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On the Condition of Nearness Between Operators (*).

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Sunto. – *Siano A e B due applicazioni definite su un insieme $\mathcal{B}$ e a valori in un spazio di Banach $\mathcal{B}_1$. Si espone una teoria delle applicazioni A e B vicine. Si dimostra, ad esempio, che A è iniettiva, o surgettiva, o bigettiva se e solo se A è vicina ad una B con queste proprietà (cfr. Appendice). Si danno condizioni sufficienti per la vicinanza nel caso generale e poi nel caso particolare in cui $\mathcal{B}_1$ è un spazio di Hilbert. Ulteriori condizioni sufficienti si danno quando A e B sono applicazioni differenziali non variazionali, del 2° ordine, definite su un aperto $\Omega \subset \boldsymbol{R}^n$, di classe C^1. Ricordata una opportuna definizione di operatore ellittico quasi-base* (cfr. [1] *e* [2]) *si dimostra, per questi operatori un teorema di isomorfismo $H^{2,q} \cap H_0^{1,q}(\Omega) \to \to L^q(\Omega)$, con $q > 1$, valevole anche quando Ω non è convesso. Questo risultato migliora un precedente risultato di [3]. L'ultimo paragrafo è dedicato agli operatori parabolici.*

Summary. – *Let A and B be two mappings defined on a set $\mathcal{B}$ taking values in a Banach space $\mathcal{B}_1$. We present a theory of nearness of mappings A and B. We shall prove, for instance, that A is injective, or surjective or bijective if and only if A is near B with these properties (see Appendix). We shall give sufficient conditions for the nearness in the general case and then in the particular case wherein $\mathcal{B}_1$ is a real Hilbert space. We shall give further sufficient conditions when A and B are non variational differential mappings of second order, defined on an open set $\Omega \subset \boldsymbol{R}^n$, of class C^1. After recalling an appropriate definition of a quasi-basic elliptic operator* (*see* [1] *and* [2]) *we prove, for these operators an isomorphism theorem $H^{2,q} \cap H_0^{1,q}(\Omega) \to L^q(\Omega)$, with $q > 1$, valid also when Ω* is not convex. This result improves an earlier result of [3]. *The final section is dedicated to parabolic operators.*

0. – Introduction.

Many problems of analysis (and not only of analysis) lead one to consider a situation of the following type:

Suppose

$\mathcal{B}$ is a set

$\mathcal{B}_1$ is a real Banach space.

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$A: \mathcal{B} \to \mathcal{B}_1$ is a mapping. We ask whether A is

$$\text{injective or surjective or bijective.} \tag{0.1}$$

Investigating on the above problem (0.1) one is led, in the last years, to a new existential method based on the notion of *near maps*. Let B be another mapping $\mathcal{B} \to \mathcal{B}_1$.

DEFINITION 1. – *We shall say that A is near B if there exist two positive constants α and K, with $K \in (0, 1)$, such that $\forall u, v \in \mathcal{B}$, we have*

$$\|B(u) - B(v) - \alpha[A(u) - A(v)]\|_{\mathcal{B}_1} \leq K\|B(u) - B(v)\|_{\mathcal{B}_1}. \tag{0.2}$$

The theory of near mappings already has a sufficient amount of propositions which allows one to deduce interesting consequences. First of all one can prove the following theorem.

THEOREM 1. – *The mapping $A: \mathcal{B} \to \mathcal{B}_1$ is injective, or surjective, or bijective, if and only if it is near to a mapping $B: \mathcal{B} \to \mathcal{B}_1$ which is injective, or surjective, or bijective.*

Infact, the theorem has been proved, in the literature, only in the particular case wherein $\mathcal{B}$ and $\mathcal{B}_1$ are real Hilbert spaces and B is a bijection (see [2] section 2 and [1] theorem 3.1). However, we can also prove now in the version stated in the text (see Appendix).

As an example we have the following result.

THEOREM 2. – *If $B: \mathcal{B} \to \mathcal{B}_1$ is bijective and if A is near to B then, $\forall f \in \mathcal{B}_1$, $\exists_1 u \in \mathcal{B}$ such that*

$$A(u) = f \tag{0.3}$$

and for this u we have the estimate

$$\|B(u) - B(v)\|_{\mathcal{B}_1} \leq \frac{\alpha}{1 - K}\|f - A(v)\|_{\mathcal{B}_1}, \qquad \forall v \in \mathcal{B}_1. \tag{0.4}$$

The estimate (0.4) follows easily from the nearness condition (0.2). In this formulation the central points are

the choice of B,

the choice of the sets $\mathcal{B}$ and $\mathcal{B}_1$.

