is the inverse of F^{k+i} by 2). $W^s_{\text{loc}}(\tau(k))$, $j > k$, can be obtained by considering the transversality of F^{j-k} to $W^s_{\text{loc}}(\tau(j))$ and (2.2) follows similarly. Therefore, there is an immersion from $W^u_{\text{loc}}(0) \times N$ into $W^u(0)$ and an immersion from $W^s_{\text{loc}}(0) \times N$ into $W_s(0)$. Both cover O_τ but are not necessarily injective. Briefly, we say that $W^u(0)$ is transverse to $W^s(0)$ along O_τ if no ambiguity can arise.

EXAMPLE 2.3. – Let us consider the interval map F: $[0, 1] \to [0, 1]$, $F(x) = \mu x(1-x)$, $0 < \mu \leq 4$. The map F is not invertible and has a fixed point $x_0 = 1 -$

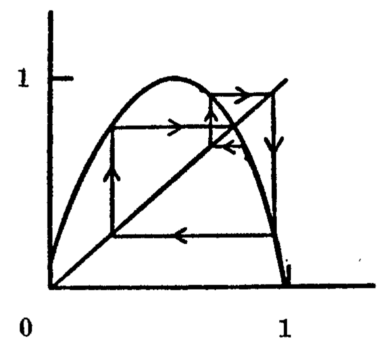

Figure 2.1

$-1/\mu$, $\mu > 1$, which is hyperbolic if $\mu \neq 3$. When $\mu = 4$, an homoclinic orbit is plotted in fig. 2.1, which hits x_0 after a finite number of iterates of F, an observation previously made by BLOCK [1]. It is easy to check that the homoclinic orbit is transverse.

Example 2.3 is a special case of snap-back repellers defined by MAROTTO [10] which will be discussed later.

3. – Stable and unstable manifolds.

In this section, we state and prove the existence of local stable and unstable manifolds $W^s_{\text{loc}}(0)$ and $W^u_{\text{loc}}(0)$ of a hyperbolic fixed point of a map. The existence of the local stable manifold follows from [7] with very little change needed. For a diffeomorphism F, the existence of the local unstable manifold follows from the existence of the local stable manifold of F^{-1}. However, if F is noninvertible, a direct proof for the existence of the local unstable manifold is needed (see [5]). In spite of the fact that the result may be known to some people, we give the proof for completeness.

THEOREM 3.1. – *Let X, Y and $Z = X \times Y$ be Banach spaces and A, B be linear continuous maps in X and Y respectively, with $\sigma(A) < 1$ and $\sigma(B) > 1$. Suppose that $\|A\|$, $\|B^{-1}\| \leqslant \lambda$ for some constant $0 < \lambda < 1$. Suppose U is an open neighborhood of 0 in Z and f_1: $U \to X$, f_2: $U \to Y$ are C^k $(k \geqslant 1)$ maps with $f_i(0) = 0$, $Df_i(0) = 0$, $i = 1, 2$. Consider F: $U \to Z$,*

$$F: \begin{cases} x_1 = Ax_0 + f_1(x_0, y_0)\,, & (3.1) \\ y_1 = By_0 + f_2(x_0, y_0)\,. & (3.2) \end{cases}$$

Then there exist open balls C_1, D_1 centered at 0 in X, Y respectively, and a unique C^k map h_1: $C_1 \to D_1$ with $h_1(0) = 0$, $Dh_1(0) = 0$ such that

$$F(\text{graph } h_1) \subset \text{graph } h_1\,.$$

The restriction of F to graph h_1 is a contraction. Moreover, if $F^n(z) \in C_1 \times D_1$ for $n \geqslant 0$, $z \in$ graph h_1.

There also exist open balls C_2, D_2 centered at 0 in X, Y respectively, and a unique C^k map h_2: $D_2 \to C_2$ with $h_2(0) = 0$, $Dh_2(0) = 0$ such that the restriction of F^{-1} from graph h_z into itself a well-defined single valued C^k contraction; thus, a diffeomorphism onto F^{-1}(graph h_2) with the inverse F as an expansion. Moreover, if $z \in C_2 \times D_2$ and the negatively infinite trajectory $F^{-n}(z) \in C_2 \times D_2$, $n \geqslant 0$ exists, $z \in$ graph h_2.

For the proof of the last part of the theorem, we consider the Banach space l of the bounded, negatively infinite sequences in Z; that is, $l = \{z_{-i}, i \geqslant 0\}$, with the norm $|\{z_i\}|_l = \sup_{i \geqslant 0} |z_{-i}|_z$ Suppose $g \in G^r(Z)$ with all the derivatives being bounded in any bounded set of Z. The map Πg: $l \to l$ is defined as $\Pi g(z)(-i) = g(z(-i))$, $i \geqslant 0$ for $\{z_{-i}\} \in l$. Unfortunately, since continuity does not imply uniform continuity in infinite dimensional Banach spaces, Πg is not C^r even for $r = 0$. The remedy is to consider a subspace $l_0 \subset l$, $\{z_{-i}\} \in l_0$ if and only if $z_{-i} \to 0$ as $i \to \infty$. The following lemma is very elementary and can be easily proved by induction, but works as well as the lemmas in [7], [8] for composition maps.

LEMMA 3.2. – *Let g: $Z \to Z$, $g \in C^r$ and $g(0) = 0$. Then Πg: $l_0 \to l_0$ is C^r and $(\Pi g)^{(k)} = \Pi g^{(k)}$, $k \leqslant r$.*

PROOF OF THEOREM 3.1. – For any $\varepsilon > 0$ and any Banach space E, let

$$B_\varepsilon^E = \{x \in E: |x| < \varepsilon\} .$$

For $\varepsilon > 0$ sufficiently small and any $y \in B_\varepsilon^y$, $\gamma \in B_\varepsilon^{l_0}$, define

$$G(y, \gamma)(-n) = \gamma(-n) - \Big(\sum_{i=n+1}^{\infty} A^{i-n-1} f_1(\gamma(-i)), B^{-n} y - \sum_{i=1}^{n} B^{-n-1+i} f_2(\gamma(-i))\Big) . \tag{3.3}$$

It is not difficult to show that $G(y, \gamma)(-n) \to 0$ as $n \to \infty$. Thus, G: $B_\varepsilon^y \times B_\varepsilon^{l_0} \to l_0$. Lemma 3.2 implies that $G \in C^r$. It is clear that $G(0, 0) = 0$. Applying the Implicit Function Theorem to the equation

$$G(y, \gamma) = 0 \tag{3.4}$$

in a neighborhood of $y = 0$, $\gamma = 0$, we have a unique C^r map Φ: $B_{\varepsilon_1}^y \to B_{\varepsilon_2}^{l_0}$, $\Phi(0) = 0$, for some $\varepsilon_1, \varepsilon_2 > 0$ which solves (3.4) as $\gamma = \Phi(y)$ in $B_{\varepsilon_1}^y \times B_{\varepsilon_2}^{l_0}$. Let P: $l_0 \to Z$ be the projection taking γ to $\gamma(0)$, h_2: $B_{\varepsilon_1}^y \to X$ defined as

$$P\Phi(y) = \Phi(y)(0) = \Big(\sum_{i=1}^{\infty} A^{i-1} f_1(\Phi(y)(-i)), y\Big) = (h_2(y), y)$$

is C^r with $h_2(0) = 0$. The Implicit Function Theorem also enables us to compute $D\Phi(0)$ by computing $DG(0, 0)$ and thus, conclude that $Dh_2(0) = 0$. It is easy to check, from (3.3), that

$$\Phi(y)(-n) = F(\Phi(y)(-n-1)) , \qquad n \geqslant 0 .$$

We have obtained that for $y_0 \in B^y_{\varepsilon_1}$, $x_0 = h_2(y_0)$, $z_0 = (x_0, y_0)$, there exists $z_{-i} \in F^{-i} z_0$, $i \geqslant 0$ defined as $z_{-i} = \Phi(y_0)(-i)$ and $|z_{-i}| < \varepsilon_2$. Since Φ is continuous, there exists $\varepsilon_3 < \varepsilon_1$ such that $y_0 \in B^y_{\varepsilon_3}$ implies $|z_{-i}| < \varepsilon_1$, especially $|y_{-1}| \leqslant \varepsilon_1$. We shall see very soon (see (i), (ii) below) that

$$G(y_{-1}, \{z_{-i-1}, i \geqslant 0\}) = 0 .$$

Thus, $y_{-1} \in B^y_{\varepsilon_1}$ and $x_{-1} = h_2(y_{-1})$. From

$$y_0 = By_{-1} + f_2(h_2(y_{-1}), y_{-1}) ,$$

using the Implicit Function Theorem, one concludes that, if ε_1 is sufficiently small, $|y_{-1}| \leqslant \tilde{\lambda} |y_0|$, $0 < \tilde{\lambda} < 1$, and, thus, $y_{-1} \in B^y_{\varepsilon_3}$. This completes the proof that F^{-1} is a contraction on graph h_2, $|y| < \varepsilon_3$.

Let C_2, D_2 be open balls in X, Y such that $C_2 \times D_2 \subset B^z_{\varepsilon_2}$, $D_2 \subset B^y_{\varepsilon_3}$ and $h_2(D_2) \subset C_2$. Then the restriction of h_2 on D_2 satisfies all the assertions except that we have to verify that

if $\{z_{-i}, i \geqslant 0\}$ is a negatively infinite trajectory in $B^z_{\varepsilon_2}$, then

(i) $\{z_{-i} \cdot i \geqslant 0\} \in l_0$;

(ii) $G(y_0, \{z_{-i}\}) = 0$.

For any $\theta > 0$, there exists $\varepsilon_2 > 0$ such that $\|Df_1\|$, $\|Df_2\| \leqslant \theta$ if $|z| \leqslant \varepsilon_2$. Let $\{z_{-i}, i \geqslant 0\}$ be a negatively infinite trajectory in $B^z_{\varepsilon_2}$. By induction

$$\begin{aligned} x_{-i} &= A^k x_{-i-k} + A^{k-1} f_1(z_{-i-k}) + \dots + f_1(z_{-i-1}) , \\ y_{-1} &= B^{-i} y_0 - B^{-i} f_2(z_{-1}) - \dots - B^{-1} f_2(z_{-i}) . \end{aligned}$$

Let $k \to \infty$,

$$\begin{aligned} x_{-i} &= \sum_{j=0}^{\infty} A^j f_1(z_{-i-j-1}) , \\ y_{-i} &= B^{-i} y_0 - \sum_{j=1}^{i} B^{-i+j-1} f_2(z_{-j}) , \end{aligned} \tag{3.5}$$

Then,

$$|z_{-i}| \leqslant \lambda^i |y_0| + \theta \sum_{j=1}^{i} \lambda^{i-j+1} |z_{-j}| + \theta \sum_{j=0}^{\infty} \lambda^i |z_{-i-j-1}| .$$

Suppose $\delta = \overline{\lim\limits_{i\to\infty}} |z_{-i}| > 0$. Then for any $\xi > 1$, there exists $i_0 > 0$ such that $|z_{-i}| \leqslant \xi\delta$ for $i \geqslant i_0$, and

$$|z_{-i}| \leqslant \lambda^i |y_0| + \theta\lambda^i \sum_{j=1}^{i_0} \lambda^{-j+1} |z_{-j}| + \frac{2\theta}{1-\lambda} \cdot \xi \cdot \delta . \tag{3.6}$$

If $2\theta/(1-\lambda) < 1$, we can choose $\xi > 1$ such that $2\theta/(1-\lambda)\cdot\xi < 1$. Let $i \to \infty$ in (3.6),

$$\overline{\lim_{i\to\infty}} |z_{-i}| \leqslant \frac{2\theta}{1-\lambda} \cdot \xi \cdot \delta .$$

The contradiction shows that $\delta = 0$. Therefore, $\{z_{-i}, i \geqslant 0\} \in l_0$, together with (3.5) imply (ii).

4. – Some basic lemmas.

