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On the Stokes Problem in Lipschitz Domains (*).

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1. - Introduction.

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$ with a Lipschitz continuous boundary $\partial\Omega$. Then for given $f \in W^{-1,q}(\Omega)^n$, $g \in L^q(\Omega)$, $\phi \in W^{1-(1/q),q}(\partial\Omega)^n$, $1 < q < \infty$, satisfying the compatibility condition

$$\int_\Omega g\,dx = \int_{\partial\Omega} \phi \cdot N\,d\sigma, \tag{1.1}$$

we are interested in the existence and uniqueness of a solution pair $(u, p) \in W^{1,q}(\Omega)^n \times L^q(\Omega)$ satisfying $\int_\Omega p\,dx = 0$, the Stokes equations

$$-\Delta u + \nabla p = f, \qquad \operatorname{div} u = g \text{ in } \Omega, \qquad u = \phi \text{ on } \partial\Omega, \tag{1.2}$$

and the inequality

$$\|u\|_{W^{1,q}(\Omega)^n} + \|p\|_{L^q(\Omega)} \leq C(\|f\|_{W^{-1,q}(\Omega)^n} + \|g\|_{L^q(\Omega)} + \|\phi\|_{W^{1-(1/q),q}(\partial\Omega)}) \tag{1.3}$$

where $C = C(\Omega, n, q) > O$ is a constant and N denotes the exterior normal to $\partial\Omega$. Our aim is to prove this property under the assumption that the Lipschitz constant L of the boundary $\partial\Omega$ is sufficiently small (smaller than a constant depending only on n and q); in particular such a pair (u, p) always exists if Ω is a bounded domain with $\partial\Omega \in C^1$. Such a result is well known for a more regular domain. CATTABRIGA [4] proved this result if $\partial\Omega \in C^2$ and $n = 3$; see GALDI-SIMADER [8] for all $n \geq 2$. Concerning the integral equation approach see LADYZENSKAJA [14] and DEURING-VON WAHL-WEIDEMEIER [6]. AMROUCHE-GIRAULT [3] proved the existence and uniqueness of the solution pair (u, p) together with (1.3) under the assumption $\partial\Omega \in C^{1,1}$, their proof rests on the ADN-theory [2], see TEMAM [16], and on Giga's result in [9]. Our assumption $\partial\Omega \in C^{0,1}$

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together with the smallness of L seems to be optimal for general domains. Examples for non-smoothness can be taken from the elliptic theory [10] by setting $p = 0$.

Our hypothesis enables us to solve the Stokes problem above for domains having edges and corners provided the opening angles are close to π. If $q = 2$ the above result on the existence and uniqueness of (u, p) holds for arbitrary Lipschitz domains without smallness condition on L. Furthermore, there are many results on special domains with edges and corners; see KELLOGG-OSBORN [12] for $n = 2$ and DAUGE [5] for $n = 3$.

The method of our proof is selfcontained and rather elementary, it rests on the halfspace result and on localization and perturbation techniques. However, although similar, our method is not completely parallel to that of CATTABRIGA [4]; let us analyze the difference in the basic step where the smoothness of $\partial\Omega$ is involved. Cattabriga's assumption $\partial\Omega \in C^2$ is really needed only in the case $1 < q < 2$ where the uniqueness property that $u \in W_0^{1,q}(\Omega)^n$, $p \in L^q(\Omega)$, $\int_\Omega p\,dx = 0$, $-\Delta u + \nabla p = 0$, $\operatorname{div} u = 0$ implies $u = 0$, $p = 0$ is not as trivial as for $q \geq 2$. This uniqueness assertion is needed for proving the a priori estimate (1.3) by localization and compactness arguments. In order to prove this uniqueness result, Cattabriga improves the regularity of (u, p) above by the second order derivatives of u which requires the C^2-regularity of $\partial\Omega$; this leads to $\nabla u \in L^2$ and $u = 0$. Instead of Cattabriga's argument we use a regularity property for the localized equations which enables us to consider two different exponents q and s simultaneously, see Section 3. The localized equations can be considered as equations on the halfspace or the «bended» halfspace. So we have first to treat the Stokes problem in these unbounded domains.

SOME NOTATIONS. - Let $1 < q < \infty$ and let q' be defined by $1/q + 1/q' = 1$. We use the Lebesgue space $L^q(\Omega)$ with norm $\|u\|_{L^q(\Omega)} = \|u\|_q$ and the usual Sobolev spaces $W^{1,q}(\Omega)$ and $W_0^{1,q}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\nabla u\|_q}$ where $C_0^\infty(\Omega)$ means the space of all smooth functions having a compact support in Ω. The norm in $W^{1,q}(\Omega)$ is given by $\|u\|_{W^{1,q}(\Omega)} = \|u\|_{L^q(\Omega)} + \|\nabla u\|_{L^q(\Omega)}$ where $\|\nabla u\|_{L^q(\Omega)} = (\|\partial_1 u\|_q^q + \ldots + \|\partial_n u\|_q^q)^{1/q}$, $\nabla u = (\partial_1 u, \ldots, \partial_n u)$, $\partial_\nu = \partial/\partial x_\nu$ $(\nu = 1, \ldots, n)$. Furthermore we denote $\operatorname{div} u = \partial_1 u_1 + \ldots + \partial_n u_n$ and $\Delta u = \partial_1^2 u + \ldots + \partial_n^2 u$. Let $L^q(\Omega)^n$, $C_0^\infty(\Omega)^n, \ldots$ be the corresponding spaces of vector fields $u = (u_1, \ldots, u_n)$. Let

$$W^{-1,q}(\Omega) = [W_0^{1,q'}(\Omega)]^*$$

denote the dual space of $W_0^{1,q'}(\Omega)$. For a functional $f\colon v \to [f, v]$ from $W^{-1,q}(\Omega)$ the norm is defined by

$$\|f\|_{W^{-1,q}(\Omega)} = \sup_{0 \neq v \in C_0^\infty(\Omega)} |[f, v]| / \|\nabla v\|_{q'}.$$

The usual trace space $W^{1-(1/q),q}(\partial\Omega)$ is well defined if $\partial\Omega$ is Lipschitz continuous N denotes the outward normal vector to $\partial\Omega$ and $\int_{\partial\Omega} \ldots d\sigma$ the surface integral.

$u|_{\partial\Omega} \in W^{1-(1/q),q}(\partial\Omega)$ means the trace of $u \in W^{1,q}(\Omega)$. Let $\langle u, v\rangle = \int_\Omega u\cdot v\,dx$ denote the $L^q - L^{q'}$-pairing for scalar fields, vector fields or matrices.

A bounded domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, is called a Lipschitz domain with Lipschitz constant $L > 0$ if the following is true.

To each $x \in \partial\Omega$ there exists an open ball B with center x and a function $\omega: D \to \mathbb{R}$ on some domain $D \subseteq \mathbb{R}^{n-1}$ such that (sfter some appropriate rotation and translation of the coordinate system depending on x) it holds

$$\operatorname{graph}\omega = B \cap \partial\Omega\,, \qquad \{(y', y_n) \in \mathbb{R}^n : y' \in D,\ \delta + \omega(y') \geq y_n \geq \omega(y')\} \subseteq \mathbb{R}^n \setminus \Omega$$

$$\text{for some } \delta > 0,\ |\omega(y_2') - \omega(y_1')| \leq L|y_2' - y_1'| \qquad \text{for all } y_2', y_1' \in D\,.$$

The function ω with the latter property is called a Lipschitz function. Let us put $x = (x', x_n)$ and $\nabla = (\nabla', \partial_n)$ where $x' = (x_1, \ldots, x_{n-1})$, $\nabla' = (\partial_1, \ldots, \partial_{n-1})$. It is well known that $\omega: D \to \mathbb{R}$ is a Lipschitz function if and only if ω is continuous and $\nabla'\omega \in L^\infty(D)^{n-1}$ in the sense of distribution. We get $|\omega(y_2') - \omega(y_1')| \leq$ $\leq (\|\nabla'\omega\|_{L^\infty(D)^{n-1}})|y_2' - y_1'|$, $y_2', y_1' \in D$, for such a function.

