Annali di Matematica pura ed applicata
(IV), Vol. CLXVII (1994), pp. 117-146

The Green Relations Approach to Congruences on Completely Regular Semigroups (*).

MARIO PETRICH

Abstract. – *On the congruence lattice* $\mathcal{C}(S)$ *of a completely regular semigroup S the following mappings are considered* $\chi_p : \rho \to \rho \cap \mathcal{P}$ *and* $\chi_P : \rho \vee \mathcal{P}$, *where* $\mathcal{P}$ *is any of the Green relations* $\mathcal{H}$, $\mathcal{L}$, $\mathcal{R}$ *or* $\mathcal{D}$. *The equivalence relations* $\mathcal{P}^\wedge$ *and* $\mathcal{P}^\vee$ *induced by these maps represent the main object of study in the paper. The former is a complete* $\wedge$*-congruence whereas the latter is a complete congruence on* $\mathcal{C}(S)$. *In particular* $\mathcal{H}^\wedge$, $\mathcal{H}^\vee$, $\mathcal{L}^\vee$, $\mathcal{R}^\vee$ *coincide with the kernel, trace, left trace and right trace relations on* $\mathcal{C}(S)$, *respectively. All essential properties known for the latter relations carry over to the new relations* $\mathcal{P}^\wedge$ *and* $\mathcal{P}^\vee$. *In addition, some interesting interplays of these provide for more richness in the theory of congruences on completely regular than is the case for the kernel-trace approach to congruences on regular semigroups.*

1. – Introduction and summary.

The study of congruences on regular semigroups experienced a considerable boost when it was realized that a congruence is completely determined by its kernel and its trace. This represented an essential improvement over the earlier knowledge that a congruence on a regular semigroup is completely determined by its idempotent classes. The eruption of information concerning congruences on regular semigroups when approached from their kernel-trace aspect has been both surprizing and profound. It has been most effective when the (congruence) pair $(\ker \rho, \operatorname{tr} \rho)$ could be described abstractly in a relatively simple and/or explicit manner, as for example for inverse semigroups.

Nevertheless, the description of congruences on semigroups belonging to a special class for which there is a sufficiently transparent structure theorem should be based on that structure theorem and the ingredients figuring in it. The typical example for this is the class of completely simple semigroups and the Rees theorem. For the congruences in this case are best represented as (admissible) triples following the ingre-

(*) Entrata in Redazione il 22 giugno 1992.
Indirizzo dell'A.: Department of Mathematics, Simon Fraser University, Burnaby, BC V5A 1S6, Canada.

dients figuring in the Rees (structure) theorem. Even though this is a relatively rare case, the Rees theorem being very explicit and transparent, the same idea was used in [4] for completely regular semigroups, for they are best represented as semilattices of completely simple semigroups. In this case, a congruence can be represented by a (congruence) aggregate which consists of a congruence on the underlying semilattice and the congruences on completely simple components.

The last discussed approach to congruences on completely regular semigroups can be thought of as splitting a congruence ρ along the Green relation $\mathcal{D}$ in the sense of decomposing it into $\rho \cap \mathcal{D}$ and $\rho \vee \mathcal{D}$, as explained in [4]. This obviously suggests substituting $\mathcal{D}$ by any Green's relation $\mathcal{P}$, which is the main idea of the present paper. Not only do we thus generalize the approach taken in [4] but with the case $\mathcal{P} = \mathcal{H}$, recover essentially the kernel-trace approach as well. We can thus view the new point of view as an extension of the kernel-trace approach. By introducing also Green's relations $\mathcal{L}$ and $\mathcal{R}$ into the play, we arrive at a systematic study of congruences through their relationship with the Green relations. This paper is also related to [3]. The results therein may provide the clue as to the possibility of extending or generalizing the achievements of this paper to regular semigroups.

We now briefly summarize the content of various sections. Terminology and notation are taken care of in Section 2. From now on $\mathcal{P}$ stands for any of the Green relations $\mathcal{H}$, $\mathcal{L}$, $\mathcal{R}$ or $\mathcal{D}$ on a completely regular semigroup. The mapping $\rho \to \rho \cap \mathcal{P}$ is discussed in Section 3 and the properties of the induced relation in Section 4. The same pattern is followed in Sections 5 and 6 for the mapping $\rho \to \rho \vee \mathcal{P}$ and the induced relation. Further properties of both relations as well as applications to some greatest and least congruences are discussed in Section 7. In Section 8, a related diagram and a resulting network of congruences are considered. The paper concludes in Section 9 with a discussion of relationship with the kernel-trace approach to congruences.

2. – Terminology and notation.

On any set X, ε and ω denote the equality and the universal relations, respectively; if needed for clarity, we shall affix the subscript X to these symbols. We denote by $\Pi(X)$ the partition lattice of $X$with meet $\cap$ and join $\vee$ and will think of its members also as equivalence relations on X. If $\rho \in \Pi(X)$ and $x \in X$, then $x\rho$ is the ρ-class containing x. If $\lambda, \rho \in \Pi(X)$ are such that $\lambda \supseteq \rho$, then λ/ρ is the member of $\Pi(X/\rho)$ defined by

$$x\rho\,\lambda/\rho\, y\rho \quad \text{if} \quad x\lambda y \quad (x, y \in X).$$

For any lattice L and $\alpha, \beta \in L$, let

$$[\alpha, \beta] = \{\gamma \in L \,|\, \alpha \leq \gamma \leq \beta\}, \qquad [\alpha) = \{\gamma \in L \,|\, \alpha \leq \gamma\}.$$

Now let S be a semigroup. We write

$$\mathcal{GR} = \{\mathcal{H}, \mathcal{L}, \mathcal{R}, \mathcal{D}\}$$

for these Green's relations. If S has an identity, let $S^1 = S$, otherwise let S^1 be equal to S with an identity adjoined. We denote by $E(S)$ the set of idempotents of S and by $\mathcal{C}(S)$ the lattice of congruences on S with the operation of meet $\wedge$ and join $\vee$. Let $\rho \in \mathcal{C}(S)$. If $\mathcal{F}$ is a family of semigroups, then ρ is an *$\mathcal{F}$-congruence* if $S/\rho \in \mathcal{F}$ and ρ is *over* $\mathcal{F}$ if each idempotent ρ-class is in $\mathcal{F}$. Also

$$\ker \rho = \{a \in S \mid a \rho e \text{ for some } e \in E(S)\}, \qquad \operatorname{tr} \rho = \rho|_{E(S)}$$

are the *kernel* and the *trace* of ρ, respectively. For a relation θ on S, θ^* denotes the congruence on S generated by θ. If $\theta \in \Pi(S)$, θ^0 denotes the greatest congruence on S contained in θ; recall that for any $a, b \in S$,

$$a \theta^0 b \Leftrightarrow (x a y \,\theta\, x b y \text{ for all } x, y \in S^1).$$

Throughout the paper S shall stand for a completely regular semigroup, that is a semigroup which is the union of its (maximal) subgroups. For $a \in S$, a^{-1} stands for the inverse of a in the maximal subgroup of S containing a; $a^0 = aa^{-1} = a^{-1}a$ is the identity element of this subgroup. If $\rho \in \mathcal{C}(S)$, then ρ preserves both operations of upper -1 and 0 and

$$\ker \rho = \{a \in S \mid a \rho a^0\}.$$

Completely regular semigroups form a variety with the binary operation of product and the unary operation of inverse. We denote by $[u = v]$ the variety of completely regular semigroups determined by the identity $u = v$.

We shall need the next lemma and its corollary from the literature. Since they are used many times, we give the lemma a complete proof.

LEMMA 2.1. – *For $\rho \in \mathcal{C}(S)$ and $a, b \in S$, we have*

$$a \rho b \Leftrightarrow a^0 \operatorname{tr} \rho\, b^0, \quad ab^{-1} \in \ker \rho.$$

PROOF. – If $a \rho b$, then $a^0 \rho b^0$ and $ab^{-1} \rho b^0$ whence $a^0 \operatorname{tr}\rho\, b^0$ and $ab^{-1} \in \ker \rho$. Conversely, if $a^0 \operatorname{tr}\rho\, b^0$ and $ab^{-1} \in \ker\rho$, then

$$a = a^0 a \rho b^0 a = bb^0 b^{-1} a a^0 \rho b a^0 b^{-1} a b^0,$$

$$= ba^{-1}(ab^{-1})(ab^{-1}) b \rho ba^{-1}(ab^{-1}) b = ba^0 b^0 \rho bb^0 = b.$$

COROLLARY 2.2. – *Let $\lambda, \rho \in \mathcal{C}(S)$. If $\ker\lambda \subseteq \ker\rho$ and $\operatorname{tr}\lambda \subseteq \operatorname{tr}\rho$, then $\lambda \subseteq \rho$.*

We now introduce the basic symbolism which will be adhered to throughout the paper.

NOTATION 2.3. – *Define the mappings*

$$\chi_{\mathcal{P}}: \rho \to \rho \cap \mathcal{P}, \qquad \chi_{\mathcal{P}}: \rho \to \rho \vee \mathcal{P}, \qquad (\rho \in \mathcal{C}(S))$$

and the induced (equivalence) relations $\mathcal{P}^\wedge$ *and* $\mathcal{P}^\vee$ *on* $\mathcal{C}(S)$ *by*

$$\lambda \mathcal{P}^\wedge \rho \Leftrightarrow \lambda \cap \mathcal{P} = \rho \cap \mathcal{P}, \qquad \lambda \mathcal{P}^\vee \rho \Leftrightarrow \lambda \vee \mathcal{P} = \rho \vee \mathcal{P}.$$

In all our considerations of the Green relations $\mathcal{H}$, $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{D}$, we shall consider only $\mathcal{H}$, $\mathcal{L}$ and $\mathcal{D}$ by duality of $\mathcal{L}$ and $\mathcal{R}$ so that the latter need not be mentioned.

3. – The relation $\mathcal{P}^\wedge$.

For a Green relation $\mathcal{P}$, we have defined above a relation $\mathcal{P}^\wedge$ by

$$\lambda \mathcal{P}^\wedge \rho \Leftrightarrow \lambda \cap \mathcal{P} = \rho \cap \mathcal{P} \qquad (\lambda, \rho \in \mathcal{C}(S)).$$

In this and the next section, we shall study a number of properties of this (equivalence) relation. We have already remarked the obvious fact that $\mathcal{P}^\wedge$ is induced by the mapping χ_p: $\mathcal{C}(S) \to \Pi(S)$. The following notation will be used to characterize the image of $\mathcal{C}(S)$ in $\Pi(S)$ under χ_p.

NOTATION 3.1. – *For* $\mathcal{P} \in \mathcal{GR}$, *let*

$$\Pi_p(S) = \{\theta \in \Pi(X) \mid \theta = \theta^* \cap \mathcal{P}\}.$$

LEMMA 3.2. – *For* $\mathcal{P} \in \mathcal{GR}$ *and* $\theta \in \Pi(S)$, $\theta = \theta^* \cap \mathcal{P}$ *if and only if* $\theta = \rho \cap \mathcal{P}$ *for some* $\rho \in \mathcal{C}(S)$.

PROOF. – For the direct part, take $\rho = \theta^*$. Conversely, $\theta = \rho \cap \mathcal{P}$ implies

$$\theta = \theta \cap \mathcal{P} \subseteq \theta^* \cap \mathcal{P} \subseteq \rho \cap \mathcal{P} = \theta.$$

LEMMA 3.3. – *Let* $\mathcal{P} \in \mathcal{GR}$ *and* $\rho \in \mathcal{C}(S)$.

(i) $\rho \cap \mathcal{P} = (\rho \cap \mathcal{P})^* \cap \mathcal{P}$.

(ii) $\ker \rho = \ker (\rho \cap \mathcal{P})^*$.

PROOF. – (i) Since $\rho \cap \mathcal{P} \subseteq (\rho \cap \mathcal{P})^*$, we get $\rho \cap \mathcal{P} \subseteq (\rho \cap \mathcal{P})^* \cap \mathcal{P}$. Also $(\rho \cap \mathcal{P})^* \subseteq \rho$ implies that $(\rho \cap \mathcal{P})^* \cap \mathcal{P} \subseteq \rho \cap \mathcal{P}$.

(ii) If $a \in \ker \rho$, then $a \rho \cap \mathcal{P} a^0$ whence $a (\rho \cap \mathcal{P})^* a^0$ and thus $a \in \ker (\rho \cap \mathcal{P})^*$. Hence $\ker \rho \subseteq \ker (\rho \cap \mathcal{P})^*$ and the opposite inclusion holds since $(\rho \cap \mathcal{P})^* \subseteq \rho$.

We shall see below that the classes of $\mathcal{P}^\wedge$ are intervals. In order to describe the upper ends of these intervals, the following symbolism will come in handy.

NOTATION 3.4. – *For any $\mathcal{P} \in \mathcal{GR}$ and $\rho \in \mathcal{C}(S)$, define a relation p_ρ on S by*

$$a\, h_\rho\, b \Leftrightarrow ab^{-1} \in \ker \rho\,,$$

$$a\, l_\rho\, b \Leftrightarrow a^0 b \rho b^0 a^0 b\,, \quad ab^{-1} \in \ker \rho\,,$$

$$a\, r_\rho\, b \Leftrightarrow ab^0 \rho ab^0 a^0\,, \quad ab^{-1} \in \ker \rho\,,$$

$$a\, d_\rho\, b \Leftrightarrow ab \rho ba\,, \quad ab^{-1} \in \ker \rho\,.$$

For any relation θ on S, define a relation $\tilde{\theta}$ on S by

$$a \tilde{\theta} b \Leftrightarrow (\text{for every } u \in S,\ a\theta u \Leftrightarrow b\theta u)\,.$$

Simple verification shows that $\tilde{\theta}$ is an equivalence relation on S.
We are now ready for the first principal result of the paper.