Consider $F: Z \to Z$ defined as (3.1) and (3.2). Assume all the hypotheses of Theorem 3.1. By a C^1 change of variable, we assume that the local stable and unstable manifolds are flat, i.e., $W^s_{\mathrm{loc}}(0) = \{y = 0\}$ and $W^u_{\mathrm{loc}}(0) = \{x = 0\}$. Thus, in addition to the hypotheses in Theorem 3.1, we assume that $f_1(0, y) = 0$ and $f_2(x, 0) = 0$. Consequently,

$$f_{1y}(0, y) = 0 , \tag{4.1}$$

$$f_{2x}(x, 0) = 0 . \tag{4.2}$$

A closed ε-ball in a Banach space E with center zero is denoted by $\overline{B^E_\varepsilon}$. For any $\theta > 0$, we choose $\varepsilon > 0$ so small such that $|Df_1|, |Df_2| \leqslant \theta$ in $\overline{B^z_\varepsilon}$. We assume that $W^u_{\mathrm{loc}}(0)$, $W^s_{\mathrm{loc}}(0)$ is contained in $\overline{B^z_\varepsilon}$ and

$$\lambda + \theta < 1 . \tag{4.3}$$

DEFINITION 4.1. – A C^1 submanifold φ_s is said to be an *s-slice of size* $(\varepsilon_1, \delta, K)$, or an s-slice modeled on $\overline{B^x_{\varepsilon_1}}$, intersecting $W^u_{\mathrm{loc}}(0)$ transversally at $(0, y^*)$ with $|y^*| \leqslant \delta$ and having the inclination $\leqslant K$, if

$$\varphi_s = \{(x, y) : y = g(x),\ |x| \leqslant \varepsilon_1,\ |y^*| = |g(0)| \leqslant \delta,\ g \in C^1 \text{ and } \|Dg\| \leqslant K\} .$$

A C^1 submanifold φ_u is said to be a *u-slice of size* $(\varepsilon_1, \delta, K)$ or a u-slice modeled on $\overline{B^{y}_{\varepsilon_1}}$, intersecting $W^s_{\mathrm{loc}}(0)$ transversally at $(x^*, 0)$ with $x^* \leqslant \delta$ and having the inclination $\leqslant K$, if

$$\varphi_u = \{(x, y) : x = h(y),\ |y| \leqslant \varepsilon_1,\ |x^*| = |h(0)| \leqslant \delta,\ h \in C^1 \text{ and } \|Dh\| \leqslant K\} .$$

In all of the above, ε_1, δ, K are positive constants.

Lemma 4,2, 4.4, 4.5, 4.6 are called the Inclination Lemmas and for diffeomorphisms in R^n, see [2] and [12]. They play the same roles as Lemma 3.3, estimates (3.5) in [14]. However, those estimates are not valid in our case.

LEMMA 4.2. – *Given $K > 0$, there exist ε_1, $\delta > 0$ and $c > 1$ such that*

(i) *for any u-slice φ_u of size $(\varepsilon_0/c, \delta, K)$, $\varepsilon_0 \leqslant \varepsilon_1$, F sends φ_u diffeomorphically onto its image and $\overline{B^y_{\varepsilon_0}} \cap F(\varphi_u)$ is a u-slice of size $(\varepsilon_0, \delta, K)$;*

(ii) *for any s-slice φ_s of size $(\varepsilon_0/c, \delta, K)$, $\varepsilon_0 \leqslant \varepsilon_1$, $\overline{B^x_{\varepsilon_0}} \cap F^{-1}(\varphi_s)$ is an s-slice of size $(\varepsilon_0, \delta, K)$.*

PROOF. – (i) Let F: $(x_0, y_0) \to (x_1, y_1)$, $(x_0, y_0) \in \varphi_u$. Assume that θ is small and satisfies

$$1 < d = \frac{1 - \lambda\theta(K + 1)}{\lambda}. \tag{4.4}$$

For this θ, choose $\varepsilon > 0$ so that $|Df_1|$, $|Df_2| \leqslant \theta$ in $\overline{B^z_\varepsilon}$. Let δ, ε_1 satisfy

$$\varepsilon_1 < \frac{\varepsilon}{2}, \qquad \delta + K\varepsilon_1 < \frac{\varepsilon}{2}. \tag{4.5}$$

Then $\varphi_u \subset B^z_\varepsilon$ and

$$x_1 = A(g(y_0)) + f_1(g(y_0), y_0), \tag{4.6}$$

$$y_1 = By_0 + f_2(g(y_0), y_0). \tag{4.7}$$

Write (4.7) as

$$B^{-1}y_1 = y_0 + B^{-1}f_2(g(y_0), y_0). \tag{4.8}$$

The Lipschitz constant for $B^{-1}f_2(g(y_0), y_0)$ as a function of y_0 is bounded by $\lambda\theta(K + 1)$. By the Implicit Function Theorem, the right hand side of (4.8) defines a diffeomorphism from $y_0 \in \overline{B^y_{\varepsilon_0/c}}$ to $B^{-1}y_1$, which covers a ball of radius $(1 - \lambda\theta \cdot \cdot (K + 1))\,\varepsilon_0/c$. Therefore, y_1 covers a ball of radius

$$\frac{1 - \lambda\theta(K + 1)}{\lambda}\frac{\varepsilon_0}{c} = \frac{d}{c}\cdot\varepsilon_0 .$$

Let c, asserted in the lemma, be $c = d$. Substituting y_0 as a function of y_1 into (4.6), we have a u-slice $x_1 = g_1(y_1)$, modeled on $\overline{B^y_{\varepsilon_0}}$ and transverse to $W^s_{\text{loc}}(0)$ at $F(g(0), 0) = (g_1(0), 0)$. Since $F|W^s_{\text{loc}}(0)$ is a contraction, $g_1(0) \leqslant g(0) \leqslant \delta$. It remains to show that $\|Dg_1\| \leqslant K$.

Let (ξ, η) be a tangent vector to φ_u at (x_0, y_0), $\eta \neq 0$ and $|\xi|/|\eta| \leqslant K$. Let $(\xi', \eta') = DF(x_0, y_0)(\xi, \eta)$,

$$\frac{|\xi'|}{|\eta'|} = \frac{|(A + f_{1x})\xi + f_{1y}|\eta||}{|f_{2x}\xi + (B + f_{2y})\eta|} \leqslant \frac{(\lambda + \theta)|\xi| + \|f_{1y}\|\cdot|\eta|}{(\lambda^{-1} - \theta)|\eta| - \theta|\xi|} \leqslant \frac{(\lambda + \theta)K + \|f_{1y}\|}{d}. \tag{4.9}$$

If θ is small erough, then

$$\frac{(\lambda + \theta)K + \theta}{d} \leqslant K,$$

and (i) is proved if ε_1, δ are small so that (4.5) is valid.

(ii) Let

$$\varepsilon_1 < \frac{\varepsilon}{2}, \qquad K\varepsilon_1 + \delta < \frac{\varepsilon}{2}. \tag{4.10}$$

Let $x_1, y_1) \in \varphi_s$, an s-slice of size $(\varepsilon_1, \delta, K)$. We look for (x_0, y_0) such that $F(x_0, y_0) = (x_1, y_1)$

$$By_0 + f_2(x_0, y_0) = h(Ax_0 + f_1(x_0, y_0))$$

or

$$y_0 = -B^{-1}f_2(x_0, y_0) + B^{-1}h(Ax_0 + f_1(x_0, y_0)). \tag{4.11}$$

We use the contraction mapping principle to solve (4.11). Let H be the set of all the continuous functions form $\overline{B^x_{\varepsilon_0}}$ into $\overline{B^y_{\varepsilon/2}}$ with the distance of any two functions in H given by the supremum norm. Let $c > 1$ be such that

$$\lambda\varepsilon_0 + \theta\varepsilon_0 \leqslant \varepsilon_0/c. \tag{4.12}$$

The existence of such c is from (4.3). A continuous function $\mathcal{F}\varphi(\cdot)$ is defined on $\overline{B^x_{\varepsilon_0}}$ for any $\varphi \in H$ as

$$\mathcal{F}\varphi(x) = -B^{-1}f_2(x, \varphi(x)) + B^{-1}h\big(Ax + f_1(x, \varphi(x))\big), \tag{4.13}$$

since $f_1(0, y) = 0$, $|Ax + f_1(x, \varphi(x))| \leqslant \lambda\varepsilon_0 + \theta\varepsilon_0 \leqslant \varepsilon_0/c$ by (4.12) and h is defined on $\overline{B^x_{\varepsilon_0/c}}$. Furthermore, $\mathcal{F}\varphi \in H$ if

$$\lambda\theta\cdot\frac{\varepsilon}{2} + \lambda(K\varepsilon_1 + \delta) \leqslant \frac{\varepsilon}{2}. \tag{4.14}$$

The verification of (4.14) uses $f_2(x, 0) = 0$, (4.10) and (4.3). We observe that $\mathcal{F}\colon H \to H$ is a contraction if θ is small. Therefore, there is a unique fixed point of $\mathcal{F}$, denoted by h_0: We can show that $h_0 \in C^1(\overline{B^x_{\varepsilon_1}})$ by using the Implicit. Function Theorem locally to solve (4.11) in the neighborhood of $(x_0, h_0(x_0))$. We also see that $h_0(0) \leqslant h(0) \leqslant \delta$ since $F|W^u$ is an expansion. It remains to check that $\|Dh_0\| \leqslant K$. Suppose (ξ, η) is a nonzero tangent vector to $F^{-1}\varphi_s$ at (x_0, y_0). Then

$$B\eta + f_{2x}\cdot\xi + f_{2y}\cdot\eta = Dh(A\xi + f_{1x}\cdot\xi + f_{1y}\cdot\eta)\,,$$

$$|\eta|(1 - \lambda\theta - \lambda K\theta) \leqslant (\lambda^2 K + \lambda\theta K + \lambda\|f_{2x}\|)|\xi|\,.$$

$|\xi| = 0$ would imply that $|\eta| = 0$; thus $|\xi| \neq 0$ and

$$\frac{|\eta|}{|\xi|} \leqslant \frac{\lambda^2 K + \lambda K\theta + \lambda\|f_2 x\|}{1 - \lambda\theta(K+1)} = \frac{(\theta + \lambda)K + \lambda\|f_2 x\|}{d}\,. \tag{4.15}$$

If θ is small (ε small), $((\theta + \lambda)K + \theta)/d \leqslant K$ and (ii) is proved. This completes the proof of Lemma 4.2.

DEFINITION 4.3. – Let $\varphi_u^{(1)}$, $\varphi_u^{(2)}$ be two u-slices of size $(\varepsilon_1, \delta, K)$ and let $\varphi_s^{(1)}$, $\varphi_s^{(2)}$ be two s-slices of size $(\varepsilon_1, \delta, K)$. Define the distances with respect to the uniform norm as

$$d(\varphi_u^{(1)}, \varphi_u^{(2)}) = \sup_{|y| \leqslant \varepsilon_1} |g_1(y) - g_2(y)|\,,$$

$$d(\varphi_s^{(1)}, \varphi_s^{(2)}) = \sup_{|x| \leqslant \varepsilon_1} |h_1(x) - h_2(x)|\,,$$

where $\varphi_u^{(i)}$, $\varphi_s^{(i)}$ are graphs of g_i, h_i, $i = 1, 2$.

LEMMA 4.4. – *Given $K > 0$, the constants ε_1, δ can be chosen so that the results of Lemma 1 are true. Moreover, there is a constant $0 < \tilde{\lambda} < 1$ such that*

$$d(F^n\varphi_u^{(1)}, F^n\varphi_u^{(2)}) \leqslant (\tilde{\lambda})^n d(\varphi_u^{(1)}, \varphi_u^{(2)})\,,$$

$$d(F^{-n}\varphi_s^{(1)}, F^{-n}\varphi_s^{(2)}) \leqslant (\tilde{\lambda})^n d(\varphi_s^{(1)}, \varphi_s^{(2)})\,,$$

where F^n, F^{-n} are abbreviations for $(\overline{B^y_{\varepsilon_1}} \cap F)^n$ and $(\overline{B^x_{\varepsilon_1}} \cap F^{-1})^n$ which are defined inductively as follows: while applying on a set $V \subset Z$,

$$(\overline{B^y_{\varepsilon_1}} \cap F)^1 V = \overline{B^y_{\varepsilon_1}} \cap F(V)\,, \qquad (\overline{B^x_{\varepsilon_1}} \cap F)^{-1} V = \overline{B^x_{\varepsilon_1}} \cap F^{-1}(V)\,,$$

$$(\overline{B^y_{\varepsilon_1}} \cap F)^{n+1} V = \overline{B^y_{\varepsilon_1}} \cap F[(\overline{B^y_{\varepsilon_1}} \cap F)^n V]\,,$$

$$(\overline{B^x_{\varepsilon_1}} \cap F^{-1})^{n+1} V = \overline{B^x_{\varepsilon_1}} \cap F^{-1}[(\overline{B^x_{\varepsilon_1}} \cap F^{-1})^n V]\,, \qquad n \geqslant 1\,.$$

PROOF. – Suppose E_1, E_2 are Banach spaces and $\varphi \in C^1(E_1, E_2)$. We define $ev\colon C^1(E_1, E_2)\times E_1 \to E_2$ as $ev(\varphi, e) = \varphi(e)$. Let

$$\begin{aligned} g_t &= (t-1)g_2 + (2-t)g_1\,, \\ h_t &= (t-1)h_2 + (2-t)h_1\,, \qquad 1\leqslant t\leqslant 2\,. \end{aligned}$$

