Annali di Matematica pura ed applicata
(IV), Vol. CLXVII (1994), pp. 73-100

# A Critical Point Theory for Nonsmooth Functionals (*)(**).

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**Summary.** – *A new generalized notion of* $\|df(u)\|$ *is introduced, which allows to prove several results of critical point theory for continuous functionals. An application to variational inequalities is shown.*

## 1. – Introduction.

Several results of classical critical point theory [17, 18] have been recently extended to suitable classes of non-differentiable functionals. Of course a basic tool for such a development is constituted by a generalized notion of $\|df(u)\|$ which allows to formulate the notions of critical point and Palais-Smale sequence.

The case of locally Lipschitz continuous functionals on Banach spaces has been treated in [4]. The notions of critical point and Palais-Smale sequence are formulated in terms of the Clarke's subdifferential $\partial f(u)$ [5].

By means of the notion of slope $|\nabla f|(u)$, introduced in [9], a critical point theory for certain functionals defined on Hilbert spaces (see Definition 2.13) has been developed in [8, 10, 11, 16].

The case of functionals on Banach spaces of the form $f = f_0 + f_1$ with $f_0$ convex and $f_1$ of class $C^1$ is treated in [19]. In this case it is equivalent to use the Clarke's subdifferential or the slope, in order to state the notions of critical point and Palais-Smale sequence.

Let us point out that a general critical point theory for continuous functionals cannot be developed by means of the mentioned notions of Clarke's subdifferential and slope. Consider in fact $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x - \sqrt[3]{x^2}$. The function $f$ verifies the Palais-Smale condition in any reasonable sense and has a behaviour like in the Ambrosetti-Rabinowitz mountain pass theorem [1]. Of course the mountain pass point should be the origin, but $\partial f(0) = \emptyset$ and $|\nabla f|(0) = +\infty$.

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(*) Entrata in Redazione il 6 febbraio 1992.

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(**) Lavoro eseguito nell'ambito di un progetto nazionale di ricerca finanziato dal Ministero dell'Università e della Ricerca Scientifica e Tecnologica (40% - 1989).

The aim of this paper is to propose a new generalized notion of $\|df(u)\|$, i.e. the notion of weak slope (see Definitions (2.1) and (2.4)), which allows to develop a critical point theory for continuous functionals (see Theorems (3.7), (3.9), (3.10) and (3.12)). This new notion is conveniently related with the previous ones (see Theorems (2.11), (2.14), (2.17) and (3.13)), so that the results of [4, 19, 20] are implied by our results.

From the technical point of view, we take advantage, as in [19, 20], of the Ekeland's variational principle [2, 12], which allows us to reduce the global problem to a local one.

Let us point out that a general critical point theory for lower semicontinuous functionals seems not to be possible. Consider $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x + 1$ for $x < 0$ and $f(x) = x$ for $x \geq 0$. In this case $x = 0$ should not be considered as a mountain pass point, because the value $f(0)$ is not correct. However at the end of section 3 we suggest a procedure to treat at least some classes of lower semicontinuous functionals (for instance, that of [19]).

In the last section we show an application to an eigenvalue problem for elliptic variational inequalities. If the derivative $g'$ of the nonlinearity $g$ is subjected to a suitable lower estimate, the problem has been already solved in [3, 6, 7, 15]. By means of our techniques, we give a result under a natural estimate on $|g|$.

## 2. - The weak slope.

Throughout this section $X$ will denote a metric space endowed with the metric $d$.

(2.1) DEFINITION. - Let $f: X \to \mathbb{R}$ be a continuous function and let $u \in X$. We denote by $|df|(u)$ the supremum of the $\sigma \in [0, +\infty[$ such that there exist $\delta > 0$ and a continuous map

$$\mathcal{H}: B(u, \delta) \times [0, \delta] \to X$$

such that

$$\forall v \in B(u, \delta),\ \forall t \in [0, \delta]:\ d(\mathcal{H}(v, t), v) \leq t, \qquad f(\mathcal{H}(v, t)) \leq f(v) - \sigma t.$$

The extended real number $|df|(u)$ is called *the weak slope* of $f$ at $u$.

Let us recall a notion from [9].