1. – Some propositions on near mappings.

We shall prove here some easy, but useful, propositions.

THEOREM 3. – *If A is near B with constants α and K then A is near to all mappings βB, for $\beta > 0$, with constants $\beta\alpha$ and K.*

THEOREM 4. – *If A is near B with constants α and K then βA, $\beta > 0$, A* is near to B, *with constants α/β and K.*

In particular A is near B with constants α and K is equivalent to say that αA is near B with constants 1 and K.

THE THEOREM OF THE ker. – Given a mapping $A\colon \mathcal{B} \to \mathcal{B}_1$ we let, as is customary,

$$\ker A = \{u \in \mathcal{B}\colon A(u) = 0\}.$$

THEOREM 5. – *If A and B are two mappings near to each other $\mathcal{B} \to \mathcal{B}_1$ and if*

$$\ker A \cap \ker B \neq \emptyset \tag{1.5}$$

then

$$\ker A = \ker B. \tag{1.6}$$

In fact, if we choose $v \in \ker A \cap \ker B$, we obtain, from the nearness relation (0.2), that $\forall u \in \mathcal{B}$,

$$\|B(u) - \alpha A(u)\|_{\mathcal{B}_1} \leq K\|B(u)\|_{\mathcal{B}_1}$$

and hence, since $K \in (0, 1)$,

$$A(u) = 0 \Leftrightarrow B(u) = 0. \tag{1.7}$$

Let A, B, C be mappings $\mathcal{B} \to \mathcal{B}_1$.

THEOREM 6. – *If A is near B with constants α and K, and if C is near B with constants α' and K' then*

$$\alpha A + \alpha' C \quad \textit{is near } B \tag{1.8}$$

with constants $1/2$ and $(K + K')/2$.

PROOF. – We have, $\forall u,\ v \in \mathcal{B}$

$$\|2[B(u)-B(v)]-[\alpha A(u)+\alpha' C(u)-\alpha A(v)-\alpha' C(v)]\|_{\mathcal{B}_1} \leqslant$$
$$\leqslant \|B(u)-B(v)-\alpha[A(u)-A(v)]\|_{\mathcal{B}_1}+\|B(u)-B(v)-\alpha'[C(u)-C(v)]\|_{\mathcal{B}_1} \leqslant$$
$$\leqslant \frac{K+K'}{2}\|2[B(u)-B(v)]\|_{\mathcal{B}_1}.$$

Hence $\alpha A+\alpha' C$ is near $2B$ with constants 1 and $(K+K')/2$, and hence, by theorem 3, $\alpha A+\alpha' C$ is near B with constants $1/2$ and $(K+K')/2$.

This theorem can also be stated in other forms which can be deduced from Theorem 6 making use of the Theorems 3 and 4.

2. - Sufficient conditions for the nearness of two operators.

We have particularly simple conditions when

$\mathcal{B}_1$ *is a real Hilbert space.*

We recall the following definition of monotonicity.

DEFINITION 2. – *A is said to be monotone with respect to B if $\forall u,\ v \in \mathcal{B}$ we have*

$$(A(u)-A(v)\,|\,B(u)-B(v))_{\mathcal{B}_1} \geqslant 0\,. \tag{2.9}$$

A is said to be strictly monotone with respect to B if there exist positive constants M and ν such that $\forall u,\ v \in \mathcal{B}$ we have

$$\begin{gathered}\|A(u)-A(v)\|_{\mathcal{B}_1} \leqslant M\|B(u)-B(v)\|_{\mathcal{B}_1}\,,\\ (A(u)-A(v)\,|\,B(u)-B(v))_{\mathcal{B}_1} \geqslant \nu\|B(u)-B(v)\|^2_{\mathcal{B}_1}\,.\end{gathered} \tag{2.10}$$

These definitions of monotonocity are symmetric.
These definitions of monotonocity depend on the choice of $\mathcal{B}_1$.
The following theorems are easily proved.

THEOREM 7. – *Two mappings A and B: $\mathcal{B} \to \mathcal{B}_1$*, are near if and only if *A and B are* strictly monotone.

For a proof of this theorem see for ex. [1], Lemma 3.2, page 16.

The generalized theorem of Lax-Milgram is an easy consequence of this. Let $\mathcal{B}=\mathcal{B}_1=H$ be a real Hilbert space. Let A be a mapping $H \to H$ and let I be the identity on H. Suppose that A is strictly monotone (with respect to I). Then A is near I and since I is obviously a bijection $H \to H$ it follows that (see Theorem 2).