THEOREM 3.5. – *Let $\mathcal{P} \in \mathcal{GR}$. The mapping*

$$\chi_p : \rho \to \rho \cap \mathcal{P} \qquad (\rho \in \mathcal{C}(S))$$

is a complete $\wedge$-homomorphism of $\mathcal{C}(S)$ onto $\Pi_p(S)$ which induces $\mathcal{P}^\wedge$. Consequently $\mathcal{P}^\wedge$ is a complete $\wedge$-congruence on $\mathcal{C}(S)$ but in general, it is not a $\vee$-congruence. For each $\rho \in \mathcal{C}(S)$, we have $\rho\mathcal{P}^\wedge = [\rho_p, \rho^p]$ where

$$\rho_p = (\rho \cap \mathcal{P})^*\,, \qquad \rho^p = \tilde{p}_\rho^0\,.$$

PROOF. – Trivially χ_p is a complete $\wedge$-homomorphism. By Proposition 3.2, χ_p maps $\mathcal{C}(S)$ onto $\Pi_p(S)$. Obviously χ_p induces $\mathcal{P}^\wedge$ so that $\mathcal{P}^\wedge$ is a complete $\wedge$-congruence.

Let $S = Y \times G$ where $Y = \{0, 1\}$ is a 2-element semilattice and G is a nontrivial group. Denote by σ the least group congruence on S and by ρ the Rees congruence on S determined by the ideal $\{0\} \times G$. Then

$$(\sigma \cap \mathcal{P}) \vee (\rho \cap \mathcal{P}) = \varepsilon \vee \rho = \rho\,, \qquad (\sigma \vee \mathcal{P}) \cap \mathcal{P} = \omega \cap \mathcal{P} = \mathcal{P}\,,$$

and $\rho \neq \mathcal{P}$ since G is nontrivial. Therefore $\mathcal{P}^\wedge$ is not a $\vee$-congruence in this example. Note that here $\mathcal{P} = \mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D}$.

We now return to the general situation and let $\rho \in \mathcal{C}(S)$. By Lemma 3.3 (i), we have $(\rho \cap \mathcal{P})^* \cap \mathcal{P} = \rho \cap \mathcal{P}$, that is $(\rho \cap \mathcal{P})^* \mathcal{P}^\wedge \rho$. If $\lambda \in \mathcal{C}(S)$ is such that $\lambda \mathcal{P}^\wedge \rho$, then $(\rho \cap \mathcal{P})^* = (\lambda \cap \mathcal{P})^* \subseteq \lambda$ which establishes the minimality of ρ_p.

For the upper end, we first assume that for $\lambda \in \mathcal{C}(S)$, we have $\lambda \mathcal{P}^\wedge \rho$ and shall prove that $\lambda \subseteq \tilde{p}_\rho$. By Lemma 3.3 (ii), we get

$$\ker \lambda = \ker (\lambda \cap \mathcal{P})^* = \ker (\rho \cap \mathcal{P})^* = \ker \rho$$

Next let $a \lambda b$ and $a p_\rho u$. We consider several cases.

Let $\mathcal{P} = \mathcal{H}$. Then $au^{-1} \in \ker \rho$ so that $au^{-1} \in \ker \lambda$ whence $bu^{-1} \in \ker \lambda$ and hence $bu^{-1} \in \ker \rho$. Therefore $b\, h_\rho\, u$.

Let $\mathcal{P} = \mathcal{L}$. Then $a^0 u \rho u^0 a^0 u$ and $au^{-1} \in \ker \rho$. As above, the latter implies $bu^{-1} \in \ker \rho$. Also $a^0 u \rho \cap \mathcal{L} u^0 a^0 u$ and hence, by hypothesis, $a^0 u \lambda \cap \mathcal{L} u^0 a^0 u$ which, by hypothesis, implies $b^0 u \lambda \cap \mathcal{L} u^0 b^0 u$. But then $b^0 u \rho \cap \mathcal{L} u^0 b^0 u$ whence $b^0 u \rho u^0 b^0 u$ which together with $bu^{-1} \in \ker \rho$ gives $b\, l_\rho u$.

For $\mathcal{P} = \mathcal{D}$, the argument follows along the same lines as for $\mathcal{P} = \mathcal{L}$.

We have proved that $b\, p_\rho u$. By symmetry, we conclude that $a \widetilde{p}_\rho b$. It follows that $\lambda \subseteq \widetilde{p}_\rho$ whence $\lambda \subseteq \widetilde{p}_\rho^0$. This establishes the maximality of $\widetilde{p}_\rho^0$ and implies that $\rho \cap \mathcal{P} \subseteq \widetilde{p}_\rho^0 \cap \mathcal{P}$.

Next we show that $\widetilde{p}_\rho^0 \cap \mathcal{P} \subseteq \rho$. Let $a \widetilde{p}_\rho^0 \cap \mathcal{P} b$. Then $xay\, \widetilde{p}_\rho xby$ for all $x, y \in S^1$ and thus

$$xay\, p_\rho u \Leftrightarrow xby\, p_\rho u \qquad (x, y \in S^1,\ u \in S).$$

In particular, for $x = y = 1$ and $u = b$, we have $a\, p_\rho b$ since p_ρ is reflexive. We now consider several cases.

Let $\mathcal{P} = \mathcal{H}$. Then $a \mathcal{H} b$ so that $a^0 = b^0$. Also $a\, h_\rho b$ gives $ab^{-1} \in \ker \rho$, which together with $a^0 \rho b^0$ by Lemma 2.1 yields $a \rho b$.

Let $\mathcal{P} = \mathcal{L}$. Then $a \mathcal{L} b$ so that $a^0 = a^0 b^0$ and $b^0 = b^0 a^0$. Also $a \mathcal{L} b$ gives $a^0 b^0 \rho b^0 a^0 b^0$ and $ab^{-1} \in \ker \rho$. But then

$$a^0 = a^0 b^0 \rho b^0 a^0 b^0 = b^0$$

which together with $ab^{-1} \in \ker \rho$ by Lemma 2.1 yields $a \rho b$.

For $\mathcal{P} = \mathcal{D}$, the argument follows along the same lines as for $\mathcal{P} = \mathcal{L}$.

We have proved that $\widetilde{p}_\rho^0 \cap \mathcal{P} \subseteq \rho$ which implies that $\widetilde{p}_\rho^0 \cap \mathcal{P} \subseteq \rho \cap \mathcal{P}$. Above we have established the opposite inclusion. Therefore $\widetilde{p}_\rho^0\, \mathcal{P}^\wedge\, \rho$ which proves the maximality of ρ^P.

As an example of interplay of ρ_l, ρ_r and ρ_d, we shall prove only one result. For its proof, we need a lemma of independent interest.

LEMMA 3.6. – *For* $\rho \in \mathcal{C}(S)$, *we have*

(i) $(\rho \cap \mathcal{L})^* \vee (\rho \cap \mathcal{R})^* = ((\rho \cap \mathcal{L}) \vee (\rho \cap \mathcal{R}))^*$,

(ii) $\rho \cap \mathcal{D} = (\rho \cap \mathcal{L}) \vee (\rho \cap \mathcal{R})$,

where the second joins are taken within $\Pi(S)$.

PROOF. – (i) The left hand side is clearly contained in the right hand side. Also $\rho \cap \mathcal{L} \subseteq (\rho \cap \mathcal{L})^*$ and $\rho \cap \mathcal{R} \subseteq (\rho \cap \mathcal{R})^*$ imply that

$$(\rho \cap \mathcal{L}) \vee (\rho \cap \mathcal{R}) \subseteq (\rho \cap \mathcal{L})^* \vee (\rho \cap \mathcal{R})^*$$

whence the remaining inclusion.

(ii) The right hand side is obviously contained in the left hand side. Let $a \rho \cap \mathcal{D} b$ and let $T = D_a = D_b$. Giving T a Rees matrix representation with normalized sandwich matrix, letting

$$a = (i, g, \lambda), \qquad b = (j, h, \mu), \qquad c = (i, h, \mu),$$

and $\theta = \rho|_T$ with the admissible triple (r, N, π), we get $i\,r\,j$, $gh^{-1} \in N$ and $\lambda\,\pi\,\mu$, see ([1], III.4). It follows that $a\rho \cap \mathcal{L} c \rho \cap \mathcal{R} b$ so that $a(\rho \cap \mathcal{L})(\rho \cap \mathcal{R}) b$. Therefore $\rho \cap \mathcal{D} \subseteq (\rho \cap \mathcal{L}) \vee (\rho \cap \mathcal{R})$.

PROPOSITION 3.7. – *For* $\rho \in \mathcal{C}(S)$, *we have* $\rho_l \vee \rho_r = \rho_d$.

PROOF. – Indeed

$$\begin{aligned} \rho_l \vee \rho_r &= (\rho \cap \mathcal{L})^* \vee (\rho \cap \mathcal{R})^* \\ &= ((\rho \cap \mathcal{L}) \vee (\rho \cap \mathcal{R}))^* \qquad \text{by Lemma 3.6(i)} \\ &= \rho \cap \mathcal{D} = \rho_d \qquad \text{by Lemma 3.6(ii)}. \end{aligned}$$

4. – Properties of the relation $\mathcal{P}^\wedge$.

We shall now characterize the relation $\mathcal{P}^\wedge$ in several ways for which we need the following symbolism concerning varieties:

$$\begin{aligned} &\widehat{\mathcal{H}} = \mathcal{B} = [x = x^2], && \text{bands}, \\ &\widehat{\mathcal{L}} = \mathcal{RRB} = [x = x^2,\ xyx = yx], && \text{right regular bands}, \\ &\widehat{\mathcal{R}} = \mathcal{LRB} = [x = x^2,\ xyx = xy], && \text{left regular bands}, \\ &\widehat{\mathcal{D}} = \mathcal{S} = [x = x^2,\ xy = yx], && \text{semilattices}. \end{aligned}$$

THEOREM 4.1. – *For* $\mathcal{P} \in \mathcal{GR}$ *and* $\lambda, \rho \in \mathcal{C}(S)$, *the following statements are equivalent.*

(i) $\lambda \cap \mathcal{P} \subseteq \rho \cap \mathcal{P}$.

(ii) $\ker \lambda \subseteq \ker \rho$, $\operatorname{tr} \lambda_p \subseteq \operatorname{tr} \rho_p$.

(iii) $\lambda_p \subseteq \rho_p$.

(iv) $\lambda / (\lambda \wedge \rho) \subseteq p_\varepsilon$.

(v) $\rho|_{e\lambda}$ is a $\widehat{\mathcal{P}}$-congruence for every $e \in E(S)$.

PROOF. – (i) *implies* (ii). By Lemma 3.3(i), we get

$$\ker \lambda = \ker (\lambda \cap \mathcal{P})^* \subseteq \ker (\rho \cap \mathcal{P})^* = \ker \rho .$$

Also $\operatorname{tr} \lambda_p = \operatorname{tr} (\lambda \cap \rho)^* \subseteq \operatorname{tr} (\rho \cap \mathcal{P})^* = \operatorname{tr} \rho_p$.

(ii) *implies* (iii). Again by Lemma 3.3 (ii), we obtain

$$\ker \lambda_p = \ker(\lambda \cap \mathcal{P})^* = \ker \lambda \subseteq \ker \rho = \ker(\rho \cap \mathcal{P})^* = \ker \rho_p$$

which together with $\operatorname{tr} \lambda_p \subseteq \operatorname{tr} \rho_p$ implies that $\lambda_p \subseteq \rho_p$ by Corollary 2.2.

(iii) *implies* (iv). We consider $\mathcal{P} = \mathcal{L}$ first. Let $a, b \in S$. Then

$$\begin{aligned} a \lambda b &\Rightarrow a^0 b \,\lambda \cap \mathcal{L}\, b^0 a^0 b \\ &\Rightarrow a^0 b \,\lambda_l\, b^0 a^0 b \\ &\Rightarrow a^0 b \,\rho_l\, b^0 a^0 b \quad \text{by hypothesis} \\ &\Rightarrow a^0 b \,\rho\, b^0 a^0 b, \end{aligned}$$

$$\begin{aligned} a \lambda b &\Rightarrow ab^{-1} \in \ker \lambda \\ &\Rightarrow ab^{-1} \,\lambda \cap \mathcal{L}\, (ab^{-1})^2 \\ &\Rightarrow ab^{-1} \,\lambda_l\, (ab^{-1})^2 \\ &\Rightarrow ab^{-1} \,\rho_l\, (ab^{-1})^2 \text{ by hypothesis} \\ &\Rightarrow ab^{-1} \,\rho\, (ab^{-1})^2. \end{aligned}$$

If now $a(\lambda \wedge \rho)\, \lambda/(\lambda \wedge \rho)\, b(\lambda \wedge \rho)$, then $a \lambda b$ so by the above

$$(a(\lambda \wedge \rho))^0 (b(\lambda \wedge \rho)) = (b(\lambda \wedge \rho))^0 (a(\lambda \wedge \rho))^0 (b(\lambda \wedge \rho)),$$

$$(a(\lambda \wedge \rho))(b(\lambda \wedge \rho))^{-1} \in E(S/(\lambda \wedge \rho))$$

so that $a(\lambda \wedge \rho)\, l_\varepsilon\, b(\lambda \wedge \rho)$. Therefore $\lambda/(\lambda \wedge \rho) \subseteq l_\varepsilon$. The case $\mathcal{P} = \mathcal{D}$ follows along the same lines; the case $\mathcal{P} = \mathcal{H}$ requires only a part of the above argument with $\mathcal{H}$ replacing $\mathcal{L}$.