(2.2) DEFINITION. - Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. We define the function

$$\mathcal{G}_f: \operatorname{epi}(f) \to \mathbb{R}$$

putting

$$\operatorname{epi}(f) = \{(u, \xi) \in X \times \mathbb{R}: f(u) \leq \xi\} \quad \text{and} \quad \mathcal{G}_f(u, \xi) = \xi.$$

In the following epi$(f)$ will be endowed with the metric

$$d((u,\xi),(v,\mu)) = (d(u,v)^2 + (\xi-\mu)^2)^{1/2}.$$

Of course epi$(f)$ is closed in $X \times \mathbb{R}$ and $\mathcal{G}_f$ is Lipschitz continuous of constant 1. Consequently $|d\mathcal{G}_f|(u,\xi) \leq 1$ for every $(u,\xi) \in \mathrm{epi}(f)$.

(2.3) PROPOSITION. – *Let $f\colon X \to \mathbb{R}$ be a continuous function and let $u \in X$, $\xi \in \mathbb{R}$. Then*

$$|d\mathcal{G}_f|(u,f(u)) = \begin{cases} \dfrac{|df|(u)}{\sqrt{1+(|df|(u))^2}} & \text{if } |df|(u) < +\infty\,, \\ 1 & \text{if } |df|(u) = +\infty\,, \end{cases}$$

$$|d\mathcal{G}_f|(u,\xi) = 1 \qquad \text{if } f(u) < \xi\,.$$

PROOF. – We first demonstrate that

$$|d\mathcal{G}_f|(u,f(u)) \geq \begin{cases} \dfrac{|df|(u)}{\sqrt{1+(|df|(u))^2}} & \text{if } |df|(u) < +\infty\,, \\ 1 & \text{if } |df|(u) = +\infty\,. \end{cases}$$

If $|df|(u) = 0$, it is true. Otherwise, let $0 < \sigma < |df|(u)$ and let $\mathcal{H}\colon B(u,\delta) \times [0,\delta] \to X$ be a continuous map as in Definition (2.1).

Consider $\mathcal{K}\colon B((u,f(u)),\delta) \times [0,\delta] \to \mathrm{epi}(f)$ defined by

$$\mathcal{K}((v,\mu),t) = \left(\mathcal{H}\left(v, \frac{t}{\sqrt{1+\sigma^2}}\right), \mu - \frac{\sigma}{\sqrt{1+\sigma^2}}t\right).$$

Since

$$f\left(\mathcal{H}\left(v, \frac{t}{\sqrt{1+\sigma^2}}\right)\right) \leq f(v) - \frac{\sigma}{\sqrt{1+\sigma^2}}t \leq \mu - \frac{\sigma}{\sqrt{1+\sigma^2}}t\,,$$

actually we have $\mathcal{K}((v,\mu),t) \in \mathrm{epi}(f)$.

Of course $\mathcal{K}$ is continuous and

$$d(\mathcal{K}((v,\mu),t),(v,\mu)) = \left(d\left(\mathcal{H}\left(v, \frac{t}{\sqrt{1+\sigma^2}}\right), v\right)^2 + \left(\frac{\sigma}{\sqrt{1+\sigma^2}}t\right)^2\right)^{1/2} \leq$$

$$\leq \left(\frac{t^2}{1+\sigma^2} + \frac{\sigma^2 t^2}{1+\sigma^2}\right)^{1/2} = t\,.$$

Furthermore we have

$$\mathcal{G}_f(\mathcal{K}((v,\mu),t)) = \mu - \frac{\sigma}{\sqrt{1+\sigma^2}}t = \mathcal{G}_f(v,\mu) - \frac{\sigma}{\sqrt{1+\sigma^2}}t\,.$$

It follows that

$$|d\mathcal{G}_f|(u,f(u)) \geq \frac{\sigma}{\sqrt{1+\sigma^2}}\,,$$

from which

$$|d\mathcal{G}_f|(u,f(u)) \geq \frac{|df|(u)}{\sqrt{1+(|df|(u))^2}} \qquad \text{if } |df|(u) < +\infty$$

and

$$|d\mathcal{G}_f|(u,f(u)) = 1 \qquad \text{if } |df|(u) = +\infty\,.$$

We now demonstrate that

$$|d\mathcal{G}_f|(u,f(u)) \leq \frac{|df|(u)}{\sqrt{1+(|df|(u))^2}} \qquad \text{if } |df|(u) < +\infty\,.$$

If $|d\mathcal{G}_f|(u,f(u)) = 0$, it is true. Otherwise, let $0 < \sigma < |d\mathcal{G}_f|(u,f(u))$ and let $\mathcal{K}\colon B((u,f(u)),\delta) \times [0,\delta] \to \operatorname{epi}(f)$ be a continuous map as in Definition (2.1).