Within the proofs we use positive constants $C, C_1, \ldots$ which may change from line to line.

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2. – Main theorem and preliminary results.

Our main theorem reads as follows

2.1. THEOREM. – *Let $1 < q < \infty$ and let $\Omega \subseteq \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain of class C^1 or a bounded Lipschitz domain with sufficiently small Lipschitz constant $L > 0$ (i.e. $L \leq M$ where $M = M(n, q) > 0$ is a constant depending only on n, q). Then for each given $f \in W^{-1,q}(\Omega)^n$, $g \in L^q(\Omega)$ and $\phi \in W^{1-(1/q),q}(\partial\Omega)^n$ satisfying $\int_\Omega g\,dx =$ $= \int_{\partial\Omega} N\cdot\phi\,d\sigma$, there exists a unique pair $(u, p) \in W^{1,q}(\Omega)^n \times L^q(\Omega)$ such that $\int_\Omega p\,dx = 0$ and*

$$-\Delta u + \nabla p = f, \qquad \operatorname{div} u = g\,, \qquad u|_{\partial\Omega} = \phi\,. \tag{2.2}$$

Moreover, this pair is subject to the inequality

$$\|u\|_{W^{1,q}(\Omega)^n} + \|p\|_{L^q(\Omega)} \leq C(\|f\|_{W^{-1,q}(\Omega)^n} + \|g\|_{L^q(\Omega)} + \|\phi\|_{W^{1-(1/q),q}(\partial\Omega)})\,. \tag{2.3}$$

where $C = C(\Omega, n, q) > 0$ is a constant.

REMARKS. - *a*) Let $\widetilde{L}^q(\Omega) = \left\{v \in L^q(\Omega): \int_\Omega v\,dx = 0\right\}$ and define the operator

$$S_q: W_0^{1,q}(\Omega)^n \times \widetilde{L}^q(\Omega) \to W^{-1,q}(\Omega)^n \times \widetilde{L}^q(\Omega) \tag{2.4}$$

defined by

$$S_q(u, p) = (\langle \nabla u, \nabla \cdot \rangle - \langle p, \operatorname{div} \cdot \rangle, -\operatorname{div} u)$$

with the functional $\langle \nabla u, \nabla \cdot \rangle - \langle p, \operatorname{div} \cdot \rangle: v \to \langle \nabla u, \nabla v \rangle - \langle p, \operatorname{div} v \rangle$. Obviously, S_q is bounded and the dual operator

$$S_q^*: W_0^{1,q'}(\Omega)^n \times \widetilde{L}^{q'}(\Omega) \to W^{-1,q'}(\Omega) \times \widetilde{L}^{q'}(\Omega)$$

coincides with $S_{q'}$; we get

$$S_q^* = S_{q'} \tag{2.5}$$

which is a consequence of the symmetry property

$$\langle \nabla u, \partial v \rangle - \langle p, \operatorname{div} v \rangle - \langle \operatorname{div} u, h \rangle = \langle \nabla v, \nabla u \rangle - \langle h, \operatorname{div} u \rangle - \langle \operatorname{div} v, p \rangle$$

for all $(u, p) \in W_0^{1,q}(\Omega)^n \times \widetilde{L}^q(\Omega)$, $(v, h) \in W_0^{1,q'}(\Omega)^n \times \widetilde{L}^{q'}(\Omega)$.

Then the abstract formulation of Theorem 2.1 for $\phi = 0$ means:

$$S_q \quad \text{is an isomorphism}. \tag{2.6}$$

b) It is well known and easy to prove that the Lipschitz constant L of each C^1-domain can be chosen arbitrarily small. For this purpose we have to choose the balls B in the definition of the Lipschitz domain sufficiently small. Therefore, the assertion for C^1-domains in Theorem 2.1 is a corollary of the assertion for Lipschitz domains.

The proof of Theorem 2.1 rests on localization arguments by which the assertion is reduced to the corresponding results for the whole space $\mathbb{R}^n$, the half space

$$H = \mathbb{R}^n_- = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n: x_n < 0\}$$

and the bended halfspace

$$H_\omega = \{x = (x', x_n) \in \mathbb{R}^n: x_n < \omega(x')\}, \tag{2.7}$$

where $\omega: \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function. We only need the case that supp ω is compact which means that H_ω behaves like H for large $|x|$.

For the unbounded domains $\Omega = \mathbb{R}^n$, $= H$ or $= H_\omega$ the solution space $W_0^{1,q}(\Omega)^n \times \widetilde{L}^q(\Omega)$ above is too small to prove an existence and uniqueness result. Therefore, in this cases we define

$$\widehat{W}_0^{1,q}(\Omega)^n = \overline{C_0^\infty(\Omega)^n}^{\|\nabla v\|_q} \tag{2.8}$$

being the completion of $C_0^\infty(\Omega)^n$ under the Dirichlet norm $\|\nabla v\|_q$. If $\partial\Omega \neq \emptyset$ we can identify each Cauchy sequence (u_i) in $C_0^\infty(\Omega)^n$ with respect to $\|\nabla v\|_q$ with that element

$u \in L^q_{\mathrm{loc}}(\Omega)^n$ having the properties $\nabla u \in L^q(\Omega)^{n^2}$, $\lim\limits_i \nabla u_i = \nabla u$ in $L^q(\Omega)^{n^2}$ and $u = \lim\limits_i u_i$ in $L^q_{\mathrm{loc}}(\Omega)^n$, since we may use Poincare's inequality near the boundary $\partial\Omega$. The same is possible for $\Omega = \mathbb{R}^n$ if $1 < q < n$ by using Sobolev's inequality. However if $\Omega = \mathbb{R}^n$ and $q > n$, such a Cauchy sequence (u_i) need not converge in $L^q_{\mathrm{loc}}(\mathbb{R}^n)^n$ but there exist constants c_i such that this is true for $(u_i + c_i)$ and so we can identify (u_i) in this case with a class in $L^q_{\mathrm{loc}}(\mathbb{R}^n)^n$ whose elements differ by a constant. For simplicity we will consider $\widehat{W}_0^{1,q}(\mathbb{R}^n)^n$ for all $1 < q < \infty$ as a space of such classes; if $1 < q < n$ we find for each $u \in \widehat{W}_0^{1,q}(\mathbb{R}^n)^n$ a representative in $L^s(\mathbb{R}^n)^n$ where $1/n + 1/s = 1/q$, see [8], [13] for details.

Let

$$\widehat{W}^{-1,q}(\Omega)^n = [\widehat{W}_0^{1,q'}(\Omega)^n]^* \tag{2.9}$$

be the dual space of $\widehat{W}_0^{1,q'}(\Omega)^n$, $1/q + 1/q' = 1$. The norm of some functional $f\colon v \to [f, v]$ from $\widehat{W}^{-1,q}(\Omega)^n$ is given by

$$\|f\|_{\widehat{W}^{-1,q}(\Omega)^n} = \sup_{0 \neq v \in C_0^\infty(\Omega)^n} |[f, v]| / \|\nabla v\|_{q'} . \tag{2.10}$$

The space $\widehat{W}_0^{1,q}(\Omega)^n$ is isometric to some closed subspace of $L^q(\Omega)^{n^2}$ and therefore reflexive, we get $[\widehat{W}^{-1,q}(\Omega)^n]^* = \widehat{W}_0^{1,q'}(\Omega)^n$. If $\partial\Omega \neq \emptyset$ or if $\Omega = \mathbb{R}^n$ and $1 < q' < n$ (i.e. $q > n/(n-1)$), each $f \in C_0^\infty(\Omega)^n$ defines the functional $v \to \langle f, v\rangle = \int_\Omega f \cdot v\, dx$ which is identified with f. If $\Omega = \mathbb{R}^n$, $q' \geq n$ (i.e. $1 < q \leq n/(n-1)$) we must additionally suppose that $\int_\Omega f dx = 0$ to get a well defined functional $\langle f, \cdot\rangle \in \widehat{W}^{-1,q}(\Omega)^n$ since f must be zero on the class consisting of constants. The space of these functionals $\langle f, \cdot\rangle$ is a dense subspace of $\widehat{W}^{-1,q}(\Omega)^n$ in all cases; indeed, $\langle f, v\rangle = 0$ for all such f and given $v \in \widehat{W}_0^{1,q'}(\Omega)^n$ implies $v = 0$.