We first consider

$$\begin{aligned} x_1 &= Ag_t(y_0) + f_1(g_t(y_0), y_0)\,, \\ y_1 &= By_0 + f_2(g_t(y_0), y_0)\,, \end{aligned}$$

or

$$x_1 = A\, ev(g_t, y_0) + f_1(ev(g_t, y_0), y_0)\,, \tag{4.16}$$

$$y_1 = By_0 + f_2(ev(g_t, y_0), y_0)\,. \tag{4.17}$$

For y_1 fixed, y_0 can be solved as a function of t in (4.17), and substituted into (4.16) to obtain x_1 as a function of t. We shall estimate $\partial x_1/\partial t$ by more symmetric formulas. Assume that δy_0, δx_1, δy_1, δt are tangent vectors in the corresponding spaces, and Dg_t is the derivative of $g_t(\cdot)$. Then,

$$\left\{\begin{aligned} &\delta x_1 = A\big[ev((g_2-g_1)\,\delta t, y_0) + Dg_t\cdot\delta y_0\big] + \\ &\qquad\qquad + f_{1x}\cdot\big[ev((g_2-g_1)\,\delta t, y_0) + Dg_t\cdot\delta y_0\big] + f_{1y}\cdot\delta y^0\,, \\ &0 = \delta y_1 = B\,\delta y_0 + f_{2x}\big[ev((g_2-g_1)\,\delta t, y_0) + Dg_t\cdot\delta y_0\big] + f_{2y}\cdot\delta y_0\,. \end{aligned}\right.$$

$$\left\{\begin{aligned} &|\delta x_1|\leqslant(\lambda+\theta)\big[d(\varphi_u^{(1)}, \varphi_u^{(2)})|\delta t| + K|\delta y_0|\big] + \lambda\theta\cdot|\delta y_0|\,, \\ &|\delta y_0|\leqslant\lambda\theta\big[d(\varphi_u^{(1)}, \varphi_u^{(2)})|\delta t| + K|\delta y_0|\big] + \lambda\theta\cdot|\delta y_0|\,. \end{aligned}\right.$$

It follows that

$$\begin{aligned} \left|\frac{\partial y_0}{\partial t}\right| &\leqslant \frac{\lambda\theta}{1-\lambda\theta(K+1)}\,d(\varphi_u^{(1)}, \varphi_u^{(2)}) \leqslant \frac{\theta}{d}\,d(\varphi_u^{(1)}, \varphi_u^{(2)})\,, \\ \left|\frac{\partial x_1}{\partial t}\right| &\leqslant (\lambda+\theta)\,d(\varphi_u^{(1)}, \varphi_u^{(2)}) + \big((\lambda+\theta)K+\theta\big)\cdot\left|\frac{\partial y_0}{\partial t}\right|\,. \end{aligned}$$

Using the estimate for $|\partial y_0/\partial t|$, we find that, when θ is small, there exists $0<\tilde\lambda<1$ such that

$$\left|\frac{\partial x_1}{\partial t}\right| \leqslant \tilde\lambda d(\varphi_u^{(1)}, \varphi_u^{(2)})\,.$$

Therefore

$$|x_1(y_1, g_2) - x_1(y_1, g_1)| \leqslant \int_1^2 \left|\frac{\partial x_1}{\partial t}\right| dt \leqslant \tilde{\lambda} d(\varphi_u^{(1)}, \varphi_u^{(2)}) .$$

The first inequality in the lemma is proved.

Next consider

$$By_0 + f_2(x_0, y_0) = h_t(Ax_0 + f_1(x_0, y_0))$$

or

$$By_0 + f_2(x_0, y_0) = ev(h_t, Ax_0 + f_1(x_0, y_0)) .$$

Let x_0 be fixed and y_0 be a function of t,

$$B \cdot \delta y_0 + f_{2y} \cdot \delta y_0 = ev((h_2 - h_1)\, \delta t, Ax_0 + f_1(x_0, y_0)) + Dh_t \cdot f_{1y} \cdot \delta y_0 ,$$

$$|\delta y_0|(1 - \lambda\theta - \lambda\theta K) \leqslant \lambda d(\varphi_s^{(1)}, \varphi_s^{(2)}) \cdot |\delta t| .$$

Therefore

$$\left|\frac{\partial y_0}{\partial t}\right| \leqslant \frac{1}{d}\, d(\varphi_s^{(1)}, \varphi_s^{(2)}) ,$$

and

$$|y_0(x_0, h_2) - y_0(x_0, h_1)| \leqslant \int_1^2 \left|\frac{\partial y_0}{\partial t}\right| dt \leqslant \frac{1}{d}\, d(\varphi_s^{(1)}, \varphi_s^{(2)}) .$$

If $\tilde{\lambda} = 1/d$, the second inequality is proved. This completes the proof of the lemma.

LEMMA 4.5. – *Assume further that Df_1 and Df_2 are uniformly continuous in $\overline{B_\varepsilon^z}$. Then, for any $K > 0$, ε_1, δ can be chosen such that for any u-slice φ_u (s-slice φ_s) of size $(\varepsilon_1, \delta, K)$, $F^n\varphi_u$ is a u-slice ($F^{-n}\varphi_s$ is an s-slice) of size $(\varepsilon_1, \delta_n, K_n)$, with $\delta_n \leqslant \tilde{\lambda}^n \delta$ and $K_n \to 0$ as $n \to \infty$, where $0 < \tilde{\lambda} < 1$ and F^n, F^{-n} are abbreviations as before.*

PROOF. – Only $K_n \to 0$ has to be proved. Since $f_{1y}(0, y) = 0$, by the uniform continuity of Df_1, for any $\zeta > 0$ there is a $\xi > 0$ such that $\|f_{1y}(x, y)\| \leqslant \zeta$ if $|x| \leqslant \xi$ and $|y| \leqslant \varepsilon_1$. From Lemma 4.4, there is an $n_0 > 0$ such that $F^n \varphi_u \subset \overline{B_\xi^x} \times \overline{B_{\varepsilon_1}^y}$ for $n \geqslant n_0$. By (4.9), we obtain that $K_{n+1} \leqslant ((\lambda + \theta)K_n + \zeta)/d$, $n \geqslant n_0$. Thus,

$$K_{n_0+n} \leqslant \left(\frac{\lambda + \theta}{d}\right)^n K_{n_0} + \frac{\zeta}{d - (\lambda + \theta)} ,$$

$$\overline{\lim_{n \to \infty}}\, K_n \leqslant \frac{\zeta}{d - (\lambda + \theta)} .$$

Since ζ is arbitrary, this implies $K_n \to 0$ as $n \to \infty$.

A similar proof is applied to $F^{-n}\varphi_s$, if we consider $f_{2x}(x, 0) = 0$, uniform continuity of Df_2 in a neighborhood of 0, Lemma 4.4 for $F^{-n}\varphi_s$, and (4.15). This finishes the proof of the lemma.

The proof of Lemma 4.6 below is similar to that of Lemmas 4.2, 4.4 and 4.5 and shall be omitted. However, due to the lack of uniform continuity of the derivatives, the results concerning C^1 closeness must be formulated very carefully. Let M_1 and M_2 be C^1 submanifolds in Z. By M_2 is $C^1-\xi$ near M_1, ξ a positive number, we mean that there are Banach spaces E_1, E_2 such that $Z = E_1 \oplus E_2$ and a constant $\varrho > 0$ such that M_i is the graph of h_i: $B_\varrho^{E_1} \to E_2$, $i = 1, 2$, and

$$\|h_1 - h_2\|_{C^1(B_\varrho^{E_1}, E_2)} \leqslant \xi .$$

Conversely, we shall use $M_1 \overset{+}{\pitchfork} M_2 = q$ to denote that $M_1 \cap M_2 = \{q\}$ and $T_q M_1 \oplus \oplus T_q M_2 = Z$.

LEMMA 4.6. - *Let M_1, M_2, N be C^1 submanifolds in Z, and $M_2 \overset{+}{\pitchfork} N = q$. Suppose F: $Z \to Z$ is C^1 and the restriction F: $M_1 \to M_2$ is a diffeomorphism. Let $p \in M_1$ and $F(p) = q$. Then the following hold.*

(i) *There exist constants ξ, η, $L > 0$ and discs $U_1 \ni p$ in M_1 and $U_2 \ni q$ in M_2 such that any C^1 submanifolds $\tilde U_1$ and $\tilde U_2$, $C^1 - \xi$ near U_1, are sent diffeomorphically onto $F\tilde U_1$ and $F\tilde U_2$ which contain discs $\tilde U_1^0$ and $\tilde U_2^0$, $C^1 - \eta$ near U_2, $d(\tilde U_1^0, \tilde U_2^0) \leqslant \leqslant L d(\tilde U_1, \tilde U_2)$ where d is the distance between discs measured by the C^0 norm. Furthermore, for any $\zeta > 0$, there exists a disc $U_2^\zeta \subset U_2$ such that $F\tilde U_1$ contains a disc $C^1 - \zeta$ close to U_2^ζ if $\tilde U_1$ is sufficiently near U_1 in the C^1 norm.*

(ii) *Consider F^{-1} from a neighborhood of q into a neighborhood of p. There exist constants ξ, η, $L > 0$ and C^1 discs $V_2 \ni q$ in N and $V_1 \subset F^{-1}V_2$ with $M_1 \overset{+}{\pitchfork} V_1 = p$. For any C^1 submanifolds $\tilde V_1$ and $\tilde V_2$, $C^1 - \xi$ near V_2, $F^{-1}\tilde V_1$ and $F^{-1}\tilde V_2$ contain discs $\tilde V_1^0$ and $\tilde V_2^0$, $C^1 - \eta$ near $\tilde V_1$, $d(\tilde V_1^0, \tilde V_2^0) \leqslant L d(\tilde V_1, \tilde V_2)$ where d is measured by the C_0 norm. Furthermore, for any $\zeta > 0$, there exists a disc $V_1^\zeta \subset V_1$ such that $F^{-1}\tilde V_2$ contains a disc $C^1 - \zeta$ close to V_1^ζ if $\tilde V_2$ is sufficiently near V_2 in the C^1 norm.*

5. - Symbolic dynamics.

We now suppose that U_i, $0 \leqslant i \leqslant m$ are pairwise disjoint open sets in a Banach space Z. Let TZ be the subspace of all the trajectories of $F \in C^k(Z)$, $k \geqslant 0$, whose orbits are included in $\bigcup_{0 \leqslant i \leqslant m} U_i$. Let $S = \{U_0, \ldots, U_m\}$ be armed with the discrete topology. For each $\tau_z \in TZ$, an element $\tau_u \in \Pi_N S$ is defined as $\tau_u(n) = U_j$ if $\tau_z(n) \in U_j$. Thus J_1: $TZ \to \Pi_N S$ is defined as J_1: $\tau_z \to \tau_u$. Obviously, J_1 is continuous. Some interesting questions arise. What is the range of J_1? Is J_1 injective? If J_1 is injective, is J_1^{-1} continuous? The affirmative answer to these questions would ensure that TZ is homeomorphic to the subspace of sequences of symbols (U_i's) and ΠF, acting on TZ, is equivalent to the shift operator σ defined on the space of τ_u's via J_1.

DEFINITION 5.1. – For $S = \{U_0, \ldots, U_m\}$ and $\bar{k} > 0$ an integer, a subset $TU \subset \subset \Pi_N S$ is defined as $\tau_u \in TU$ if and only if

1) $\tau_u(i) = U_j$ implies that $\tau_u(i+1) = U_{j+1}$ for $1 \leqslant j < m$,

2) $\tau_u(i) = U_m$ implies that $\tau_u(i+j) = U_0$ for $1 \leqslant j \leqslant \bar{k}$,

3) $\tau_u(i) = U_1$ implies that $\tau_u(i-j) = U_0$ for $1 \leqslant j \leqslant \bar{k}$.

TU is a topological space with the topology induced from $\Pi_N S$.

To understand the meaning of this definition, suppose $\bigcup_{0 \leqslant i \leqslant m} U_i$ is a neighborhood of a homoclinic trajectory asymptotic to a fixed point 0 of F. Suppose $0 \in U_0$. Then to say $J_1 \tau_z \in TU$ is equivalent to saying that, if $\tau(j) \in U_0$ for some j, then it stays in U_0 for at least $\bar{k}$ iterates of F and one can leave U_0 only by going to U_1 and then march back to U_0 staying again for at least $\bar{k}$ iterates of F. The same remark applies to F^{-1}. The main theorem stated below is saying essentially that J_1 is a homeomorphism between TZ and TU if $\bigcup_{0 \leqslant i \leqslant m} U_i$ is some neighborhood of a transverse homoclinic orbit.

We are ready to state our main theorem.

