Annali di Matematica pura ed applicata
(IV), Vol. CLXVII (1994), pp. 389-402

On The Geometrical Structure of Euler-Lagrange Equations (*).

PAOLO GIUSEPPE COSTANTINI

Summary. – *We show that the Euler-Lagrange operator and the Poincaré-Cartan form arise in a very simple and natural way from the contact structure of the second order jet space, in a purely differential context, without any reference to a variational problem. By the way we obtain an intrinsic expression of the Euler-Lagrange operator.*

Introduction.

The Euler-Lagrange equations have been introduced in the analytical approach to variational calculus, and now they are the fundamental field equations of physical theories. In the last few years, a new attention has been directed to the geometrical structures related to the calculus of variations. So, the geometrical approach has been greatly developed, and many deep results have been found (for instance, see [GS], [HE], [KR], [MM1], [TA], [TU], [VI]).

The aim of this paper is to show that the Euler-Lagrange equations can be simply obtained in a direct, natural and elegant way from the contact structure of the 1-jet space, independently of a variational approach.

The literature concerning this subject is large (see, for instance, [FF], [GS], [MM1], [TA], [TU], [VI]). Our approach is very quick and fully intrinsic. In particular, we base our procedure on the momentum and the canonical splitting of the differential of forms on jets. We derive the Euler-Lagrange operator and the Poincaré-Cartan form in the same pure differential framework.

We consider first the *relative tangent valued 1-forms* on a generic fibred manifold $q: F \to M$, i.e. the sections of the tensor bundle $T^*F \underset{F}{\otimes} TM \to F$, and make use of a well known result (see for example [SA]) which associates to such a section a derivation of degree 1 on the exterior algebra over the base space M.

This derivation lifts an s-form on M to an $(s+1)$-form on F.

(*) Entrata in Redazione il 15 febbraio 1993; ricevuta in nuova versione l'1 ottobre 1993.

Indirizzo dell'A.: Via Dante Alighieri 52, 50055 Lastra a Signa (Firenze), Italy.

We then fix a fibred manifold $p: E \to B$, and apply this result to the contact morphism

$$\mathcal{D}^{k+1}: J_{k+1}E \to T^*B \underset{J_kE}{\otimes} TJ_kE$$

and to its complementary morphism

$$\theta^{k+1}: J_{k+1}E \to T^*J_kE \underset{J_kE}{\otimes} VJ_kE$$

on the k-order jet space of E, which are $p_{k+1,k}$-relative tangent valued 1-forms, and obtain the two derivations d_h and d_v, of degree 1, on the exterior algebra over J_kE, called the *horizontal* and *vertical differentials* respectively. They reveal themselves to be very powerful tools.

Starting from a lagrangian density $\mathcal{L}$, i.e. a p_B-semibasic m-form on J_1E, we make use of canonical isomorphisms and of the differentials d_h, d_v, relative to the second order jet space of E, for the construction of the fundamental objects of first order field theory. Namely, we obtain first the *Legendre form*, or *momentum form*—a p_B-semibasic m-form on J_1E—through the vertical derivative of the lagrangian. The application of the horizontal and vertical differentials to this m-form yields a *canonical splitting* of its exterior differential, hence the Euler-Lagrange operator and the Poincaré-Cartan form. Thus, these two objects arise in a very natural way.

Finally, for the sake of completeness, we link the objects previously obtained to the variational context, hence to field theory. So, we recover in a geometrical way, the well known result that the Euler-Lagrange equations are a necessary and sufficient condition for a section $s: B \to E$ to be critical, and we find an equivalent condition, involving the Poincaré-Cartan form, which allows us to easily prove the Noether theorem (see also [GS], [GA]).

0. – Preliminaries.

We assume all manifolds and maps to be smooth. We denote by $\mathcal{F}(M)$, $\mathcal{T}(M)$, $\Omega(M)$ the sheaves of local functions, vector fields and differential forms on the manifold M, respectively.

Our fundamental geometrical framework is constituted by a fibred manifold $p: E \to B$, with $\dim B = m$, $\dim E = m + n$. The typical fibred chart of E will be denoted by (x^λ, y^i), with $1 \leq \lambda \leq m$, $1 \leq i \leq n$.