Using the Hahn-Banach theorem, for each $f \in \widehat{W}^{-1,q}(\Omega)^n$ we can find a matrix $F = (F_{ij}) \in L^q(\Omega)^{n^2}$ such that $[f, v] = \langle F, \nabla v\rangle$ for all $v \in \widehat{W}_0^{1,q'}(\Omega)^n$. Moreover, F can be chosen such that $\|F\|_q = \|f\|_{\widehat{W}^{-1,q}(\Omega)^n}$. Of course, each $f \in \widehat{W}^{-1,q}(\Omega)^n$ yields a well defined distribution on Ω being identified with f; we obtain $f = -\operatorname{div} F$ in the sense of distributions where $\operatorname{div} F = (\partial_1 F_{i1} + \ldots + \partial_n F_{in})_{i=1,\ldots,n}$.

Our main result on $\mathbb{R}^n$, $\mathbb{R}^n_-$ and H_ω reads as follows. The equation $-\Delta u + \nabla p = f$ is understood in the sense of distributions.

2.3. LEMMA. – *Let $1 < q < \infty$, $n \geq 2$ and let $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n_-$. Then for each $f \in \widehat{W}^{-1,q}(\Omega)^n$ and $g \in L^q(\Omega)$ there exists a unique pair $(u, p) \in \widehat{W}_0^{1,q}(\Omega)^n \times L^q(\Omega)$ satisfying*

$$-\Delta u + \nabla p = f, \qquad \operatorname{div} u = g . \tag{2.11}$$

Moreover it holds

$$\|\nabla u\|_{L^q(\Omega)^{n^2}} + \|p\|_{L^q(\Omega)} \leq C(\|f\|_{\widehat{W}^{-1,q}(\Omega)^n} + \|g\|_{L^q(\Omega)}) \tag{2.12}$$

where $C = C(\Omega, q) > 0$ is a constant.

Furthermore, suppose additionally $1 < s < \infty$, $g \in L^s(\Omega)$ *and*

$$\sup_{0 \neq v \in C_0^\infty(\Omega)^n} (|[f, v]| / \|\nabla v\|_{L^{s'}(\Omega)^{n^2}}) < \infty . \tag{2.13}$$

Then $(u, p) \in \widehat{W}_0^{1,s}(\Omega)^n \times L^s(\Omega)$.

2.4. LEMMA. - *Let* $1 < q < \infty$, $1 < s < \infty$, $n \geq 2$, *let* $\omega: \mathbb{R}^{n-1} \to \mathbb{R}$ *be a Lipschitz function with compact support and let* $L = \|\nabla' \omega\|_{L^\infty(\mathbb{R}^{n-1})^{n-1}}$ *be sufficiently small (i.e.* $L \leq K$ *where* $K = K(n, q, s) > 0$ *is a constant).*

Then for each $f \in \widehat{W}^{-1,q}(H_\omega)^n$ *and* $g \in L^q(H_\omega)$ *there exists a unique pair* $(u, p) \in \widehat{W}_0^{1,q}(H_\omega)^n \times L^q(H_\omega)$ *satisfying*

$$-\Delta u + \nabla p = f, \qquad \operatorname{div} u = g . \tag{2.14}$$

Moreover it holds

$$\|\nabla u\|_{L^q(H_\omega)^{n^2}} + \|p\|_{L^q(H_\omega)} \leq C(\|f\|_{\widehat{W}^{-1,q}(H_\omega)^n} + \|g\|_{L^q(H_\omega)}) \tag{2.15}$$

where $C = C(\omega, n, q, s) > 0$ *is a constant.*

If additionally $g \in L^s(H_\omega)$ *and*

$$\sup_{0 \neq v \in C_0^\infty(H_\omega)^n} (|[f, v]| / \|\nabla v\|_{L^{s'}(H_\omega)^{n^2}}) < \infty , \tag{2.16}$$

then $(u, p) \in \widehat{W}_0^{1,s}(H_\omega) \times L^s(H_\omega)$.

The proofs are given in the next sections. Observe that the boundary conditions $u|_{\partial H} = 0$ and $u|_{\partial H_\omega} = 0$ are implicity contained in $u \in \widehat{W}_0^{1,q}(H)^n$ and $u \in \widehat{W}_0^{1,q}(H_\omega)^n$, respectively.

3. - The whole space $\mathbb{R}^n$, the halfspace $\mathbb{R}^n$ and the bended halfspace H_ω; proof of Lemmas 2.3 and 2.4.

PROOF OF LEMMA 2.3 FOR $\Omega = \mathbb{R}^n$. - First we assume $f \in C_0^\infty(\mathbb{R}^n)^n$ and $g \in C_0^\infty(\mathbb{R}^n)$ where we identify f with the functional $[f, \cdot]$: $v \to [f, v] = \langle f, v \rangle = \int_{\mathbb{R}^n} fv\,dx$. If $1 < q \leq n/(n-1)$ we must additionally suppose that $\int_{\mathbb{R}^n} f dx = 0$.

In this case a (smooth) solution of $-\Delta u + \nabla p = f$, $\operatorname{div} u = g$ can be sought of the form

$$u = u_1 + u_2 + h, \qquad p = p_1 + p_2 \tag{3.1}$$

where

$$h = \nabla(E * g), \qquad u_1 = E * f, \qquad u_2 = U * \nabla h,$$

$$p_1 = -Q * f, \qquad p_2 = -Q * \nabla h .$$

Here $*$ means the convolution, while E and $U = (U_{ij})$, $Q = (Q_i)$ are the fundamental solutions of the Laplacian and Stokes equations, respectively, namely

$$E(x) = \begin{cases} c_1(n)|x|^{2-n} & \text{if } n \geq 3, \\ (2\pi)^{-1}\log|x| & \text{if } n = 2, \end{cases}$$

$$U_{ij}(x) = \begin{cases} c_2(n)[\delta_{ij}|x|^{2-n} + (n-2)|x|^{-n}x_i x_j] & \text{if } n \geq 3, \\ (4\pi)^{-1}[\delta_{ij}(\log|x|) - |x|^{-2}x_i x_j] & \text{if } n = 2, \end{cases}$$

$$Q_i(x) = \begin{cases} c_3(n)|x|^{-n}x_i & \text{if } n \geq 3, \\ (2\pi)^{-1}|x|^{-2}x_i & \text{if } n = 2, \end{cases}$$

$c_1(n)$, $c_2(n)$, $c_3(n)$ being constants depending only on n.

A repeated use of the Calderon-Zygmund theorem on singular integrals, see [8; p. 302], then leads to $u \in \widehat{W}_0^{1,r}(\mathbb{R}^n)^n$, $p \in L^r(\mathbb{R}^n)$ and

$$\|\nabla u\|_{L^r(\mathbb{R}^n)^{n^2}} + \|p\|_{L^r(\mathbb{R}^n)} \leq C_1(\|f\|_{\widehat{W}^{-1,r}(\mathbb{R}^n)} + \|g\|_{L^r(\mathbb{R}^n)}) \tag{3.2}$$

for all $1 < r < \infty$, where $C_1 = C_1(n, r) > 0$ is a constant.