THEOREM 8. – *If* $A\colon H \to H$ *is strictly monotone,* $\forall f \in H\ \exists_1 u \in H$ *such that* $A(u) = f$ *and we have the estimate*

$$\|u\|_H \leq \frac{1}{\nu}\|f - A(0)\|_H$$

where ν *is the constant of monotonocity.*

If A is linear the condition of strict monotonocity with respect to I reduces to the classical condition of continuity and coercivity.

THEOREM 9. – *If* A, B, C *are mappings* $\mathcal{B} \to \mathcal{B}_1$ *and if*

(2.11) $\qquad A$ is near B *with constants* α *and* K,

(2.12) $\qquad$ *if* C is monotone with respect to B,

then $A + C$ *is near* $B + \alpha C$ *with the same constants* α *and* K.

PROOF. – $\forall u$, $v \in \mathcal{B}$ we have, by (2.11)

$$\|B(u) + \alpha C(u) - B(v) - \alpha C(v) - \alpha[A(u) + C(u) - A(v) - C(v)]\|_{\mathcal{B}_1} \leq \leq K\|B(u) - B(v)\|_{\mathcal{B}_1}. \tag{2.13}$$

On the other hand, from (2.12) we have

$$\|B(u) - B(v)\|_{\mathcal{B}_1} \leq \|B(u) + \alpha C(u) - B(v) - \alpha C(v)\|_{\mathcal{B}_1}. \tag{2.14}$$

The assertion follows from (2.13) and (2.14).

We derive from this, as a corollary, the following

THEOREM 10. – *Under the hypothesis* (2.11) *and* (2.12), $\forall f \in \mathcal{B}_1$ *the problem*

$$\begin{cases} u \in \mathcal{B} \\ A(u) + C(u) = f \end{cases} \tag{2.15}$$

has a solution (*has a unique solution*) if and only if *the problem*

$$\begin{cases} u \in \mathcal{B} \\ B(u) + \alpha C(u) = f \end{cases} \tag{2.16}$$

has a solution (*has a unique solution*).

In particular, if $f = 0$ *the two systems* (2.15) *and* (2.16) *have the* same solutions (consequence of Theorems 3 and 2) *provided that they have one solution in common.*

Infact, the Theorem 9 can be made more precise in the following manner:

THEOREM 11. – *If* A, B, C *are mappings* $\mathcal{B} \to \mathcal{B}_1$ *satisfying* (2.11) *and* (2.12) *and if*

$$\ker A \cap \ker B \cap \ker C \neq \emptyset \tag{2.17}$$

then

$$\ker (A + C) = \ker (B + \alpha C). \tag{2.18}$$

Infact, by Theorem 9, $A + C$ is near $B + \alpha C$.

This last theorem has applications, for example, in the so called *eigen value* problem.

Let $\mathcal{B}$ be a real linear space, in particular, $0 \in \mathcal{B}$.

Let $\mathcal{B}_1$ be a real Hilbert space.

Let A, B, C be three mappings $\mathcal{B} \to \mathcal{B}_1$ with $A(0) = B(0) = C(0) = 0$.

DEFINITION 3. – *We shall say that the number $\lambda \in \boldsymbol{R}$ is an eigen value of* A *with respect to C if*

$$\ker (A - \lambda C) \neq 0. \tag{2.19}$$

The Theorem 9 ensures that if

$$\begin{aligned} &\text{i) } A \text{ is near } B \text{ with constants } \alpha \text{ and } K, \\ &\text{ii) } -\lambda C \text{ is monotone with respect to } B, \end{aligned} \tag{2.20}$$

then

$$\ker (A - \lambda C) = \ker (B - \alpha\lambda C).$$

In other words, if (2.20) holds, λ is an eigen value for A with respect to C if and only if $\alpha\lambda$ is an eigen value for B with respect to C.

A concrete case of this situation has been studied in [5].

3. – Other sufficient conditions for the nearness in the case where A is a differential operator.

Suppose that A is a differential operator. For simplicity let us suppose that A is a second order operator of *quasi-basic* type. That is, we suppose that A is an operator of the type

$$A(u) = a(x, H(u)) \tag{3.19}$$

where x varies in a *bounded open* set $\Omega \subset \boldsymbol{R}^n$, of class C^2, u is a vector $\Omega \to \boldsymbol{R}^N$, N being an integer $\geqslant 1$, $H(u) = \{D_{ij}u\}$ is an $n \times n$ matrix of vectors in $\boldsymbol{R}^N$, that is an element of $\boldsymbol{R}^{n^2N}$ and finally $a(x, \xi)$ is a vector of $\boldsymbol{R}^N$, measurable in x and continuous in ξ such that $a(x, 0) = 0$.