(iv) *implies* (v). Let $e \in E(S)$. Then

$$\begin{aligned} a \in e\lambda &\Rightarrow a \lambda a^0 \\ &\Rightarrow a(\lambda \wedge \rho)\, \lambda/(\lambda \wedge \rho)\, a^0 (\lambda \wedge \rho) \\ &\Rightarrow a(\lambda \wedge \rho)\, p_\varepsilon\, a^0 (\lambda \wedge \rho) \qquad \text{by hypothesis} \\ &\Rightarrow a(\lambda \wedge \rho) \in E(S/(\lambda \wedge \rho)) \\ &\Rightarrow a \in \ker(\lambda \wedge \rho) \subseteq \ker \rho \end{aligned}$$

and thus $\rho|_{e\lambda}$ is a band congruence. Next consider $\mathcal{P} = \mathcal{L}$. Then

$$\begin{aligned} a, b \in e\lambda &\Rightarrow a\,\lambda\, b \\ &\Rightarrow a(\lambda \wedge \rho)\,\lambda/(\lambda \wedge \rho)\, b(\lambda \wedge \rho) \\ &\Rightarrow a(\lambda \wedge \rho)\, p_\varepsilon\, b(\lambda \wedge \rho) \qquad \text{by hypothesis} \\ &\Rightarrow (a(\lambda \wedge \rho))^0 (b(\lambda \wedge \rho)) = (b(\lambda \wedge \rho))^0 (a(\lambda \wedge \rho))^0 (b(\lambda \wedge \rho)) \\ &\Rightarrow a^0 b\, \rho\, b^0 a^0 b \end{aligned}$$

so that $\rho|_{e\lambda}$ is a right regular band congruence. The case $\mathcal{P} = \mathcal{D}$ follows along the same lines and $\mathcal{P} = \mathcal{H}$ is included in the first part of the proof.

(v) *implies* (i). Let $a\,\lambda \cap \mathcal{P} b$. Then $ab^{-1}\lambda b^0$ so that $ab^{-1} \in b^0\lambda$ whence $ab^{-1}\rho(ab^{-1})^0$ by hypothesis which gives $ab^{-1} \in \ker\rho$.

If $a\,\lambda \cap \mathcal{H} b$, then $a^0 = b^0$ which together with $ab^{-1} \in \ker\rho$ by Lemma 2.1 gives $a\,\rho\, b$. Suppose that $a\,\lambda \cap \mathcal{L} b$. Then $a^0 \mathcal{L} b^0$ so that $a^0 = a^0 b^0$ and $b^0 = b^0 a^0$. Also a^0, $b^0 \in a^0\lambda$ which by hypothesis yields $a^0 b\,\rho\, b^0 a^0 b$. Consequently $a^0 = a^0 b^0 \rho\, b^0 a^0 b^0 = b^0$. This together with $ab^{-1} \in \ker\rho$ by Lemma 2.1 implies that $a\,\rho\, b$.

Therefore, for $\mathcal{P} = \mathcal{H}$ and $\mathcal{P} = \mathcal{L}$, we have $\lambda \cap \mathcal{P} \subseteq \rho$. The case $\mathcal{P} = \mathcal{D}$ follows along the same lines. Thus in all cases $\lambda \cap \mathcal{P} \subseteq \rho \cap \mathcal{P}$, as required.

As an immediate consequence, we have the following characterization of the relation $\mathcal{P}^\wedge$.

COROLLARY 4.2. – *For $\mathcal{P} \in \mathcal{GR}$ and $\lambda, \rho \in \mathcal{C}(S)$, the following statements are equivalent.*

(i) $\lambda\, \mathcal{P}^\wedge \rho$.

(ii) $\ker\lambda = \ker\rho$, $\operatorname{tr}\lambda_p = \operatorname{tr}\rho_p$.

(iii) $\lambda/(\lambda \wedge \rho)$, $\rho/(\lambda \wedge \rho) \subseteq p_\varepsilon$.

(iv) $\rho|_{e\lambda}$ and $\lambda|_{e\rho}$ are $\widehat{\mathcal{P}}$—congruences for every $e \in E(S)$.

The $\mathcal{P}^\wedge$-classes of the equality relation ε and the universal relation ω can now be easily characterized. To this end, we need a lemma of independent interest.

LEMMA 4.3. – *For any $\mathcal{P} \in \mathcal{GR}$, we have $p_\varepsilon \cap \mathcal{P} = \varepsilon$.*

PROOF. – Let $a\,p_\varepsilon \cap \mathcal{P} b$. Then $ab^{-1} \in E(S)$. If $a\,h_\varepsilon \cap \mathcal{H} b$, then $a^0 = b^0$. If $a\,l_\varepsilon \cap \mathcal{L} b$, then $a^0 = a^0 b^0$, $b^0 = b^0 a^0$, and $a^0 b = b^0 a^0 b$ whence again $a^0 = b^0$. The case $\mathcal{P} = \mathcal{D}$ is similar. Therefore $a^0 = b^0$ in all cases which with $ab^{-1} \in E(S)$ by Lemma 2.1 implies that $a = b$.

COROLLARY 4.4. – *For $\mathcal{P} \in \mathcal{GR}$ and $\rho \in \mathcal{C}(S)$, the following statements are equivalent.*

(i) $\rho\, \mathcal{P}^{\wedge}\, \varepsilon$.

(ii) $\ker \rho = E(S)$, $\operatorname{tr} \rho_p = \varepsilon$.

(iii) $\rho \subseteq p_{\varepsilon}$.

(iv) ρ *is over* $\widehat{\mathcal{P}}$.

(v) $\rho \cap \mathcal{P} = \varepsilon$.

PROOF. – The equivalence of the first four items follows by specializing the statements of Corollary 4.2 to $\lambda = \varepsilon$. That (iii) implies (v) follows directly from Lemma 4.3. Clearly (v) implies (i).

The relations on a set are said to be *disjoint* if their intersection is the equality relation. Hence $\varepsilon\mathcal{P}^{\wedge}$ consists of congruences disjoint from $\mathcal{P}$; in particular ε^{P} is the greatest such.

COROLLARY 4.5. – *For* $\mathcal{P} \in \mathcal{GR}$ *and* $\rho \in \mathcal{C}(S)$, *the following statements are equivalent.*

(i) $\rho\, \mathcal{P}^{\wedge}\, \omega$.

(ii) ρ *is a band congruence and* ρ_p *is a group congruence.*

(iii) $\omega/\rho \subseteq p_{\varepsilon}$.

(iv) ρ *is a* $\widehat{\mathcal{P}}$*-congruence.*

(v) $\mathcal{P} \subseteq \rho$.

PROOF. – The equivalence of the first four items follows by specializing the statements of Corollary 4.2 to $\lambda = \omega$. Items (i) and (v) are obviously equivalent.

We conclude that $\omega\mathcal{P}^{\wedge}$ consists precisely of congruences containing ρ so that $\omega\mathcal{P}^{\wedge} = [\mathcal{P}^{*}, \omega]$.

We have seen in Theorem 3.5 that $\mathcal{P}^{\wedge}$ is always a (complete) $\wedge$-congruence but need not be a $\vee$-congruence. We shall now investigate the relationship of χ_p being a homomorphism vs. $\mathcal{P}^{\wedge}$ being a congruence. To this end, we need a lemma.

LEMMA 4.6. – *For* $\mathcal{P} \in \mathcal{GR}$ *and* $\lambda, \rho \in \mathcal{C}(S)$, *we have*

$$(\lambda \cap \mathcal{P}) \vee (\rho \cap \mathcal{P}) = (\lambda_p \vee \rho_p) \cap \mathcal{P}$$

in the lattice $\Pi_p(S)$.

PROOF. – First $\lambda \cap \mathcal{P} = \lambda_p \cap \mathcal{P} \subseteq (\lambda_p \vee \rho_p) \cap \mathcal{P}$ and similarly $\rho \cap \mathcal{P} \subseteq (\lambda_p \vee \rho_p) \cap \mathcal{P}$. Now let $\theta \in \mathcal{C}(S)$ be such that $\lambda \cap \mathcal{P}$, $\rho \cap \mathcal{P} \subseteq \theta \cap \mathcal{P}$. In view of Theorem 4.1, we have $\lambda_p \subseteq \theta_p$ and $\rho_p \subseteq \theta_p$ so that $\lambda_p \vee \rho_p \subseteq \theta$ whence $(\lambda_p \vee \rho_p) \cap \mathcal{P} \subseteq \theta \cap \mathcal{P}$. The assertion of the lemma follows.

PROPOSITION 4.7. – *The map χ_p is a homomorphism if and only if the relation $\mathcal{P}^\wedge$ is a congruence.*

PROOF. – The direct part is well known. For the converse, let $\lambda, \rho \in \mathcal{C}(S)$. Since $\lambda_p \mathcal{P}^\wedge \lambda$ and $\rho_p \mathcal{P}^\wedge \rho$, the hypothesis implies that $\lambda_p \vee \rho_p \mathcal{P}^\wedge \lambda \vee \rho$. This together with Lemma 4.6 yields $(\lambda \cap \mathcal{P}) \vee (\rho \cap \mathcal{P}) = (\lambda \vee \rho) \cap \mathcal{P}$ and χ_p is a $\vee$-homomorphism; it is always a $\wedge$-homomorphism.

5. – The relation $\mathcal{P}^\vee$.

For a Green relation $\mathcal{P}$ we have defined in Notation 2.3 a relation $\mathcal{P}^\vee$ by

$$\lambda \mathcal{P}^\vee \rho \Leftrightarrow \lambda \vee \mathcal{P} = \rho \vee \mathcal{P} \qquad (\lambda, \rho \in \mathcal{C}(S)).$$

In this and the next section, we shall study a number of properties of this (equivalence) relation. We have already remarked the obvious fact that $\mathcal{P}^\vee$ is induced by the mapping $\chi_P \colon \mathcal{C}(S) \to \Pi(S)$. The following notation will be used to characterize the image of $\mathcal{C}(S)$ in $\Pi(S)$ under χ_P.

NOTATION 5.1. – *For $\mathcal{P} \in \mathcal{GR}$, let*

$$\Pi_P(S) = \{\theta \in \Pi(S) \mid \theta = \theta^0 \vee \mathcal{P}\}.$$

LEMMA 5.2. – *For $\mathcal{P} \in \mathcal{GR}$ and $\theta \in \Pi(S)$, $\theta = \theta^0 \mathcal{P}$ if and only if $\theta = \rho \vee \mathcal{P}$* for some $\rho \in \mathcal{C}(S)$.

PROOF. – For the direct part, take $\rho = \theta^0$. Conversely, $\theta = \rho \vee \mathcal{P}$ implies

$$\theta = \theta \vee \mathcal{P} \supseteq \theta^0 \vee \mathcal{P} \supseteq \rho \vee \mathcal{P} = \theta.$$

The next result will be used repeatedly. (The formula $\rho \vee \mathcal{H} = \rho\mathcal{H}\rho$ below is due to N. R. REILLY.)

LEMMA 5.3. – *Let $\rho \in \mathcal{C}(S)$ and $\mathcal{P} \in \mathcal{GR}$. Then $\rho \vee \mathcal{P} = \rho\mathcal{P}\rho$.*

(i) $a \rho \vee \mathcal{H} b \Leftrightarrow a^0 \rho b^0$ $(a, b \in S)$.

(ii) $a \rho \vee \mathcal{L} b \Leftrightarrow a \rho ab^0$, $b \rho ba^0$ $(a, b \in S)$.

In addition $\rho \vee \mathcal{H} = \mathcal{H}\rho\mathcal{H}$.

PROOF. – If $a \mathcal{H} x \rho y \mathcal{H} b$ and $g = (x^0 y^0)^0$, then $a \rho gag \mathcal{H} gbg \rho b$. Hence $\mathcal{H}\rho\mathcal{H} \subseteq \rho\mathcal{H}\rho$ which evidently implies that $\rho\mathcal{H}\rho = \rho \vee \mathcal{H}$.

(i) If $a \rho \vee \mathcal{H} b$, there exists a sequence

$$a \rho x_1 \mathcal{H} x_2 \rho \ldots x_n \mathcal{H} b$$

whence

$$a^0 \rho x_1^0 = x_2^0 \rho \dots x_n^0 = b^0$$

so that $a^0 \rho b^0$. Conversely, if $a^0 \rho b^0$, then $a \mathcal{H} a^0 \rho b^0 \mathcal{H} b$ so that $a \rho \vee \mathcal{H} b$. The last assertion of the lemma is now evident.

(ii) Indeed,

$$\begin{aligned}
a \rho\mathcal{L}\rho b &\Leftrightarrow a \rho x \mathcal{L} y \rho b \quad \text{for some } x, y \in S \\
&\Rightarrow a\rho = x\rho \,\mathcal{L}\, y\rho = b\rho \\
(1) \qquad &\Rightarrow a\rho \,\mathcal{L}\, b\rho \\
&\Rightarrow a\rho = (a\rho)(b\rho)^0, \quad b\rho = (b\rho)(a\rho)^0 \\
(2) \qquad &\Rightarrow a \rho ab^0, \quad b \rho ba^0 .
\end{aligned}$$

Let $x = ab^0$ and $y = ba^0 b^0$. Then

$$x = ab^0 = a(a^0 b^0) = a(a^0 b^0)^{-1}(a^0 b^{-1})(ba^0 b^0) \in Sy ,$$

$$y = ba^0 b^0 = (ba^{-1})(ab^0) \in Sx$$

so that $x \mathcal{L} y$. Now (2) yields $a \rho x \mathcal{L} y \rho b$ so that $a \rho\mathcal{L}\rho b$. Therefore all the above statements are equivalent and part (1) shows that $\rho\mathcal{L}\rho$ is an equivalence relation. But then $\rho \vee \mathcal{L} = \rho\mathcal{L}\rho$ and part (2) gives the first assertion.

The formula $\rho \vee \mathcal{D} = \rho\mathcal{D}\rho$ was proved in ([3], Proposition 8.1 and [4], Lemma 2.1 (v)).

We shall need the following symbolism.

NOTATION 5.4. – *Let the upper bar denote the permutation $(H\,D)(L\,R)$. For $\mathcal{P} \in \mathcal{GR}$, we define a relation $\leqslant_P$ by: for $e, f \in E(S)$,*

$$e \leqslant_L f \Leftrightarrow e = ef, \qquad e \leqslant_R f \Leftrightarrow e = fe ,$$

$$\leqslant_H = \leqslant_L \cap \leqslant_R, \qquad e \leqslant_D f \Leftrightarrow e \in SfS .$$

We are now ready for the principal result of this section.

