Some Transformations and Reduction Formulas for Multiple q-Hypergeometric Series (*).

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Summary. – *Some simple ideas of G. E. Andrews* ([2], [3]) *are used here to derive a transformation formula for a general multiple q-series with essentially arbitrary terms. As applications of* (*or motivated by*) *this q-series transformation, several transformation and reduction formulas for q-hypergeometric series in two and more variables are presented. Relevant connections of the various q-identities considered here with a number of known results are also indicated.*

1. – For real or complex q, $|q| < 1$, let

$$(1.1)\qquad (\lambda)_\mu \equiv (\lambda; q)_\mu = \prod_{j=0}^{\infty} \left(\frac{1-\lambda q^j}{1-\lambda q^{\mu+j}}\right)$$

for arbitrary λ and μ, so that

$$(1.2)\qquad (\lambda)_n = \begin{cases} 1, & \text{if } n = 0, \\ (1-\lambda)(1-\lambda q)\dots(1-\lambda q^{n-1}), & \forall n \in \{1, 2, 3, \dots\}, \end{cases}$$

and

$$(1.3)\qquad (\lambda)_\infty = \prod_{j=0}^{\infty} (1-\lambda q^j).$$

In terms of a bounded multiple sequence $\{\Omega(m_1, \dots, m_n)\}$, we define

$$(1.4)\qquad F(z_1, \dots, z_n) = \sum_{m_1,\dots,m_n=0}^{\infty} \Omega(m_1, \dots, m_n) \frac{z_1^{m_1}}{(q)_{m_1}} \cdots \frac{z_n^{m_n}}{(q)_{m_n}},$$

provided that the multiple q-series converges absolutely. Also let

$$(1.5)\qquad M = m_1 + \dots + m_n \quad \text{and} \quad K = k_1 + \dots + k_p.$$

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Then it is easily observed that

$$(1.6)\qquad \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\lambda_1)_M \dots (\lambda_p)_M}{(\mu_1)_M \dots (\mu_p)_M} \Omega(m_1, \dots, m_n) \frac{z_1^{m_1}}{(q)_{m_1}} \dots \frac{z_n^{m_n}}{(q)_{m_n}} =$$

$$= \frac{(\lambda_1)_\infty \dots (\lambda_p)_\infty}{(\mu_1)_\infty \dots (\mu_p)_\infty} \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\mu_1 q^M)_\infty \dots (\mu_p q^M)_\infty}{(\lambda_1 q^M)_\infty \dots (\lambda_p q^M)_\infty} \Omega(m_1, \dots, m_n) \frac{z_1^{m_1}}{(q)_{m_1}} \dots \frac{z_n^{m_n}}{(q)_{m_n}} .$$

By Heine's theorem [8, p. 92, Equation (3.2.2.11)]

$$(1.7)\qquad \frac{(\lambda z)_\infty}{(z)_\infty} = \sum_{m=0}^{\infty} \frac{(\lambda)_m}{(q)_m} z^m , \qquad |z| < 1 ,$$

(1.6) readily yields the multiple q-series transformation

$$(1.8)\qquad \sum_{m_1,\dots,m_n=0}^{\infty} \frac{(\lambda_1)_M \dots (\lambda_p)_M}{(\mu_1)_M \dots (\mu_p)_M} \Omega(m_1, \dots, m_n) \frac{z_1^{m_1}}{(q)_{m_1}} \dots \frac{z_n^{m_n}}{(q)_{m_n}} =$$

$$= \frac{(\lambda_1)_\infty \dots (\lambda_p)_\infty}{(\mu_1)_\infty \dots (\mu_p)_\infty} \sum_{k_1,\dots,k_p=0}^{\infty} F(z_1 q^K, \dots, z_n q^K) \prod_{i=1}^{p} \left\{ (\mu_i/\lambda_i)_{k_i} \frac{\lambda_i^{k_i}}{(q)_{k_i}} \right\} ,$$

where $F(z_1, \dots, z_n)$ is defined by (1.4), and M and K are given by (1.5).