DEFINITION 4. – *We shall say that the vector $a(x, \xi)$ is* elliptic *if it satisfies the following condition:*

(A) *There exist three positive constants α, γ, δ with $\gamma + \delta < 1$, such that $\forall \xi, \tau \in \boldsymbol{R}^{n^2N}$ and $\forall x \in \Omega$ we have*

$$\| \sum_i \xi_{ii} - \alpha[a(x, \xi + \tau) - a(x, \tau)] \|_N \leq \gamma\|\xi\| + \delta\| \sum_i \xi_{ii} \|_N . \tag{3.20}$$

For an analysis of the condition (A) see for example [1].

The following theorem can easily be proved (see [1] and [3])

THEOREM 12. – *If the vector $a(x, \xi)$ is elliptic and the open set Ω is convex then the operator $a(x, H(u))$ is near the operator Δu, both operators being understood as operators $H^2 \cap H_0^1(\Omega) \to L^2(\Omega)$.*

The proof is made easy in view of the fact that, if Ω is convex, we have the following classical estimate of C. Miranda - G. Talenti:

$$\forall u \in H^2 \cap H_0^1(\Omega) \text{ we have } \int_\Omega \|H(u)\|^2 dx \leq \int_\Omega \|\Delta u\|^2 dx .$$

If Ω is not convex, the condition of ellipticity (A) is not in general sufficient to gaurentee the nearness stated in Theorem 12.

We shall now prove that, even if Ω is not convex, we have the nearness asserted in the Theorem 12, provided that a some what more restrictive condition than the condition (A) is imposed on the vector $a(x, \xi)$. This new condition, that we shall call the condition (A_2) depends on the geometry of Ω.

4. – The case of non convex Ω.

Let $A(u)$ as before be the differential operator (3.19).

Let the open set Ω be bounded and of class C^2, from the theory of linear differential operators we know that there exists a constant $C(2) \geq 1$ such that, $\forall u \in H^2 \cap H_0^1(\Omega)$, we have

$$\int_\Omega \|H(u)\|^2 dx \leq C^2(2) \int_\Omega \|\Delta u\|^2 dx . \tag{4.21}$$

Then, we shall impose the following condition on the vector $a(x, \xi)$:

(A_2) *There exist three positive constants α, γ, δ, with $\gamma + \delta < 1$, such that $\forall \xi, \tau \in \boldsymbol{R}^{n^2N}$ and $\forall x \in \Omega$ we have*

$$\left\| \sum_i \xi_{ii} - \frac{\alpha}{C(2)} [a(x, \xi + \tau) - a(x, \tau)] \right\|_N \leq \frac{\gamma}{C(2)} \|\xi\| + \delta\| \sum_i \xi_{ii} \|_N . \tag{4.22}$$

The condition (A_2) is more restrictive than the condition (A) since $C(2) \geq 1$, and hence, if (A_2) holds, the vector $a(x, \xi)$ is again elliptic.

THEOREM 13. – *If the vector $a(x, \xi)$ satisfies the condition (A_2) then the operator $a(x, H(u))$ is near Δu considered as an operator $H^2 \cap H_0^1(\Omega) \to L^2(\Omega)$.*

PROOF. – $\forall u, v \in H^2 \cap H_0^1(\Omega)$ and $\forall \varepsilon > 0$, we have

$$\mathcal{A} = \int_\Omega \left\| \Delta u - \frac{\alpha}{C(2)} [a(x, H(u+v)) - a(x, H(v))] \right\|^2 \leq$$

$$\leq \int_\Omega (1+\varepsilon) \frac{\gamma^2}{C^2(2)} \|H(u)\|^2 \left(1 + \frac{1}{\varepsilon}\right) \delta^2 \|\Delta u\|^2 dx .$$

Making use of (4.21), we get

$$\mathcal{A} \leq \left[(1+\varepsilon)\gamma^2 + \left(1 + \frac{1}{\varepsilon}\right)\delta^2\right] \int_\Omega \|\Delta u\|^2 dx .$$