We recall a few notation and results on the geometry of jet spaces. We are involved with the *k-jet space* J_kE of $p: E \to B$, where $k \geq 0$ is an integer, with the natural fibrings ([MM2], pag. 172-3)

$$p_{k,B}: J_kE \to B\,,$$
$$p_{k,h}: J_kE \to J_hE\,, \qquad 0 \leq h < k\,.$$

For $k = 1$, we set

$$p_B \equiv p_{1,B}: J_1E \to B\,,$$

$$p_E \equiv p_{1,0}: J_1E \to E\,.$$

The fibred charts on J_kE induced by a fibred chart (x^λ, y^i) of E are denoted by

$$(x^\lambda, y^i, y^i_\lambda, y^i_{\lambda_2\lambda_2}, \dots, y^i_{\lambda_1\dots\lambda_k}), \qquad 1 \leqslant \lambda_1 \leqslant \dots \leqslant \lambda_k \leqslant m\,,$$

and the local bases for the vector fields and the 1-forms on J_kE are denoted respectively by

$$(\partial_\lambda, \partial_i, \partial^\lambda_i, \dots, \partial_i^{\lambda_1\dots\lambda_k}), \qquad (d^\lambda, d^i, d^i_\lambda, \dots, d^i_{\lambda_1\dots\lambda_k}),$$

with the same restrictions on the indices.

Let $s: B \to E$ be a local section of $p: E \to B$. We have the *k-jet prolongation of s*

$$j_k s: B \to J_kE\,,$$

with coordinate expression

$$(x^\lambda, y^i, y^i_\lambda, y^i_{\lambda_1\lambda_2}, \dots, y^i_{\lambda_1\dots\lambda_k}) \circ j_k s = (x^\lambda, s^i, \partial_\lambda s^i, \partial_{\lambda_1\lambda_2} s^i, \dots, \partial_{\lambda_1\dots\lambda_k} s^i),$$

where

$$\partial_{\lambda_1\dots\lambda_k} \equiv \frac{\partial^k}{\partial x^{\lambda_1}\dots\partial x^{\lambda_k}}\,.$$

The *contact structure* on J_kE is the pair of complementary natural linear fibred morphisms over J_kE ([MM2], pag. 182)

(1) $$\mathcal{D}^{k+1}: J_{k+1}E \to T^*B \underset{J_kE}{\otimes} TJ_kE\,,$$

(2) $$\theta^{k+1}: J_{k+1}E \to T^*J_kE \underset{J_kE}{\otimes} VJ_kE\,,$$

with coordinate expressions

(3) $$\mathcal{D}^{k+1} = d^\mu \otimes \mathcal{D}^{k+1}_\mu\,,$$

(4) $$\theta^{k+1} = \theta^i \otimes \partial_i + \theta^i_{\lambda_1} \otimes \partial^{\lambda_1}_i + \dots + \theta^i_{\lambda_1\dots\lambda_k} \otimes \partial_i^{\lambda_1\dots\lambda_k}$$

where we have set

(5) $$\mathcal{D}^{k+1}_\mu \equiv (\partial_\mu + y^i_\mu\partial_i + y^i_{\mu\lambda_1}\partial^{\lambda_1}_i + \dots + y^i_{\mu\lambda_1\dots\lambda_k}\partial_i^{\lambda_1\dots\lambda_k}): J_{k+1}E \to TJ_kE\,,$$

$1 \leqslant \mu_1 \leqslant m$, $1 \leqslant \lambda_1 \leqslant \dots \leqslant \lambda_k \leqslant m$,

(6) $$\begin{cases} \theta^i \equiv d^i - y^i_\mu d^\mu: J_1E \to T^*E\,, \\ \dots\dots\dots \\ \theta^i_{\lambda_1\dots\lambda_k} \equiv (d^i_{\lambda_1\dots\lambda_k} - y^i_{\lambda_1\dots\lambda_k\mu} d^\mu): J_{k+1}E \to T^*J_kE\,. \end{cases}$$

In order to define prolongations on J_1E of vector fields on E, we recall the natural *exchange fibred morphism over* $J_1E \underset{E}{\times} TE$ ([MM1], pag. 33)

$$r\colon J_1TE \to TJ_1E\,, \tag{7}$$

with coordinate expression

$$(x^\lambda, y^i, y^i_\lambda; \dot{x}^\lambda, \dot{y}^i, \dot{y}^i_\lambda)\circ r = (x^\lambda, y^i, y^i_\lambda; \dot{x}^\lambda, \dot{y}^i, \dot{y}^i_\lambda - y^i_\mu \dot{x}^\mu_\lambda)\,,$$

where $(x^\lambda, y^i, \dot{x}^\lambda, \dot{y}^i; y^i_\lambda, \dot{x}^\lambda_\mu, \dot{y}^i_\lambda)$ and $(x^\lambda, y^i, y^i_\lambda; \dot{x}^\lambda, \dot{y}^i, \dot{y}^i_\lambda)$ are the charts induced on J_1TE and TJ_1E by (x^λ, y^i), respectively. Indeed, if $\boldsymbol{u}\colon E \to TE$ is a local vector field with coordinate expression $\boldsymbol{u} = u^\lambda \partial_\lambda + u^i \partial_i$, then we obtain the projectable vector field ([MM2], pag. 195)

$$\boldsymbol{w} \equiv r\circ J_1\boldsymbol{u}\colon J_1E \to TJ_1E\,, \tag{8}$$

with coordinate expression

$$\boldsymbol{w} = u^\lambda \partial_\lambda + u^i\partial_i + (\partial_\lambda u^i - y^i_\mu \partial_\lambda u^\mu + y^j_\lambda \partial_j u^i)\,\partial^\lambda_i\,. \tag{9}$$

1. – The horizontal and vertical differentials.