Assume now, f and g satisfy the assumptions of the theorem. Then the density property of $C_0^\infty(\mathbb{R}^n)^n$ in $\widehat{W}^{-1,q}(\mathbb{R}^n)^n$ explained above and the estimate (3.2) with $q = r$ yield the existence of a pair $(u, p) \in \widehat{W}_0^{1,q}(\mathbb{R}^n)^n \times L^q(\mathbb{R}^n)$ satisfying (2.11). From (2.13) we know that $f|_{C_0^\infty(\mathbb{R}^n)^n}$ extends by continuity to a functional $\tilde{f} \in \widehat{W}^{-1,s}(\mathbb{R}^n)^n$. Therefore, the same argument as above yields a pair $(\tilde{u}, \tilde{p}) \in \widehat{W}_0^{1,s}(\mathbb{R}^n)^n \times L^s(\mathbb{R}^n)$ satisfying $-\Delta\tilde{u} + \nabla\tilde{p} = \tilde{f}$, $\operatorname{div}\tilde{u} = g$. We will show that $u = \tilde{u}$ and $p = \tilde{p}$; this proves $u \in \widehat{W}_0^{1,s}(\mathbb{R}^n)^n$, $p \in L^s(\mathbb{R}^n)$ and also the uniqueness of (u, p) by choosing $q = s$. For this purpose we put $w = u - \tilde{u}$ and $\psi = p - \tilde{p}$. Then we get

$$\langle\nabla w, \nabla v\rangle - \langle\psi, \operatorname{div} v\rangle = 0 \quad \text{for all } v \in C_0^\infty(\mathbb{R}^n)^n \tag{3.3}$$

and $\operatorname{div} w = 0$. Setting in particular $v = \nabla\chi$, $\chi \in C_0^\infty(\mathbb{R}^n)$ we conclude from (3.3) that

$$\langle\psi, \Delta\chi\rangle = 0 \quad \text{for all } \chi \in C_0^\infty(\mathbb{R}^n).$$

Then Weyl's lemma yields that ψ is harmonic on $\mathbb{R}^n$, it holds $\psi \in C^\infty(\mathbb{R}^n)$ and $\Delta\psi = 0$. The mean value property for harmonic functions leads to

$$\psi(x) = c_2(n)R^{-n}\int_{B_R(x)}(p(y) - \tilde{p}(y))\,dy \tag{3.4}$$

where $c_2(n)$ depends only on n; we set $B_R(x) = \{y \in \mathbb{R}^n : |x - y| < R\}$, $R > 0$. Applying Hölder's inequality yields

$$|\psi(x)| \leq c_3 R^{-n}\left[R^{n/q'}\left(\int_{B_R(x)}|p(y)|^q\,dy\right)^{1/q} + R^{n/s'}\left(\int_{B_R(x)}|\tilde{p}(y)|^s\,dy\right)^{1/s}\right]$$

with $c_3 = c_3(n, q, s)$. Recalling $p \in L^q(\mathbb{R}^n)$, $\tilde{p} \in L^s(\mathbb{R}^n)$ and letting $R \to \infty$ we obtain $\psi = 0$. Now from (3.3) we get that ∇w is harmonic and the same argument as for ψ yields $\Delta w = 0$. Therefore, w is constant and so zero as element in $\widehat{W}_0^{1,q}(\mathbb{R}^n)^n$. This proves the assertion of Lemma 2.3 for $\Omega = \mathbb{R}^n$.

In the next proof we use the notation

$$C_0^\infty((\bar{\Omega})) = \{u|_\Omega : u \in C_0^\infty(\mathbb{R}^n)\} \tag{3.5}$$

for unbounded domains $\Omega \subseteq \mathbb{R}^n$.

PROOF OF LEMMA 2.3 FOR $\Omega = \mathbb{R}^n_-$. – For the halfspace $H = \mathbb{R}^n_-$ we first assume that $f \in C_0^\infty(\overline{H})^n$ and $g \in C_0^\infty(\overline{H})$ where the identify f with the functional $v \to \langle f, v\rangle$. We consider extensions to C_0^∞-functions on $\mathbb{R}^n$ which are again denoted by f, g, respectively. Following [4; p. 323] we then look for a (smooth) solution to (2.11) of the form

$$u = \hat{u} + W, \qquad p = \hat{p} + S$$

where $\hat{u} = (\hat{u}_i)_{i=1,\dots,n}$ ia defined by $\hat{u}_i = h_i + \sum_{j=1}^n U_{ij} * (f_j - \partial_j g)$ with h and U_{ij} as before, $W = (W_i)$ by

$$W_i(x) = \sum_{j=1}^n \left[\int_{\partial H} K_{ij}(x' - y', x_n) A_j(y', 0)\, dy' - \int_{\partial H} K_{ij}(x' - y', x_n) h_j(y', 0)\, dy' \right]$$

with $A_i = \sum_{j=1}^n U_{ij} * (f_j - \partial_j g)$, $x = (x', x_n)$, $\hat{p}$ by

$$\hat{p} = -\sum_{j=1}^n Q_j * (f_j - \partial_j g),$$

$$S(x) = \sum_{j=1}^n \left[\partial_j \int_{\partial H} k(x' - y', x_n) A_j(y', 0)\, dy' + \int_{\partial H} k(x' - y', x_n) h_j(y', 0)\, dy' \right],$$

and

$$K_{ij}(x' - y', x_n) = c_1(n) \frac{x_n (x_i - y_i)(x_j - y_j)}{(|x' - y'|^2 + x_n^2)^{(n+2)/2}}, \qquad y_n = 0,$$

$$k(x' - y', x_n) = c_2(n) \frac{x_n}{(|x' - y'|^2 + x_n^2)^{n/2}}, \qquad y_n = 0.$$

Following the proof given in [4; pp. 323-326] which is based on a well known variant of the Calderon-Zygmund theorem [2; Theorem 3.3], see also [8], we obtain $u \in \widehat{W}_0^{1,r}(H)^n$, $p \in L^r(H)$ and

$$\|\nabla u\|_{L^r(H)^{n^2}} + \|p\|_{L^r(H)} \leq C_1(\|f\|_{\widehat{W}^{-1,q}(H)^n} + \|g\|_{L^q(H)}) \tag{3.6}$$

for all $1 < r < \infty$ where $C_1 = C_1(n, r) > 0$. Using the density of $C_0^\infty(H)^n$ in $\widehat{W}^{-1,q}(H)^n$ and that of $C_0^\infty(H)$ in $L^q(H)$ we get the existence of a pair $(u, p) \in \widehat{W}_0^{1,q}(H)^n \times L^q(H)$

satisfying (2.11) and (2.12). Using (2.13) we find in the same way a pair $(\tilde{u}, \tilde{p}) \in \widehat{W}_0^{1,s}(H)^n \times L^s(H)$ satisfying (2.11) and (2.12) with q replaced by s. To show $u = \tilde{u}$, $p = \tilde{p}$ we put $w = u - \tilde{u}$, $\psi = p - \tilde{p}$ and get $\operatorname{div} w = 0$ and

$$\langle \nabla w, \nabla v \rangle - \langle \psi, \operatorname{div} v \rangle = 0$$

even for all $v \in \widehat{W}_0^{1,q'}(H)^n \cap \widehat{W}_0^{1,s'}(H)^n$. Now we take $F \in C_0^\infty(H)^n$ and construct a solution pair (v, χ) of (2.11) with f replaced by F and $g = 0$. Thus we get $\operatorname{div} v = 0$, $v \in \widehat{W}_0^{1,r}(H)^n$, $\chi \in L^r(H)$ for all $1 < r < \infty$ and

$$\langle \nabla v, \nabla v^* \rangle - \langle \chi, \operatorname{div} v^* \rangle = \langle F, v^* \rangle$$

even for all $v^* = v_1^* + v_2^*$ with $v_1^* \in \widehat{W}_0^{1,q}(H)^n$, $v_2^* \in \widehat{W}_0^{1,s}(H)^n$. Setting $w = v^*$ yields

$$\langle F, w \rangle = \langle \nabla v, \nabla w \rangle - \langle \chi, \operatorname{div} w \rangle = \langle \nabla w, \nabla v \rangle - \langle \chi, \operatorname{div} v \rangle = 0$$

for all $F \in C_0^\infty(H)^n$ which leads to $w = 0$. This yields $\langle \psi, \operatorname{div} v \rangle = 0$ for all $v \in C_0^\infty(H)^n$ which shows that $\Delta\psi = 0$. Since $\psi = p - \tilde{p}$, ψ cannot be a constant unless it is zero.