Assuming $\varepsilon = \delta/\gamma$ we have

$$\mathcal{A} \leq (\gamma + \delta)^2 \int_\Omega \|\Delta u\|^2 dx .$$

Then we again have the following

THEOREM 14. – *Under the hypothesis of the theorem* 13, $\forall f \in L^2(\Omega)$ *the Dirichlet problem*

$$\begin{cases} u \in H^2 \cap H_0^1(\Omega), \\ a(x, H(u)) = f \quad in \ \Omega , \end{cases} \tag{4.23}$$

has a unique solution u and we have the following estimate

$$\|H(u)\|_{L^2(\Omega)} \leq \frac{\alpha}{1 - (\gamma + \delta)} \|f\|_{L^2(\Omega)} . \tag{4.24}$$

Hence, if (A_2) holds, then we have proved that the operator $a(x, H(u))$ is an isomorphism $H^2 \cap H_0^1(\Omega) \to L^2(\Omega)$, without the hypothesis that Ω is convex.

5. – An $L^q(\Omega)$, $q > 1$, regularity result.

Making use of the procedure followed in the preceeding section we can also prove that, if in Dirichlet problem (4.23) we have $f \in L^q(\Omega)$, $q > 1$, then the solution u belongs to $H^{2,q} \cap H_0^{1,q}(\Omega)$.

This result is known in the literature under the following hypothesis:

(i) $a(x, \xi)$ satisfies the condition (A),

(ii) Ω is convex,

(iii) q is sufficiently close to 2 ($2 \leqslant q < q_0$).

(See, for instance [3] Theorem 3.)

Here we prove that the result is true $\forall q > 1$, and without the hypothesis (ii), provided that the hypothesis (A) on the vector $a(x, \xi)$ is replaced by a slightly more restrictive hypothesis (A_q), which allows us to verify that the operator $a(x, H(u))$ is near Δu considered as an operator $H^{2,q} \cap H_0^{1,q}(\Omega) \to L^q(\Omega)$.

It is known, from the linear theory that, if Ω is of class C^2 and $q > 1$, then $\forall u \in H^{2,q} \cap H_0^{1,q}(\Omega)$ we have the estimate

$$\int_\Omega \|H(u)\|^q dx \leqslant C^q(q) \int_\Omega \|\Delta u\|^q dx\,, \qquad C(q) \geqslant 1\,. \tag{5.25}$$

We shall impose the following condition on the vector $a(x, \xi)$:

(A_q) *There exist three constants* α, γ, δ, *with* $\gamma + \delta < 1$ *such that* $\forall x \in \Omega$ *and* $\forall \xi, \tau \in \boldsymbol{R}^{n^2N}$

$$\Big\| \sum_i \xi_{ii} - \frac{\alpha}{C(q)} [a(x, \xi + \tau) - a(x, \tau)] \Big\|_N \leqslant \frac{\gamma^\beta}{C(q)} \|\xi\| + \delta^\beta \Big\| \sum_i \xi_{ii} \Big\|_N \tag{5.26}$$

where $\beta = (q + 1)/q$.

The condition (A_q) is more restrictive than the condition (A) because

$$\gamma^\beta < \gamma\,, \qquad \delta^\beta < \delta\,, \qquad C(q) \geqslant 1$$

and hence, if (A_q) holds, then the vector $a(x, \xi)$ is again elliptic.

If the vector $a(x, \xi)$ satisfies (A_q) then, $\forall u$, $v \in H^{2,q} \cap H_0^{1,q}(\Omega)$ and $\forall \varepsilon > 0$, we have

$$\mathfrak{a} = \int_\Omega \|C(q)\Delta u - \alpha[a(x, H(u + v)) - a(x, H(v))]\|_N^q dx \leqslant$$

$$\leqslant \int_\Omega (\gamma^\beta \|H(u)\| + \delta^\beta \|C(q)\Delta u\|)^q dx \leqslant \int_\Omega (1 + \varepsilon)^q \gamma^{\beta q} \|H(u)\|^q + \left(1 + \frac{1}{\varepsilon}\right)^q \delta^{\beta q} \|C(q)\Delta u\|^q dx\,.$$

And, from (5.25), we have

$$\mathfrak{a} \leqslant \left\{(1 + \varepsilon)^q \gamma^{\beta q} + \left(1 + \frac{1}{\varepsilon}\right)^q \delta^{\beta q}\right\} \int_\Omega \|C(q)\Delta u\|^q dx\,.$$

Choosing $\varepsilon = \delta/\gamma$ we get

$$\mathfrak{a} \leqslant (\gamma + \delta)^{q+1} \int_\Omega \|C(q)\Delta u\|^q dx\,. \tag{5.27}$$

Since $(\gamma + \delta)^{q+1} < 1$, the estimate (5.27) proves that $a(x, H(u))$ is near the operator $C(q)\Delta u$, and hence near Δu, considered as operators $H^{2,q} \cap H_0^{1,q}(\Omega) \to L^q(\Omega)$.