THEOREM 5.5. – *For $\mathcal{P} \in \mathcal{GR}$. The mapping*

$$\chi_P : \rho \to \rho \vee \mathcal{P} \qquad (\rho \in \mathcal{C}(S))$$

is a complete homomorphism of $\mathcal{C}(S)$ onto $\Pi_P(S)$ which induces $\mathcal{P}^\vee$. Consequently $\mathcal{P}^\vee$ is a complete congruence on $\mathcal{C}(S)$. For each $\rho \in \mathcal{C}(S)$, we have $\rho\mathcal{P}^\vee = [\rho_P, \rho^P]$ where

$$\rho_P = (\rho \cap \leqslant_{\bar{P}})^*, \qquad \rho^P = (\rho \vee \mathcal{P})^0 .$$

PROOF. – Clearly χ_P is a $\vee$-homomorphism. In order to show that it is a $\wedge$-homomorphism, we let $\mathcal{F} \subseteq \mathcal{C}(S)$ and remark that the inclusion

$$(3) \qquad (\bigwedge_{\rho \in \mathcal{F}} \rho) \vee \mathcal{P} \subseteq \bigwedge_{\rho \in \mathcal{F}} (\rho \vee \mathcal{P})$$

holds trivially. For the opposite inclusion, we let $a \bigwedge_{\rho \in \mathcal{F}} (\rho \vee \mathcal{P})\, b$ so that $a \rho \vee \mathcal{P} b$ for all $\rho \in \mathcal{F}$ and consider several cases.

Let $\mathcal{P} = \mathcal{H}$. By Lemma 5.3 (i), we have $a^0 \rho b^0$ for all $\rho \in \mathcal{F}$ and hence $a^0 \bigwedge_{\rho \in \mathcal{F}} \rho\, b^0$ so that $a(\bigwedge_{\rho \in \mathcal{F}} \rho) \vee \mathcal{H} b$ again by Lemma 5.3 (i). For $\mathcal{P} = \mathcal{L}$, the argument follows along the same lines by using Lemma 5.3 (ii). This proves the opposite inclusion for $\mathcal{P} = \mathcal{H}$ and $\mathcal{P} = \mathcal{L}$ so that χ_P is a complete homomorphism for $\mathcal{P} \in \{\mathcal{H}, \mathcal{L}\}$; for $\mathcal{P} = \mathcal{D}$ this was proved in ([4], Theorem 4.3). In particular, $\mathcal{P}^\vee$ is a complete congruence on $\mathcal{C}(S)$.

Now let $\rho \in \mathcal{C}(S)$. We consider first the lower end of $\rho\mathcal{P}^\vee$. Clearly

$$\rho \vee \mathcal{P} \supseteq (\rho \cap \leqslant_P)^* \vee \mathcal{P}.$$

For the opposite inclusion, we let $a \rho \vee \mathcal{P} b$ and consider several cases.

Let $\mathcal{P} = \mathcal{H}$. By Lemma 5.3 (i), we have $a^0 \rho b^0$ which yields $(a^0 b^0)^0 \rho \cap \leqslant_D a^0$ and $(a^0 b^0)^0 \rho \cap \leqslant_D b^0$ so that

$$a^0 (\rho \cap \leqslant_D)^* (a^0 b^0)^0 (\rho \cap \leqslant_D)^* b^0$$

and hence $a^0 (\rho \cap \leqslant_D)^* b^0$. Now Lemma 5.3 (i) gives that $a (\rho \cap \leqslant_D)^* \vee \mathcal{H} b$.

Next let $\mathcal{P} = \mathcal{L}$. By Lemma 5.3 (ii), we have $a \rho ab^0$ and $b \rho ba^0$. Letting $e = a^0$ and $f = b^0$, we first obtain $e \rho ef$ and $f \rho fe$ which yields

$$(4) \qquad (ef)^0 \rho \cap \leqslant_R e, \qquad (fe)^0 \rho \cap \leqslant_R f.$$

The second part of (4) gives $(fe)^0 (\rho \cap \leqslant_R)^* f$ whence

$$(5) \qquad (fe)^0 f (\rho \cap \leqslant_R)^* f.$$

Furthermore

$$[(fe)^0 f](ef)^0 = (fe)^{-1} fef(ef)^0 = (fe)^{-1} fef = (fe)^0 f,$$
$$(ef)^0 [(fe)^0 f] = (ef)^{-1} ef(fe)^0 f = (ef)^{-1} fe(fe)^0 f = (ef)^{-1} fef = (ef)^0$$

and thus $(ef)^0 \mathcal{L} (fe)^0 f$. Now by the first part of (4) and (5), we get

$$a \mathcal{L} e (\rho \cap \leqslant_R)^* (ef)^0 \mathcal{L} (fe)^0 f (\rho \cap \leqslant_R)^* f \mathcal{L} b$$

so that $a (\rho \cap \leqslant_R)^* \vee \mathcal{L} b$.

For $\mathcal{P} \in \{\mathcal{H}, \mathcal{L}\}$, this proves that $\rho \vee \mathcal{P} \subseteq (\rho \cap \leqslant_{\bar{P}})^* \vee \mathcal{P}$ whence the equality so that $\rho \mathcal{P}^\vee (\rho \cap \leqslant_{\bar{P}})^*$.

To show minimality of $(\rho \cap \leqslant_{\bar{P}})^*$, we let $\lambda \in \mathcal{C}(S)$ be such that $\lambda \mathcal{P}^\vee \rho$ and $e \rho \cap \leqslant_{\bar{P}} f$.

Let $\mathcal{P} = \mathcal{H}$. Then $\lambda \vee \mathcal{H} = \rho \vee \mathcal{H}$ and $e \rho \cap \leqslant_D f$. Hence $e \rho f$ which by Lemma 5.3 (i) and $\lambda \vee \mathcal{H} = \rho \vee \mathcal{H}$ implies that $e \lambda f$. Thus $\rho \cap \leqslant_D \subseteq \lambda$. Next let $\mathcal{P} = \mathcal{L}$. Then $\lambda \vee \mathcal{L} = \rho \vee \mathcal{L}$ and $e \rho \cap \leqslant_R f$. Hence $e \rho ef$ and $f \rho fe$ which by Lemma 5.3 (ii) and $\lambda \vee \mathcal{L} = \rho \vee \mathcal{L}$ implies $f \lambda fe$. But $e \leqslant_R f$ gives $fe = e$ so that $e \lambda f$. Therefore $\rho \cap \leqslant_R \subseteq \lambda$. We thus have $\rho \cap \leqslant_{\overline{P}} \subseteq \lambda$ whence $(\rho \cap \leqslant_{\overline{P}})^* \subseteq \lambda$ which establishes the minimality of ρ_P for $\mathcal{P} \in \{\mathcal{H}, \mathcal{L}\}$. The case $\mathcal{P} = \mathcal{D}$ follows from ([4], Theorem 4.3 (i)).

It remains to consider the upper end. Since $\rho \subseteq \rho \vee \mathcal{P}$, we have $\rho \subseteq (\rho \vee \mathcal{P})^0$ whence $\rho \vee \mathcal{P} \subseteq (\rho \vee \mathcal{P})^0 \vee \mathcal{P}$. Conversely, $(\rho \vee \mathcal{P})^0 \subseteq \rho \vee \mathcal{P}$ implies that $(\rho \vee \mathcal{P})^0 \vee \mathcal{P} \subseteq \rho \vee \mathcal{P}$. Therefore $\rho \vee \mathcal{P} = (\rho \vee \mathcal{P})^0 \vee \rho$, that is $\rho \mathcal{P}^{\vee} (\rho \vee \mathcal{P})^0$. If $\lambda \in \mathcal{C}(S)$ is such that $\lambda \mathcal{P}^{\vee} \rho$, then $\lambda \subseteq \lambda \vee \mathcal{P} = \rho \vee \mathcal{P}$ whence $\lambda \subseteq (\rho \vee \mathcal{P})^0$. This proves the maximality of ρ^P.

As an example of interplay of ρ^L, ρ^R and ρ^H, we shall prove only one result. For its proof, we need a lemma of independent interest.

LEMMA 5.6. – *For* $\rho \in \mathcal{C}(S)$, *we have*

(i) $(\rho \vee \mathcal{L})^0 \wedge (\rho \vee \mathcal{R})^0 = ((\rho \vee \mathcal{L}) \cap (\rho \vee \mathcal{R}))^0$,

(ii) $\rho \vee \mathcal{H} = (\rho \vee \mathcal{L}) \cap (\rho \vee \mathcal{R})$.

PROOF. – (i) Since $(\rho \vee \mathcal{L})^0 \subseteq \rho \vee \mathcal{L}$ and $(\rho \vee \mathcal{R})^0 \subseteq \rho \vee \mathcal{R}$, we have

$$(\rho \vee \mathcal{L})^0 \vee (\rho \vee \mathcal{R})^0 \subseteq (\rho \vee \mathcal{L}) \cap (\rho \vee \mathcal{R})$$

whence the inclusion

$$(\rho \vee \mathcal{L})^0 \vee (\rho \vee \mathcal{R})^0 \subseteq ((\rho \vee \mathcal{L}) \cap (\rho \vee \mathcal{R}))^0 .$$

The opposite inclusion is obvious.

(ii) Let $a (\rho \vee \mathcal{L}) \cap (\rho \vee \mathcal{R}) b$. By Lemma 5.3 (ii) and its dual, we have $a \rho ab^0$ and $b \rho a^0 b$ whence $a^0 \rho a^0 b^0 \rho b^0$. But then Lemma 5.3 (i) gives $a \rho \vee \mathcal{H} b$. Therefore $(\rho \vee \mathcal{L}) \cap (\rho \vee \mathcal{R}) \subseteq \rho \vee \mathcal{H}$ and the opposite inclusion holds trivially.

PROPOSITION 5.7. – *For* $\rho \in \mathcal{C}(S)$, *we have* $\rho^L \wedge \rho^R = \rho^H$.

PROOF. – Indeed,

$$\begin{aligned} \rho^L \wedge \rho^R &= (\rho \vee \mathcal{L})^0 \wedge (\rho \vee \mathcal{R})^0 \\ &= ((\rho \vee \mathcal{L}) \cap (\rho \vee \mathcal{R}))^0 \qquad \text{by Lemma 5.6(i)} \\ &= (\rho \vee \mathcal{H})^0 \qquad \text{by Lemma 5.6(ii)} \\ &= \rho^H . \end{aligned}$$

6. - Properties of the relation $\mathcal{P}^{\vee}$.

We shall now characterize the relation $\mathcal{P}^{\vee}$ in several ways for which we need the following symbolism concerning varieties:

$$\begin{array}{ll} \check{\mathcal{H}} = \mathcal{G} = [x^0 = y^0], & \text{groups}, \\ \check{\mathcal{L}} = \mathcal{L}\mathcal{G} = [x^0 y^0 = x^0], & \text{left groups}, \\ \check{\mathcal{R}} = \mathcal{R}\mathcal{G} = [x^0 y^0 = y^0], & \text{right groups}, \\ \check{\mathcal{D}} = \mathcal{C}\mathcal{S} = [(xyx)^0 = x^0], & \text{completely simple semigroups}. \end{array}$$

THEOREM 6.1. - *For $\mathcal{P} \in \mathcal{G}\mathcal{R}$ and $\lambda, \rho \in \mathcal{C}(S)$, the following statements are equivalent.*

(i) $\lambda \vee \mathcal{P} \subseteq \rho \vee \mathcal{P}$.

(ii) $\lambda \cap \leqslant_{\bar{P}} \subseteq \rho \cap \leqslant_{\bar{P}}$.

(iii) $\lambda_P \subseteq \rho_P$.

(iv) $\lambda^P \subseteq \rho^P$.

(v) $\lambda/(\lambda \wedge \rho) \subseteq \mathcal{P}$.

(vi) $\rho|_{e\lambda}$ *is a* $\check{\mathcal{P}}$*-congruence for every* $e \in E(S)$.

PROOF. - (i) *implies* (ii). By hypothesis $\lambda \subseteq \rho \vee \mathcal{P}$. We let $e, f \in E(S)$ be such that $e \lambda \cap \leqslant_{\bar{P}} f$ and distinguish several cases.

For $\mathcal{P} = \mathcal{H}$, $e \lambda f$ implies $e \rho \vee \mathcal{H} f$ so by Lemma 5.3 (i), we get $e \rho f$ and thus $e\rho \cap \leqslant_D f$. Let $\mathcal{P} = \mathcal{L}$. Then $e\lambda \cap \leqslant_R f$ so that $e \rho \vee \mathcal{L} f$ and $e = fe$. By Lemma 5.3 (ii), $f\rho fe$ and thus $e\rho \cap \leqslant_R f$. Finally let $\mathcal{P} = \mathcal{D}$. Then $\lambda \subseteq \rho \vee \mathcal{D}$ and $e\lambda \cap \leqslant_H f$ so that $e\rho \vee \mathcal{D} f$. By Lemma 5.3, we have $e \rho x \mathcal{D} y \rho f$ for some $x, y \in S$, whence $e\rho \mathcal{D} f\rho$. Also $e \leqslant_H f$ implies $e\rho \leqslant_H f\rho$ which then gives $e\rho = f\rho$. Therefore $e\rho \cap \leqslant_H f$.

(ii) *implies* (iii). This follows directly from the expression for $(\ )_P$.