Let $\delta' > 0$ be such that $\delta' \leq \delta\sqrt{1-\sigma^2}$ and $d(v,u)^2 + |f(v)-f(u)|^2 < \delta^2$ for every $v \in B(u,\delta')$.

Consider $\mathcal{H}\colon B(u,\delta') \times [0,\delta'] \to X$ defined by

$$\mathcal{H}(v,t) = \mathcal{K}_1\left((v,f(v)),\ \frac{t}{\sqrt{1-\sigma^2}}\right),$$

where $\mathcal{K}_1$ is the first component of $\mathcal{K}$.

Of course $\mathcal{H}$ is continuous and

$$d(\mathcal{H}(v,t),v)^2 = d\left(\mathcal{K}_1\left((v,f(v)),\ \frac{t}{\sqrt{1-\sigma^2}}\right),v\right)^2 \leq$$

$$\leq \frac{t^2}{1-\sigma^2} - \left|\mathcal{K}_2\left((v,f(v)),\ \frac{t}{\sqrt{1-\sigma^2}}\right) - f(v)\right|^2 \leq \frac{t^2}{1-\sigma^2} - \frac{\sigma^2 t^2}{1-\sigma^2} = t^2.$$

Moreover

$$f(\mathcal{H}(v,t)) = f\left(\mathcal{H}_1\left((v,f(v)),\ \frac{t}{\sqrt{1-\sigma^2}}\right)\right) \leq \mathcal{H}_2\left((v,f(v)),\ \frac{t}{\sqrt{1-\sigma^2}}\right) =$$

$$= \mathcal{G}_f\left(\mathcal{H}\left((v,f(v)),\ \frac{t}{\sqrt{1-\sigma^2}}\right)\right) \leq \mathcal{G}_f(v,f(v)) - \frac{\sigma}{\sqrt{1-\sigma^2}}t = f(v) - \frac{\sigma}{\sqrt{1-\sigma^2}}t.$$

It follows

$$|df|(u) \geq \frac{\sigma}{\sqrt{1-\sigma^2}}$$

hence

$$|d\mathcal{G}_f|(u,f(u)) \leq \frac{|df|(u)}{\sqrt{1+(|df|(u))^2}} \quad \text{if } |df|(u) < +\infty.$$

Finally, if $f(u) < \xi$, there exists $\delta > 0$ such that $\mu \geq f(v) + \delta$ whenever $(v,\mu) \in B((u,\xi),\delta)$. If we define $\mathcal{H}\colon B((u,\xi),\delta) \times [0,\delta] \to \operatorname{epi}(f)$ by $\mathcal{H}((v,\mu),t) = (v, \mu - t)$, we find immediately that $|d\mathcal{G}_f|(u,\xi) = 1$. ∎

The previous proposition allows us to define in a consistent way the weak slope also in the lower semicontinuous case.

If $f\colon X \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function and $b \in \mathbb{R}$, we set

$$\mathcal{D}(f) = \{u \in X\colon f(u) < +\infty\},$$

$$f^b = \{u \in X\colon f(u) \leq b\}.$$

(2.4) DEFINITION. - Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and let $u \in \mathcal{D}(f)$. We set

$$|df|(u) = \begin{cases} \dfrac{|d\mathcal{G}_f|(u,f(u))}{(1-(|d\mathcal{G}_f|(u,f(u)))^2)^{1/2}} & \text{if } |d\mathcal{G}_f|(u,f(u)) < 1, \\ +\infty & \text{if } |d\mathcal{G}_f|(u,f(u)) = 1. \end{cases}$$

Since the above definition is indirect, let us give a criterion to obtain a lower estimate of $|df|(u)$.

(2.5) PROPOSITION. - *Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and let $u \in \mathcal{D}(f)$. Let us assume that there exist $\delta > 0$, $b > f(u)$, $\sigma > 0$ and a continu-*

*ous* map $\mathcal{H}\colon (B(u,\delta)\cap f^b)\times[0,\delta]\to X$ *such that*

$$\forall v\in B(u,\delta)\cap f^b,\quad \forall t\in[0,\delta]\colon d(\mathcal{H}(v,t),v)\le t,\qquad f(\mathcal{H}(v,t))\le f(v)-\sigma t\,.$$

*Then* $|df|(u)\ge\sigma$.

PROOF. – The case $|d\mathcal{G}_f|(u,f(u))=1$ is trivial. Let us assume $|d\mathcal{G}_f|(u,f(u))<1$.