It follows from this that

THEOREM 15. – *If the vector $a(x, \xi)$ satisfies the condition (A_q) and if $f \in L^q(\Omega)$, $q > 1$ the solution* u *of the Dirichlet problem* (4.23) *belongs to $H^{2,q} \cap H_0^{1,q}(\Omega)$ and we have the estimate*

$$\|H(u)\|_{L^q(\Omega)} \leqslant \frac{\alpha}{1 - (\gamma + \delta)^\beta} \|f\|_{L^q(\Omega)}. \tag{5.28}$$

6. – Parabolic operators of second order.

Still remaining in the case of a second order differential operator of basic type, the Theorem 9 allows us to study also some other operators which are not elliptic in the sense that they do not satisfy the condition (A). For example, the operators $A = A_1 + A_2$ where A_1 satisfies the condition (A) but A_2 is just monotone with respect to Δ. Or the operator $A = A_1 + A_2$ where A_1 is elliptic in the sense of (A) only with respect to a block of $k < n$ variables.

An example of the first case is given by the operator considered in page 7 of [1]

$$a(x, H(u)) = \Delta u + \frac{2(n-1)}{(n-2)} D_{nn} u, \qquad \text{with } n \geqslant 3. \tag{6.29}$$

This operator does not satisfy the condition (A) and does not satisfy, as is well known, the condition of Cordes. However it is elliptic in the classical sense and consequently is an isomorphism $H^2 \cap H_0^1(\Omega) \to L^2(\Omega)$.

We can reobtain this risult from the Theorem 9. Infact, setting

$$A_1(u) = \Delta u \quad \textit{and} \quad A_2(u) = \frac{2(n-1)}{(n-2)} D_{nn} u$$

A_1 satisfies (A) and $A_2(u)$ is trivially monotone with respect to Δu. Hence, by Theorem 9, the operator (6.29), and more generally, the operator

$$a(x, H(u)) + \frac{1}{\alpha} \frac{2(n-1)}{(n-2)} D_{nn} u$$

with $a(x, \xi)$ elliptic, is an isomorphism $H^2 \cap H_0^1(\Omega) \to L^2(\Omega)$.

An example of the second case is given by the second order, for simplicity, quasi-basic type parabolic operators

$$a(x, t, H(u)) - \frac{\partial u}{\partial t}. \tag{6.30}$$

Let Ω be an open set in $\boldsymbol{R}^n$, of class C^2 and x be a point in it. Let Q be the cylinder $\Omega \times (0, T)$, with $T > 0$, and $X = (x, t)$ be a point in it. Let $a(X, \xi)$ be a vector in $\boldsymbol{R}^N$, which measurable in $X \in Q$, continuous in $\xi \in \boldsymbol{R}^{n^2 N}$, elliptic in ξ (that, is satisfies the condition (A)) and satisfies the condition $a(X, 0) = 0$.

We shall denote by $W_0^{2,1}(Q)$ the real Hilbert space

$$W_0^{2,1}(Q) = \left\{u \in L^2(0, T, H^2 \cap H_0^1(\Omega)) : \frac{\partial u}{\partial t} \in L^2(Q),\ u(x, 0) = 0\right\} \tag{6.31}$$

provided with the norm

$$\|u\|^2_{W_0^{2,1}(Q)} = \int_Q \|H(u)\|^2 + \beta^2 \left\| \frac{\partial u}{\partial t} \right\|^2 dX, \qquad \text{with } \beta > 0\,. \tag{6.32}$$

If Ω is convex or if the vector $a(X, \xi)$ satisfies the more restrictive condition (A_2) introduced in (3.20), then the operator $A_1(u) = a(X, H(u))$ is near the operator Δu, considered as a mapping

$$W_0^{2,1}(Q) \to L^2(Q)$$

with constants α and $K = (\gamma + \delta)$ (see Theorem 12).

Instead, the operator

$$A_2(u) = -\frac{\partial u}{\partial t}$$

$W_0^{2,1}(Q) \to L^2(Q)$ is monotone with respect to Δu (as can easily be checked, see for instance [4]).