From $u = \tilde{u}$, $p = \tilde{p}$ we conclude $(u, p) \in \widehat{W}_0^{1,s}(H)^n \times L^s(H)$. The uniqueness of (u, p) follows by considering $q = s$ in the calculation above. This completes the proof of Lemma 2.3.

PROOF OF LEMMA 2.4. – We show that the operator S_q in (2.4) for H_ω after some transformation of the coordinates differs from the corresponding operator for $H = \mathbb{R}_-^n$ only by a «small» perturbation. Then Lemma 2.4 will be proved by applying Kato's perturbation criterion in the following formulation.

3.1. LEMMA (KATO [11]). – *Consider Banach spaces X, Y and two bounded linear operators A and B from X to Y. Suppose A has a bounded inverse from Y to X and*

$$\|Bv\| \leq C\|Av\|, \qquad v \in X$$

with a constant satisfying $0 < C < 1$. *Then* $A + B\colon X \to Y$ *is bijective with a bounded inverse.*

The proof of this lemma is easy, If I denotes the identity in Y we get $\|BA^{-1}\| \leq \leq C < 1$ for the operator norm and writing $A + B = (I + BA^{-1})A$ we get the desired operator

$$(A + B)^{-1} = [(I + BA^{-1})A]^{-1} = A^{-1} \sum_{\nu=0}^{\infty} (-1)^\nu (BA^{-1})^\nu .$$

Going back to the proof of Lemma 2.4, we first define the transformation $x \to \hat{x}$ from H_ω to H by $\hat{x}_1 = x_1, \dots, \hat{x}_{n-1} = x_{n-1}$, $\hat{x}_n = x_n - \omega(x')$. We write $x = (x', x_n) \in H_\omega$, $\hat{x} = (\hat{x}', \hat{x}_n) \in H$, $\nabla = (\nabla', \partial_n)$ with $\partial_i = \partial/\partial x_i$, $\hat{\nabla} = (\hat{\nabla}', \hat{\nabla}_n)$ with $\hat{\partial}_i = \partial/\partial \hat{x}_i$, $i = = 1, \dots, n$. If the functions u, $\hat{u}$ are related by $u(x) = \hat{u}(\hat{x})$ we obtain a transformation

$u \to \widehat{u}$ of functions defined on H_ω to functions defined on H. We obtain

$$
\begin{aligned}
&(\partial_i u)(x) = ((\widehat{\partial}_i - \omega_i \widehat{\partial}_n)\widehat{u})(\widehat{x}) \quad \text{for } 1, \dots, n-1,\\
&(\widehat{\partial}_i \widehat{u})(\widehat{x}) = ((\partial_i - \omega_i \partial_n) u)(x) \quad \text{for } 1, \dots, n-1,\\
&(\partial_n u)(x) = (\widehat{\partial}_n \widehat{u})(\widehat{x})
\end{aligned}
\tag{3.7}
$$

where $\omega_i = \partial_i \omega$ $(i = 1, \dots, n-1)$. Correspondingly we get a transformation $f \to \widehat{f}$ of functionals for H_ω to functionals for H by setting $[f, v] = [\widehat{f}, \widehat{v}]$. In this proof we distinguish $\|\cdot\|$, $\langle\cdot,\cdot\rangle$ by writing $\|\cdot\|_{q, H_\omega}$, $\|\cdot\|_{q, H}$, $\langle\cdot,\cdot\rangle_{H_\omega}$ and $\langle\cdot,\cdot\rangle_H$. Observe that the Jacobian of the transformation $x \to \widehat{x}$ is one.

Furthermore we write $u = (u', u_n)$, $\widehat{u} = (\widehat{u}', \widehat{u}_n)$ and $\widehat{\operatorname{div}}\, \widehat{u} = \widehat{\partial}_1 \widehat{u}_1 + \dots + \widehat{\partial}_n \widehat{u}$. Using (3.7) we easily get

$$
\|\nabla u\|_{q, H_\omega} \leq C_1 \|\widehat{\nabla} \widehat{u}\|_{q, H} \leq C_2 \|\nabla u\|_{q, H_\omega}
\tag{3.8}
$$

for all $u \in \widehat{W}_0^{1,q}(H_\omega)^n$ and correspondingly with q replaced by s; C_1, C_2 are constants only depending on ω, q. The transformation $u \to \widehat{u}$ yields an isomorphism from $\widehat{W}_0^{1,q}(H_\omega)^n$ to $\widehat{W}_0^{1,q}(H)^n$. Correspondingly, $f \to \widehat{f}$ yields an isomorphism from $\widehat{W}^{-1,q}(H_\omega)^n$ to $\widehat{W}^{-1,q}(H)^n$.

Next we transform the equations (2.14) from H_ω to H. An elementary calculation yields

$$
\begin{aligned}
\langle \nabla u, \nabla v\rangle_{H_\omega} - \langle p, \operatorname{div} v\rangle_{H_\omega} &= \langle \widehat{\nabla} \widehat{u}, \widehat{\nabla} \widehat{v}\rangle_H - \langle \widehat{p}, \widehat{\operatorname{div}}\, \widehat{v}\rangle_H - \langle \widehat{\nabla}' \widehat{u}, (\widehat{\nabla}\omega)\widehat{\partial}_n \widehat{v}\rangle_H -\\
&- \langle (\widehat{\nabla}'\omega)\widehat{\partial}_n \widehat{u}, \widehat{\nabla}' \widehat{v}\rangle_H + \langle (\widehat{\nabla}'\omega)\widehat{\partial}_n \widehat{u}, (\widehat{\nabla}'\omega)\widehat{\partial}_n \widehat{v}\rangle_H + \langle \widehat{p}, (\widehat{\nabla}'\omega)(\widehat{\partial}_n \widehat{v}')\rangle_H,
\end{aligned}
$$

and $\operatorname{div} u = \widehat{\operatorname{div}}\, \widehat{u} - (\widehat{\nabla}'\omega)\cdot(\widehat{\partial}_n \widehat{u}')$.

The abstract formulation for the first assertion in Lemma 2.4 means that the operator $S_q \colon \widehat{W}_0^{1,q}(H_\omega)^n \times L^q(H_\omega) \to \widehat{W}^{-1,q}(H_\omega)^n \times L^q(H_\omega)$ defined by

$$
S_q(u, p) = (\langle \nabla u, \nabla\cdot\rangle_{H_\omega} - \langle p, \operatorname{div}\cdot\rangle_{H_\omega}, -\operatorname{div} u)
$$

is an isomorphism. Let $S_{q,H}$ be the corresponding operator with H_ω replaced by H. The calculation above shows the following representation

$$
S_q(u, p) = S_{q,H}(\widehat{u}, \widehat{p}) + B(\widehat{u}, \widehat{p})
\tag{3.9}
$$

where the perturbation $B(\widehat{u}, \widehat{p})$ is given by

$$
\begin{aligned}
B(\widehat{u}, \widehat{p}) = (- \langle \widehat{\nabla}' \widehat{u}, (\widehat{\nabla}'\omega)\widehat{\partial}_n \cdot\rangle_H - \langle \widehat{\nabla}' \widehat{\partial}_n \widehat{u}, \widehat{\nabla}' \cdot\rangle_H +\\
+ \langle (\widehat{\nabla}'\omega)\widehat{\partial}_n \widehat{u}, (\widehat{\partial}'\omega)\widehat{\partial}_n \cdot\rangle_H + \langle \widehat{p}, (\widehat{\nabla}'\omega)(\widehat{\partial}_n \cdot)\rangle_H, -(\widehat{\nabla}'\omega)(\widehat{\partial}_n \widehat{u}')).
\end{aligned}
$$

This expression yields

$$
\|B(\widehat{u}, \widehat{p})\|_{\widehat{W}^{-1,q}(H)^n \times L^q(H)} \leq C_1 \|\nabla'\omega\|_\infty (\|\widehat{\nabla}\widehat{u}\|_{q,H} + \|\widehat{p}\|_{q,H}) + C_2 \|\nabla'\omega\|_\infty \|\widehat{\nabla}\widehat{u}\|_{q,H},
$$

setting $(f, g) = S_{q,H}(\widehat{u}, \widehat{p})$ in (2.12) leads to

$$
\|\widehat{\nabla}\widehat{u}\|_{q,H} + \|\widehat{p}\|_{q,H} \leq C_3 \|S_{q,H}(\widehat{u}, \widehat{p})\|_{\widehat{W}^{-1,q}(H)^n \times L^q(H)}
$$

and so we obtain the estimate

$$\|B(\hat{u}, \hat{p})\|_{\widehat{W}^{-1,q}(H)^n \times L^q(H)} \leq C_4(\|\nabla' \omega\|_\infty + \|\nabla' \omega\|_\infty^2)\|S_{q,H}(\hat{u}, \hat{p})\|_{\widehat{W}^{-1,q}(H)^n \times L^q(H)} \tag{3.10}$$

where $C_4 = C_4(n, q)$ depends only on n, q.