In this section we deal with a generic fibred manifold $q\colon F \to M$, which later will be specified.

DEFINITION 1. – A section

$$\chi\colon F \to \Lambda^s(T^*F) \underset{F}{\otimes} TM$$

is said to be a *relative tangent valued s-form on F* (*vector-valued s-form along q*: [SA], pag. 75). We denote the sheaf of q-relative tangent valued 1-forms by $\mathcal{T}^*(F) \otimes \mathcal{T}(M)$

We shall be concerned with the case of relative tangent valued 1-forms. The coordinate expression of a relative tangent valued 1-form on F is

$$\chi = (\chi^\mu_\lambda d^\lambda + \chi^\mu_i d^i)\otimes \partial_\mu\,,$$

where $\chi^\mu_\lambda, \chi^\mu_i \in \mathcal{F}(F)$.

We have a natural contraction between relative tangent valued 1-forms on F and s-forms on M, which yields 1-forms on F; its coordinate expression is, for $\chi = (\chi^\mu_\lambda d^\lambda + \chi^\mu_i d^i)\otimes\partial_\mu \in \mathcal{T}^*(F)\otimes\mathcal{T}(M)$ and $\alpha = \alpha_{\lambda_1\ldots\lambda_s} d^{\lambda_1}\wedge\ldots\wedge d^{\lambda_s} \in \mathcal{T}^*(M)$:

$$\alpha \,\lrcorner\, \chi = (\alpha_{\mu\lambda_2\ldots\lambda_s}\chi^\mu_\lambda d^\lambda + \alpha_{\mu\lambda_2\ldots\lambda_s}\chi^\mu_i d^i)\wedge d^{\lambda_2}\wedge\ldots\wedge d^{\lambda_s}.$$

The following proposition is the specialization of Proposition 3.4.4 and Proposition 3.4.7 of [SA], pag. 78-79, to the case of relative tangent valued 1-forms.

PROPOSITION 1. – *Let* $\chi: F \to T^*F \underset{F}{\otimes} TM$ *a relative tangent-valued 1-form on F. Then, we obtain:*

i) *a derivation of degree* 0

$$i_\chi : \Omega(M) \to \Omega(F), \tag{10}$$

characterized, for $\alpha: M \to \Lambda^s(T^*M)$ *by the relation*

$$i_\chi \alpha = \chi \rfloor \alpha ;$$

ii) *a derivation of degree* 1

$$d_\chi \equiv i_\chi \circ d - d \circ i_\chi : \Omega(M) \to \Omega(F), \tag{11}$$

where d is the exterior derivative on the exterior algebra of F.

If $\chi_1, \chi_2 \in \mathcal{T}^*(F) \otimes \mathcal{T}(M)$, then we have

$$i_{\chi_1 + \chi_2} = i_{\chi_1} + i_{\chi_2}, \qquad d_{\chi_1 + \chi_2} = d_{\chi_1} + d_{\chi_2}.$$

Now, let us go back to the fibred manifold $p: E \to B$. By recalling the canonical inclusions

$$J_{k+1}E \underset{B}{\times} T^*B \hookrightarrow T^*J_{k+1}E,$$

$$VJ_kE \hookrightarrow TJ_kE,$$

we can easily see that the contact morphism (1), (2) of $J_{k+1}E$ are relative tangent valued 1-forms with respect to the fibring $p_{k+1,k}: J_{k+1}E \to J_kE$. We can then apply the abogve Proposition 1 to these 1-forms and obtain the two derivations of degree 0

$$i_h \equiv i_{\mathcal{D}^{k+1}} : \Omega(J_kE) \to \Omega(J_{k+1}E), \tag{12}$$

$$i_v \equiv i_{\theta^{k+1}} : \Omega(J_kE) \to \Omega(J_{k+1}E), \tag{13}$$

and the two derivations of degree 1

$$d_h \equiv i_h \circ d - d \circ i_h : \Omega(J_kE) \to \Omega(J_{k+1}E), \tag{14}$$

$$d_v \equiv i_v \circ d - d \circ i_v : \Omega(J_kE) \to \Omega(J_{k+1}E). \tag{15}$$

DEFINITION 2. – The two derivations d_h (14) and d_v (15) are said to be the *horizontal* and *vertical differentials*, respectively.