(iii) *implies* (iv). The expression for $(\ )^P$ is clearly order preserving. Consequently

$$\lambda^P = (\lambda_P)^P \subseteq (\rho_P)^P = \rho^P.$$

(iv) *implies* (v). The hypothesis implies that $\lambda \subseteq \lambda^P \subseteq \rho^P \subseteq \rho \vee \mathcal{P}$. Let $a \lambda b$. For $\mathcal{P} = \mathcal{H}$, by Lemma 5.2 (i), we get $a^0 \rho b^0$ which together with $a^0 \lambda b^0$ implies $a(\lambda \wedge \rho) \mathcal{H} b(\lambda \wedge \rho)$ so that $\lambda/(\lambda \wedge \rho) \subseteq \mathcal{H}$. For $\mathcal{P} = \mathcal{L}$, by Lemma 5.3 (ii), we get $a \rho ab^0$ and $b \rho ba^0$ which together with $a \lambda ab^0$ and $b \lambda ba^0$ gives

$$a(\lambda \wedge \rho) = a(\lambda \wedge \rho)(b(\lambda \wedge \rho))^0, \qquad b(\lambda \wedge \rho) = b(\lambda \wedge \rho)(a(\lambda \wedge \rho))^0$$

so that $a(\lambda \wedge \rho) \mathcal{L} b(\lambda \wedge \rho)$ which proves that $\lambda/(\lambda \wedge \rho) \subseteq \mathcal{L}$. Finally let $\mathcal{P} = \mathcal{D}$. As above, $a\,\rho \vee \mathcal{D}\, b$ implies $a\rho\, \mathcal{D}\, b\rho$; we also have $a\lambda = b\lambda$ so that $a(\lambda \wedge \rho)\, \mathcal{D}\, b(\lambda \wedge \rho)$. Consequently $\lambda/(\lambda \wedge \rho) \subseteq \mathcal{D}$.

(v) *implies* (vi). Let $e, f \in E(S)$ be such that $e\,\lambda f$. Then

$$e(\lambda \wedge \rho)\,\lambda/(\lambda \wedge \rho)\, f(\lambda \wedge \rho)$$

and the hypothesis implies that $e(\lambda \wedge \rho)\,\mathcal{P} f(\lambda \wedge \rho)$. If $\mathcal{P} = \mathcal{H}$, then $e(\lambda \wedge \rho) = f(\lambda \wedge \rho)$ whence $e\,\rho f$ and thus $\rho|_{e\lambda}$ is a group congruence. If $\mathcal{P} = \mathcal{L}$, then $e(\lambda \wedge \rho)\,\mathcal{L} f(\lambda \wedge \rho)$ whence $e(\lambda \wedge \rho) = e(\lambda \wedge \rho) f(\lambda \wedge \rho)$ so that $e\,\rho\, ef$ and $\rho|_{e\lambda}$ is a left group congruence. If $\mathcal{P} = \mathcal{D}$, then $e(\lambda \wedge \rho)\,\mathcal{D} f(\lambda \wedge \rho)$ so that

$$e(\lambda \wedge \rho) = x(\lambda \wedge \rho) f(\lambda \wedge \rho) y(\lambda \wedge \rho)$$

for some $x, y \in S$ whence $e\,\rho\, xfy$ and analogously $f\rho\, uev$ for some $u, v \in S$ which proves that $\rho|_{e\lambda}$ is a completely simple congruence.

(vi) *implies* (i). We let $a\,\lambda\, b$ and consider several cases. If $\mathcal{P} = \mathcal{H}$, then $a^0 \in b^0\lambda$ so, by hypothesis, $a^0 \rho\, b^0$ which by Lemma 5.3 (i) yields $a\,\rho \vee \mathcal{H} b$. If $\mathcal{P} = \mathcal{L}$, we similarly get $a^0 \rho\, a^0 b^0$ and $b^0 \rho\, b^0 a^0$ so that

$$a\,\mathcal{L}\, a^0 \rho\, a^0 b^0 a^0\, \mathcal{L}\, b^0 a^0 \rho\, b^0\, \mathcal{L}\, b$$

and thus $a\,\rho \vee \mathcal{L}\, b$. If $\mathcal{P} = \mathcal{D}$, then we similarly get $a^0 \rho\, x b^0 y$ and $b^0 \rho\, u a^0 v$ for some $x, y, u, v \in S$ whence

$$a\,\mathcal{D}\, a^0 \rho\, x b^0 y\, \rho\, x(ua^0 v)\, b^0 y\, \mathcal{D}\, (ua^0 v)(xb^0 y)\, \rho\, (ua^0 v)\, a^0\, \mathcal{D}\, ua^0 v\, \rho\, b^0$$

and thus $a\,\rho \vee \mathcal{D}\, b$.

Therefore, in all cases $\lambda \subseteq \rho \vee \mathcal{P}$ whence $\lambda \vee \mathcal{P} \subseteq \rho \vee \mathcal{P}$.

As an immediate consequence, we have the following characterization of the relation $\mathcal{P}^{\vee}$.

COROLLARY 6.2. – *For $\mathcal{P} \in \mathcal{GR}$ and $\lambda, \rho \in \mathcal{C}(S)$, the following statements are equivalent.*

(i) $\lambda \mathcal{P}^{\vee} \rho$.

(ii) $\lambda \cap \leqslant_{\bar{\mathcal{P}}} = \rho \cap \leqslant_{\bar{\mathcal{P}}}$.

(iii) $\lambda/(\lambda \wedge \rho),\ \rho/(\lambda \wedge \rho) \subseteq \mathcal{P}$.

(vi) $\rho|_{e\lambda}$ *and* $\lambda|_{e\rho}$ *are* $\check{\mathcal{P}}$*-congruence for every* $e \in E(S)$.

The $\mathcal{P}^{\vee}$-classes of the equality relation ε and the universal relation ω can now be easily characterized. Compare the next corollary with ([3], Theorem 2.18) for $\mathcal{P} = \mathcal{R}$ and with ([3], Corollary 2.21) for $\mathcal{P} = \mathcal{H}$

COROLLARY 6.3. – *For $\mathcal{P} \in \mathcal{GR}$ and $\lambda, \rho \in \mathcal{C}(S)$, the following statements are equivalent.*

(i) $\rho\, \mathcal{P}^{\vee}\, \varepsilon$.

(ii) $\rho \cap \leqslant_{\bar{P}} = \varepsilon$.

(iii) $\rho \subseteq \mathcal{P}$.

(vi) ρ *is over* $\check{\mathcal{P}}$.

PROOF. – This follows from Corollary 6.2 by letting $\lambda = \varepsilon$.

We conclude that $\varepsilon\mathcal{P}^{\vee}$ consists precisely of congruences contained in $\mathcal{P}$ so that $\varepsilon\mathcal{P}^{\vee} = [\varepsilon, \mathcal{P}^0]$.

COROLLARY 6.4. – *For* $\mathcal{P} \in \mathcal{G}\mathcal{R}$ *and* $\rho \in \mathcal{C}(S)$, *the following statements are equivalent.*

(i) $\rho\, \mathcal{P}^{\vee}\, \omega$.

(ii) $\leqslant_{\bar{P}} \subseteq \rho$.

(iii) $\omega/\rho \subseteq \mathcal{P}$.

(vi) ρ *is a* $\check{\mathcal{P}}$*-congruence.*

(v) $\rho \vee \mathcal{P} = \omega$.

PROOF. – The equivalence of the first four items follows from Corollary 6.2 by letting $\lambda = \omega$. Clearly (v) is a reformulation of (i).

In particular, ω_P is the least congruence θ on S for which $\theta \vee \mathcal{P} = \omega$.

We shall now characterize the $\mathcal{P}^{\vee}$-class of $\mathcal{P}^*$. To this end, we need some preparation which is of independent interest.

LEMMA 6.5. – *For* $\mathcal{P} \in \mathcal{G}\mathcal{R}$, $\rho \in \mathcal{C}(S)$ *and* $a, b \in S$, *have*

$$a\, \rho \vee \mathcal{P}\, b \Leftrightarrow a\rho\, \mathcal{P}\, b\rho\,.$$

PROOF. – Indeed, by Lemma 5.2, we have

$$a\, \rho \vee \mathcal{H}\, b \Leftrightarrow a^0\, \rho\, b^0 \Leftrightarrow (a\rho)^0 = (b\rho)^0 \Leftrightarrow a\rho\, \mathcal{H}\, b\rho\,,$$

$$\begin{aligned} a\, \rho \vee \mathcal{L}\, b &\Leftrightarrow a\, \rho\, ab^0,\ b\, \rho\, ba^0 \\ &\Leftrightarrow a\rho = (a\rho)(b\rho)^0, \quad b\rho = (b\rho)(a\rho)^0 \Leftrightarrow a\rho\, \mathcal{L}\, b\rho\,, \end{aligned}$$

$$\begin{aligned} a\, \rho \vee \mathcal{D}\, b &\Rightarrow a\, \rho\, x\, \mathcal{D}\, y\, \rho\, b \text{ for some } x, y \in S \Rightarrow a\rho\, \mathcal{D}\, b\rho \\ &\Rightarrow a\rho = (x\rho)(b\rho)(y\rho), \quad b\rho = (u\rho)(a\rho)(v\rho) \text{ for some } x, y, u, v \in S \\ &\Rightarrow a\, \rho\, xby, \quad b\, \rho\, uav \\ &\Rightarrow a\, \rho\, x\, \mathcal{D}\, b \quad \text{as at the end of the proof of Theorem 6.1.} \end{aligned}$$

COROLLARY 6.6. – *For $\mathcal{P} \in \mathcal{GR}$ and $\rho \in \mathcal{C}(S)$, we have*

$$\rho \vee \mathcal{P} \text{ is a congruence} \Leftrightarrow \mathcal{P} \text{ is a congruence on } S/\rho .$$

PROOF. – Using Lemma 6.5, for any $a, b, c \in S$, we get

$$a \rho \vee \mathcal{P} b \Rightarrow ca \rho \vee \mathcal{P} cb , \quad ac \rho \vee \mathcal{P} bc$$

if and only if

$$a\rho \mathcal{P} b\rho \Rightarrow (c\rho)(a\rho) \mathcal{P} (c\rho)(a\rho) , \quad (a\rho)(c\rho) \mathcal{P} (b\rho)(c\rho)$$

whence the assertion.

We can now give the desired characterization.

PROPOSITION 6.7. – *For $\mathcal{P} \in \mathcal{GR}$ and $\rho \in \mathcal{C}(S)$, the following statements are equivalent.*

(i) $\rho \mathcal{P}^{\vee} \mathcal{P}^*$.

(ii) $\rho \vee \mathcal{P} = \mathcal{P}^*$.

(iii) $\rho \subseteq \mathcal{P}^*$ *and* $\rho \vee \mathcal{P}$ *is a congruence.*

(iv) $\rho \subseteq \mathcal{P}^*$ *and* $\mathcal{P}$ *is a congruence on* S/ρ.

PROOF. – Item (ii) is a reformulation of item (i). Items (iii) and (iv) are equivalent by Corollary 6.6. Item (ii) trivially implies item (iii).

(iii) *implies* (ii). First $\rho = \mathcal{P}^*$ implies $\rho \vee \mathcal{P} \subseteq \mathcal{P}^*$. Conversely, since $\mathcal{P} \subseteq \rho \vee \mathcal{P}$ and the latter is a congruence, we also have $\mathcal{P}^* \subseteq \rho \vee \mathcal{P}$.

7. – Properties of $\mathcal{P}^{\wedge}$- and $\mathcal{P}^{\vee}$-classes and of their ends.

The main result here provides an isomorphism of the part of a $\mathcal{P}^{\wedge}$- or $\mathcal{P}^{\vee}$-class above a given congruence ρ and a certain $\mathcal{P}^{\wedge}$- or $\mathcal{P}^{\vee}$-class of the equality congruence on S/ρ. We also study greatest congruences over $\mathcal{V}$ and least $\mathcal{V}$-congruences for some familiar varieties $\mathcal{V}$. This is illustrated by two diagrams. Recall the notation $[\alpha)$ from Section 2.

THEOREM 7.1. – *Let $\mathcal{P} \in \mathcal{GR}$, $+ \in \{\wedge, \vee\}$ and $\rho \in \mathcal{C}(S)$. Then the mapping*

$$\varphi \colon \lambda \to \lambda/\rho \qquad (\lambda \in \rho\mathcal{P}^{+} \cap [\rho))$$

is an isomorphism of $\rho\mathcal{P}^{+} \cap [\rho)$ onto $\varepsilon_{S/\rho}\mathcal{P}^{+}$.

PROOF. – It is well known and easy to prove that the mapping

$$\psi \colon \lambda \to \lambda/\rho \qquad (\lambda \in [\rho))$$

is an isomorphism of $[\rho)$ onto $\mathcal{C}(S/\rho)$. It thus suffices to prove that for any $\lambda \in [\rho)$, we have $\lambda \in \rho\mathcal{P}^+$ if and ony if $\lambda/\rho \in \varepsilon_{S/\rho}\mathcal{P}^+$, or equivalently, that $\lambda\,\mathcal{P}^+\,\rho$ if and only if $\lambda/\rho\,\mathcal{P}^+\,\varepsilon_{S/\rho}$. Let $\lambda \in [\rho)$.

Assume that $\lambda\,\mathcal{P}^\wedge\,\rho$ so that $\lambda \cap \mathcal{P} = \rho \cap \mathcal{P}$; we must show that $\lambda/\rho \cap \mathcal{P} = \varepsilon_{S/\rho}$. Let $a, b \in S$ be such that $a\rho\,\lambda/\rho \cap \mathcal{P}\,b\rho$. Then $a\,\lambda\,b$ and, by Lemma 6.5, $a\,\rho \vee \mathcal{P}\,b$. It follows that $ab^{-1}\lambda\,b^0$ so that $ab^{-1}\lambda \cap \mathcal{P}(ab^{-1})^2$. Now $\lambda \cap \mathcal{P} = \rho \cap \mathcal{P}$ implies that $ab^{-1}\rho\,(ab^{-1})^2$ and thus $ab^{-1} \in \ker\rho$. In order to show that $a^0\rho\,b^0$, we consider several cases.

If $\mathcal{P} = \mathcal{H}$, then $a\,\rho \vee \mathcal{H}\,b$ and hence $a^0\rho\,b^0$ by Lemma 5.2 (i).

Let $\mathcal{P} = \mathcal{L}$. Then $a\,\rho \vee \mathcal{L}\,b$ and hence, by Lemma 5.2 (ii), $a\,\rho\,ab^0$ and $b\,\rho\,ba^0$. Further, $a\,\lambda\,b$ implies that $ab\,\lambda \cap \mathcal{L}\,b^0ab$ which by $\lambda \cap \mathcal{L} = \rho \cap \mathcal{L}$ implies that $ab\,\rho\,b^0ab$. It follows that $a\,\rho\,ab^0\rho\,b^0ab^0$ whence $a\,\rho\,b^0a$ and finally $a^0\rho\,b^0a^0$. Since by the above also $b^0\rho\,b^0a^0$, we conclude that $a^0\rho\,b^0$.