THEOREM 5.2. – *Let X, Y and $Z = X \times Y$ be Banach spaces, $F\colon Z \to Z$ defined as in Theorem* 3.1 *with DF uniformly continuous in a neighborhood of the hyperbolic fixed point of F. Assume that the local stable and unstable manifolds are $X^s_{\text{loc}}(0) = \{y = 0\}$, $W^u_{\text{loc}}(0) = \{x = 0\}$, that* (4.1), (4.2) *are satisfied and $W^u_{\text{loc}}(0) \neq \{0\}$. Suppose τ^Γ_z is a homoclinic trajectory and $\tau^\Gamma_z(i) \to 0$ as $i \to \pm\infty$. Let $N > 0$ be an integer with $\tau^\Gamma_z(-N) \in W^u(0)$ and $\tau^\Gamma_z(N) \in W^s_{\text{loc}}(0)$, where $W^u_{\text{loc}}(0)$ and $W^s_{\text{loc}}(0)$ are contained in $\bar{B}^z_\varepsilon$ and* (4.3) *is valid in $\bar{B}^z_\varepsilon$. Assume that the following conditions are satisfied.*

1) *F^{2N} sends a disc $O_1 \cap W^u_{\text{loc}}(0)$ centered at $\tau^\Gamma_z(-N)$ diffeomorphically onto $O_2 = F^{2N} O_1$, containing $\tau^\Gamma_z(N)$.*

2) $O_2 \mathrel{\overset{+}{\pitchfork}} W^s_{\text{loc}}(0) = \tau^\Gamma_z(N)$.

Then τ^Γ_z is a transverse homoclinic trajectory. Furthermore, there exist pairwise disjoint open subsets $U_0, \ldots, U_m$, $m \geqslant 2$, in Z, and an integer $\bar{k} > 0$ such that $0 \in U_0$, $O_{\tau^\Gamma_z} \subset \bigcup_{0 \leqslant i \leqslant m} U_i$ and such that J_1 is an homeomorphism between TZ and TU defined in Definition 5.1. *ΠF acting on TZ is equivalent to σ acting on TU via J_1.*

The open set $\bigcup_{0 \leqslant i \leqslant m} U_i$ is called the extended neighborhood of $O_{\tau^\Gamma_z}$ with U_0 the « body » and $\bigcup_{1 \leqslant i \leqslant m} U_i$ the « handle ».

Before proving Theorem 5.2, we give a symbolization consisting of infinitely many symbols for a subset of TU.

DEFINITION 5.3. – Let

$$TZ_0 = \{\tau_z\colon \tau_z \in TZ,\ \tau_z(0) \in U_0 \text{ and } \tau_z(-1) \in U_m\}\,,$$
$$TU_0 = \{\tau_u\colon \tau_u \in TU,\ \tau_u(0) = U_0 \text{ and } \tau_u(-1) = U_m\}\,.$$

The set TZ_0 (TU_0) is both open and closed in TZ (TU). We observe that

$$\bigcup_i \sigma^i(TZ_0) = TZ\diagdown\{(\ldots, 0][0, \ldots)\}\,.$$
$$\bigcup_i \sigma^i(TU_0) = TU\diagdown\{(\ldots, U_0][U_0, \ldots)\}\,.$$

and $J_1(TZ_0) \subset TU_0$, $TZ_0 \supset J_1^{-1}(TU_0)$. Therefore,

$$J_1\colon TZ\diagdown\{(\ldots, 0][0, \ldots)\} \to TU\diagdown\{(\ldots, U_0][U_0, \ldots)\}$$

is a homeomorphism if and only if $J_1\colon TZ_0 \to TU_0$ is a homeomorphism, since $J_1\tau_z = \tau_u$ if and only if $J_1(\sigma\tau_z) = \sigma\tau_u$ and σ is a homeomorphism on both TZ and TU.

Let $[\bar{k}, +\infty]$ be the space of all the integers $\geq \bar{k} > 0$ furnished with the discrete topology and compactified by $+\infty$. Let $\Pi_N[\bar{k}, +\infty]$ be the product space. For any $\tau_{\bar{N}} = (\ldots, k_{-i}, \ldots, k_{-1}][k_0, \ldots, k_j, \ldots) \in \Pi_N[\bar{k}, +\infty]$, a corresponding element $\tau_u \in TU_0$ is defined as:

1) $\tau_u(l) = U_m$ if and only if
   (A) $l = -\sum_{i=1}^{j} k_{-i} - jm - 1$, $j = 0, 1, \ldots$, provided $l \neq -\infty$;
   (B) $l = \sum_{i=0}^{j} k_i + jm + m - 1$, $j = 0, 1, \ldots$, provided $l \neq +\infty$.

2) $\tau_u(l - i) = U_{m-i}$, $0 \leq i \leq m$, for all l defined by (A) or (B).

3) $\tau_u(i) = U_0$ if not defined by 1) and 2).

Accordingly, $\tilde{J}_2\colon \Pi_N[\bar{k}, +\infty] \to TU_0$ is defined, continuous and onto.

DEFINITION 5.4. – A quotient space $T\bar{N} = \Pi_N[\bar{k}, +\infty]/\sim$ is defined if

$$\tau_{\bar{N}}^{(1)} = (\ldots, k_{-i}^{(1)}, \ldots, k_{-1}^{(1)}][k_0^{(1)}, \ldots, k_j^{(1)}, \ldots) \sim \tau_{\bar{N}}^{(2)} = (\ldots, k_{-i}^{(2)}, \ldots, k_{-1}^{(2)}][k_0^{(2)}, \ldots, k_j^{(2)}, \ldots)$$

means that there exist $-\infty \leq n_1 \leq -1$ and $0 \leq n_2 \leq +\infty$ such that $k_{n_1}^{(j)} = k_{n_2}^{(j)} = +\infty$, $j = 1, 2$, and $k_i^{(1)} = k_i^{(2)}$ for $n_1 < i < n_2$.

Thus, the map J_2: $T\bar{N} \to TU_0$, $J_2[\tau_{\bar{N}}] = \tilde{J}_2\tau_{\bar{N}}$ is well defined, continuous, injective and onto. It is easy to check that a basis $\mathcal{B}$ for the topology in $T\bar{N}$ is

$$\mathcal{B} = \left\{[B]: B = \{\tau_{\bar{N}}: \tau_{\bar{N}}[-i-1, j+1] \in (\geqslant k, k_{-i}, \ldots, k_{-1}][k_0, \ldots, k_j, \geqslant k)\}\right\}.$$

where $k_{-i}, \ldots, k_{-1}, k_0, \ldots, k_j$ are integers, and $k \geqslant \bar{k}$ is an integer, $\geqslant k$ stands for $[k, +\infty] \subset [\bar{k}, +\infty]$.

THEOREM 5.5. – *TU_0 and $T\bar{N}$ are both compact and Hausdorff. J_2 is a homeomorphism from $T\bar{N}$ onto TU_0.*

The proof of Theorem 5.5 is elementary and is omitted.

PROOF OF THEOREM 5.2. – We first show that when O_2 is small, F sends O_2 diffeomorphically onto a disc $\pitchfork W^s_{\text{loc}}(0)$ at $\tau^\Gamma_z(N+1)$. Let $O_2 = \{(x_0, y_0): x_0 = g(y_0)\}$ with the inclination K_0. Consider

$$y_1 = By_0 + f_2(g(y_0), y_0) .$$

Since $f_{2x}(\tau^\Gamma_z(N)) = 0$, for any $\tilde{\theta} > 0$, we may let O_2 be sufficiently small so that $\|f_{2x}(x, y)\| \leqslant \tilde{\theta}$ in O_2. Thus, $\|df_2/dy_0\| \leqslant K_0\tilde{\theta} + \theta$, and y_0 can be solved as a C^1 function of y_1 if $\lambda(K_0\tilde{\theta} + \theta) < 1$. Substituting into (4.6), x_1 is a C^1 function of y_1. Therefore, by induction, F^iO_2 contains a C^1 disc $\pitchfork W^s_{\text{loc}}(0)$ at $\tau^\Gamma_z(N+1)$, $i \geqslant 0$, with the inclination K_i, and is diffeomorphic to a disc of O_2. We give estimates on K_i's. Let (ξ_i, η_i), $|\eta_i| \neq 0$ be a tangent vector to a small disc contained in F^iO_2, on which we assume that $\|f_{2x}(x, y)\| \leqslant \tilde{\theta}_i$.

$$\frac{|\xi_{i+1}|}{|\eta_{i+1}|} = \frac{|(A + f_{1x})\xi_i + f_{1y}\eta_i|}{|f_{2x}\xi_i + (B + f_{2y})\eta_i|} \leqslant \frac{(\lambda + \theta)|\xi_i| + \theta(\eta_i|}{(\lambda^{-1} - \theta)|\eta_i| - \tilde{\theta}_i|\xi_i|} \leqslant \frac{(\lambda + \theta)K_i + \theta}{d_i},$$

where $d_i = (1 - \lambda(K_i\tilde{\theta}_i + \theta))/\lambda$. There exists a constant d_∞ such that $d_i \geqslant d_\infty > 1$ for all $i \geqslant 0$ provided that the disc contained in F^iO_2 is sufficiently small, and $\tilde{\theta}_i$ is sufficiently small, since we have $\lambda + \theta < 1$. Therefore,

$$K_{i+1} \leqslant \frac{(\lambda + \theta)K_i + \theta}{d_\infty},$$

and

$$K_i < K_0 + \frac{\theta}{d_\infty - (\lambda + \theta)} \triangleq K_\infty, \qquad i \geqslant 0 .$$

This completes the proof of the transversality of the homocline trajectory τ^Γ_z.

We next consider $F^{-2N}W^s_{\text{loc}}(0)$ in a neighborhood of $\tau^\Gamma_z(-N)$. From Lemma 4.6, it contains a C^1 disc $\pitchfork W^u_{\text{loc}}(0)$ at $\tau^\Gamma_z(-N)$ and is denoted by R_1. Analoguous to what has been done in Lemma 1.2, we obtain that $F^{-i}R_1$ contains a disc $\pitchfork W^u_{loc}(0)$ at $\tau^\Gamma_z(-N-i)$, $i\geqslant 0$, with the inclination $< K_{1\infty}$ for come constant $K_{1\infty}>0$. The key to the proof is (4.1) and $|f_{1y}(x, y)|$ being arbitrarily small in some sufficiently small neighborhood of each $\tau^\Gamma_z(-N-i)$.

We now construct $U_0, \dots, U_\infty$ and $\bar{k}$ as asserted in the theorem. Suppose that ε_1, ε_2 are positive constants such that for u-slices of size $(\varepsilon_2, \varepsilon_1, K_\infty)$ and s-slices of size $(\varepsilon_1, \varepsilon_2, K_{1\infty})$, Lemma 4.2-4.6 are valid. Assume that only a finite number of points of $O_{\tau^\Gamma_z}$, denoted by $q_1, \dots, q_{m-1}$, $m\geqslant 2$, are outside $\tilde{U}=B^x_{\varepsilon_1}\times B^y_{\varepsilon_2}$. There exist an open neighborhood V_i for each q_i such that

$$\begin{aligned}
&V_i\cap V_j=\emptyset\,, && 1\leqslant i\neq j\leqslant m-1\,,\\
&V_i\cap \tilde{U}=\emptyset\,, && 1\leqslant i\leqslant m-1\,,\\
&FV_i\subset V_{i+1}\,, && 1\leqslant i\leqslant m-2\,,\\
&FV_{m-1}\subset \tilde{U}\,,\quad F\tilde{U}\cap V_i=\emptyset\,, && 2\leqslant i\leqslant m-1\,.
\end{aligned}$$

Let $p_0, p_1\in O_{\tau^\Gamma_z}\cap\tilde{U}$, $Fp_1=q_1$, $Fq_{m-1}=p_0$, and $F^m p_1=p_0$.