PROPOSITION 2. – *The horizontal and vertical differentials* (14) *and* (15) *fulfill the following properties* (*see* [SA], *pag.* 216-217):

a) $d_h + d_v = p^*_{k+1,k} \circ d$;

b) $(j_{k+1}s)^* \circ d_v = 0$;

c) $d \circ (j_k s)^* = (j_{k+1} s)^* \circ d_h$;

d) $d_h^2 = d_v^2 = 0$;

e) $d_h \circ d_v + d_v \circ d_h = 0$.

The two differentials (14) e (15) are characterized by the following formulas, for each $f \in \mathcal{F}(J_k E)$:

$$(16) \qquad \begin{cases} d_h f = (\partial_\mu f + y^i_\mu \partial_i f + \ldots + y^i_{\lambda_1 \ldots \lambda_k \mu} \partial_i^{\lambda_1 \ldots \lambda_k} f)\, d^\mu \equiv (\mathcal{D}^k_\mu \cdot f)\, d^\mu, \\ d_h d^\lambda = 0, \\ d_h d^i_{\lambda_1 \ldots \lambda_k} = -d^i_{\lambda_1 \ldots \lambda_k \mu} \wedge d^\mu; \end{cases}$$

and

$$(17) \qquad \begin{cases} d_v f = \partial_i f\, \theta^i + \partial_i^\mu f\, \theta^i_\mu + \ldots + \partial_i^{\lambda_1 \ldots \lambda_k} f\, \theta^i_{\lambda_1 \ldots \lambda_k}, \\ d_v d^\lambda = 0, \\ d_v d^i_{\lambda_1 \ldots \lambda_k} = d^i_{\lambda_1 \ldots \lambda_k \mu} \wedge d^\mu. \end{cases}$$

Hence, in particular,

$$(18) \qquad \begin{cases} d_h x^\lambda = i_h d^\lambda = d^\lambda, \\ d_h y^i_{\lambda_1 \ldots \lambda_k} = i_h d^i_{\lambda_1 \ldots \lambda_k} = y^i_{\lambda_1 \ldots \lambda_k \mu} d^\mu, \\ d_v x^\lambda = i_v d^\lambda = 0, \\ d_v y^i_{\lambda_1 \ldots \lambda_k} = i_v d^i_{\lambda_1 \ldots \lambda_k} = d^i_{\lambda_1 \ldots \lambda_k} - y^i_{\lambda_1 \ldots \lambda_k \mu} d^\mu = \theta^i_{\lambda_1 \ldots \lambda_k}. \end{cases}$$

2. – Fundamental objects.

The next step is the construction of the fundamental geometrical structures of field theory in a purely differential way; our main geometrical object is the fibred manifold $p: E \to B$, and our main tools are the horizontal and vertical differentials (14), (15), and further canonical isomorphisms.

We start from a *lagrangian density*

$$\mathcal{L}: J_1 E \to \Lambda^m (T^* B),$$

i.e. a p_B-semibasic m-form on $J_1 E$. Its coordinate expression is $\mathcal{L} = l\omega$, where

$$l: J_1 E \to \boldsymbol{R}$$

is a local function on $J_1 E$, called the *lagrangian function*, and $\omega \equiv d^1 \wedge \ldots d^m$.

The direct application of d_h and d_v to $\mathcal{L}$ gives no new and interesting objects; indeed, by using Proposition 2*c*) and *a*), we can easily see that

$$(19) \qquad d_h \mathcal{L} = 0, \qquad d_v \mathcal{L} = d\mathcal{L},$$

where we have still denoted the pullback of $d\mathcal{L}$ over J_2E by $d\mathcal{L}$ (we will always omit the indication of obvious pullbacks). It is interesting to see this locally. From the expressions (16)-(17), we see that the differentials d_h and d_v act only on the lagrangian function l, for they both vanish on the base components d^λ, and we see that $d_h l \wedge \omega = 0$, and $d_v l \wedge \omega = d\mathcal{L}$. In this case, we have no new terms of second order.

The coordinate expression of $d\mathcal{L}$ is

$$d\mathcal{L} = \partial_i l d^i \wedge \omega + \pi^\lambda_i d^i_\lambda \wedge \omega \,, \tag{20}$$

where we have set $\pi^\lambda_i \equiv \partial^\lambda_i l$.

Now, let us consider $\mathcal{L}$ as a fibred morphism $\mathcal{L}\colon J_1E \to \Lambda^m(T^*B)$ over $p\colon E \to B$. Then, we can perform the vertical derivative

$$V_E\mathcal{L}\colon V_E J_1 E \to V\Lambda^m(T^*B)\,, \tag{21}$$

where V_E is the vertical functor with respect to the base space E (for the sake of simplicity, we have omitted to write explicitly the pullback $p^*(T^*B)$).