Finally, let $\mathcal{P} = \mathcal{D}$. Then $a\,\rho \vee \mathcal{D}\,b$ so that, by Lemma 5.2 (ii), $a\,\rho\,x\,\mathcal{D}\,y\,\rho\,b$ for some $x, y \in S$. Now $a\,\lambda\,b$ implies $ab\,\lambda \cap \mathcal{D}\,ba$ which by $\lambda \cap \mathcal{P} = \rho \cap \mathcal{P}$ implies that $ab\,\rho\,ba$. Hence $xy\,\rho\,yx$. Let $T = D_x = D_y$, $\theta = \rho|_T$ and $\overline{T} = T/\theta$. Then $\overline{x}\overline{y} = \overline{y}\overline{x}$ so that $\overline{x}\,\mathcal{H}\,\overline{y}$. It follows that $x^0\theta\,y^0$ whence $x^0\rho\,y^0$ and finally $a^0\rho\,b^0$.

We have proved that in all cases $a^0\rho\,b^0$ which together with $ab^{-1} \in \ker\rho$ by Lemma 2.1 implies that $a\,\rho\,b$. Therefore $a\rho = b\rho$, as required.

Conversely, assume that $\lambda/\rho\,\mathcal{P}^\wedge\,\varepsilon_{S/\rho}$, that is $\lambda/\rho \cap \mathcal{P} = \varepsilon_{S/\rho}$; we must show that $\lambda\,\mathcal{P}^\wedge\,\rho$. Let $a\,\lambda \cap \mathcal{P}\,b$. Then $a\rho\,\lambda/\rho \cap \mathcal{P}\,b\rho$ and thus, by hypothesis, $a\rho\,\varepsilon_{S/\rho}\,b\rho$ and hence $a\,\rho\,b$. Consequently $\lambda \cap \mathcal{P} \subseteq \rho$ and since $\lambda \supseteq \rho$, we obtain $\lambda \cap \mathcal{P} = \rho \cap \mathcal{P}$, that is $\lambda\,\mathcal{P}^\wedge\,\rho$.

Now suppose that $\lambda\,\mathcal{P}^\vee\,\rho$ so that $\lambda \vee \mathcal{P} = \rho \vee \mathcal{P}$; we must show that $\lambda/\rho\,\mathcal{P}^\vee\,\varepsilon_{S/\rho}$. Let $a, b \in S$ be such that $a\rho\,\lambda/\rho\,b\rho$. Then $a\,\lambda\,b$ which together with $\lambda \vee \mathcal{P} = \rho \vee \mathcal{P}$ implies that $a\,\rho \vee \mathcal{P}\,b$. By Lemma 6.5, we get $a\rho\,\mathcal{P}\,b\rho$. Hence $\lambda/\rho \subseteq \mathcal{P}$ which implies that $\lambda/\rho \vee \mathcal{P} = \varepsilon_{S/\rho}$, that is $\lambda/\rho\,\mathcal{P}^\vee\,\varepsilon_{S/\rho}$.

Conversely, assume that $\lambda/\rho\,\mathcal{P}^\vee\,\varepsilon_{S/\rho}$, that is $\lambda/\rho \subseteq \mathcal{P}$; we must show that $\lambda\,\mathcal{P}^\vee\,\rho$. We let $a\,\lambda \vee \mathcal{P}\,b$ and consider several cases. Recall that $\lambda \supseteq \rho$ so that $\lambda \cap \mathcal{P} \supseteq \rho \cap \mathcal{P}$.

Let $\mathcal{P} = \mathcal{H}$. Then $a\,\lambda \vee \mathcal{H}\,b$ and thus $a^0\lambda\,b^0$ by Lemma 5.2 (i). Hence $(a\rho)^0\lambda/\rho\,(b\rho)^0$ which together with $\lambda/\rho \subseteq \mathcal{H}$ implies $(a\rho)^0\mathcal{H}(b\rho)^0$ so that $a^0\rho\,b^0$. By Lemma 5.2 (i), we get $a\,\rho \vee \mathcal{H}\,b$. Consequently $\lambda \vee \mathcal{H} \subseteq \rho \vee \mathcal{H}$.

Assume next that $\mathcal{P} \neq \mathcal{H}$. By Lemma 5.2, we have $a\,\lambda\,x\,\mathcal{P}\,y\,\lambda\,b$ for some $x, y \in S$. Hence

$$a\rho\,\lambda/\rho\,x\rho\,\mathcal{P}\,y\rho\,\lambda/\rho\,b\rho$$

which together with $\lambda/\rho \subseteq \mathcal{P}$ gives $a\rho\,\mathcal{P}\,b\rho$. Now Lemma 6.5 yields $a\,\rho \vee \mathcal{P}\,b$. Consequently $\lambda \vee \mathcal{P} \subseteq \rho \vee \mathcal{P}$.

Therefore in all cases $\lambda\,\mathcal{P}^\vee\,\rho$ as required.

COROLLARY 7.2. – *For $\mathcal{P} \in \mathcal{GR}$, $\alpha \in \{p, P\}$ and $\rho \in \mathcal{C}(S)$, we have*

(i) $\rho^{\alpha}/\rho = (\varepsilon_{S/\rho})^{\alpha}$,

(ii) $\rho = \rho^{\alpha} \Leftrightarrow (\varepsilon_{S/\rho})^{\alpha} = \varepsilon_{S/\rho}$.

For a small set of varieties $\mathcal{V}$, we can now easily describe the greatest congruence on S over $\mathcal{V}$ as well as the least $\mathcal{V}$-congruence on S. This will be achieved in the next two propositions. Beside the notation for varieties introduced at the outset of Sections 4 and 6, we shall also need

$$\mathcal{LZ} = [xy = x], \qquad \text{left zero semigroups},$$

$$\mathcal{RB} = [xyx = x], \qquad \text{rectangular bands}.$$

Compare the next lemma with ([3], Corollary 3.10).

LEMMA 7.3. – *The following statements concerning* $\rho \in \mathcal{C}(S)$ *are equivalent.*

(i) ρ *is over* $\mathcal{LZ}$.

(ii) $\rho \cap \mathcal{H} = \varepsilon$, $\rho \subseteq \mathcal{L}$.

(iii) $\rho \cap \mathcal{R} = \varepsilon$, $\rho \subseteq \mathcal{D}$.

PROOF. – Taking into account that $\mathcal{LZ} = \mathcal{B} \cap \mathcal{LG} = \mathcal{LRB} \cap \mathcal{CS}$, we obtain

$$\begin{aligned} \rho \text{ is over } \mathcal{LZ} &\Leftrightarrow \rho \text{ is over both } \mathcal{B} \text{ and } \mathcal{LG} \\ &\Leftrightarrow \rho \text{ is over both } \mathcal{LRB} \text{ and } \mathcal{CS} \end{aligned}$$

By Corollary 4.4, we have

$$\rho \text{ is over } \mathcal{B} \Leftrightarrow \rho \cap \mathcal{H} = \varepsilon,$$

$$\rho \text{ is over } \mathcal{LRB} \Leftrightarrow \rho \cap \mathcal{R} = \varepsilon,$$

and by Corollary 6.3, we have

$$\rho \text{ is over } \mathcal{LG} \Leftrightarrow \rho \subseteq \mathcal{L},$$

$$\rho \text{ is over } \mathcal{CS} \Leftrightarrow \rho \subseteq \mathcal{D}.$$

The assertion of the lemma follows by combining these statements.

Compare item (i) of the next lemma with ([3], Theorem 3.8).

LEMMA 7.4. – *Let* $\rho \in \mathcal{C}(S)$.

(i) ρ *is over* $\mathcal{RB} \Leftrightarrow \rho \cap \mathcal{H} = \varepsilon$, $\rho \subseteq \mathcal{D}$.

(ii) ρ *is over* $\mathcal{S} \Leftrightarrow \rho \cap \mathcal{L} = \rho \cap \mathcal{R} = \varepsilon$.

PROOF. – (i) Taking into account that $\mathcal{RB} = \mathcal{B} \cap \mathcal{CS}$, we get

$$\rho \text{ is over } \mathcal{RB} \Leftrightarrow \rho \text{ is over both } \mathcal{B} \text{ and } \mathcal{CS},$$

$$\rho \text{ is over } \mathcal{RB} \Leftrightarrow \rho \cap \mathcal{H} = \varepsilon \quad \text{by Corollary 4.4},$$

$$\rho \text{ is over } \mathcal{CS} \Leftrightarrow \rho \subseteq \mathcal{D} \quad \text{by Corollary 6.3}$$

and the assertion follows.

(ii) Taking into account that $\mathcal{S} = \mathcal{LRB} \cap \mathcal{RRB}$, we get

$$\rho \text{ is over } \mathcal{S} \Leftrightarrow \rho \text{ is over both } \mathcal{LRB} \text{ and } \mathcal{RRB},$$

$$\rho \text{ is over } \mathcal{LRB} \Leftrightarrow \rho \cap \mathcal{R} = \varepsilon \quad \text{by Corollary 4.4}$$

$$\rho \text{ is over } \mathcal{RRB} \Leftrightarrow \rho \cap \mathcal{L} = \varepsilon \quad \text{by the dual of Corollary 4.4}$$

and the assertion follows.

NOTATION 7.5. – *For a variety $\mathcal{V}$ of completely regular semigroups, denote by $\tau_{\mathcal{V}}$ the greatest congruence on S over $\mathcal{V}$, if it exists.*

LEMMA 7.6. – *Let $\mathcal{U}$ and $\mathcal{V}$ be varieties of completely regular semigroups and assume that $\tau_{\mathcal{U}}$, $\tau_{\mathcal{V}}$ and $\tau_{\mathcal{U} \cap \mathcal{V}}$ exist. Then $\tau_{\mathcal{U}} \wedge \tau_{\mathcal{V}} = \tau_{\mathcal{U} \cap \mathcal{V}}$*

PROOF. – Since $\tau_{\mathcal{U} \cap \mathcal{V}}$ in over $\mathcal{U} \cap \mathcal{V}$, it is also over both $\mathcal{U}$ and $\mathcal{V}$ so that $\tau_{\mathcal{U} \cap \mathcal{V}} \subseteq \tau_{\mathcal{U}}$ and $\tau_{\mathcal{U} \cap \mathcal{V}} \subseteq \tau_{\mathcal{V}}$ whence $\tau_{\mathcal{U} \cap \mathcal{V}} \subseteq \tau_{\mathcal{U}} \wedge \tau_{\mathcal{V}}$. Conversely, $\tau_{\mathcal{U}}$ is over $\mathcal{U}$ and $\tau_{\mathcal{V}}$ is over $\mathcal{V}$ so that $\tau_{\mathcal{U}} \wedge \tau_{\mathcal{V}}$ is over $\mathcal{U} \cap \mathcal{V}$ whence $\tau_{\mathcal{U}} \wedge \tau_{\mathcal{V}} \subseteq \tau_{\mathcal{U} \cap \mathcal{V}}$.

PROPOSITION 7.7. – *The greatest congruence on S over*

$$\begin{array}{rcl}
\mathcal{B} & \textit{is} & \varepsilon^h, \\
\mathcal{LRB} & \textit{is} & \varepsilon^r, \\
\mathcal{S} & \textit{is} & \varepsilon^d = \varepsilon^l \wedge \varepsilon^r, \\
\mathcal{CS} & \textit{is} & \varepsilon^D = \mathcal{D}, \\
\mathcal{LG} & \textit{is} & \varepsilon^L = \mathcal{L}^0, \\
\mathcal{G} & \textit{is} & \varepsilon^H = \mathcal{H}^0, \\
\mathcal{LZ} & \textit{is} & \varepsilon^h \wedge \varepsilon^L = \varepsilon^r \wedge \varepsilon^D, \\
\mathcal{RB} & \textit{is} & \varepsilon^h \wedge \varepsilon^D.
\end{array}$$

PROOF. – This follows easily from Corollaries 4.4 and 6.3 and Lemmas 7.3, 7.4 and 7.6.

The above proposition uses the upper ends of the $\mathcal{P}^\wedge$- and $\mathcal{P}^\vee$-classes of ε and is based upon (greatest) congruences over certain varieties $\mathcal{V}$ of completely regular semigroups. We now consider the dual situation which will use the lower ends of the $\mathcal{P}^\wedge$- and $\mathcal{P}^\vee$-classes of ω and will be based upon (least) $\mathcal{V}$-congruences.

LEMMA 7.8. – *The following statements concerning $\rho \in \mathcal{C}(S)$ are equivalent.*

(i) *ρ is a $\mathcal{LZ}$-congruence.*

(ii) $\mathcal{H} \subseteq \rho$, $\rho \vee \mathcal{L} = \omega$.

(iii) $\mathcal{R} \subseteq \rho$, $\rho \vee \mathcal{D} = \omega$.

PROOF. – Taking into account that $\mathcal{LZ} = \mathcal{B} \cap \mathcal{LG} = \mathcal{LRB} \cap \mathcal{CS}$, we obtain

$$\begin{aligned} \rho \text{ is } a\ \mathcal{LZ}\text{-congruence} &\Leftrightarrow \rho \text{ is both a } \mathcal{B}\text{- and a } \mathcal{LG}\text{-congruence}, \\ &\Leftrightarrow \rho \text{ is both a } \mathcal{LRB}\text{- and a } \mathcal{CS}\text{-congruence}. \end{aligned}$$

By Corollary 4.5, we have

$$\rho \text{ is a } \mathcal{B}\text{-congruence} \Leftrightarrow \mathcal{H} \subseteq \rho,$$

$$\rho \text{ is a } \mathcal{LRB}\text{-congruence} \Leftrightarrow \mathcal{R} \subseteq \rho,$$

and by Corollary 6.4

$$\rho \text{ is a } \mathcal{LG}\text{-congruence} \Leftrightarrow \rho \vee \mathcal{L} = \omega,$$

$$\rho \text{ is a } \mathcal{CS}\text{-congruence} \Leftrightarrow \rho \vee \mathcal{D} = \omega.$$

The assertion of the lemma follows by combining these statements.