If $K = C_4(\|\nabla' \omega\|_\infty + \|\nabla' \omega\|_\infty^2) < 1$ we may apply Lemma 3.1 and conclude that $S_{q,H} + B$ is an isomorphism from $\widehat{W}_0^{1,q}(H)^n \times L^q(H)$ to $\widehat{W}^{-1,q}(H)^n \times L^q(H)$. Due to (3.9) and (3.8) we now obtain that S_q above is an isomorphism. This proves the first assertion of Lemma 2.4. To prove the last assertion we consider the intersections

$$X = [\widehat{W}_0^{1,q}(H)^n \times L^q(H)] \cap [\widehat{W}_0^{1,s}(H)^n \times L^s(H)]$$

with

$$\|(u, p)\|_X = \|(u, p)\|_{\widehat{W}_0^{1,q}(H_\omega)^n \times L^q(H_\omega)} + \|(u, p)\|_{\widehat{W}_0^{1,s}(H)^n \times L^s(H)}$$

and

$$Y = [\widehat{W}^{-1,q}(H)^n \times L^q(H)] \cap [\widehat{W}^{-1,q}(H)^n \times L^s(H)]$$

with

$$\|(f, g)\|_Y = \|(f, g)\|_{\widehat{W}^{-1,q}(H)^n \times L^q(H)} + \|(f, g)\|_{\widehat{W}^{-1,s}(H)^n \times L^s(H)}.$$

For the definition of Y observe we have to identify two functionals which coincide on $C_0^\infty(H)^n$.

The same calculation as above for (3.10) now yields

$$\|B(\hat{u}, \hat{p})\|_Y \leq K\|S_{q,H}(\hat{u}, \hat{p})\|_Y \tag{3.11}$$

with $K = C_4(\|\nabla' \omega\|_\infty + \|\nabla' \omega\|_\infty^2)$ where $C_4 = C_4(n, q, s)$ also depends an s. We get $0 < K < 1$ if $\|\nabla' \omega\|_\infty$ is sufficiently small. The abstract version of the last assertion of Lemma 2.3 for $\Omega = H$ means that the operator $S_{q,H}$ is an isomorphism from X to Y. Using (3.11) we conclude from Lemma 3.1 as before that S_q is an isomorphism from X to Y now with H replaced by H_ω. This proves the last assertion of Lemma 2.4.

4. - Proof of Theorem 2.1.

According to Remark 2.2, *b)* it is sufficient to consider a bounded Lipschitz domain $\Omega \subseteq \mathbb{R}^n$ with Lipschitz constant L which fulfills some smallness condition.

We will use the well known localization procedure and apply locally the lemmas on $\mathbb{R}^n$ and H_ω; this leads to the desired result. For this purpose we choose open balls $B_1, B_2, \dots, B_k \subseteq \mathbb{R}^n$ covering the closure $\overline{\Omega}$ of Ω, i.e. $\overline{\Omega} \subseteq \bigcup_{i=1}^k B_i$. Furthermore we choose functions $\varphi_i \in C_0^\infty(B_i)$, $i = 1, \dots, k$, with $0 \leq \varphi_i \leq 1$ and $\sum_{i=1}^k \varphi_i(x) = 1$ for all $x \in \overline{W}$. According to the definition of Lipschitz domains we can choose $B_1, \dots, B_k$ as follows.

There is some k' with $1 < k' \leq k$ such that $B_i \cap \partial\Omega \neq \emptyset$ for $i = 1, 2, \dots, k'$ and $\overline{B}_i \subseteq \Omega$ for $i = k' + 1, \dots, k$. For each $i = 1,, \dots k'$ we can find some Lipschitz continuous function $\omega^i : D^i \to \mathbb{R}$ with compact support and $\|\nabla\omega^i\|_\infty \leq L$ such that $B_i \cap \Omega \subseteq H_{\omega^i}$ and $B_i \cap \partial\Omega \subseteq \partial H_{\Omega^i}$ (after some appropriate rotation and translation of the coordinate system depending on i); see (2.7) for the definition of H_{ω^i}. Put $\Omega_i = B_i \cap \Omega$ for $i = 1, \dots k$.

First we consider the case $\phi = 0$ in the equations (2.2). Let $(u, p) \in W_0^{1,q}(\Omega)^n \times \times L^q(\Omega)$ with $\int_\Omega p\,dx = 0$, put $f = -\Delta u + \nabla p$, $g = \operatorname{div} u$ and multiply the equations

$$-\Delta u + \nabla p = f, \qquad \operatorname{div} u = g \tag{4.1}$$

by the cut off function φ_i, $i = 1, \dots, k$. This yields the local equations

$$-\Delta(\varphi_i u) + \nabla(\varphi_i p) = f_i, \qquad \operatorname{div}(\varphi_i u) = g_i, \tag{4.2}$$

with $f_i = \varphi_i f - (\Delta\varphi_i)u - 2(\nabla\varphi_i)(\nabla u) + (\nabla\varphi_i)p$ and $g_i = \varphi_i g + (\nabla\varphi_i)u$. For $i = 1,, \dots k'$ we may treat (4.2) as equations on H_{ω^i} and apply Lemma 2.4 and for $i = k' + 1, \dots, k$ we get equations which can be considered as equations on $\mathbb{R}^n$ or on H (after some translation). We carry out this procedure in several steps.

In the first step we prove the a priori estimate

$$\|\nabla u\|_{L^q(\Omega)^{n^2}} + \|p\|_{L^q(\Omega)} \leq C(\|f\|_{W^{-1,q}(\Omega)^n} + \|g\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)^n} + \|p\|_{W^{-1,q}(\Omega)}) \tag{4.3}$$

containing two additional terms on the right compared with (2.3); they will be removed later on by some compactness argument.

Assuming $L \leq K(n, q, q)$ with $q = s$ in Lemma 2.4 we may apply (2.15) for $i = 1,, \dots k'$ and obtain

$$\|\nabla(\varphi_i u)\|_{L^q(H_{\omega^i})^{n^2}} + \|\varphi_i p\|_{L^q(H_{\omega^i})} \leq C(\|f_i\|_{\hat{W}^{-1,q}(H_{\omega^i})^n} + \|g_i\|_{L^q(H_{\omega^i})}).$$

To estimate the expressions on the right we consider test functions $v \in C_0^\infty(H_{\omega^i})^n$, apply Poincare's inequality on $\Omega_i = B_i \cap \Omega \subseteq H_{\omega^i}$ (suppressing a rotation and translation possibly needed depending on i) and use the estimates

$$\|\nabla(\varphi_i v)\|_{L^{q'}(\Omega)^{n^2}} \leq C\|\nabla v\|_{L^{q'}(H_{\omega^i})^{n^2}},$$

$$\|(\nabla\varphi_i)v\|_{L^{q'}(\Omega)^n} \leq C\|\nabla v\|_{L^{q'}(H_{\omega^i})^{n^2}},$$

$$\|(\Delta\varphi_i)v\|_{L^{q'}(\Omega)^n} \leq C\|\nabla v\|_{L^{q'}(H_{\omega^i})^{n^2}},$$

$$|\langle u, (\Delta\varphi_i)v\rangle| \leq C\|u\|_{L^q(\Omega)^n}\|\nabla v\|_{L^{q'}(H_{\omega^i})},$$

$$|\langle \nabla u, (\nabla\varphi_i)v\rangle| \leq C\|u\|_{L^q(\Omega)^n}\|\nabla v\|_{L^{q'}(H_{\omega^i})},$$

$$|\langle p, (\nabla\varphi_i)v\rangle| \leq C\|p\|_{W^{-1,q}(\Omega)}\|\nabla v\|_{L^q(H_{\omega^i})}.$$