For various special values of the coefficients $\Omega(m_1, \dots, m_n)$, the function $F(z_1, \dots, z_n)$ defined by (1.4) can be expressed in a closed form. Indeed, in every such situation, the result (1.8) would simplify considerably, and we shall be led to a transformation or reduction formula for a multiple q-hypergeometric series. Some of these special cases of (1.8) will be presented in the next section.

2. – Denoting, as usual, a q-hypergeometric series with r numerator and s denominator parameters by ${}_r\Phi_s$ (*cf.* [8, p. 90, Equation (3.2.1.11)]), let

$$\Phi^{r:u_1;\dots;u_n}_{s:v_1;\dots;v_n}$$

denote an analogous q-hypergeometric series in n variables in which there are r numerator and s denominator parameters of the type $(\lambda)_M$, and u_j numerator and v_j denominator parameters of the type $(\lambda)_{m_j}$ $(j = 1, \dots, n)$ [*cf.* the definitions (1.2), (1.4) and (1.5)]. Now set

$$(2.1)\qquad \Omega(m_1, \dots, m_n) = \prod_{j=1}^{n} \left\{ \frac{(\alpha_j)_{m_j} (\beta_j)_{m_j}}{(\alpha_j \beta_j z_j)_{m_j}} \right\}$$

in (1.4) and apply the q-Gauss theorem [8, p. 97, Equation (3.3.2.5)]

$$(2.2)\qquad {}_2\Phi_1 \left[\begin{matrix} a, b; \\ \\ c; \end{matrix} \; q, \frac{c}{ab} \right] = \frac{(c/a)_\infty (c/b)_\infty}{(c)_\infty (c/ab)_\infty} ,$$

so that

$$F(z_1, \dots, z_n) = \prod_{j=1}^{n} \left\{ \frac{(\alpha_j z_j)_\infty (\beta_j z_j)_\infty}{(z_j)_\infty (\alpha_j \beta_j z_j)_\infty} \right\}, \tag{2.3}$$

and then the q-series transformation (1.8) reduces to the multiple q-hypergeometric identity:

$$\Phi^{p:2;\dots;2}_{p:1;\dots;1}\left[\begin{matrix}(\lambda_p): & \alpha_1, \beta_1; & \dots; & \alpha_n, \beta_n; \\ (\mu_p): & \alpha_1\beta_1 z_1; & \dots; & \alpha_n\beta_n z_n;\end{matrix}\ q;\ z_1, \dots, z_n\right] = \tag{2.4}$$

$$= \frac{(\lambda_1)_\infty \cdots (\lambda_p)_\infty}{(\mu_1)_\infty \cdots (\mu_p)_\infty} \prod_{j=1}^{n} \left\{ \frac{(\alpha_j z_j)_\infty (\beta_j z_j)_\infty}{(z_j)_\infty (\alpha_j \beta_j z_j)_\infty} \right\} \cdot$$

$$\cdot \Phi^{2n:1;\dots;1}_{2n:0;\dots;0}\left[\begin{matrix}(z_n), (\alpha_n\beta_n z_n): & \mu_1/\lambda_1; & \dots; & \mu_p/\lambda_p; \\ (\alpha_n z_n), (\beta_n z_n): & -; & \dots; & -;\end{matrix}\ q;\ \lambda_1, \dots, \lambda_p\right],$$

where, for convenience, (ω_n) abbreviates the array of n parameters $\omega_1, \dots, \omega_n$, with similar interpretations for (λ_p), *et cetera.*