It follows now, from Theorem 9, that the operator (6.30) is near the operator $\Delta u - \alpha(\partial u/\partial t)$. We conclude that $\forall f \in L^2(Q)$ the Cauchy-Dirichlet problem

$$u \in W_0^{2,1}(Q), \qquad a(X, H(u)) - \frac{\partial u}{\partial t} = f \quad \text{in } Q \tag{6.33}$$

has a unique solution since this happens, as is well known, for the linear problem

$$u \in W_0^{2,1}(Q), \qquad \Delta u - \alpha \frac{\partial u}{\partial t} = f \quad \text{in } Q\,. \tag{6.34}$$

This theorem, together with some regularity results for the solution u, is proved in [4].

We shall give here a generalization of this considering parabolic operators which do not depend linearly on $\partial u/\partial t$:

$$a(X, H(u)) - b\left(X, u, Du, \frac{\partial u}{\partial t}\right) \tag{6.35}$$

where $B(u) = b(X, u, Du, \partial u/\partial t)$ is a vector in $\boldsymbol{R}^N$, which is measurable in X and continuous in the other variables and with a controlled growth condition, which means that

$$u \in W_0^{2,1}(Q) \Rightarrow B(u) \in L^2(Q).$$

Suppose further that

$$\text{(6.36)} \qquad \begin{aligned} &\text{i) } B(0) = 0, \\ &\text{ii) } -B(u) \text{ is monotone with respect to } \Delta u. \end{aligned}$$

The condition (ii) means that $\forall u,\ v \in W_0^{2,1}(Q)$ we have

$$\text{(6.37)} \qquad -(\Delta(u-v)\,|\,B(u)-B(v))_{L^2(Q)} \geqslant 0.$$

Then, it follows, from Theorem 9, that $a(X, H(u)) - B(u))$ is near the operator $\Delta u - \alpha B(u)$ where α is the constant of nearness of the vector $a(X, \xi)$ to the vector Δu. Infact, $\forall u,\ v \in W_0^{2,1}(Q)$ we have

$$\mathfrak{a} = \|\Delta(u-v) - \alpha[B(u)-B(v)] - \alpha[a(X, H(u)) - a(X, H(v)) - B(u) + B(v)]\|_{L^2(Q)} =$$

$$= \|\Delta(u-v) - \alpha[a(X, H(u)) - a(X, H(v))]\|_{L^2(Q)} \leqslant K\|\Delta(u-v)\|_{L^2(Q)}.$$

Moreover, by the assumption (ii), we have

$$\|\Delta(u-v)\|_{L^2(Q)} \leqslant \|\Delta(u-v) - \alpha[B(u)-B(v)]\|_{L^2(Q)}.$$

We then conclude that $\forall u,\ v \in W_0^{2,1}(Q)$ we have

$$\text{(6.38)} \qquad \mathfrak{a} \leqslant K\|\Delta(u-v) - \alpha[B(u)-B(v)]\|_{L^2(Q)}.$$

Thus we have the following

THEOREM 16. – *If* $\xi \to a(X, \xi)$ *is elliptic and* $B(u)$ *satisfies* (i) *and* (ii), *then* $\forall f \in L^2(Q)$, *the Cauchy-Dirichlet problem*

$$\text{(6.39)} \qquad \begin{cases} u \in W_0^{2,1}(Q), \\ a(X, H(u)) - B(u) = f \quad \textit{in } Q, \end{cases}$$

has a solution (has a unique solution) if the problem

$$\text{(6.40)} \qquad \begin{cases} u \in W_0^{2,1}(Q), \\ \Delta u - \alpha B(u) = f \quad \textit{in } Q. \end{cases}$$

has a solution (has a unique solution).

We can further say that, if $f \equiv 0$, the problems (6.39) and (6.40) have the same solutions, in view of the hypothesis that

$$B(0) = 0.$$

Infact, we can assume $v = 0$ in the nearness relation (6.38) and find that

$\forall u \in W_0^{2,1}(Q)$ we have

$$\|\Delta u - \alpha B(u) - \alpha[a(X, H(u)) - B(u)]\|_{L^2(Q)} \leq K\|\Delta u - \alpha B(u)\|_{L^2(Q)}.$$

The assertion follows from this since $K \in (0, 1)$.

7. – Appendix.

PROOF OF THEOREM 1. – The *only if part follows from the trivial observation that any mapping A: $\mathcal{B} \to \mathcal{B}_1$ is near to itself. It is enough to take in (0.2)*

$$0 < \alpha < 2 \quad \text{and} \quad K = |1 - \alpha|.$$

The proof of the if part is not as simple and is proved in various propositions each of which, it is better to discuss separately.