Its coordinate expression is

$$V_E\mathcal{L} = (\pi^\lambda_i \check{y}^i_\lambda)\,\omega\,,$$

where $\check{y}^i_\lambda$ are the fibre components of the linear chart $(x^\lambda, y^i, y^i_\lambda, \check{y}^i_\lambda)$ induced on $V_E J_1 E$ by (x^λ, y^i).

Then, by taking into account the canonical isomorphism

$$V\Lambda^m(T^*B) \simeq \Lambda^m(T^*B) \underset{B}{\times} \Lambda^m(T^*B)\,, \tag{22}$$

projecting $V_E\mathcal{L}$ to the second factor and taking the pullback over J_1E, we obtain the section

$$\check{d}\mathcal{L} \equiv pr_2 \circ V_E\mathcal{L}\colon J_1E \to V^*E \underset{J_1E}{\otimes} (\Lambda^m(T^*B) \underset{B}{\otimes} TB)\,, \tag{23}$$

In coordinates we have:

$$\check{d}\mathcal{L} = \pi^\lambda_i \check{d}^i_\lambda \otimes \omega\,, \tag{24}$$

where we have denoted by $\check{d}^i_\lambda$ the *vertical 1-forms* induced by the fibred chart $(x^\lambda, y^i, y^i_\lambda)$; they are to be kept distinct from the forms d^i_λ on T^*J_1E.

We would like to apply the horizontal and vertical differentials to the vertical derivative $\check{d}\mathcal{L}$ of $\mathcal{L}$; but this is not possible in a direct way, for $\check{d}\mathcal{L}$ is not a differential form. Hence, we have to transform $\check{d}\mathcal{L}$ in an «equivalent» differential form, that is in a differential form which carries the same content of $\check{d}\mathcal{L}$; this content is essentially given by the derivatives $\pi^\lambda_i \equiv \partial^\lambda_i l$.

Thus, we have the following result.

REMARK 3. – The following composition of natural maps

$$J_1E \underset{E}{\times} (V^*E \underset{E}{\otimes} (\Lambda^m(T^*B) \underset{B}{\otimes} TB)) \xrightarrow{\langle,\rangle} J_1E \underset{E}{\times} (V^*E \underset{E}{\otimes} \Lambda^{m-1}(T^*B)) \xrightarrow{\theta^{1*}} \tag{25}$$

$$\xrightarrow{\theta^{1*}} T^*E \underset{E}{\otimes} \Lambda^{m-1}(T^*B) \hookrightarrow T^*E \underset{E}{\otimes} \Lambda^{m-1}(T^*E) \xrightarrow{A} \Lambda^m(T^*E),$$

turns out to be an injective fibred morphism over $p_E: J_1E \to E$.

The section $\check{d}\mathcal{L}$ (23) can then be characterized by the p_E-semibasic m-form

$$\Pi: J_1E \to \Lambda^m(T^*E), \tag{26}$$

The coordinate expression of Π is

$$\Pi = \pi_i^\lambda \theta^i \wedge \omega_\lambda = \pi_i^\lambda (d^i \wedge \omega_\lambda - y_\lambda^i \omega), \tag{27}$$

where we have set

$$\omega_\lambda \equiv i_{\partial_\lambda} = (-1)^{\lambda-1} d^1 \wedge \ldots \wedge \widehat{d^\lambda} \wedge \ldots \wedge d^m. \tag{28}$$

This expression confirms the equivalence of Π and $\check{d}\mathcal{L}$ the fundamental content carried by $\check{d}\mathcal{L}$, constituted by the derivatives π_i^λ, is maintained in (27).

DEFINITION 3. – The p_E-semibasic m-form Π (26) is said to be the *legendre form*, or the *momentum form*, associated with $\mathcal{L}$.

Now, let us apply the horizontal and vertical differentials to the Legendre form (26). Unlike (19), the result is now «non trivial»: from the coordinate expressions (27) and (16)-(17) for the action of d_h and d_v on functions, we see that the presence of the term $\pi_i^\lambda d^i \wedge \omega_\lambda$ yields at least a second-order term $y_{\lambda\mu}^i$.

THEOREM 4. – *The (pullback of the) differential of the Legendre form Π (26), associated with the lagrangian $\mathcal{L}$, splits canonically on J_2E into a sum of two $p_{2,1}$-semibasic $(m+1)$-forms:*

$$d\Pi = d_h\Pi + d_v\Pi. \tag{29}$$

PROOF. – This theorem is an immediate consequence of Proposition 2*a*). q.e.d.

The coordinate expressions of $d_h\Pi$, $d_v\Pi$ are:

$$d_h\Pi = -(\varpi_\lambda^2 \cdot \pi_i^\lambda) d^i \wedge \omega - \pi_i^\lambda d_\lambda^i \wedge \omega, \tag{30}$$

$$d_v\Pi = \partial_j \pi_i^\lambda \theta^j \wedge \theta^i \wedge \omega_\lambda + \partial_j^\mu \pi_i^\lambda \theta_\mu^j \wedge \theta^i \wedge \omega_\lambda. \tag{31}$$

These expressions show that $d_h\Pi$ and $d_v\Pi$ are m-semibasic over E.