Next we set

$$\Omega(m_1, \dots, m_n) = (\alpha_1)_{m_1} \cdots (\alpha_n)_{m_n} \tag{2.5}$$

in (1.4) and (1.8), and apply Heine's theorem (1.7), or (alternatively) we let $\beta_j \to 0$ $(j = 1, \dots, n)$ in (2.4) above, and we obtain the special case (*cf.* [4, p. 49, Equation (1)])

$$\Phi^{p:1;\dots;1}_{p:0;\dots;0}\left[\begin{matrix}(\lambda_p): & \alpha_1; & \dots; & \alpha_n; \\ (\mu_p): & -; & \dots; & -;\end{matrix}\ q;\ z_1, \dots, z_n\right] = \tag{2.6}$$

$$= \frac{(\lambda_1)_\infty \cdots (\lambda_p)_\infty}{(\mu_1)_\infty \cdots (\mu_p)_\infty} \prod_{j=1}^{n} \left\{ \frac{(\alpha_j z_j)_\infty}{(z_j)_\infty} \right\} \cdot$$

$$\cdot \Phi^{n:1;\dots;1}_{n:0;\dots;0}\left[\begin{matrix}(z_n): & \mu_1/\lambda_1; & \dots; & \mu_p/\lambda_p; \\ (\alpha_n z_n): & -; & \dots; & -;\end{matrix}\ q;\ \lambda_1, \dots, \lambda_p\right].$$

By setting

$$\Omega(m_1, \dots, m_n) = 1 \tag{2.7}$$

in (1.4) and (1.8), and using the elementary identity [8, p. 92, Equation (3.2.2.14)]

$$\sum_{m=0}^{\infty} \frac{z^m}{(q)_m} = \frac{1}{(z)_\infty}, \tag{2.8}$$

or (alternatively) by further specializing (2.4) and (2.6) when $\alpha_j \to 0$ and $\beta_j \to 0$ $(j = 1, \ldots, n)$, we find that

$$\text{(2.9)} \qquad \Phi^{p:0;\ldots;0}_{p:0;\ldots;0}\begin{bmatrix}(\lambda_p): & -; & \ldots; & -; & \\ & & & & q; \quad z_1, \ldots, z_n \\ (\mu_p): & -; & \ldots; & -; & \end{bmatrix} =$$

$$= \frac{(\lambda_1)_\infty \ldots (\lambda_p)_\infty}{(z_1)_\infty \ldots (z_n)_\infty (\mu_1)_\infty \ldots (\mu_p)_\infty} \Phi^{n:1;\ldots;1}_{0:0;\ldots;0}\begin{bmatrix}(z_n): & \mu_1/\lambda_1; & \ldots; & \mu_p/\lambda_p; & \\ & & & & q; \quad \lambda_1, \ldots, \lambda_p \\ —: & —; & \ldots; & —; & \end{bmatrix},$$

provided that each of the multiple q-hypergeometric series *terminates.*

By applying one or the other of several known q-hypergeometric summation theorems (see, for example, [8], pp. 247-248) in place of the q-Gauss theorem (2.2), we can similarly deduce from (1.8) a number of special q-hypergeometric transformations analogous to (2.4), (2.6) and (2.9).

3. – In terms of the q-Lauricella function

$$\text{(3.1)} \qquad \Phi^{(n)}_D[\alpha, \beta_1, \ldots, \beta_n; \quad \gamma; \quad z_1, \ldots, z_n] =$$

$$= \Phi^{1:1;\ldots;1}_{1:0;\ldots;0}\begin{bmatrix}\alpha: & \beta_1; & \ldots; & \beta_n; & \\ & & & & q; \quad z_1, \ldots, z_n \\ \gamma: & -; & \ldots; & -; & \end{bmatrix},$$

the special case of (2.6) when $p = 1$ would immediately yield ANDREWS' formula ([2, p. 621, Equation (4.1)]; see also [3, p. 207, Theorem 5])

$$\text{(3.2)} \qquad \Phi^{(n)}_D[\lambda, \alpha_1, \ldots, \alpha_n; \quad \mu; \quad z_1, \ldots, z_n] =$$

$$= \frac{(\lambda)_\infty}{(\mu)_\infty} \prod_{j=1}^{n} \left\{\frac{(\alpha_j z_j)_\infty}{(z_j)_\infty}\right\} {}_{n+1}\Phi_n\begin{bmatrix}z_1, \ldots, z_n, \mu/\lambda; & \\ & q, \lambda \\ \alpha_1 z_1, \ldots, \alpha_n z_n; & \end{bmatrix}.$$