LEMMA 7.9. – *Let $\rho \in \mathcal{C}(S)$.*

(i) *ρ is a $\mathcal{RB}$-congruence* $\Leftrightarrow \mathcal{H} \subseteq \rho$, $\rho \vee \mathcal{D} = \omega$.

(ii) *ρ is a $\mathcal{S}$-congruence* $\Leftrightarrow \mathcal{D} \subseteq \rho$.

PROOF. – (i) Taking into account that $\mathcal{RB} = \mathcal{B} \cap \mathcal{CS}$, we get

$$\rho \text{ is a } \mathcal{RB}\text{-congruence} \Leftrightarrow \rho \text{ is both a } \mathcal{B} \text{ and a } \mathcal{CS}\text{-congruence},$$

$$\rho \text{ is a } \mathcal{B}\text{-congruence} \quad \Leftrightarrow \mathcal{H} \subseteq \rho \quad \text{by Corollary 4.5},$$

$$\rho \text{ is a } \mathcal{CS}\text{-congruence} \quad \Leftrightarrow \rho \vee \mathcal{D} = \omega \quad \text{by Corollary 6.4}$$

and the assertion follows.

(ii) It is well known that $\mathcal{D}$ is the least semilattice congruence on S.

NOTATION 7.10. – *For a variety $\mathcal{V}$ of completely regular semigroups, denote by $\theta_{\mathcal{V}}$ the least $\mathcal{V}$-congruence on S.*

It is well known that for varieties $\mathcal{U}$ and $\mathcal{V}$, we have $\theta_{\mathcal{U}} \wedge \theta_{\mathcal{U}} = \theta_{\mathcal{U} \vee \mathcal{V}}$. This can be used for describing $\theta_{\mathcal{U} \vee \mathcal{V}}$ when $\theta_{\mathcal{U}}$ and $\theta_{\mathcal{V}}$ are known. We thus state briefly some of these «basic $\theta_{\mathcal{V}}$».

PROPOSITION 7.11. – *The least*

$$\begin{aligned} &\mathcal{B}\text{-congruence is } \omega_h = \mathcal{H}^*, \\ &\mathcal{L}\mathcal{R}\mathcal{B}\text{-congruence is } \omega_r = \mathcal{R}^*, \\ &\mathcal{S}\text{-congruence is } \omega_d = \mathcal{D}, \\ &\mathcal{C}\mathcal{S}\text{-congruence is } \omega_D, \\ &\mathcal{L}\mathcal{G}\text{-congruence is } \omega_L, \\ &\mathcal{G}\text{-congruence is } \omega_H. \end{aligned}$$

The following result will be useful in drawing the diagrams of some of the congruences we have encountered.

LEMMA 7.12. – *Let $\mathcal{P} \in \mathcal{G}\mathcal{R}$ and $\lambda, \rho \in \mathcal{C}(S)$ be such that $\lambda \cap \mathcal{P} = \varepsilon$ and $\rho \vee \mathcal{P} = \omega$. Then $\lambda \subseteq \rho$.*

PROOF. – Let $a \lambda b$. Then $ab^{-1} \lambda b^0$ so that $ab^{-1} \lambda \cap \mathcal{P} (ab^{-1})^2$ which by hypothesis implies that $ab^{-1} \in E(S)$. Let $e = a^0$ and $f = b^0$. The hypothesis implies that $e \rho \vee \mathcal{P} f$. In order to show that $e \rho f$, we consider several cases.

If $\mathcal{P} = \mathcal{H}$, then $e \rho \vee \mathcal{H} f$ whence $e \rho f$ by Lemma 5.2 (i). Next let $\mathcal{P} = \mathcal{L}$. Then $ef \lambda \cap \mathcal{L} fef$ which by hypothesis implies that $ef = fef$. Also by hypothesis $e \rho \vee \mathcal{L} f$ which by Lemma 5.2 (ii) implies that $e \rho ef$ and $f \rho fe$. Therefore $e \rho ef = fef \rho f$.

Finally, let $\mathcal{P} = \mathcal{D}$. Then $ef \lambda \cap \mathcal{D} fe$ which by hypothesis yields $ef = fe$. Also by hypothesis $e \rho \vee \mathcal{D} f$ which by Lemma 5.2 (iii) implies that $e \rho x \mathcal{D} y \rho f$ for some $x, y \in S$. Hence $x^0 y^0 \rho ef = fe \rho y^0 x^0$ which gives $x^0 = y^0$ since $x^0 \mathcal{D} y^0$. But then $e \rho f$.

In all cases we have $a^0 \rho b^0$ which together with $ab^{-1} \in E(S) \subseteq \ker \rho$ by Lemma 2.1 implies that $a \rho b$. Therefore $\lambda \subseteq \rho$.

COROLLARY 7.13. – *For $\mathcal{P} \in \mathcal{G}\mathcal{R}$ we have*

$$\varepsilon^p \subseteq \omega_P, \qquad \varepsilon^P = \mathcal{P}^0 \subseteq \mathcal{P}^* = \omega_p .$$

PROOF. – The first inclusion follows directly from Lemma 7.12. The first equality follows from Theorem 5.5 and the second from Theorem 3.5.

Notice the duality of the statements in Corollary 7.13. In Diagram 1, we apply Notations 7.5 and 7.10. Full lines indicate true meets and joins.

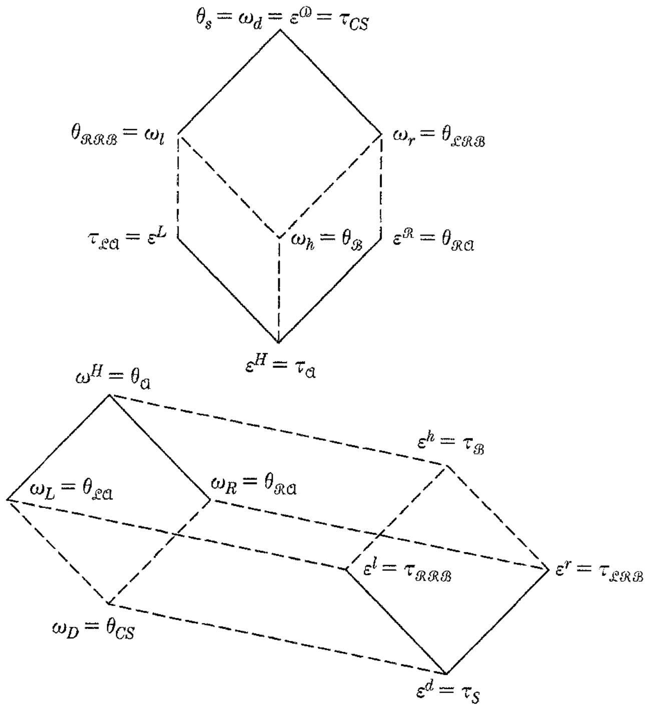


Diagram 1.

8. - A diagram and a network.

The diagram in question represent the $\wedge$-subsemilattice of $\Pi(\mathcal{C}(S))$ generated by the set

$$\{\mathcal{P}^{\wedge} \mid \mathcal{P} \in \mathcal{G}\mathcal{R}\} \cup \{\mathcal{P}^{\vee} \mid \mathcal{P} \in \mathcal{G}\mathcal{R}\}.$$

The network announced is obtained by fixing $\mathcal{P} \in \mathcal{G}\mathcal{R}$ and $\rho \in \mathcal{C}(S)$ and then forming the sequence $\rho_p, \rho_P, \rho_{pP}, \rho_{Pp}, \ldots$. We thereby obtain a sublattice of $\mathcal{C}(S)$. Particularly interesting networks are obtained by letting $\rho = \omega$ and varying $\mathcal{P}$ over all of $\mathcal{G}\mathcal{R}$.

PROPOSITION 8.1.

(i) *For* $\mathcal{P} \in \mathcal{G}\mathcal{R}$, *we have* $\mathcal{P}^{\wedge} \cap \mathcal{P}^{\vee} = \varepsilon$.

(ii) $\mathcal{L}^{\wedge} \cap \mathcal{R}^{\wedge} = \mathcal{D}^{\wedge}$.

(iii) $\mathcal{L}^{\vee} \cap \mathcal{R}^{\vee} = \mathcal{H}^{\wedge}$.

PROOF. - (i) If $\lambda\, \mathcal{P}^{\wedge} \cap \mathcal{P}^{\vee}\, \rho$, then by Corollaries 4.2 and 6.2 and Lemma 4.3, we get,

$$\lambda/(\lambda \wedge \rho),\ \rho/(\lambda \wedge \rho) \subseteq p_{\varepsilon} \cap \mathcal{P} = \varepsilon$$

which evidently implies that $\lambda = \rho$. The case $\mathcal{P} = \mathcal{D}$ forms the content of ([4], Corollary 3.2).

(ii) We must show that for any $\lambda, \rho \in \mathcal{C}(S)$,

$$\lambda \cap \mathcal{L} = \rho \cap \mathcal{L}, \quad \lambda \cap \mathcal{R} = \rho \cap \mathcal{R} \Leftrightarrow \lambda \cap \mathcal{D} = \rho \cap \mathcal{D}. \tag{6}$$

If the left hand side of (6) holds, then by Lemma 3.6 (ii) we obtain

$$\lambda \cap \mathcal{D} = (\lambda \cap \mathcal{L}) \vee (\lambda \cap \mathcal{R}) = (\rho \cap \mathcal{L}) \vee (\rho \cap \mathcal{R}) = \rho \cap \mathcal{D}.$$

The opposite implication in (6) is trivial.

(iii) The argument here follows along the same lines using Lemma 5.6 (ii).

Proposition 8.1 (i) indicates that a congruence ρ on S is uniquely determined by its $\mathcal{P}^{\wedge}$- and $\mathcal{P}^{\vee}$-classes. It thus should be possible to express ρ by means of $\rho \cap \mathcal{P}$ and $\rho \vee \mathcal{P}$. This we do in the next result.

PROPOSITION 8.2. – *Let* $\rho \in \mathcal{C}(S)$ *and* $a, b \in S$. *Then*

$$a \rho b \Leftrightarrow a (\rho \vee \mathcal{H})^0 b, \quad ab^{-1} \in \ker(\rho \cap \mathcal{H})^*$$

$$\Leftrightarrow a (\rho \vee \mathcal{L})^0 b, \quad a^0 b \rho \cap \mathcal{L} b^0 a^0 b, \quad ab^{-1} \in \ker(\rho \cap \mathcal{L})^*$$

$$\Leftrightarrow a \rho \vee \mathcal{D} b, \quad ab \rho \wedge \mathcal{D} ba, \quad ab^{-1} \in \ker(\rho \cap \mathcal{D}).$$

PROOF. – Let $a \rho b$. Then $\rho \subseteq (\rho \vee \mathcal{H})^0$ and thus $a (\rho \vee \mathcal{H})^0 b$. Also $ab^{-1} \rho b^0$ whence $ab^{-1} \rho \cap \mathcal{H} (ab^{-1})^2$ so that $ab^{-1} \in \ker(\rho \cap \mathcal{H})^*$.

Let $a (\rho \vee \mathcal{H})^0 b$ and $ab^{-1} \in \ker(\rho \cap \mathcal{H})^*$. Then $a \rho \vee \mathcal{H} b$ so, by Lemma 5.2 (i), we have $a^0 \rho b^0$ whence $a^0 b \rho b^0 a^0 b$. Since we always have $a^0 b \mathcal{L} b^0 a^0 b$, it follows that $a^0 b \rho \cap \mathcal{L} b^0 a^0 b$. Since $\mathcal{H} \subseteq \mathcal{L}$, the hypothesis also implies that $a (\rho \vee \mathcal{L})^0 b$ and $ab^{-1} \in \ker(\rho \cap \mathcal{L})^*$.

Let $a (\rho \vee \mathcal{L})^0 b$, $a^0 b \rho \cap \mathcal{L} b^0 a^0 b$ and $ab^{-1} \in \ker(\rho \cap \mathcal{L})^*$. By Lemma 5.2 (ii), we have $a^0 \rho a^0 b^0$ and $b^0 \rho b^0 a^0$. By hypothesis, we have $a^0 b^0 \rho b^0 a^0 b^0$ whence $a^0 \rho a^0 b^0 \rho b^0 a^0 b^0 \rho b^0$. Also $(\rho \cap \mathcal{L})^* \subseteq \rho$ so that $ab^{-1} \in \ker \rho$, which together with $a^0 \rho b^0$ yields $a \rho b$ by Lemma 2.1.

The equivalence of $a \rho b$ and the last expression is the content of ([4], Lemma 3.1).

Of course, Proposition 8.2 actually implies Proposition 8.1 (i). In Diagram 2, we represent the $\wedge$-sublattice of $\Pi(\mathcal{C}(S))$ generated by $\{\mathcal{P}^{\wedge} \mid \mathcal{P} \in \mathcal{GR}\} \cup \{\mathcal{P}^{\vee} \mid \mathcal{P} \in \mathcal{GR}\}$ using the relations established in Proposition 8.1.

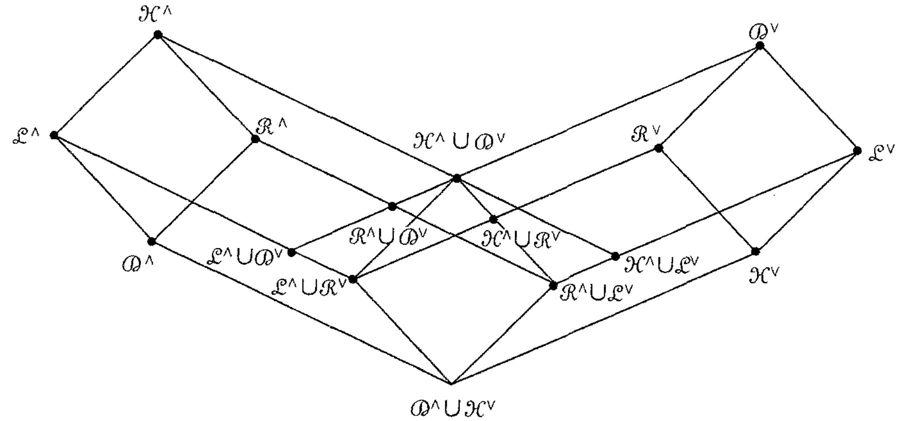


Diagram 2.