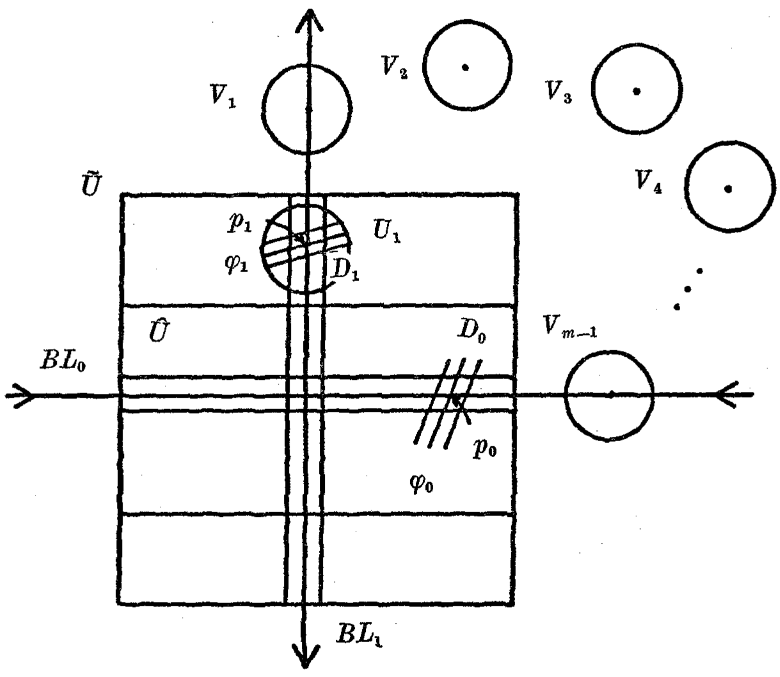

Figure 5.1

We have shown that it is legitimate to assume that $F^m W^u_{loc}(0)$ contains a C^1 disc $\varphi_0\pitchfork W^s_{loc}(0)$ at p_0 with φ_0 being a u-slice of size $(\varepsilon_3, \varepsilon_1-\eta, K_\infty-\eta)$ and that $F^{-n}W^s_{loc}(0)$ contains a C^1 disc $\varphi_1\pitchfork W^u$ at p_1 with φ_1 being an s-slice of size $(\varepsilon_4, \varepsilon_2-\eta, K_{1\infty}-\eta)$ with some constants $\eta>0$, $0<\varepsilon_3<\varepsilon_2$, $0<\varepsilon_4<\varepsilon_1$. By Lemma 4.6, if $0<\eta_1<\eta$, ε_3, ε_4 are small enough, the F^m image of any u-slice C^1 near $W^u_{\text{loc}}(0)$ contains a u-slice $C^1-\eta_1$ near φ_0, and hence, a u-slice of size $(\varepsilon_3, \varepsilon_1 - \eta + \eta_1, K_\infty)$, and the F^{-m} image of any s-slice C^1 near $W^s_{\text{loc}}(0)$ contains an s-slice $C^1-\eta_1$ near φ_1, and hence, an s-slice of size $(\varepsilon_4, \varepsilon_2-\eta+\eta_1, K_{1\infty})$. We denote the family of all u-slices of size $(\varepsilon_3, \varepsilon_1-\eta+\eta_1, K_\infty)$ by $\mathcal{U}$ and the family of all s-slices of size $(\varepsilon_4, \varepsilon_2-\eta+\eta_1, K_{1\infty})$ by $\mathcal{S}$. We may assume that $\overline{\mathcal{U}}\subset\tilde{U}$ and $\overline{\mathcal{S}}\subset\tilde{U}$ in the point set sense. We use $\overline{W}$ or ClW to denote the closure of a set W.

Consider $BL_1(\bar{k}) = \bigcup_{k\geqslant\bar{k}} (B^y_{\varepsilon_2}\cap F)^k\mathcal{U}$ for a positive integer $\bar{k}$. When $\bar{k}$ is large, $BL_1(\bar{k})$ is C^1 near $B^y_{\varepsilon_2}$ and $F^m(BL_1(\bar{k}))$ is $C^1-\eta_1$ near φ_0. Similarly, consider $BL_0(\bar{k}) = \bigcup_{k\geqslant\bar{k}} (B^x_{\varepsilon_1}\cap F^{-1})^k\mathcal{S}$. When $\bar{k}$ is large, $BL_0(\bar{k})$ is C^1 near $B^x_{\varepsilon_1}$ and $F^{-m}(BL_0(\bar{k}))$ is $C^1-\eta_1$ near φ_1. If η_1 is small and $\bar{k}$ is large, $F^m(BL_1(\bar{k}))\ \overset{+}{\pitchfork}\ BL_0(\bar{k})$. The intersection is denoted by D_0. Also $F^{-m}(BL_0(\bar{k}))\ \overset{+}{\pitchfork}\ BL_1(\bar{k})$ and the intersection is denoted by D_1. We may assume that $\overline{D}_0, \overline{D}_1\subset\tilde{U}$ and $F\overline{D}_1\subset V_1$. It is clear that $F^mD_1=\overline{D}_0$ and $F^{-m}D_0=D_1$ if restricted to a neighborhood of p_1.

It also follows from Lemma 4.6 and 4.2 that if $\bar{k}$ is large enough, F^{m+k} (F^{-m-k}), $k\geqslant\bar{k}$ are Lipschitz contractions in the C^0 norm, on u-slices in $\mathcal{U}$ into u-slices near φ_0 $(BL_0(\bar{k})$ into $BL_0(\bar{k}))$, with the Lipschitz constant $\leqslant\bar{\lambda}$, $0<\bar{\lambda}<1$.

Let $\hat{U}=\tilde{U}\cap F^{-1}\tilde{U}$. Then $\hat{U}$ is open and $p_1\notin\hat{U}$ since $q_1\notin\tilde{U}$. If $\bar{k}$ is large, the distance between $\overline{D}_1$ and $\hat{U}$ is positive. It is also clear that

$$\tilde{U}\cap F^{-1}\hat{U}\subset\tilde{U}\cap F^{-1}\tilde{U}=\hat{U}\,,\qquad \tilde{U}\cap F^{-1}\hat{U}=\hat{U}\cap F^{-1}\hat{U}\,.$$

By induction, we have $(\hat{U}\cap F^{-1})^{\bar{k}-1}\hat{U}=(\tilde{U}\cap F^{-1})^{\bar{k}}\,\tilde{U}$. Clearly, $D_0\subset(\tilde{U}\cap F^{-1})^{\bar{k}}\,\tilde{U}$. We claim that $\overline{D}_0\subset(\tilde{U}\cap F^{-1})^{\bar{k}}\,\tilde{U}$, since $\overline{\mathcal{U}}$ and $\overline{\mathcal{S}}\subset\tilde{U}$. Therefore, $\overline{D}_0\subset(\hat{U}\cap F^{-1})^{\bar{k}-1}\hat{U}$ and $\overline{D}_1\subset F^{-m}(\overline{D}_0)\subset F^{-m}(\hat{U}\cap F^{-1})^{\bar{k}-1}\hat{U}$. The last set is open so there is an open neighborhood U_1 of $\overline{D}_1$ such that $U_1\cap\hat{U}=\emptyset$, $U_1\subset\tilde{U}$, $FU_1\subset V_1$ and $U_1\subset F^{-m}\cdot(\hat{U}\cap F^{-1})^{\bar{k}-1}\hat{U}$. We claim that $U_0=\hat{U}$, U_1, $U_i=V_{i-1}$, $2\leqslant i\leqslant m$, associated with $\bar{k}$ (U_1 depends on $\bar{k}$) fulfill all the requirenents of the theorem.

We first show that $J_1(TZ)\subset TU$. For this, only condition 2) in Definition 5.1 has to be checked. Suppose $\tau_z\in TZ$ with $\tau_z(-1)\in U_m$, $\tau_z(0)\in U_0$, then $\tau_z(-m)\in U_1\subset F^{-m}(\hat{U}\cap F^{-1})^{\bar{k}-1}\hat{U}$. This implies that $\tau_z(0)\in(\hat{U}\cap F^{-1})^{\bar{k}-1}\hat{U}$. Hence, for $1\leqslant j\leqslant\bar{k}-1$, $\tau_z(j)\in(\hat{U}\cap F^{-1})^{k-1-j}\hat{U}\subset\hat{U}=U_0$. Therefore,

$$J_1(TZ)\diagdown\{(\dots,0][0,\dots)\}\subset TU\diagdown\{(\dots,U_0][U_0,\dots)\}\,.$$

This, together with $J_1(\dots,0][0,\dots)=(\dots,U_0][U_0,\dots)$, implies $J_1(TZ)\subset TU$.

It remains to show that $J_1(TZ)\supset TU$ and J_1^{-1} is single valued and continuous. It suffices to prove the following assertions:

(i) J_1^{-1} is well defined, single valued and continuous on $TU\diagdown\{(\dots,U_0][U_0,\dots)\}$;

(ii) $J_1^{-1}(\dots,U_0][U_0,\dots)=(\dots,0][0,\dots)$ and J_1^{-1} is continuous at $(\dots,U_0][U_0,\dots)$.

For (ii), by Theorem 3.1, $J_1^{-1}(\ldots, U_0][U_0, \ldots)$ must lie on $W^s_{\mathrm{loc}}(0)$ and $W^u_{\mathrm{loc}}(0)$; hence, identically equal to zero. It follows from Lemma 4.4 that if $\tau_z \in TZ$ such that $\tau_z[-i, i] = (\underbrace{U_0, \ldots, U_0}_{i}][\underbrace{U_0, \ldots, U_0}_{i+1})$, then $\tau_z(0)$ lies on s-slices $C(\tilde{\lambda})^i$ near W^s and u-slices $C(\tilde{\lambda})^i$ near $W^u(0 < \tilde{\lambda} < 1)$ in the C^0 norm. Therefore, $\tau_z(0)$ is in a ball of radius $2C(\tilde{\tau})^i$ centered at 0. $\tau_z(0) \to 0$ as $i \to \infty$. Therefore, J_1^{-1} is continuous at $(\ldots, U_0][U_0, \ldots)$.

For (i), it suffices to show that $J_3 \triangleq J_1^{-1} J_2$ is well defined, single valued and continuous on $T\bar{N}$, since by Theorem 5.5, $J_2\colon T\bar{N} \to TU_0$ is a homeomorphism. Also, see the comment after Definition 5.3. It is now clear that we have to show that $J_1^{-1}\tilde{J}_2$ is well defined, single valued and continuous on $\Pi_N[\bar{k}, +\infty]$.

$$\begin{tikzcd}
\prod\limits_N [\bar{k}, +\infty] \arrow[d, "\pi"'] \arrow[dr, "\tilde{J}_2"] & \\
T\bar{N} \arrow[r, shift left, "J_2"] \arrow[dr, "J_3"'] & TU_0 \arrow[l, shift left, "J_2^{-1}"] \\
& TZ_0 \arrow[u, "J_1"']
\end{tikzcd}$$

Let $\tau_{\bar{N}} = (\ldots, k_{-i}, \ldots, k_{-1}][k_0, \ldots, k_j, \ldots) \in \Pi_N[\bar{k}, +\infty]$. Assume that $k_n \neq +\infty$ for all n. The other cases can be proved similarly. If $\tau_z \in J_1^{-1}\tilde{J}_2\tau_{\bar{N}}$, it is necessary that $\tau_z(0) \in D_0$. Let $Z(k_{-i}, \ldots, k_{-1}][k_0, \ldots, k_j)$, denote the subset of D_0 such that for each $z \in Z(k_{-i}, \ldots, k_{-1}][k_0, \ldots, k_j)$, there exists a finite trajectory $\overset{0}{\tau}_z$ with $J_1 \overset{0}{\tau}_z = (k_{-i}, \ldots, k_{-1}][k_0, \ldots, k_j)$ and $\overset{0}{\tau}_z(0) = z$. Evidently,

$$F^{k_{-1}+m} Z(k_{-i}, \ldots, k_{-2}][k_{-1}, \ldots, k_j) = Z(k_{-i}, \ldots, k_{-1}][k_0, \ldots, k_j)\,.$$

We claim that $z(k_{-N}, \ldots, k_{-1}][k_0, \ldots, k_{N-1})$ is contained in a set of s-slices $\subset BL_0(\bar{k})$ $\big($a set of u-slices $\subset F^m(BL_1(\bar{k}))\big)$ in which the distance between any two of them is $\leqslant C(\tilde{\lambda})^N$. This is clearly true for $N = 1$. For $N = 2$, the assertion follows from

$$F^{k_{-1}+m} Z(k_{-2}][k_{-1}, k_0, k_1) = Z(k_{-2}, k_{-1}][k_0, k_1]\,,$$
$$F^{k_0+m} Z(k_{-2}, k_{-1}][k_0, k_1) = Z(k_{-2}, k_{-1}, k_0][k_1)\,,$$

and the contractiveness of $F^{k_{-1}+m}$ on u-slices and F^{-k_0-m} on s-slices considered. It follows by induction that the assertion is valid for general N. We have shown that

$$\mathrm{Cl}\, Z(k_{-N}, \ldots, k_{-1}][k_0, \ldots, k_{N-1}) \subset \text{a closed ball of radius} \leqslant C_1(\tilde{\lambda})^N . \tag{5.1}$$

It is easy to see that $\tau_z(0) \in \bigcap_{i,j>0} \mathrm{Cl}\, Z(k_{-i}, \ldots, k_{-1}][k_0, \ldots, k_j)$. Similarly,

$$\tau_z(-l) \in \bigcap_{i,j>0} \mathrm{Cl}\, Z(k_{-i}, \ldots, k_{-n-1}][k_{-n}, \ldots, k_j)\,, \qquad l = \sum_{\alpha=1}^{n} k_{-\alpha} + nm\,, \quad n = 0, 1, \ldots . \tag{5.2}$$