ANDREWS [3, p. 209, Corollary 5.3] applies the case $n = 2$ of his formula (3.2) and HALL's identity for ${}_3\Phi_2$ [5, p. 276] to give an alternative proof of a result of AL-SALAM [1, p. 457, Equation (9)] which we recall here in the *corrected* form:

$$\text{(3.3)} \qquad \Phi^{(1)}[\beta\beta'/\alpha, \beta', \beta; \quad \beta\beta'; \quad x, y] =$$

$$= \frac{(\beta' x/\alpha)_\infty (\beta y/\alpha)_\infty}{(x)_\infty (y)_\infty} \Phi^{(1)}[\alpha, \beta, \beta'; \quad \beta\beta'; \quad \beta' x/\alpha, \beta y/\alpha],$$

where we have used Jackson's notation $\Phi^{(1)}$, instead of $\Phi^{(2)}_D$, for the q-Appell function.

A simple proof of Al-Salam's result (3.3), using only Heine's transformation [8, p. 97, Equation (3.3.2.3)]

$$(3.4)\qquad {}_2\Phi_1\begin{bmatrix} a, b; & \\ & q, z \\ c\ ; & \end{bmatrix} = \frac{(abz/c)_\infty}{(z)_\infty}\,{}_2\Phi_1\begin{bmatrix} c/a, c/b; & \\ & q, \dfrac{abz}{c} \\ c\ ; & \end{bmatrix},$$

runs as follows. Indeed, by expressing $\Phi^{(1)}$ as an infinite series of ${}_2\Phi_1$ and applying (3.4) term-by-term, we have the q-hypergeometric transformations

$$(3.5)\qquad \Phi^{(1)}[\alpha, \beta, \beta';\ \gamma;\ x, y] = \frac{(\alpha\beta x/\gamma)_\infty}{(x)_\infty}\,\Phi^{1:1;2}_{1:0;1}\begin{bmatrix} \gamma/\beta: & \gamma/\alpha; & \alpha, \beta'; & \\ & & & q; \dfrac{\alpha\beta x}{\gamma}, y \\ \gamma\ : & —; & \gamma/\beta\ ; & \end{bmatrix}$$

and

$$(3.6)\qquad \Phi^{(1)}[\alpha, \beta, \beta';\ \gamma;\ x, y] = \frac{(\alpha\beta' y/\gamma)_\infty}{(y)_\infty}\,\Phi^{1:2;1}_{1:1;0}\begin{bmatrix} \gamma/\beta': & \alpha, \beta; & \gamma/\alpha; & \\ & & & q;\ x, \dfrac{\alpha\beta' y}{\gamma} \\ \gamma\ : & \gamma/\beta'; & —; & \end{bmatrix}$$

In particular, if $\gamma = \beta\beta'$, (3.5) and (3.6) readily give

$$(3.7)\qquad \Phi^{(1)}[\alpha, \beta, \beta';\ \beta\beta';\ x, y] = \frac{(\alpha x/\beta')_\infty}{(x)_\infty}\,\Phi^{(1)}[\beta', \beta\beta'/\alpha, \alpha;\ \beta\beta';\ \alpha x/\beta', y]$$

and

$$(3.8)\qquad \Phi^{(1)}[\alpha, \beta, \beta';\ \beta\beta';\ x, y] = \frac{(\alpha y/\beta)_\infty}{(y)_\infty}\,\Phi^{(1)}[\beta, \alpha, \beta\beta'/\alpha;\ \beta\beta';\ x, \alpha y/\beta]\,.$$