This leads to

$$\|f_i\|_{\widehat{W}^{-1,q}(H_{\omega^i})^n} =$$

$$= \sup_{0\neq v\in C_0^\infty(H_{\omega^i})^n} \{ |\langle f, \varphi_i v\rangle - \langle u, (\nabla\varphi_i)v\rangle - 2\langle \nabla u, (\nabla\varphi_i)v\rangle + \langle p, (\nabla\varphi_i)v\rangle| / \|\nabla v\|_{L^{q'}(H_{\Omega^i})^{n^2}} \} \leq$$

$$\leq C_1\left(\sup_{0\neq v\in C_0^\infty(H_{\omega^i})^n} |\langle f, \varphi_i v\rangle| / \|\nabla(\varphi_i v)\|_{L^{q'}(\Omega)^{n^2}}\right) + C_2(\|u\|_{L^q(\Omega)^n} + \|p\|_{W^{-1,q}(\Omega)}) \leq$$

$$\leq C_1\|f\|_{W^{-1,q}(\Omega)^n} + C_2(\|u\|_{L^q(\Omega)^n} + \|p\|_{W^{-1,q}(\Omega)})$$

and

$$\|g_i\|_{L^q(H_{\omega^i})} \leq C_1\|g\|_{L^q(\Omega)} + C_2\|u\|_{L^q(\Omega)^n}.$$

For $i = k' + 1, \dots, k$ we get the same estimates with H_{ω^i} replaced by H and applying Lemma 2.3 instead of Lemma 2.4. In this case no smalles assumption on L is needed. Summing up these estimate over $i = 1, 2, \dots, k', \dots, k$ we obatin the desired estimate (4.3).

The next step yields the uniqueness property for (2.2). We will show that

$$1 < q < \infty, \qquad (u, p) \in W_0^{1,q}(\Omega)^n \times L^q(\Omega), \qquad \int_\Omega p\,dx = 0,$$

$$-\Delta u + \nabla p = 0, \qquad \operatorname{div} u = 0,$$

implies $u = 0$ and $p = 0$.

For $i = 1, \dots, k$ the local equations (4.2) now have the form

$$-\Delta(\varphi_i u) + \nabla(\varphi_i p) = f_i, \qquad \operatorname{div}(\varphi_i u) = g_i \tag{4.4}$$

with $f_i = -(\Delta\varphi_i)u - 2(\nabla\varphi_i)(\nabla\varphi_i)p$, $g_i = (\nabla\varphi_i)\cdot u$. Applying the regularity property in the Lemmas 2.3 and 2.4 concerning the exponent s we can show in a number of steps that

$$u \in W_0^{1,2}(\Omega)^n, \qquad p \in L^2(\Omega). \tag{4.5}$$

Then we conclude that $0 = \langle -\Delta u + \nabla p, u\rangle = \|\nabla u\|^2_{L^2(\Omega)}$, $\nabla u = 0$, $\nabla p = 0$ and $p = 0$ using $\int_\Omega p\,dx = 0$. So it remains to show (4.5). This property is clear if $q \geq 2$ since Ω is bounded. So we assume now $1 < q < 2$.

First we consider $i = 1, \dots, k'$. In the following we have to apply Lemma 2.4 for finitely many exponents $s_1, \dots, s_m$ (depending on q, n) instead of s. Therefore we can find some $\widetilde{K} = \widetilde{K}(n, q) > 0$ such that for $L \leq \widetilde{K}$ in the following the assertion of Lemma 2.4 is applicable.

Now we choose $s_1 > 1$ such that $1/n + 1/s_1 = 1/q$. Then Sobolev's embedding theorem yields $\|u\|_{L^{s_1}(\Omega)^n} \leq C\|\nabla u\|_{L^q(\Omega)^{n^2}}$. Defining $q' = q/(q-1)$, $s_1' = s_1/(s_1 - 1)$ we get $1/n + 1/q' = 1/s_1'$, $q' > s_1'$, $s_1 > q$. Applying again Sobolev's embedding theorem and

Poincare's inequality on $\Omega_i \subseteq H_{\omega^i}$ we obtain for $v \in C_0^\infty(H_{\omega^i})^n$:

$$|\langle u, (\Delta\varphi_i)v\rangle| \leq C_1 \|u\|_{L^q(\Omega_i)^n} \|\nabla v\|_{L^{q'}(H_{\omega^i})^{n^2}},$$

$$|\langle u, (\Delta\varphi_i)v\rangle| \leq C_2 \|u\|_{L^q(\Omega_i)^n} \|\nabla v\|_{L^{s_1'}(H_{\omega^i})^{n^2}},$$

$$|\langle \nabla u, (\nabla\varphi_i)v\rangle| \leq C_3 \|\nabla u\|_{L^q(\Omega_i)^{n^2}} \|\nabla v\|_{L^{q'}(H_{\omega^i})^{n^2}},$$

$$|\langle \nabla u, (\nabla\varphi_i)v\rangle| \leq C_4 \|\nabla u\|_{L^q(\Omega_i)^{n^2}} \|\nabla v\|_{L^{s_1'}(H_{\omega^i})^{n^2}},$$

$$|\langle p, (\nabla\varphi_i)v\rangle| \leq C_5 \|p\|_{L^q(\Omega_i)} \|\nabla v\|_{L^{q'}(H_{\omega^i})^{n^2}},$$

$$|\langle p, (\nabla\varphi_i)v\rangle| \leq C_6 \|p\|_{L^q(\Omega_i)} \|\nabla v\|_{L^{s_1'}(H_{\omega^i})^{n^2}}.$$

This leads to $f_i \in \widehat{W}^{-1,q}(H_{\omega^i})^n$ and $f_i \in \widehat{W}^{-1,s_1}(H_{\omega^i})^n$. In the same way we get $g_i \in L^q(H_{\omega^i}) \cap L^{s_1}(H_{\omega^i})$. The application of Lemma 2.4 now yields

$$(\varphi_i u, \varphi_i p) \in [W_0^{1,q}(H_{\omega^i})^n \times L^q(H_{\omega^i})] \cap [W_0^{1,s_1}(H_{\omega^i})^n \times L^{s_1}(H_{\omega^i})].$$

For $i = k'+1, \dots, k$ we conclude in the same way using Lemma 2.3 for the halfspace H instead of Lemma 2.4. This yields

$$(\varphi_i u, \varphi_i p) \in [W_0^{1,q}(H)^n \times L^q(H)] \cap [W_0^{1,s_1}(H)^n \times [L^{s_1}(H)]$$

for $i = k'+1, \dots, k$.

Therefore we have $(u, p) \in [W_0^{1,q}(\Omega)^n \times L^q(\Omega)] \cap [W_0^{1,s_1}(\Omega)^n \times L^{s_1}(\Omega)]$. If $s_1 \geq 2$ it follows the desired result (4.4). If $s_1 < 2$ we repeat this procedure with q replaced by $s_1 > q$ and with s_1 replaced by s_2 defined by $1/n + 1/s_2 = 1/s_1$, this yields $(u, p) \in W_0^{1,s_2}(\Omega)^n \times L^{s_2}(\Omega)$. So we obtain (4.5) in a finite number of steps.