The right hand side of (5.2) is a singleton set since it is the intersection of descending closed sets with estimates (5.1). Therefore τ_z is unique if it exists,

Conversely, define τ_z formally by (5.2) on a sequence of infinitely many $-l$'s and choose the values of τ_z between each of the $-l$'s and after $\tau_z(0)$ by the map F. We can verify that τ_z is a trajectory in TZ and $\tilde{J}_2(\ldots, k_{-i}, \ldots, k_{-1}][k_0, \ldots, k_j, \ldots) = J_1\tau_z$. We start with

$$\begin{aligned} F^{m+k_{-n}}\tau_z(-l) &= F^{m+k_{-n}} \bigcap_{i,j>n} \mathrm{Cl}\, Z(k_{-i}, \ldots, k_{-n-1}][k_{-n}, \ldots, k_j) \subset \\ &\subset \bigcap_{i,j>n} F^{m+k_{-n}}\, \mathrm{Cl}\, Z(k_{-i}, \ldots, k_{-n-1}][k_{-n}, \ldots, k_j) \subset \\ &\subset \bigcap_{i,j>n} \mathrm{Cl}\, F^{m+k_{-n}}\, Z(k_{-i}, \ldots, k_{-n-1}][k_{-n}, \ldots, k_j) = \\ &= \bigcap_{i,j>n} \mathrm{Cl}\, Z(k_{i-i}, \ldots, k_{-n}][k_{-n+1}, \ldots, k_j)\,. \end{aligned}$$

Since the last is a singleton set, all the inclusions are equalities. This proves the consistency of the definition of τ_z on the $-l$'s. The only thing unpleasant is that

$$\tau_z(-l) \in \mathrm{Cl}\, Z(k_{-i}, \ldots, k_{-n-1}][k_{-n}, \ldots, k_j)\,,$$

not

$$\tau_z(-l) \in Z(k_{-i}, \ldots, k_{-n-1}][k_{-n}, \ldots, k_j)\,.$$

But,

$$\mathrm{Cl}\, Z(k_{-i}, \ldots, k_{-n-1}][k_{-n}, \ldots, k_j) \subset \mathrm{Cl}\, Z[k_{-n}, \ldots, k_j) \subset Z[k_{-n}, \ldots, k_{j-1})$$

due to the continuity of the forward iterates of F. Therefore, the iterates of F on $\tau_z(-l)$ must stay in the «body» for $k_{-n}, \ldots, k_{j-1}$ times before leaving the «body» for the «handle». Since j can be arbitrarily large, $\tau_z \in TZ$ and $\tilde{J}_2(\ldots, k_{-i}, \ldots, k_{-1}]\cdot[k_0, \ldots, k_j, \ldots) = J_1\tau_z$.

The continuity of $J_1^{-1}\tilde{J}_2$ follows from (5.1). This completes the proof of the theorem.

COROLLARY 5.6 (SIL'NIKOV [14]). – *$T\bar{N}$ is homeomorphic to TZ_0 via the map $J_3 = J_1^{-1} J_2$.*

COROLLARY 5.7. – *Suppose the distance between U_i, U_j, $0 \leqslant i < j \leqslant m$ is positive. Let $J_1 \tau_z^{(\beta)} = \tau_u^{(\beta)}$, $\beta = 1, 2$. Then, $\tau_z^{(1)}(i) \to \tau_z^{(2)}(i)$ as $i \to +\infty$ $(-\infty)$ if and only if $\tau_u^{(1)}(i) = \tau_u^{(2)}(i)$ for $i \geqslant n$ $(\leqslant n)$, where n is some constant.*

PROOF. – Necessity is trivial. Sufficiency follows from estimate (5.1).

6. – Further consequences.

Throughout this section, we assume the hypotheses of Corollary 5.7 are satisfied. The above results are generalizations of the work of Sil'nicov [14] on diffeomorphisms in R^n. We generalized it to C^k maps in Banach spaces, and refined the argument by showing that the extended neighborhood and $\bar{k}$ can be associated in such a way that all the trajectories in the neighborhood can be symbolized precisely by TU, depending on $\bar{k}$. Note that, in the notation of Sil'nikov's original work, trajectories in N^+, N^-, $N^{\pm}$, and N, i.e., asymptotic to 0 in the positive direction, negative direction, both directions, and not asymptotic to 0 at all, are symbolized distinctly. However, our work shows that trajectories in any of the four subsets are dense, a phenomenon concealed by his original symbolization. To illustrate, we show that the trajectories that are asymptotic to 0 in both directions are dense in TZ. Given $\tau_z \in TZ$, $J_1 \tau_z = \tau_u = (\ldots, U_{\alpha_{-i}}, \ldots, U_{\alpha_{-1}}][U_{\alpha_0}, \ldots, U_{\alpha_j}, \ldots)$. Let $\tau_u^{(n)} = (\ldots, U_0, U_0, U_{\alpha_{-n}}, \ldots, U_{\alpha_{-1}}][U_{\alpha_0}, \ldots, U_{\alpha_n}, U_0, U_0, \ldots)$, $n \geqslant 1$ and $\tau_z^{(n)} = J_1^{-1} \tau_u^{(n)}$. By Corollary 5.7, $\tau_z^{(n)}$ is asymptotic to 0 in both directions for each $n \geqslant 1$. Furthermore, $\tau_z^{(n)} \to \tau_z$ since $\tau_u^{(n)} \to \tau_u$.

All the significance of the symbolizations for diffeomorphisms discussed by Sil'nikov and Smale hold true in our case. For example, there are countably many trajectories that are periodic or homoclinic to 0 in TZ. TZ is topologically transitive, i.e., there is a trajectory $\tau_z \in TZ$ such that $\sigma^n \tau_z$, $n = 0, \pm 1, \ldots$, is dense in TZ. We infer that each trajectory in TZ is unstable from the instability of TU, since given any $\tau_u \in TU$, we can construct $\tau_u^{(l)}$ such that $\tau_u^{(l)}(-\infty, l] = \tau_u(-\infty, l]$ and $\tau_u^{(l)}(i) \neq \tau_u(i)$ for infinitely many $i > l$. From Corollary 5.7,

$$\lim_{i \to \infty} \sup |\tau_z^{(l)}(i) - \tau_z(i)| \geqslant \varepsilon > 0 ,$$

where $\tau_z^{(l)} = J_1^{-1} \tau_u^{(l)}$ and $\tau_z = J_1^{-1} \tau_u$, ε is a constant independent of l. But $\tau_z^{(l)}(0) \to \tau_z(0)$ as $l \to \infty$. This proves the instability of each trajectory.

The following is a counterpart to Smale's invariant, Cantor like set near a homoclinic point [16].

COROLLARY 6.1. – *There exist an integer $k > 0$ and a subset of trajectories $TZ(k)$ of F^k in a neighborhood of $O_{\tau_z^\Gamma}$ such that F^k acting on $TZ(k)$ is invariant and equivalent to the shift map on the doubly infinite sequence of two symbols.*

PROOF. – By Theorem 5.2, it suffices to examine σ^k on TU. Let $k \geq \bar{k} + m$ be any fixed integer. If, by the symbol s_0, we mean $\{\underbrace{U_0, \dots, U_0}_{k\text{-fold}}\}$ and the symbol s_1, $\{\underbrace{U_0, \dots, U_0}_{k\text{-fold}}, U_1, \dots, U_m\}$, a subset of TU is defined and is invariant under σ^k.

Comparing our results with other papers, one finds that the invariant set of trajectories under ΠF are discussed instead of the invariant set of points under F. For F being diffeomorphic, define P as the projection $\mathbb{P}$: $TZ \to Z$, $\mathbb{P}\tau_z = \tau_z(0)$. Then P is a homeomorphism from TZ onto $TZ(0) \overset{\text{def}}{=} P(TZ)$. ΠF: $TZ \to TZ$ is equivalent to F: $TZ(0) \to TZ(0)$, via P.

$$\begin{array}{ccc} TZ & \xrightarrow{\Pi P} & TZ \\ \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle P} \\ TZ(0) & \xrightarrow{F} & TZ(0)\,. \end{array}$$

Therefore, the symbolizations for the point set $TZ(0)$, invariant under F is induced from that of TZ, or F: $TZ(0) \to TZ(0)$ is equivalent to a shift homeomorphism σ: $TU \to TU$.

Another interesting case is the appearance of a snap-back repeller named after MAROTTO [10]. An expanding fixed point 0 of a C^1 map F: $Z \to Z$ is said to be a snap-back repeller if there is a point $z_0 \in W^u_{\text{loc}}(0)$ with $z_0 \neq 0$, and an integer $n \geq 1$ such that $F^n(z_0) = 0$ and $DF^i(z_0)$ is an isomorphism onto Z, for $1 \leq i \leq n$. It is easy to see that there is a transverse homoclinic trajectory τ_z^Γ passing through z_0 and hitting 0 after finite iterates of F. And it can be treated as a special case of Theorem 5.2 with $W^s_{\text{loc}}(0) = \{0\}$. However, the results are nicer if we consider positive trajectories τ_z^+ and τ_u^+. Let $U_0, \dots, U_m$ be open sets containing $O_{\tau_z^\Gamma}$ and $0 \in U_0$. Let $S = \{U_0, \dots, U_m\}$ and $\bar{k} > 0$ an integer. A subset $TU^+ \subset \Pi_{N^+} S$ is defined on $\tau_u^+ \in TU^+$ if and only if 1) and 2) but 3) of Definition 5.1 hold. TU^+ is a topological space with the topology induced from $\Pi_{N^+} S$. The semishift operator σ^+ is defined on TU^+ as $\sigma^+ \tau_u^+(i) = \tau_u^+(i+1)$, $i \in N^+$. σ^+ is continuous, surjective but not injective. Let $TZ_+ \subset \Pi_{N^+} Z$ be the set of all the positive trajectories whose orbits are contained in $\bigcup\limits_{0 \leq i \leq m} U_i$. TZ^+ is a topological space with the topology induced from $\Pi_{N^+} Z$. Let P be the projection from TZ^+ to $TZ^+(0) \overset{\text{def}}{=} P(TZ^+) \subset Z$, defined as $P\tau_z^+ = \tau_z^+(0) \in Z$ for any $\tau_z^+ \in TZ^+$. It is obvious that P is a homeomorphism. Let J_1: $TZ^+ \to TU^+$ be defined as $\tau_u^+(i) = (J_1 \tau_z^+)(i) = U_j$ if $\tau_z^+(i) \in U_j$, $0 \leq j \leq m$, $i \geq 0$.

THEOREM 6.2. – *Suppose F: $Z \to Z$ is C^1 with 0 as a snap-back repeller. Then there exist open sets $U_0, \ldots, U_m$ and an integer $\bar{k} > 0$ such that $\bigcup_{0 \leqslant i \leqslant m} U_i$ contains the homoclinic orbit and $0 \in U_0$. Furthermore, J_1: $TZ_+ \to TU^+$ is a homeomorphism and the following diagram commutes.*

$$\begin{array}{ccc} TZ^+(0) & \xrightarrow{F} & TZ^+(0) \\ P\uparrow & & \uparrow P \\ TZ^+ & \xrightarrow{\Pi F} & TZ^+ \\ J_1\downarrow & & \downarrow J_1 \\ TU^+ & \xrightarrow{\sigma^+} & TU_+ . \end{array}$$

The proof of Theorem 6.2 is similar to that of Theorem 5.2. One only has to observe that the s-slices are points in Z and the u-slices coincide with $W^u_{\mathrm{loc}}(0)$. We don't ask that DF be uniformly continuous in the neighborhood of 0 since the Inclination Lemmas are trivially true in this case. We obtain that, when a snap-back repeller appears, the above symbolic dynamics can be used to discuss trajectories, positive trajectories and invariant point sets in a neighborhood of the homoclinic orbit.

7. – Chaotic behavior.

We have shown that trajectories in TZ have very complicated behavior—the motion of $F^i \tau_z(0)$ is quite unpredictable except that it must stay in U_0 for at least $\bar{k}$ iterates of F before leaving U_0 for the « handle ». We shall show that this kind of motion implies chaos described by LI and YORKE [9], [10], [17]; that is, if TZ is homeomorphic to TU via J_1, then there exists chaos in the following sense:

1) There exists $k > 0$ such that for each integer $p \geqslant k$, F has a trajectory of period p.