Applying the transformations (3.7) and (3.8) successively, we obtain the q-hypergeometric identity

$$(3.9)\qquad \Phi^{(1)}[\alpha, \beta, \beta';\ \beta\beta';\ x, y] = \frac{(\alpha x/\beta')_\infty(\alpha y/\beta)_\infty}{(x)_\infty(y)_\infty}\,\Phi^{(1)}[\beta\beta'/\alpha, \beta', \beta;\ \beta\beta';\ \alpha x/\beta', \alpha y/\beta]\,,$$

which immediately yields (3.3) upon replacing x and y by $\beta' x/\alpha$ and $\beta y/\alpha$, respectively.

4. – Making use of JACKSON's transformation [6, p. 145, Equation (4)]

$$(4.1)\qquad {}_2\Phi_1\begin{bmatrix} a, b; & \\ & q, z \\ c\ ; & \end{bmatrix} = \frac{(az)_\infty}{(z)_\infty}\sum_{m=0}^{\infty} q^{\frac{1}{2}m(m-1)}\frac{(a)_m(c/b)_m}{(c)_m(az)_m}\frac{(-bz)^m}{(q)_m},$$

instead of (3.4), it is not difficult to prove the reduction formulas

$$(4.2)\qquad \Phi^{(1)}[\alpha, \beta, \beta';\ \beta\beta';\ x, y] = \frac{(\alpha x)_\infty}{(x)_\infty}\,{}_3\Phi_2\begin{bmatrix} \alpha, \beta', \beta x/y; & \\ & q, y \\ \beta\beta',\ \alpha x\ ; & \end{bmatrix}$$

and

$$(4.3)\qquad \Phi^{(1)}[\alpha, \beta, \beta';\quad \beta\beta';\quad x, y] = \frac{(\alpha y)_\infty}{(y)_\infty}\, {}_3\Phi_2\left[\begin{matrix}\alpha, \beta, \beta' y/x; \\ \\ \beta\beta', \alpha y\ ;\end{matrix}\quad q, x\right].$$

Now we recall SEARS' transformation [7, p. 167, Equation (8.3)]

$$(4.4)\qquad {}_4\Phi_3\left[\begin{matrix}a, b, c, q^{-n}; \\ \\ d, e, f\ ;\end{matrix}\quad q, q\right] = \frac{(e/c)_n(de/ab)_n}{(e)_n(de/abc)_n}\, {}_4\Phi_3\left[\begin{matrix}d/a, d/b, c, q^{-n}\ ; \\ \\ d, cq^{1-n}/e, cq^{1-n}/f;\end{matrix}\quad q, q\right],$$

where $abc = def\, q^{n-1}$ $(n = 0, 1, 2, \ldots)$. Setting $f = abc\, q^{1-n}/de$ and letting $n \to \infty$, (4.4) reduces to the known identity [7, p. 174, Equation (10.1)]

$$(4.5)\qquad {}_3\Phi_2\left[\begin{matrix}a, b, c; \\ \\ d, e\ ;\end{matrix}\quad q, \frac{de}{abc}\right] = \frac{(e/c)_\infty(de/ab)_\infty}{(e)_\infty(de/abc)_\infty}\, {}_3\Phi_2\left[\begin{matrix}d/a, d/b, c; \\ \\ d, de/ab\ ;\end{matrix}\quad q, \frac{e}{c}\right].$$