Let $\delta'\in\,]0,\delta]$ be such that $\mu\le b$ for $(v,\mu)\in B((u,f(u)),\delta')$ and let us define

$$\mathcal{K}\colon B((u,f(u)),\delta')\times[0,\delta']\to \mathrm{epi}(f)$$

by

$$\mathcal{K}((v,\mu),t)=\left(\mathcal{H}\left(v,\frac{t}{\sqrt{1+\sigma^2}}\right),\ \mu-\frac{\sigma}{\sqrt{1+\sigma^2}}t\right).$$

Following the proof of Proposition (2.3), we obtain that $\mathcal{K}((v,\mu),t)\in\mathrm{epi}(f)$,

$$d(\mathcal{K}((v,\mu),t),(v,\mu))\le t\,,$$

and

$$\mathcal{G}_f(\mathcal{K}((v,\mu),t))=\mathcal{G}_f(v,\mu)-\frac{\sigma}{\sqrt{1+\sigma^2}}t\,.$$

Since $\mathcal{K}$ is evidently continuous,

$$|d\mathcal{G}_f|(u,f(u))\ge\frac{\sigma}{\sqrt{1+\sigma^2}}\,,$$

which can be rewritten

$$\sigma^2\le\frac{(|d\mathcal{G}_f|(u,f(u)))^2}{1-(|d\mathcal{G}_f|(u,f(u)))^2}=(|df|(u))^2.\qquad\blacksquare$$

As we shall see in the next result, the weak slope is lower semicontinuous with respect to the graph topology.

(2.6) PROPOSITION. – *Let* $f\colon X\to\mathbb{R}\cup\{+\infty\}$ *be a lower semicontinuous function and let* $u\in\mathcal{D}(f)$. *If* $(u_h)$ *is a sequence converging to* $u$ *with* $(f(u_h))$ *converging to* $f(u)$, *it is*

$$|df|(u)\le\liminf_h|df|(u_h)\,.$$

PROOF. – First we treat the case in which $f\colon X\to\mathbb{R}$ is continuous.

If $|df|(u)=0$, the assertion is true. Otherwise, let $0<\sigma<|df|(u)$ and let $\mathcal{H}\colon B(u,\delta)\times[0,\delta]\to X$ be as in Definition (2.1).

Since we have eventually $u_h \in B(u, \delta/2)$, we can consider the restriction of $\mathcal{H}$ to $B(u_h, \delta/2) \times [0, \delta/2]$. It follows that $|df|(u_h) \geq \sigma$, hence the result.

The general case can be reduced to the previous one by means of the function $\mathcal{G}_f$. ■

Now let us describe a case in which it is possible to compute the weak slope of a sum of two functions.

(2.7) PROPOSITION. – *Let $f_0 : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, $f_1 : X \to \mathbb{R}$ a locally Lipschitz continuous function and let $f = f_0 + f_1$. Let $u \in \mathcal{D}(f_0)$ and let us assume that*

$$\lim_{r \to 0^+} \left( \sup \left\{ \frac{|f_1(v) - f_1(w)|}{d(v, w)} : v, w \in B(u, r), v \neq w \right\} \right) = 0 .$$

*Then for every $\xi \geq f(u)$ we have*

$$|d\mathcal{G}_f|(u, \xi) = |d\mathcal{G}_{f_0}|(u, \xi - f_1(u)) .$$

*In particular, $|df|(u) = |df_0|(u)$.*

PROOF. – Given $\xi \geq f(u)$, let us show that

$$|d\mathcal{G}_f|(u, \xi) \geq |d\mathcal{G}_{f_0}|(u, \xi - f_1(u)) .$$

If $|d\mathcal{G}_{f_0}|(u, \xi - f_1(u)) = 0$, it is obvious. Otherwise, let $0 < \sigma < |d\mathcal{G}_{f_0}|(u, \xi - f_1(u))$. Let $\varepsilon > 0$ and let

$$\mathcal{H}: B((u, \xi - f_1(u)), \delta) \times [0, \delta] \to \operatorname{epi}(f_0)$$

be as in Definition (2.1). Without loss of generality, we can assume that $f_1$ is Lipschitz continuous of constant $\varepsilon$ in $B(u, 2\delta)$. Let $\delta' \in ]0, \delta]$ be such that $(v, \mu - f_1(v)) \in B((u, \xi - f_1(u)), \delta)$ for every $(v, \mu) \in B((u, \xi), \delta')$ and let $\mathcal{K}: B((u, \xi), \delta') \times [0, \delta'] \to \operatorname{epi}(f)$ be defined by

$$\mathcal{K}((v, \mu), t) =$$

$$= \left( \mathcal{H}_1\left((v, \mu - f_1(v)), \frac{t}{1+\varepsilon}\right), \mathcal{H}_2\left((v, \mu - f_1(v)), \frac{t}{1+\varepsilon}\right) + f_1\left(\mathcal{H}_1\left((v, \mu - f_1(v)), \frac{t}{1+\varepsilon}\right)\right)\right).$$