We recall two lemmas concerning the *transport of the structure.*

LEMMA 1. – *If B: $\mathcal{B} \to \mathcal{B}_1$ is injective then $\mathcal{B}$ is a metric space with the induced metric*

$$d_{\mathcal{B}}(u, v) = \|B(u) - B(v)\|_{\mathcal{B}_1}. \tag{7.41}$$

This is obvious to check.

LEMMA 2. – *If B: $\mathcal{B} \to \mathcal{B}_1$ is bijective then $\mathcal{B}$ with the induced metric* (7.41) *is a complete metric space*

PROOF. – Let $\{u_n\}$ be a Cauchy sequence in $\{\mathcal{B}, d_{\mathcal{B}}\}$, which means that $\{B(u_n)\}$ is a Cauchy sequence in $\mathcal{B}_1$, and hence $\exists\, \mathcal{U} \in \mathcal{B}_1$ such that

$$\|B(u_n) - \mathcal{U}\|_{\mathcal{B}_1} \to 0.$$

Setting $u = B^{-1}(\mathcal{U})$ then $u \in \mathcal{B}$ and

$$d_{\mathcal{B}}(u_n, u) = \|B(u_n) - \mathcal{U}\|_{\mathcal{B}_1} \to 0.$$

This proves that $\{\mathcal{B}, d_{\mathcal{B}}\}$ is complete.

We shall now prove the *if* part of the Theorem 1.

B is injective: It follows, from the condition (2) and since $K \in (0, 1)$, that $\forall u, v \in \mathcal{B}$

$$B(u) - B(v) = 0 \Leftrightarrow A(u) - A(v) = 0$$

and hence A is also injective.

B is bijective: $\forall f \in \mathcal{B}_1$, solving the equation

$$A(u) = f, \qquad u \in \mathcal{B} \tag{7.42}$$

is equivalent to solving the equation

(7.43) $$B(u) = B(u) - \alpha A(u) + \alpha f = F(u), \qquad u \in \mathcal{B}.$$

But, $\forall u \in \mathcal{B}$ we have $F(u) \in \mathcal{B}_1$ and hence $\exists_1 \mathcal{U} = \mathcal{T}(u) \in \mathcal{B}$ such that

(7.44) $$B(\mathcal{U} = F(u)).$$

We thus construct a mapping $\mathcal{T}\colon \mathcal{B} \to \mathcal{B}$ which is a contraction of $\{\mathcal{B}, d_{\mathcal{B}}\}$ into itself. Infact, if $u, v \in \mathcal{B}$ and $\mathcal{U} = \mathcal{T}(u)$, $\mathcal{V} = \mathcal{T}(v)$ then

$$d_{\mathcal{B}}(\mathcal{U}, \mathcal{V}) = \|B(\mathcal{U}) - B(\mathcal{V})\|_{\mathcal{B}_1} = \|F(u) - F(v)\| =$$
$$= \|B(u) - B(v) - \alpha[A(u) - A(v)]\|_{\mathcal{B}_1} \leq K\|B(u) - B(v)\|_{\mathcal{B}_1} = K d_{\mathcal{B}}(u, v).$$

On the other hand, by lemma 2, the metric space $\{\mathcal{B}, d_{\mathcal{B}}\}$ is complete. Hence, by the contraction mapping theorem, $\exists_1 u \in \mathcal{B}$ which solves (7.43), and hence $\exists_1 u \in \mathcal{B}$ which solves (7.42). We have thus proved that A is also bijective.

B is surjective: We define an equivalence relation $\mathcal{R}_{\mathcal{B}}$ on $\mathcal{B}$ by

$$u \mathcal{R}_{\mathcal{B}} v \Leftrightarrow B(u) = B(v).$$

We denote by $\{u\}_{\mathcal{B}}$ the equivalence class of u and let $X = \mathcal{B}/\mathcal{R}_{\mathcal{B}}$.

Define the mappings B^* and A^*, $X \to \mathcal{B}_1$ as follows

$$B^*(\{u\}_{\mathcal{B}}) = B(u)$$

$$A^*(\{u\}_{\mathcal{B}}) = A(u)$$

A^* and B^* are again mappings near to each other with constants α and K and the mapping B^* is bijective. Hence A^* is also bijective, that is, A is surjective.

This last point of the above proof was communicated to me by Dr. A. TARSIA, for which I thank him.

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