The $(m+1)$-forms $d_h\Pi$ and $d_v\Pi$ play a fundamental role in the construction of the basic objects of our purely differential approach to field theory. First, we can characterize the Euler-Lagrange operator in the following way.

THEOREM 5. – *The* $(m+1)$*-form*

$$\mathcal{E} \equiv d\mathcal{L} + d_h \Pi \colon J_2 E \to \Lambda^{m+1}(T^* E) \tag{32}$$

is the unique combination (up to a scalar factor) of $d\mathcal{L}$ *and* $d_h \Pi$ *which is* p_E*-semibasic. Its coordinate expression is*

$$\mathcal{E} = (\partial_i l - \mathcal{D}^2_\lambda \cdot \pi^\lambda_i)\, d^i \wedge \omega = (\partial_i l - \partial_\lambda \pi^\lambda_i - y^j_\lambda \partial_j \pi^\lambda_i - y^j_{\lambda\mu} \partial^\mu_j \pi^\lambda_i)\, d^i \wedge \omega \,. \tag{33}$$

PROOF. – It suffices to compare the expressions (20) for $d\mathcal{L}$ and (30) for $d_h \Pi$. q.e.d.

DEFINITION 4. – The $(m+1)$-form $\mathcal{E}$ (32) is said to be the *Euler-Lagrange form.*

REMARK 6. – Formula (32) can be rewritten as

$$d\mathcal{L} = \mathcal{E} - d_h \Pi \,. \tag{34}$$

Hence, we can say that the differential $d\mathcal{L}$ of the lagrangian $\mathcal{L}$ splits canonically on $J_2 E$ into a sum of p_E-semibasic $(m+1)$-form $\mathcal{E}$ and in a $p_{2,1}$-semibasic $(m+1)$-form $d_h \Pi$.

We can regard the Euler-Lagrange form in other equivalent and useful ways.

PROPOSITION 7. – *We can regard in a natural way the Euler-Lagrange form* $\mathcal{E}$ *as a fibred morphism over* E

$$\check{\mathcal{E}} \colon J_2 E \to V^* E \underset{E}{\otimes} \Lambda^m(T^* B) \,. \tag{35}$$

PROOF. – The image of the Euler-Lagrange form is contained in a subspace of $\Lambda^{m+1}(T^* E)$, which turns out to be isomorphic to $V^* E \underset{E}{\otimes} \Lambda^m(T^* B)$. q.e.d.

DEFINITION 5. – The fibred morphism $\check{\mathcal{E}}$ (35) is said to be the *Euler-Lagrange operator* associated with the lagrangian $\mathcal{L}$.

Its coordinate expression is

$$\hat{\mathcal{E}} = (\partial_i l - \mathcal{D}^2_\lambda \cdot \pi^\lambda_i)\, d^i \otimes \omega = (\partial_i l - \partial_\lambda \pi^\lambda_i - y^j_\lambda \partial_j \pi^\lambda_i - y^j_{\lambda\mu} \partial^\mu_j \pi^\lambda_i)\, d^i \otimes \omega \,. \tag{36}$$

So far we have been involved with the horizontal term $d_h \Pi$ of the canonical splitting (29); now let us investigate the role of the vertical term $d_v \Pi$.

THEOREM 8. – *Then sum*

$$\mathcal{E} + d_v \Pi \tag{37}$$

is an exact form. Moreover, among all potentials of (37), *we have the distinguished canonical* p_E*-semibasic m-form*

$$\Theta \equiv \mathcal{L} + \Pi : J_1 E \to \Lambda^m (T^* E), \tag{38}$$

which depends only on the objects so far introduced.

PROOF. – From (32) and the canonical splitting (29), we obtain, on $J_2 E$,

$$\mathcal{E} + d_v \Pi = d\mathcal{L} + d_h \Pi + d_v \Pi = d(\mathcal{L} + \Pi) = d\Theta . \quad \text{q.e.d.}$$

DEFINITION 6. – The p_E semibasic m-form Θ is said to be the *Poincaré-Cartan form* associated with the lagrangian $\mathcal{L}$. Its coordinate expression is

$$\Theta = l\omega + \pi_i^\lambda \theta^i \wedge \omega = \pi_i^\lambda d^i \wedge \omega_\lambda + (l - y_\lambda^i \pi_i^\lambda)\, \omega . \tag{39}$$

REMARK 9. – The differential $d\Theta$ of the Poincaré-Cartan form Θ splits canonically on $J_2 E$ into the sum of $\mathcal{E}$ and $d_v \Pi$:

$$d\Theta = \mathcal{E} + d_v \Pi . \tag{40}$$

3. – Lagrangian field theory.