Combining (4.2) and (4.5), we find that

$$(4.6)\qquad \Phi^{(1)}[\alpha, \beta, \beta';\quad \beta\beta';\quad x, y] = \frac{(\alpha y/\beta)_\infty}{(y)_\infty}\frac{(\beta x)_\infty}{(x)_\infty}\, {}_3\Phi_2\left[\begin{matrix}\beta\beta'/\alpha, \beta, \beta x/y; \\ \\ \beta\beta',\ \beta x\ ;\end{matrix}\quad q, \frac{\alpha y}{\beta}\right] =$$

$$= \frac{(\alpha y/\beta)_\infty}{(y)_\infty}\frac{(\alpha x/\beta')_\infty}{(x)_\infty}\Phi^{(1)}[\beta\beta'/\alpha, \beta', \beta;\quad \beta\beta';\quad \alpha x/\beta', \alpha y/\beta],$$

whence Al-Salam's result (3.3) follows at once if we replace x and y by $\beta' x/\alpha$ and $\beta y/\alpha$, respectively.

In view of the reduction formula (4.2) or (4.3), Al-Salam's result (3.3) is essentially equivalent to the familiar q-hypergeometric identity (4.5) which, in turn, is equivalent to the aforementioned identity of HALL [5] employed by ANDREWS [3].

5. – Yet another interesting proof of Al-Salam's result (3.3) would make a rather straightforward use of the q-hypergeometric expansion

$$(5.1)\qquad \Phi^{1:r;u}_{1:s;v}\left[\begin{matrix}\alpha: & (a_r); & (c_u); \\ \\ \gamma: & (b_s); & (d_v);\end{matrix}\quad q;\quad x, y\right] =$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha)_n(\gamma/\alpha)_n \prod_{j=1}^{r} (a_j)_n \prod_{j=1}^{u} (c_j)_n}{(\gamma q^{n-1})_n(\gamma)_{2n} \prod_{j=1}^{s} (b_j)_n \prod_{j=1}^{v} (d_j)_n}\, q^{n(n-1)}\frac{(\alpha x y)^n}{(q)_n}\cdot$$

$$\cdot {}_{r+1}\Phi_{s+1}\left[\begin{matrix}(a_r)\, q^n, \alpha q^n\ ; \\ \\ (b_s)\, q^n, \gamma q^{2n};\end{matrix}\quad q, x\right] {}_{u+1}\Phi_{v+1}\left[\begin{matrix}(c_u)\, q^n, \alpha q^n\ ; \\ \\ (d_v) q^n, \gamma q^{2n};\end{matrix}\quad q, y\right],$$

which incidentally is an obvious special case of one of several general classes of q-series transformations given elsewhere by us [9].

The case $r = u = 1$ and $s = v = 0$ of (5.1) leads us to

$$(5.2)\qquad \Phi^{(1)}[\beta\beta'/\alpha, \beta', \beta;\quad \beta\beta';\quad x, y] = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta\beta'/\alpha)_n(\beta')_n(\beta)_n}{(\beta\beta' q^{n-1})_n(\beta\beta')_{2n}} q^{n(n-1)} \frac{(\beta\beta' xy/\alpha)^n}{(q)_n} \cdot {}_2\Phi_1\left[\begin{matrix}\beta' q^n, \beta\beta' q^n/\alpha; \\ \beta\beta' q^{2n}\;;\end{matrix}\; q, x\right] {}_2\Phi_1\left[\begin{matrix}\beta q^n, \beta\beta' q^n/\alpha; \\ \beta\beta' q^{2n}\;;\end{matrix}\; q, y\right],$$

and, by virtue of Heine's transformation (3.4), we thus have

$$(5.3)\qquad \Phi^{(1)}[\beta\beta'/\alpha, \beta', \beta;\quad \beta\beta';\quad x, y] = \frac{(\beta' x/\alpha)_\infty}{(x)_\infty} \frac{(\beta y/\alpha)_\infty}{(y)_\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta\beta'/\alpha)_n(\beta)_n(\beta')_n}{(\beta\beta' q^{n-1})_n(\beta\beta')_{2n}} q^{n(n-1)} \frac{\left(\alpha\cdot\frac{\beta' x}{\alpha}\cdot\frac{\beta y}{\alpha}\right)^n}{(q)_n} \cdot {}_2\Phi_1\left[\begin{matrix}\beta q^n, \alpha q^n; \\ \beta\beta' q^{2n}\;;\end{matrix}\; q, \frac{\beta' x}{\alpha}\right] {}_2\Phi_1\left[\begin{matrix}\beta' q^n, \alpha q^n; \\ \beta\beta' q^{2n}\;;\end{matrix}\; q, \frac{\beta y}{\alpha}\right].$$