2) There exists a subset of uncountably many trajectories CHAOS $\subset TZ$ such that,

a) for every $\tau_z^{(1)}, \tau_z^{(2)} \in$ CHAOS with $\tau_z^{(1)} \neq \tau_z^{(2)}$,

$$\limsup_{i \to \pm\infty} |\tau_z^{(1)}(i) - \tau_z^{(2)}(i)| > 0 \; ; \tag{7.1}$$

b) for $\tau_z^{(1)} \in$ CHAOS and $\tau_z^{(2)}$ being periodic in TZ, (7.1) is valid,

c) $\tau_z^{(1)}, \tau_z^{(2)} \in$ CHAOS implies that

$$\liminf_{i\to\pm\infty} |\tau_z^{(1)}(i) - \tau_z^{(2)}(i)| = 0 \, .$$

3) $\Pi F(\mathrm{CHAOS}) = \mathrm{CHAOS}$.

The ideas of the proof presented here are essentially from [9], [10].

PROOF. - 1) Let $k = \bar{k} + m$. Let

$$\tau_u = (\ldots, \text{repeat}, \underbrace{U_0, \ldots, U_0, U_1, \ldots, U_m}_{p\text{-fold}}, \text{repeat}, \ldots)$$

then $\tau_z = J_1^{-1}\tau_u \in TZ$ is a trajectory of F with the period $= p$.

2) Let

$$s_0 = \{\underbrace{U_0, \ldots, U_0}_{k\text{-fold}}\} \, , \qquad s_1 = \{\underbrace{U_0, \ldots, U_0, U_1, \ldots, U_m}_{k\text{-fold}}\} \, , \qquad k = \bar{k} + m \, .$$

For each $w \in (0, 1)$, choose an element $\tau_u^w \in TU$, composed by s_0 and s_1 such that

$$\tau_u^w \in \left\{ \begin{array}{l} (\ldots; S_{\alpha_{-i}}; \ldots; S_{\alpha_{-1}}][S_{\alpha_1}; S_{\alpha_2}; \ldots; S_{\alpha_j}; \ldots): \\ \qquad \alpha_i = 1 \text{ only if } i = \pm n^2, \; n = 1, 2, \ldots; \text{ and } \lim\limits_{n\to\infty} \dfrac{R^{\pm}(\tau_u^w, n^2)}{n} = w \end{array} \right\},$$

where $R^{\pm}(\tau_u^w, n^2)$ is the number of α_i's which equals 1 for $(1 \leqslant i \leqslant n^2)(-n^2 \leqslant i \leqslant -1)$ respectively.

Let $CHS = \{\sigma^i \tau_u^w \colon w \in (0, 1), \; i \in N\}$. Evidently, $\sigma(CHS) = CHS$. Therefore, if $\mathrm{CHAOS} = J_1^{-1}(CHS)$, $\Pi F(\mathrm{CHAOS}) = \mathrm{CHAOS}$. The assertion 3) is proved. In proving 2), we only consider the case $i \to +\infty$. We first show that a) is true for $\tau_z^{(1)} = J_1^{-1}(\tau_u^w)$ and $\tau_z^{(2)} = J_1^{-1}(\sigma^j \tau_u^w)$, $j \neq 0$. Since $w \neq 0$, there exist infinitely many integers n such that $\tau_u^w(kn^2 - 1) = U_m$, $\sigma^j \tau^w(kn^2 - 1) = \tau_u^w(kn^2 + j - 1)$. If n is sufficiently large, $kn^2 + j - 1$ is not of the form $kl^2 - 1$ for any integer l; thus $\sigma^j \tau_u^w(kn^2) \neq U_m$. This shows (7.1) is valid in this case. Obviously a) is also true for $\tau_z^{(1)} = J_1^{-1}(\sigma^i \tau_u^w)$, $\tau_z^{(2)} = J_1^{-1}(\sigma^j \tau_u^w)$, $i \neq j$. We next show that a) is true for $\tau_z^{(1)} = J_1^{-1}(\sigma^i \tau_u^{w_1})$ and $\tau_z^{(2)} = J_1^{-1}(\sigma^i \tau_u^{w_2})$, $w_1 \neq w_2$. Let $\tilde{R}^+(\tau_u, kn^2)$ be the number of $\tau_u(l)$ which equals U_m for $1 \leqslant l \leqslant kn^2$. We observe that

$$\lim_{n\to\infty} \frac{\tilde{R}^+(\sigma^i \tau_u^w, kn^2)}{n} = w \, . \tag{7.2}$$

For any given $K > 0$, there exists an $l > K$ such that $\sigma^i \tau_u^{w_1}(l) \neq \sigma^i \tau_u^{w_2}(l)$. Otherwise, from (7.2), one would have $w_1 = w_2$, contradicting the fact that $w_1 \neq w_2$.

The proof of *a*) is completed. *b*) Can be proved similarly. To prove *c*), notice that for any $\tau_u \in CHS$, the length of the successive i's such that $\tau_u(i) = U_0$ approaches $+\infty$ as $i \to +\infty$. Therefore, for $\tau_u^{(1)}, \tau_u^{(2)} \in CHS$, the length of successive i's such that $\tau_u^{(1)} = \tau_u^{(2)}(i) = U_0$ approaches $+\infty$ as $i \to +\infty$. *c*) is true by (5.1). This completes the proof of the existence of chaos.

The work of Li and Yorke indicated that Period 3 implies chaos in R. Marotto pointed out this is not the case in R^2. He proved that Snap-back Repeller implies chaos in R^n. Our work shows that the transverse homoclinic trajectory implies chaos in Banach spaces.

8. – Flows.

Noninvertible maps also arise form the Poincaré mapping of noninvertible flows. The Poincaré map can either be the return map around a periodic trajectory for an autonomous flow or the period map of a periodic flow. Both cases are discussed in this section.

Let X be a Banach space and $T(t, s)$, $t \geqslant s$ in R be a semigroup of nonlinear maps in X. We assume that

1) $T(t, s)$ is strongly continuous in t, s;
2) $T(s, s) = I$;
3) $T(t, u)T(u, s) = T(t, s)$, $t \geqslant u \geqslant s$;
4) There are constants $\alpha \geqslant 0$, $k \geqslant 1$ such that $T(t, s)x$ is C^k jointly in t and x for $t > s + \alpha$.

Examples of abstract evolution equations with $\alpha = 0$ may be found in [4]. For delay equations under some general conditions, $\alpha = k\gamma$, where $\gamma > 0$ is the delay [3].

We say that $T(t, s)$ is periodic of period $\omega > 0$ if $T(t, s) = T(t + \omega, s + \omega)$. If we do not assume that ω is the least period, then we may assume $\omega > \alpha$. The period map $F = T(\omega, 0)$ is then C^k on X. If $\xi(t)$ is a periodic trajectory of $T(t, s)$ with the period ω; that is, $T(t, s)\xi(s) = \xi(t)$, $t \geqslant s$ in R, $\xi(t + \omega) = \xi(t)$, then $\xi(0)$ is a fixed point of F. Conversely, any fixed point of F can be used to define a periodic trajectory. One can define homoclinic trajectories of $T(t, s)$ asymptotic to $\xi(t)$ in the obvious way and relate them to homoclinic trajectories of F asymptotic to $\xi(0)$.

We next assume that the semigroup is autonomous; i.e., $T(t, s) = T(t - s)$, $t \geqslant s$ in R. Let $\xi(t)$ be a periodic trajectory of least period $\omega > 0$ of $T(t)$, $t \geqslant 0$; that is, $T(t)\xi(s) = \xi(t + s)$ for all $t \geqslant 0$, $s \in R$, $\xi(t + \omega) = \xi(t)$ for all t and $\xi(t) \neq \xi(0)$, $0 < t < \omega$. Replacing ω by $n\omega$, we may assume $\omega > \alpha$. Let $X_1 \subset X$ be a codimension one hyperplane transversal to the periodic trajectory at $x = \xi(0)$. There exists a neighborhood U of $\xi(0)$ in X_1 such that for every $x \in U$, there is a unique $t = t(x)$ near ω such that $T(t(x))x \in X_1$. The map F: $U \to X_1$ is defined as $F(x) = T(t(x))x$

and is C^k. It is clear that $\xi(0)$ is a fixed point of F. Suppose $x = p(t)$ is a homoclinic trajectory of $T(t)$ asymptotic to $x = \xi(t)$. There is a constant $\tau > \alpha/2$ such that for $|t| > \tau$, $x = p(t)$ is near the orbit of $x = \xi(t)$ and intersects $U \subset X_1$ successively as $t \to \pm\infty$. Let $q_1 = p(t_1)$ and $q_2 = p(t_2)$, $q_1, q_2 \in U$, with $t_1 < -\tau$ and $t_2 > \tau$. $F^{-n} q_1$ and $F^n q_2$, $n \geq 0$, are defined as the intersections of $p(t)$ with X_1 and agree with the definition of F given before. Obviously, $F^n q_2 \to \xi(0)$ and $F^{-n} q_1 \to \xi(0)$ as $n \to \infty$. Assume that there are open sets U_1 and $U_2 \subset U$ such that $U_1 \cap U_2 = \emptyset$, $q_1 \in U_1$ and $\left(\bigcup_{n=0}^{\infty} F^n q_2\right) \cap \left(\bigcup_{n=1}^{\infty} Fz^n q_1\right) \subset U_2$. We redefine F in U_1 as $Fq_1 = q_2$ and $Fx = y$ for $x \in U_1$ and $y \in U_2$ such that $u = T(t(x))\,x$ with a unique $t = t(x)$ near $t_2 - t_1 > \alpha$. This could be done if U_1 is sufficiently small so that the flow issuing from U_1 meets U_2 transversely in a uniquely determined time $\hat{t} = t(x)$ near $t_2 - t_1$. Thus, $F\colon U_1 \cap U_2 \to X_1$ is C^k with a fixed point $\xi(0)$ and a homoclinic trajectory $\{F^{-n} q_1, F^n q_2, n \geq 0\}$.

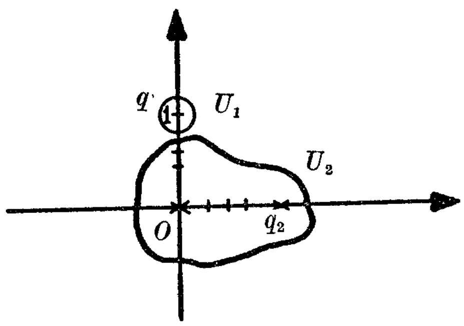

Figure 8.1

DEFINITION 8.1. – Suppose $T(t, s)$ satisfies hypertheses 1)-4) and is either periodic or autonomous. Suppose that $x = \xi(t)$ is a periodic trajectory with the Poincaré map F defined previously. It is said to be a *hyperbolic period trajectory* if

$$\sigma\big(DF(\xi(0))\big) \cap \{|\lambda| = 1\} = \emptyset .$$

Note that the map F can be different if we take other hyperplanes transversal to the periodic trajectory, e.g., in the periodic flow case, the section can be $(t^*) \times X \subset R \times T$ and the map is $T(t^* + \omega, t^*)$. Thus, we shall justify that Definition 8.1 is independent of the Poincaré section chosen. Also, if $\omega < \alpha$, there is no unique way to choose $n\omega > \alpha$ with integers $n > 0$. We shall prove Definition 8.1 is independent of n.

The *stable set* $W^s(\xi(\cdot))$ and *unstable set* $W^u(\xi(\cdot))$ of $x = \xi(t)$ is defined in the usual way. The existence of the *local stable manifold* $W^s_{\mathrm{loc}}(\xi(\cdot)) \subset W^s(\xi(\cdot))$ and local unstable manifold $W^u_{\mathrm{loc}}(\xi(\cdot)) \subset W^u(\xi(\cdot))$ in a neighborhood of the orbit of a hyperbolic period trajectory $x = \xi(t)$ shall be proved in Theorem 8.3.

DEFINITION 8.2. – A homoclinic trajectory $x = p(t)$ of $T(t, s)$ in a Banach space X, asymptotic to a periodic trajectory $x = \xi(t)$ of $T(t, s)$ is said to be a *transverse homoclinic trajectory* if

1) the periodic trajectory $x = \xi(t)$ is hyperbolic;

2) for any sufficiently large pair $s, t > 0$ such that $p(-s) \in W^u_{\text{loc}}(\xi(\cdot))$ and $p(t) \in W^s_{\text{loc}}(\xi(\cdot))$, $T(t, -s)$ sends a disc containing $p(-s)$ in $W^u_{\text{loc}}(\xi(\cdot))$ diffeomorphically onto its image which is transversal to $W^s_{\text{loc}}(\xi(\cdot))$ at $p(t)$.