So far we have made use of a *lagrangian*, but no variational principle, or variational problem has been mentioned: the lagrangian was simply a p_B-semibasic m-form on the 1-jet space of a given fibred manifold $p: E \to B$, and theory developed in Section 2 is nothing but a consequence of the geometrical structure of 1-jet space. In this section we see how our objects are related to the usual lagrangian field theory.

We refer to our given pair $(E, \mathcal{L})$ as the *physical field*, and set the related variational principle.

DEFINITION 7. – The *action* associated to the physical field $(E, \mathcal{L})$ and to the section $s: B' \to E$, where $B' \subset B$ is a compact submanifold of maximal dimension, is defined to be the integral

$$\mathcal{A}_s \equiv \int_{B'} (j_1 s)^* \mathcal{L} .$$

DEFINITION 8. – A *variation of the field E* is defined to be a pair $(E', \boldsymbol{u})$, where $E' \equiv p^{-1}(B') \subset E$, $B' \subset B$ is a compact submanifold of maximal dimension with boundary, and $\boldsymbol{u}: E' \to TE' \subset TE$ is a *projectable* vector field such that

$$\boldsymbol{u}|_{\partial E'} = 0 . \tag{41}$$

The *1-jet prolongation of* $(E', \boldsymbol{u})$ is the pair $(J_1 E', \boldsymbol{w})$, where

$$\boldsymbol{w} \equiv r \circ J_1 \boldsymbol{u}$$

is the 1-jet prolongation of $\boldsymbol{u}$ (8).

By considering the 1-parameter group of fibred diffeomorphisms of the total space E generated by $\boldsymbol{u}$, we can see that our definition of variation is actually the infinitesimal version of the usual one.

DEFINITION 9. – A section $s: B \to E$ is said to be *critical* if

$$\int_{B'} (j_1 s)^* L_{\boldsymbol{w}} \mathcal{L} = 0 , \tag{42}$$

where $L_{\boldsymbol{w}}$ is the Lie derivative with respect to the vector field $\boldsymbol{w}$ (8), for all variations $(E', \boldsymbol{u})$ of E.

So, we can recover in a geometrical language the standard analytic procedure for deriving the Euler-Lagrange equations.

LEMMA 10. – *The following conditions are equivalent:*

i) *A section* $s: B \to E$ *is critical;*

(43) ii) $(j_2 s)^* i_{\boldsymbol{u}} \mathcal{E} = 0$,

for all vector fields $\boldsymbol{u}$ *of* E.

PROOF. – By using a well known identity for the Lie derivative of a differential form, the canonical splitting (34) of $d\mathcal{L}$, the Stokes theorem and the boundary condition (41), we obtain

$$\int_{B'} (j_1 s)^* L_{\boldsymbol{w}} \mathcal{L} = \int_{B'} (j_1 s)^* (i_{\boldsymbol{w}} d\mathcal{L} + d i_{\boldsymbol{w}} \mathcal{L}) = \int_{B'} (j_2 s)^* i_{\boldsymbol{u}} \mathcal{E} - \int_{B'} (j_2 s)^* i_{\boldsymbol{w}} d_h \Pi .$$

Now it is easy to see, by a straightforward calculation, that

$$(j_2 s)^* i_{\boldsymbol{w}} d_h \Pi = d(j_1 s)^* i_{\boldsymbol{u}} \Pi .$$

By using the Stokes theorem with the boundary condition (41), we see that $\int_{B'} d(j_1 s)^* i_{\boldsymbol{u}} \Pi$ vanishes. Hence, we have

$$\int_{B'} (j_1 s)^* L_{\boldsymbol{w}} \mathcal{L} = \int_{B'} (j_2 s)^* i_{\boldsymbol{u}} \mathcal{E} . \tag{44}$$

i) $\Rightarrow$ ii) The definition of critical section and (44) imply

$$\int_{B'} (j_2 s)^* i_u \mathcal{E} = 0\,,$$

hence, for well known theorems on integration theory,

$$(j_2 s)^* i_u \mathcal{E} = 0\,,$$

for all variational vector fields (i.e. such that the boundary condition (41) holds). Moreover, we remark that we can omit the boundary condition (41) by considering a standard technique of extension of local vector fields. So, (43) holds for all projectable vector fields on E.

ii) $\Rightarrow$ i It follows immediately from (44). q.e.d.

THEOREM 11. – *Condition* (43) *is equivalent to the Euler-Lagrange equation*:

$$\check{\mathcal{E}} \circ j_2 s = 0\,, \tag{45}$$

PROOF. – It follows easily from the above Lemma, by expressing the Euler-Lagrange form in terms of the Euler-Lagrange operator. q.e.d.