On interpreting the second member of (5.3) by means of (5.1), we arrive once again at Al-Salam's result (3.3).

6. – In view of the demonstrated usefulness of such q-hypergeometric transformations as (3.5) and (3.6), we conclude by recording the following generalizations:

$$(6.1)\qquad \Phi^{1:1;r}_{1:0;s}\left[\begin{matrix}\alpha: & \beta; & (a_r); \\ \gamma: & -; & (b_s);\end{matrix}\; q;\quad x, y\right] = \frac{(\alpha\beta x/\gamma)_\infty}{(x)_\infty} \Phi^{1:1;r+1}_{1:0;s+1}\left[\begin{matrix}\gamma/\beta: & \gamma/\alpha; & \alpha, (a_r)\;; \\ \gamma\;: & —; & \gamma/\beta, (b_s);\end{matrix}\; q;\quad \frac{\alpha\beta x}{\gamma}, y\right],$$

$$(6.2)\qquad \Phi^{1:r;1}_{1:s;0}\left[\begin{matrix}\alpha: & (a_r); & \beta'; \\ \gamma: & (b_s); & -;\end{matrix}\; q;\quad x, y\right] = \frac{(\alpha\beta' y/\gamma)_\infty}{(y)_\infty} \Phi^{1:r+1;1}_{1:s+1;0}\left[\begin{matrix}\gamma/\beta': & \alpha, (a_r)\;; & \gamma/\alpha; \\ \gamma\;: & \gamma/\beta', (b_s); & —\;;\end{matrix}\; q;\quad x, \frac{\alpha\beta' y}{\gamma}\right],$$

which can be established by applying the proof of (3.5) and (3.6) *mutatis mutandis.*

Various q-hypergeometric transformations, analogous to (6.1) and (6.2), follow similarly from (4.1), and (for example) we have

$$(6.3)\qquad \Phi^{1:1;r}_{1:0;s}\left[\begin{matrix}\alpha: & \beta; & (a_r); \\ & & & q; x, y \\ \gamma: & -; & (b_s); \end{matrix}\right] =$$

$$= \frac{(\alpha x)_\infty}{(x)_\infty} \sum_{m,n=0}^{\infty} q^{\frac{1}{2}m(m-1)} \frac{(\alpha)_{m+n}(\gamma/\beta)_{m+n} \prod_{j=1}^{r} (a_j)_n}{(\gamma)_{m+n}(\alpha x)_{m+n}(\gamma/\beta)_n \prod_{j=1}^{s} (b_j)_n} \frac{(-\beta x)^m}{(q)_m} \frac{y^n}{(q)_n} =$$

$$= \frac{(\beta x)_\infty}{(x)_\infty} \sum_{m,n=0}^{\infty} q^{mn+\frac{1}{2}m(m-1)} \frac{(\gamma/\alpha)_m(\alpha)_n(\beta)_m \prod_{j=1}^{r} (a_j)_n}{(\gamma)_{m+n}(\beta x)_m \prod_{j=1}^{s} (b_j)_n} \frac{(-\alpha x)^m}{(q)_m} \frac{y^n}{(q)_n},$$

with, of course, two similar transformations written by symmetry.

The first q-hypergeometric transformation in (6.3) would lead us, when $r = 1$, $s = 0$, $a_1 = \beta'$ and $\gamma = \beta\beta'$, to the reduction formula (4.2).

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