Note that in the forgoing definitions $W^s_{\text{loc}}(\xi(\cdot)) = \{\xi(\cdot)\}$ as well as $x = p(t)$ hits the orbit $O_{\xi(\cdot)}$ at some finite t is allowed. It is also clear that $W^u_{\text{loc}}(\xi(0))$ and $W^s_{\text{loc}}(\xi(0))$ of the fixed point $\xi(0)$ of F are precisely the intersections of $W^u_{\text{loc}}(\xi(\cdot))$ and $W^s_{\text{loc}}(\xi(\cdot))$ with the Poincaré section. Another observation is that $x = p(t)$ is a transverse homoclinic trajectory if and only if it induces a transverse homoclinic trajectory on the Poincaré section for the fixed point $\xi(0)$ of the map F. There is a geometric explanation for Definition 8.2, that is, there are two narrow strips locally diffeomorphic to $W^u_{\text{loc}}(\xi(\cdot))$ and $W^s_{\text{loc}}(\xi(\cdot))$ respectively (Immersed image of $W^u_{\text{loc}}(\xi(0)) \times R$ and $W^s_{\text{loc}}(\xi(0)) \times R$, not necessarily injective), attached to $x = p(t)$ and intersect transversely along $x = p(t)$. See fig. 8.2 for the illustration of the unstable strip.

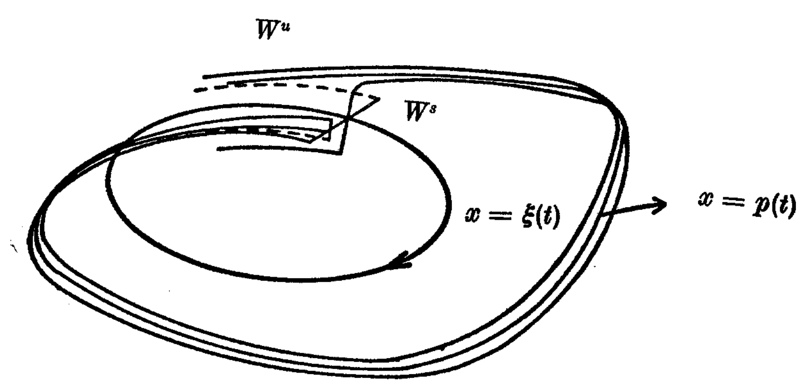

Figure 8.2

THEOREM 8.3. – *Let $x = \xi(t)$ be a periodic trajectory with the period $\omega > 0$, for $T(t, s)$ satisfying conditions 1)-4). Then in both the following cases, $T(t, s) = T(t - s)$ or $T(t, s) = T(t + \omega, s + \omega)$, the definition of the hyperbolicity of $x = \xi(t)$ is independent of the integer n, $n\omega > \alpha$, or the Poincaré section chosen. Moreover if $T(t, s)x$ is C^k jointly in t, s and x for $t > s + \alpha$, the local stable and unstable manifolds $W^s_{\text{loc}}(\xi(\cdot))$ and $W^u_{\text{loc}}(\xi(\cdot))$ exist and are C^k submanifolds in X for the autonomous case and in $R \times X$ for the periodic case.*

PROOF. – Only the proof for the periodic flow shall be given. Let $F_1 = T(n_1\omega, 0)$, $F_2 = T(n_2\omega, 0)$ where n_1 and n_2 are integers with $n_1\omega > \alpha$, $n_2\omega > \alpha$.

$$F_1^{n_2} = F_2^{n_1} \stackrel{\text{def}}{=} F_2 .$$

$\xi(0)$ is a hyperbolic fixed point of F^i if and only if it is a hyperbolic fixed point of F_1 and F_2. This shows that the definition of the hyperbolicity is independent of the way the period is multiplied.

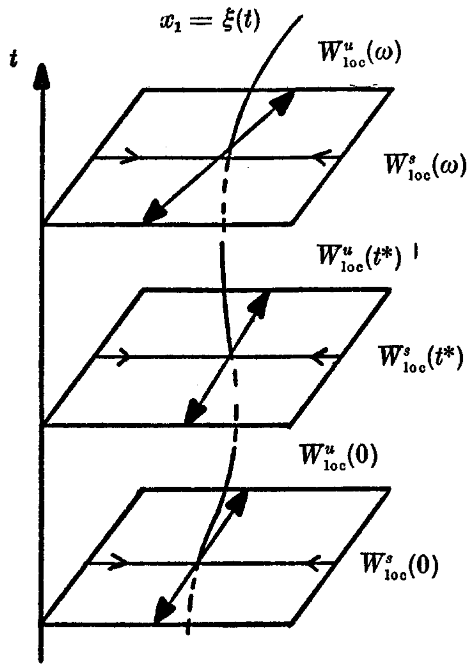

Figure 8.3

Assume that $T(\xi, 0)$ has $\xi(0) = \xi(\omega)$ as a hyperbolic fixed point. The existence of the local C^k stable and unstable manifolds $W^s_{loc}(0)$ and $W^u_{loc}(0)$ of $T(\omega, 0)$ on the section $\{0\} \times X \subset R \times X$ follow from Theorem 3.1. Periodicity implies that $W^u_{loc}(\omega) = W^u_{loc}(0)$ and $W^s_{loc}(\omega) = W^s_{loc}(0)$. Take a section $\{t^*\} \times X$ and, without loss of generality, assume that $\alpha < t^*$ and $\alpha < \omega - t^*$. Let the stable and unstable sets for $x = \xi(t)$, $W^s(\xi(\cdot))$ and $W^u(\xi(\cdot))$, intersect $\{t^*\} \times X$ in $W^s(t^*)$ and $W^u(t^*)$. Obviously, $W^u(t^*) = T(t^*, 0)W^u(0)$ and $W^s(t^*) = [T(\omega, t^*)]^{-1}W^s(\omega)$. It is easy to show that $T(t^*, 0)$ is a C^k embedding from $W^u_{loc}(0)$ into $W^u(t^*)$ with $[T(\omega, 0)]^{-1}T(\omega, t^*)$ as the inverse. Therefore $W^u_{loc}(t^*) \stackrel{\text{def}}{=} T(t^*, 0)W^u_{loc}(0)$ is a C^k submanifold in $\{t^*\} \times X$ and $W^u_{loc}(\omega) = T(\omega, t^*)W^u_{loc}(t^*)$. Also $TW^u_{loc}(\omega) = DT(\omega, t^*)TW^u_{loc}(t^*)$. Now let $Y \subset X$ be such that $DT(\omega, t^*) \cdot Y \subset TW^s_{loc}(\omega)$. Y is a linear closed subset since $TW^s_{loc}(\omega)$ is. It is easy to see that $Y \oplus TW^u_{loc}(t^*) = X$. We write $x \in X$ as $x = (x_1, y_1)$ where

$x_1 \in TW^u_{\mathrm{loc}}(t^*)$ and $y_1 \in Y$, and use the Implicit Function Theorem to solve $T(\omega, t^*) \cdot$ $\cdot W^s_{\mathrm{loc}}(t^*) \subset W^s_{\mathrm{loc}}(\omega)$. We obtain that

$$W^s_{\mathrm{loc}}(t^*) = (\xi(t^*) + (x_1, y_1) \colon x_1 = g(y_1),\ g \in C^k(B^\varepsilon_y),\ g(0) = 0,\ Dg(0) = 0\}$$

for some $\varepsilon > 0$. Thus, $W^s_{\mathrm{loc}}(t^*)$ is a C^k submanifold in $(t^*) \times X$ and $TW^s_{\mathrm{loc}}(t^*) = Y$. The proof of the invariance of $W^s_{\mathrm{loc}}(t^*)$ and $W^u_{\mathrm{loc}}(t^*)$ under $T(t^* + \omega, t^*)$ is easy and is omitted. Estimates for the spectra of $DT(t^* + \omega, t^*)$ on $TW^s_{\mathrm{loc}}(t^*)$ and $[DT(t^* + \omega, t^*)]^{-1}$ on $TW^u_{\mathrm{loc}}(t^*)$ can be obtained by considering

$$[T(t^* + \omega, t^*)]^n = T(t^*, 0) \cdot [T(\omega, 0)]^{n-1} \cdot T(\omega, t^*)$$

and

$$[T(t^* + \omega, t^*)|W^u_{\mathrm{loc}}(t^*)]^{-n} = [T(\omega, t^*)|W^u_{\mathrm{loc}}(t^*)]^{-1}[T(\omega, 0)|W^u_{\mathrm{loc}}(0)]^{-n+1} \cdot [T(t^*, 0)|_{W^u_{\mathrm{loc}}(0)}]^{-1}$$

and using $|\sigma(L)| \leqslant \lim_{n\to\infty} (\|L^n\|)^{1/n}$ for a linear bounded operator L. Consequently, $\xi(t^*)$ is a hyperbolic fixed point under $T(t^* + \omega, t^*)$ and $W^u_{\mathrm{loc}}(t^*)$, $W^s_{\mathrm{loc}}(t^*)$ are precisely the local unstable and stable manifolds under $T(t^* + \omega, t^*)$, due to the uniqueness. Thus, the definition of the hyperbolicty for the periodic trajectory of flows is independent of the cross-sections chosen.

The local unstable set of $x = \xi(t)$ is a neighborhood of $t = t^*$ is determined by

$$W^u_{\mathrm{loc}}(\xi(\cdot)) = \{(t, T(t, 0)x) \colon t \in (t^* - \varepsilon, t^* + \varepsilon),\ x \in W^u_{\mathrm{loc}}(0)\} \subset R \times X\,,$$

for some $\varepsilon > 0$. It is clearly a C^k submanifold modeled on $R \times TW^u_{\mathrm{loc}}(0)$. The local stable set of $x = \xi(t)$ in a neighborhood of $t = t^*$ is determined by

$$W^s_{\mathrm{loc}}(\xi(\cdot)) = \{(t, y) \colon T(\omega, t)y \subset W^s_{\mathrm{loc}}(\omega),\ t \in (t^* - \varepsilon, t^* + \varepsilon)\} \subset R \times X\,,$$

for some $\varepsilon > 0$. Using the local coordinates $R \times TW^u_{\mathrm{loc}}(t^*) \times Y$, and the Implicit Function Theorem, one shows that $W^s_{\mathrm{loc}}(\xi(\cdot))$ is a C^k submanifold modeled on $R \times Y = R \times TW^s_{\mathrm{loc}}(t^*)$. The proof of Theorem 8.3 is completed.

REFERENCES

[1] L. BLOCK, *Homoclinic points of mapping on the interval*, Proc. Amer. Math. Soc., **72** (1978), pp. 576-580.

[2] J. GUCKENHEIMER - J. MOSER - S. E. NEWHOUSE, *Dynamical Systems*, C.I.M.E. Lectures, Bressanone, Italy (1978), Birkhäuser, 1980.

[3] J. HALE, *Theory of functional differential equations*, Springer-Verlag, 1977.

[4] D. HENRY, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Math., Vol. **840**, Springer-Verlag, 1981.

[5] M. W. HIRSCH - C. C. PUGH - M. SHUB, *Invariant manifolds*, Lecture Notes in Math. Vol. **583**, Springer-Verlag, 1977.

[6] P. HOLMES - J. MARSDEN, *A partial differential equation with infinitely many periodic orbits: chaotic oscillations of a forced beam*, Archive Rational Mech. Anal., **76** (1981), pp. 135-166.

[7] M. C. IRWIN, *On the stable manifold theorem*, Bull. London Math. Soc., **2** (1970), pp. 196-198.

[8] M. C. IRWIN, *On the smoothness of the composition map*, Quart. J. Math. Oxford, (2), **23** (1972), pp. 113-133.

[9] T.-Y. LI - J. A. YORKE, *Period three implies chaos*, Amer. Math. Monthly, **82** (1975), pp. 985-9992.

[10] F. R. MAROTTO, *Snap-back repeller imply chaos in R^n*, J. Math. Anal. and Appl., **63** (1978), pp. 199-223.

[11] J. MOSER, *Stable and random motion in dynamical systems*, Princeton University Press. Princeton, N. J., 1973.

[12] J. PALIS - W. DE MELO, *Geometric theory of dynamical systems*, Springer-Verlag, 1982.

[13] K. J. PALMER, *Exponential dichotomies and transversal homoclinic points*, J. Diff. Equations, to appear.

[14] L. P. ŠIL'NIKOV, *On a Poincaré-Birkhoff problem*, Math. USSR-Sbornik, **3**, no. 3 (1967), pp. 353-371.

[15] S. SMALE, *Diffeomorphisms with many periodic points*, Differential and Combinatorial Topology, Princeton Univ. Press, Princeton, N. J. (1965), pp. 63-80.

[16] S. SMALE, *Differentiable dynamical systems*, Bull. Amer. Math. Soc., **73** (1967), pp. 747-817.

[17] H.-O. WALTHER, *Homoclinic solution and chaos in $\dot{x}(t) = f(x(t-k))$*, J. Nonlinear Analysis, **5** (1981), pp. 775-788.