The coordinate expression of this equation is

$$(\partial_i l - \mathcal{D}^2_\lambda \cdot \pi^\lambda_i) \circ j_2 s \equiv (\partial_i l - \partial_\lambda \pi^\lambda_i - \partial_\lambda s^j \partial_j \pi^\lambda_i - \partial_{\lambda\mu} s^j \partial^\mu_j \pi^\lambda_i) \circ j_1 s = 0\,. \tag{46}$$

We have another important geometrical characterization of critical sections.

THEOREM 12. – *Condition* (43) *for a section s to be critical is equivalent to the following condition*:

$$(j_1 s)^* i_w d\Theta = 0\,, \tag{47}$$

for all local projectable vector fields $\boldsymbol{u}\colon E \to TE$ (*see* [MM1], *pag.* 42).

PROOF. – From the canonical splitting (40) of $d\Theta$ on J_2E, we obtain

$$(j_1 s)^* i_w d\Theta = (j_2 s)^* i_u \mathcal{E} + (j_2 s)^* i_w d_v \Pi\,.$$

Moreover, the coordinate expression (31) for $d_v \Pi$ tells us that the coordinate expression of the contraction $i_w d_v \Pi$ will contain at least one of the 1-forms θ^i, θ^i_λ, whose pullback by $j_2 s$ vanishes. So, the second term vanishes. q.e.d.

For completeness of our exposition, we recover the well known results on symmetries and on Noether theorem, by using the Poincaré-Cartan form (see also [GA).

DEFINITION 10. – A *symmetry* of the Poincaré-Cartan form Θ is a projectable vector field

$$u: E \to TE$$

such that, for all critical sections $s: B \to E$,

$$(j_1 s)^* L_w \theta = 0 .$$

DEFINITION 11. – A *conserved current* is an $(m-1)$-form

$$\mathfrak{J}: J_1 E \to \Lambda^{m-1}(T^* J_1 E)$$

such that, for all critical sections $s: B \to E$,

$$(j_1 s)^* d\mathfrak{J} = 0 .$$

THEOREM 13 (Noether). – *If*

$$u: E \to TE$$

is a symmetry of Θ, then

$$i_u \Theta: J_1 E \to \Lambda^{m-1}(T^* E)$$

is a conserved current.

PROOF. – Let $s: B \to E$ be a critical section. Definition 10 and Theorem 12 give

$$(j_1 s)^* L_w \theta = (j_1 s)^* i_w d\theta + (j_1 s)^* d i_u \Theta = (j_1 s)^* d i_u \Theta = 0 \qquad \text{q.e.d.}$$

Acknowledgements. – The author wishes to thank Prof. MARCO MODUGNO and Prof. LUIGI MANGIAROTTI for stimulating discussions.

REFERENCES

[FF] FERRARIS M.. - FRANCAVIGLIA M, *On the Global Structure of Lagrangian and Hamiltonian Formalism in Higher Order Calculus of Variations,* Proceedings of the Meeting «Geometry and Physics», Florence, October 12-15, 1982.

[GA] GARCIA P. L, *The Poincaré-Cartan Invariant in the Calculus of Variations,* Symposia Mathematica, 14 (1974), pp. 219-246.

[GS] GOLDSCHMIDT H. - STERNBERG S., *The Hamiltonian-Cartan formalism in the calculus of variations,* Ann. Inst. Fourier, Grenoble, 23 (1) (1973), pp. 203-267.

[HE] HERMANN R., *Geometry, Physics and Systems,* M. Dekker, Inc., New York (1973).

[KR] KRUPKA D., *Some geometric aspects of variational problems in fibered manifolds,* Folia Fac. Scie. Nat. Univ. Purkyniane Brunensis, 14 , Opus 10 (1973), pp. 1-65.

[MM1] MANGIAROTTI S. - MODUGNO M., *Some results on the calculus of variations on jet spaces,* Ann. Inst. H. Poincaré, **39**, n. 1 (1983), pp. 29-43.

[MM2] MANGIAROTTI S. - MODUGNO M, *New operators on jet spaces,* Ann. Fac. Sci. Toulouse, **5** (1983), pp. 171-198.

[SA] SAUNDERS D. J., *The Geometry of Jet Bundles,* Cambridge University Press (1989).

[TA] TAKENS F., *A global version of the inverse problem of the calculus of variations,* J. Diff. Geom., **14** (1979), pp. 543-572.

[TU] TULCZYJEW W. M., *The Euler-Lagrange Resolution,* Lectures Notes in Mathematics, n. **836**, pp. 22-48, Springer-Verlag, Berlin (1980).

[VI] VINOGRADOV A. M, *Local symmetries and conservation laws,* Acta Appl. Math., **2** (1984), pp. 21-78.