The next lemma provides a further interesting relationship among the notation introduced.

LEMMA 8.3. – *For* $\mathcal{P} \in \mathcal{GR}$ *and* $\rho \in \mathcal{C}(S)$, *we have*

$$\rho = \rho^p \wedge \rho^P = \rho_p \vee \rho_P .$$

PROOF. – Indeed,

$$(\rho^p \wedge \rho^P) \cap \mathcal{P} = (\rho^p \cap \mathcal{P}) \cap \rho^P = (\rho \cap \mathcal{P}) \cap \rho^P = \rho \cap \mathcal{P},$$

$$(\rho^p \wedge \rho^P) \vee \mathcal{P} \subseteq \rho^P \vee \mathcal{P} = \rho \vee \mathcal{P} \subseteq (\rho^p \wedge \rho^P) \vee \mathcal{P}$$

and equality prevails. Now Proposition 8.1(i) implies that $\rho^p \wedge \rho^P = \rho$. The argument for $\rho = \rho_p \vee \rho_P$ follows along the same lines.

The lattice generated by a min $\mathcal{P}$-network is described by our next result. Monotonicity of the operators sub p and sub P plays an essential role in its proof.

THEOREM 8.4. – *For* $\mathcal{P} \in \mathcal{GR}$, $\rho \in \mathcal{C}(S)$ *and* $n \geq 0$, *we have*

(i) $\rho_{(pP)^n} \wedge \rho_{(Pp)^n} = \rho_{(pP)^n p} \vee \rho_{(Pp)^n P}$,

(ii) $\rho_{(pP)^n p} \vee \rho_{(Pp)^n P} = \rho_{(pP)^{n+1}} \wedge \rho_{(Pp)^{n+1}}$,

where $\rho_{(\,)^0} = \rho$.

PROOF. – Let $a = pP$ and $b = Pp$.

(i) The case $n = 0$ amounts to $\rho = \rho_p \vee \rho_P$ which was established in Lemma 8.3. Hence assume that $n > 0$. By Proposition 8.1 (i), to show the desired equality, it suffices to prove that the two sides of the given equation have the same intersections and joins with $\mathscr{P}$. Indeed, on the one hand,

$$\begin{aligned}(\rho_{a^n} \wedge \rho_{b^n}) \cap \mathscr{P} &= \rho_{a^n} \cap (\rho_{b^n} \cap \mathscr{P}) \\ &= \rho_{a^n} \cap (\rho_{b^{n-1}p} \cap \mathscr{P}) && \text{by definition of sub } p \\ &= (\rho_{a^n} \wedge \rho_{b^{n-1}p}) \cap \mathscr{P} \\ &= \rho_{a^n} \cap \mathscr{P} && \text{since } \rho_{b^{n-1}p} = \rho_{Pa^{n-1}} \supseteq \rho_{a^n},\end{aligned}$$

and on the other hand,

$$(\rho_{a^n p} \vee \rho_{b^n P}) \cap \mathscr{P} \supseteq \rho_{a^n p} \cap \mathscr{P} = \rho_{a^n} \cap \mathscr{P} \quad \text{by definition of sub } p$$

$$\begin{aligned}(\rho_{a^n p} \vee \rho_{b^n P}) \cap \mathscr{P} &\subseteq (\rho_{a^n p} \vee \rho_{b^n}) \cap \mathscr{P} \\ &= (\rho_{a^n p}) \cap \mathscr{P} && \text{since } \rho_{a^n p} = \rho_{pb^n} \subseteq \rho_{b^n}, \\ &= \rho_{a^n} \cap \mathscr{P} && \text{by definition of sub } p\end{aligned}$$

and therefore

$$(\rho_{a^n} \wedge \rho_{b^n}) \cap \mathscr{P} = (\rho_{a^n p} \vee \rho_{b^n P}) \cap \mathscr{P}. \tag{7}$$

Further, on the one hand,

$$(\rho_{a^n p} \wedge \rho_{b^n}) \vee \mathscr{P} \subseteq \rho_{b^n} \vee \mathscr{P},$$

$$\begin{aligned}(\rho_{a^n} \wedge \rho_{b^n}) \vee \mathscr{P} &\supseteq (\rho_{a^n} \wedge \rho_{b^n P}) \vee \mathscr{P} \\ &= \rho_{b^n P} \vee \mathscr{P} && \text{since } \rho_{b^n P} = \rho_{Pa^n} \subseteq \rho_{a^n}, \\ &= \rho_{b^n} \vee \mathscr{P} && \text{by definition of sub } P\end{aligned}$$

and on the other hand,

$$\begin{aligned}(\rho_{a^n p} \vee \rho_{b^n P}) \vee \mathscr{P} &= \rho_{a^n p} \vee (\rho_{b^n P} \vee \mathscr{P}) \\ &= (\rho_{a^n} \vee \rho_{b^n}) \vee \mathscr{P} && \text{by definition of sub } P, \\ &= \rho_{b^n} \vee \mathscr{P} && \text{since } \rho_{a^n p} = \rho_{P_{pb^n}} \subseteq \rho_{b^n}\end{aligned}$$

and therefore

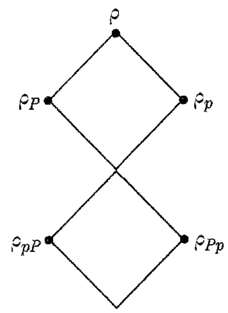
$$(\rho_{a^n p} \vee \rho_{b^n P}) \vee \mathscr{P} = (\rho_{a^n p} \vee \rho_{b^n P}) \vee \mathscr{P}. \tag{8}$$

The desired equality follows by (7) and (8) in view of Proposition 8.1 (i).

(ii) The argument here is of the same general type as above and is omitted.

DEFINITION 8.5. – *For* $\mathscr{P} \in \mathcal{GR}$ *and* $\rho \in \mathcal{C}(S)$, *we call the set* $\{\rho, \rho_p, \rho_P, \rho_{pP}, \rho_{Pp}, \dots\}$ *the* min $\mathscr{P}$-network for ρ.

Now the min $\mathcal{P}$-network for ρ together with the meets and joins provided by Theorem 8.4 forms the sublattice of $\mathcal{C}(S)$ generated by this network. Its diagram has a very simple form as follows.



As an example of min $\mathcal{P}$-networks, we consider $\rho = \omega$. To this end, we use Notation 7.10 and in addition we need the following varieties:

$$\mathcal{B}\mathcal{G} = [(xy)^0 = (x^0 y^0)^0], \qquad \text{cryptogroups (bands of groups)},$$

$$\mathcal{T} = [x = y], \qquad \text{trivial semigroups}.$$

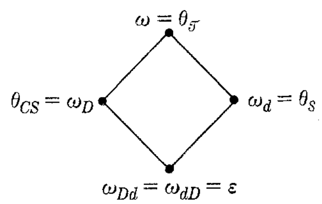
The min $\mathcal{H}$-network of ω:

$$\omega = \theta_{\mathcal{T}},\ \omega_H = \theta_{\mathcal{G}},\ \omega_h = \theta_{\mathcal{B}},\ \omega_{hH} = \theta_{\mathcal{B}\mathcal{G}},\ \ldots,$$

the min $\mathcal{L}$-network of ω:

$$\omega = \theta_{\mathcal{T}},\ \omega_L = \theta_{\mathcal{L}\mathcal{G}},\ \omega_l = \theta_{\mathcal{R}\mathcal{R}\mathcal{B}},\ \ldots,$$

the min $\mathcal{D}$-network of ω:



By monotonicity, a min $\mathcal{D}$-network has at most 4 vertices for any $\rho \in \mathcal{C}(S)$. One may also consider the following networks:

$$\rho,\ \rho_l,\ \rho_r,\ \rho_{lr},\ \rho_{rl},\ \ldots,$$

$$\rho,\ \rho_L,\ \rho_R,\ \rho_{LR},\ \rho_{RL},\ \ldots.$$

9. – Relationship with kernels and (left and right) traces.

For a congruence ρ on a regular semigroup, the *left* and the *right traces* of ρ were defined in ([2], Definition 4.1) by

$$\operatorname{ltr}\rho = \operatorname{tr}(\rho \vee \mathcal{L})^0, \qquad \operatorname{rtr}\rho = \operatorname{tr}(\rho \vee \mathcal{R})^0,$$

respectively. The following relations on the congruence lattice play an important role in the study of congruences on (completely) regular semigroups:

$$\lambda K \rho \Leftrightarrow \ker\lambda = \ker\rho, \qquad \lambda T \rho \Leftrightarrow \operatorname{tr}\lambda = \operatorname{tr}\rho,$$

$$\lambda T_l \rho \Leftrightarrow \operatorname{ltr}\lambda = \operatorname{ltr}\rho, \qquad \lambda T_r \rho \Leftrightarrow \operatorname{rtr}\lambda = \operatorname{rtr}\rho.$$

For numerous properties of these relations, see [2] and [3]. We mention here only that K is a complete $\wedge$-congruence, but generally not a $\vee$-congruence, whereas the others are all complete congruences on $\mathcal{C}(S)$. Going back to our completely regular semigroup S, we have the following relationships with some of the concepts studied so far.

Compare the next proposition with ([3], Theorems 6.3 and 6.12).

PROPOSITION 9.1. – $K = \mathcal{H}^\wedge$, $T = \mathcal{H}^\vee$, $T_l = \mathcal{L}^\vee$, $T_r = \mathcal{R}^\vee$.

PROOF. – Let $\lambda, \rho \in \mathcal{C}(S)$. Assume that $\lambda K \rho$. Let $a \, \lambda \cap \mathcal{H} \, b$. Then $ab^{-1} \in \ker\lambda$ and $a^0 = b^0$. The hypothesis implies that $ab^{-1} \in \ker\rho$ which together with $a^0 = b^0$ by Lemma 2.1 implies that $a \rho b$. Hence $\lambda \cap \mathcal{H} \subseteq \rho \cap \mathcal{H}$ and equality follows by symmetry. Therefore $\lambda \mathcal{H}^\wedge \rho$.

Conversely, suppose that $\lambda \mathcal{H}^\wedge \rho$. Let $a \in \ker\lambda$. Then $a \, \lambda \cap \mathcal{H} \, a^0$ which by hypothesis yields $a \rho a^0$ so that $a \in \ker\rho$. Thus $\ker\lambda \subseteq \ker\rho$ and equality follows by symmetry. Therefore $\lambda K \rho$.

Suppose next that $\lambda T \rho$. Let $a \, \lambda \vee \mathcal{H} \, b$. By Lemma 5.2 (i), we have $a^0 \lambda b^0$ which by hypothesis yields $a^0 \rho b^0$. Again by Lemma 5.2 (i), we conclude that $a \, \rho \vee \mathcal{H} \, b$. Consequently $\lambda \vee \mathcal{H} \subseteq \rho \vee \mathcal{H}$ and equality follows by symmetry. Therefore $\lambda \mathcal{H}^\vee \rho$.

Conversely, assume that $\lambda \mathcal{H}^\vee \rho$. Let $e \operatorname{tr}\lambda f$. Then $e \, \lambda \vee \mathcal{H} f$ so by hypothesis $e \, \rho \vee \mathcal{H} f$. Now Lemma 5.2 (i) implies that $e \rho f$ so that $e \operatorname{tr}\lambda f$. Thus $\operatorname{tr}\lambda \subseteq \operatorname{tr}\rho$ and equality follows by symmetry. Therefore $\lambda T \rho$.

It is proved in ([3], Theorem 6.3) that $T_r = \mathcal{R}^\vee$ (in our present notation) in the wider context of regular semigroups. That $T_l = \mathcal{L}^\vee$ now follows by duality.

Beside establishing a connection between the relations $\mathcal{P}^\wedge$ and $\mathcal{P}^\vee$ for $\mathcal{P} \in \mathcal{GR}$ with the relations K, T, T_l and T_r, and thus between the Green relations approach to congruences and the kernel-trace approach for completely regular semigroups, the above proposition may serve as a source of further ideas as to possible properties of our relations $\mathcal{P}^\wedge$ and $\mathcal{P}^\vee$. In fact, many properties known for the relations K, T, T_l, T_r automatically hold for $\mathcal{H}^\wedge$, $\mathcal{H}^\vee$, $\mathcal{L}^\vee$ and $\mathcal{R}^\vee$ in view of the above proposition, but they could possibly be extended to either $\mathcal{P}^\wedge$ or $\mathcal{P}^\vee$. Somewhat in the opposite direction, one could try to extend the results we have seen here for congruences on completely regular semigroups to the more general context of regular (or only inverse) semigroups.

In addition to the identifications in Proposition 9.1, we also have that $U = \mathcal{D}^\vee$ and thus $V = \mathcal{D}^\vee \cap \mathcal{H}^\wedge$ in the notation of [3]. Finally $\mathcal{H}^\wedge \cap \mathcal{H}^\vee = \varepsilon$ by Proposition 9.1 since $K \cap T = \varepsilon$. These results shed further light on Diagram 2.

REFERENCES

[1] J. M. HOWIE, *An Introduction to Semigroup Theory*, Academic Press, London (1976).
[2] F. PASTIJN - M. PETRICH, *Congruences on regular semigroups*, Trans. Amer. Math. Soc., **295** (1986), pp. 607- 633.
[3] F. PASTIJN - M. PETRICH, *The congruence lattice of a regular semigroup*, J. Pure Appl. Algebra, 53 (1988), pp. 93-123.
[4] M. PETRICH, *Congruences on completely regular semigroups*, Canad. J. Math., 41 (1989), pp. 439-461.