In the next step of our proof we show the a priori estimate (2.3) with $\phi = 0$. For this purpose we show by a compactness argument that the terms $\|u\|_{L^q(\Omega)^n}$ and $\|p\|_{W^{-1,q}(\Omega)}$ on the right of (4.3) may be omitted. We argue by contradiction. Suppose the estimate

$$\|\nabla u\|_{L^q(\Omega)^{n^2}} + \|p\|_{L^q(\Omega)} \leq C(\|f\|_{W^{-1,q}(\Omega)^n} + \|g\|_{L^q(\Omega)}) \tag{4.6}$$

is not true for all $(u, p) \in W_0^{1,q}(\Omega)^n \times L^q(\Omega)$ with $\int_\Omega p\,dx = 0$ where $f = -\nabla u + \nabla p$, $g = \operatorname{div} u$. Then we can choose $(u_i, p_i) \in W_0^{1,q}(\Omega)^n \times L^q(\Omega)$ with $\int_\Omega p_i\,dx = 0$ for $i = 1, 2, \dots$, such that

$$\lim_i \|f_i\|_{W^{-1,q}(\Omega)^n} = 0, \quad \lim_i \|g_i\|_{L^q(\Omega)} = 0,$$

$$\|\nabla u_i\|_{L^q(\Omega)^{n^2}} + \|p_i\|_{L^q(\Omega)} = 1, \qquad i = 1, 2, \dots,$$

where $f_i = -\Delta u_i + \nabla p_i$, $g_i = \operatorname{div} u_i$. We can single out a sub-sequence converging weakly in $W_0^{1,q}(\Omega)^n \times L^q(\Omega)$; we may assume that the sequence itself converges weakly to some element $(u, p) \in W_0^{1,q}(\Omega)^n \times L^q(\Omega)$. Since Ω is bounded we know that $(u, p) = \lim_i (u_i, p_i)$ holds strongly in $L^q(\Omega)^n \times W^{-1,q}(\Omega)$ and therefore that

$$\|u\|_{L^q(\Omega)^n} = \lim_i \|u_i\|_{L^q(\Omega)^n}, \qquad \|p\|_{W^{-1,q}(\Omega)} = \lim_i \|p_i\|_{W^{-1,q}(\Omega)}.$$

Applying estimate (4.3) yields

$$1 \leq C(\|f_i\|_{W^{-1,q}(\Omega)^n} + \|g_i\|_{L^q(\Omega)} + \|u_i\|_{L^q(\Omega)^n} + \|p_i\|_{W^{-1,q}(\Omega)})$$

and letting $i \to \infty$ leads to

$$1 \leq C(\|u\|_{L^q(\Omega)^n} + \|p\|_{W^{-1,q}(\Omega)}) \tag{4.7}$$

from which we conclude that $(u, p) \neq (0, 0)$.

Since $\lim_i \|f_i\|_{W^{-1,q}(\Omega)^n} = 0$, $\lim_i \|g_i\|_{L^q(\Omega)} = 0$ we obtain

$$\lim_i (\langle \nabla u_i, \nabla v \rangle - \langle p_i, \operatorname{div} v \rangle) = \langle \nabla u, \nabla v \rangle - \langle p, \operatorname{div} v \rangle = 0$$

for all $v \in C_0^\infty(\Omega)^n$ and $\operatorname{div} u = 0$. The weak convergence of p_i to p in $L^q(\Omega)$ and $\int_\Omega p_i\, dx = 0$ $(i = 1, \ldots)$ yields $\int_\Omega p\, dx = 0$. From the uniqueness assertion above we conclude that $(u, p) = (0, 0)$ being a contradiction to (4.5). This proves (2.3) for $\phi = 0$.

In the next step we show that the operator S_q in (2.4) is an isomorphism. For this purpose we use a duality argument. The inequality (2.3) with $\gamma = 0$ means that

$$\|(u, p)\|_{W_0^{1,q}(\Omega)^n \times \tilde{L}^q(\Omega)} \leq C\|S_q(u, p)\|_{W^{-1,q}(\Omega)^n \times \tilde{L}^q(\Omega)} \tag{4.8}$$

holds for all $(u, p) \in W_0^{1,q}(\Omega)^n \times \tilde{L}^q(\Omega)$ with $\tilde{L}^q(\Omega)$ as in (2.4). From the well known closed range theorem we conclude that the dual operator $S_q^* = S_{q'}$ in (2.5) is surjective and since (4.6) holds for all $1 < q < \infty$ we see that S_q is surjective and therefore bijective for all $1 < q < \infty$. This shows that S_q is an isomorphism and so Theorem 2.1 is proved for the special case $\phi = 0$.

To prove Theorem 2.1 in the general case $\phi \in W^{1-1/q,q}(\partial\Omega)^n$ where $\int_{\partial\Omega} N \cdot \phi\, d\sigma = \int_\Omega g\, dx$ we use the well known extension operator

$$E\colon W^{1-(1/q),q}(\partial\Omega)^n \to W^{1,q}(\Omega)^n$$

which is continuous and has the property $E(\phi)|_{\partial\Omega} = \phi$ for all $\phi \in W^{1-(1/q),q}(\partial\Omega)$. In particular we have

$$\|E(\phi)\|_{W^{1,q}(\Omega)^n} \leq C\|\phi\|_{W^{1-(1/q),q}(\partial\Omega)} \tag{4.9}$$

where $C = C(\Omega, n, q) > 0$ is a constant. Using the assertion of Theorem 2.1 for $\phi = 0$ we find a unique pair $(\tilde{u}, p) \in W_0^{1,q}(\Omega)^n \times L^q(\Omega)$ with $\int_\Omega p\, dx = 0$ such that

$$-\Delta\tilde{u} + \nabla p = f + \Delta E(\phi), \qquad \operatorname{div}\tilde{u} = g - \operatorname{div} E(\phi).$$

This is possible since

$$\int_\Omega (g - \operatorname{div} E(\phi))\, dx = \int_\Omega g\, dx - \int_{\partial\Omega} N \cdot E(\phi)\, d\sigma = 0\,.$$

Moreover we get from (2.3) for $\phi = 0$ and (4.9):

$$\|\nabla \widetilde{u}\|_{L^q(\Omega)^{n^2}} + \|p\|_{L^q(\Omega)} \leqslant C_1 (\|f\|_{W^{-1,q}(\Omega)^n} + \|\Delta E(\phi)\|_{W^{-1,q}(\Omega)^n} + \|g\|_{L^q(\Omega)} + \|\operatorname{div} E(\phi)\|_{L^q(\Omega)}) \leqslant$$
$$\leqslant C_2 (\|f\|_{W^{-1,q}(\Omega)^n} + \|g\|_{L^q(\Omega)} + \|\phi\|_{W^{1-(1/q),q}(\partial\Omega)^n}).$$

Setting $u = \widetilde{u} + E(\phi)$ we obtain $u|_{\partial\Omega} = \widetilde{u}|_{\partial\Omega} + E(\phi)|_{\partial\Omega} = \phi$, $\operatorname{div} u = \operatorname{div} \widetilde{u} + \operatorname{div} E(\phi) = g$ and

$$-\Delta u + \nabla p = -\Delta\widetilde{u} - \Delta E(\phi) + \nabla p = f.$$

Finally, we get $\|\widetilde{u}\|_{W^{1,q}(\Omega)^n} \leqslant C\|\nabla\widetilde{u}\|_{L^q(\Omega)^{n^2}}$ since $\widetilde{u}|_{\partial\Omega} = 0$ and

$$\|u\|_{W^{1,q}(\Omega)^n} + \|p\|_{L^q(\Omega)} \leqslant \|\widetilde{u}\|_{W^{1,q}(\Omega)^n} + \|E(\phi)\|_{W^{1,q}(\Omega)^n} + \|p\|_{L^q(\Omega)} \leqslant$$
$$\leqslant C(\|f\|_{W^{-1,q}(\Omega)^n} + \|g\|_{L^q(\Omega)} + \|\phi\|_{W^{1-(1/q),q}(\partial\Omega)^n})$$

which is the estimate (2.3). The pair (u, p) constructed above is unique; this also follows from the uniqueness assertion proved before. This completes the proof of Theorem 2.1.

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