It is, applying the triangular inequality,

$$d\big(\mathcal{K}((v, \lambda + f_1(v)), (1+\varepsilon)s), (v, \lambda + f_1(v))\big) =$$

$$= d\Big(\mathcal{H}((v, \lambda), s), \big(v, \lambda + f_1(v) - f_1(\mathcal{H}_2((v, \lambda), s))\big)\Big) \leq$$

$$\leq d(\mathcal{H}((v, \lambda), s), (v, \lambda)) + |f_1(\mathcal{H}_1((v, \lambda), s)) - f_1(v)| \leq s + \varepsilon s = (1+\varepsilon)s .$$

Furthermore, it is

$$\mathcal{G}_f(\mathcal{H}((v,\mu),t)) = \mathcal{H}_2\left((v,\mu - f_1(v)), \frac{t}{1+\varepsilon}\right) + f_1\left(\mathcal{H}_1\left((v,\mu - f_1(v)), \frac{t}{1+\varepsilon}\right)\right) \leqslant$$

$$\leqslant \mu - f_1(v) - \sigma\frac{t}{1+\varepsilon} + f_1\left(\mathcal{H}_1\left((v,\mu - f_1(v)), \frac{t}{1+\varepsilon}\right)\right) \leqslant$$

$$\leqslant \mathcal{G}_f(v,\mu) - \left(\frac{\sigma}{1+\varepsilon} - \varepsilon\right)t.$$

Hence

$$|d\mathcal{G}_f|(u,\xi) \geqslant \frac{\sigma}{1+\varepsilon} - \varepsilon$$

and, since $\varepsilon$ can be made arbitrarily small,

$$|d\mathcal{G}_f|(u,\xi) \geqslant \sigma,$$

which implies

$$|d\mathcal{G}_f|(u,\xi) \geqslant |d\mathcal{G}_{f_0}|(u,\xi - f_1(u)).$$

The opposite inequality is obtained by replacing the function $f_0$ with the function $f$ and the function $f_1$ with the function $(-f_1)$. ∎

In the following of this section we want to compare the notion of weak slope with other notions in the literature.

(2.8) DEFINITION (see [9]). – Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and let $u \in \mathcal{D}(f)$. We define

$$|\nabla f|(u) = \begin{cases} \limsup\limits_{v \to u} \dfrac{f(u) - f(v)}{d(u,v)} & \text{if } u \text{ is not a local minimum}, \\ 0 & \text{if } u \text{ is a local minimum}. \end{cases}$$

The extended real number $|\nabla f|(u)$ is called *the (strong) slope* of $f$ at $u$. It is readily seen that $|df|(u) \leqslant |\nabla f|(u)$.

(2.9) DEFINITION (see [9]). – Let $X$ be a Banach space, $A$ an open subset of $X$ and $f\colon A \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function. For every $u \in \mathcal{D}(f)$ we denote by $\partial^- f(u)$ the (possibly empty) set of $\alpha$'s in $X'$ such that

$$\liminf_{v \to u} \frac{f(v) - f(u) - \langle \alpha, v - u\rangle}{\|v - u\|} \geqslant 0.$$

The elements of $\partial^- f(u)$ are called *subdifferentials* of $f$ at $u$.

The following properties are easily verified.

(2.10) PROPOSITION. – *Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and let $u \in \mathcal{D}(f)$. Then the following facts hold:*

*(a) if $g\colon A \to \mathbb{R}$ is Fréchet differentiable at u, it is*

$$\partial^-(f+g)(u) = \{\alpha + dg(u)\colon \alpha \in \partial^- f(u)\}\,;$$

*(b) if $\alpha \in \partial^- f(u)$, it is for every $w \in X$*

$$\langle \alpha, w \rangle \leq \liminf_{t \to 0^+} \frac{f(u+tw) - f(u)}{t}\,;$$

*(c) if A and f are convex, $\partial^- f(u)$ agrees with the subdifferential of convex analysis;*

*(d) if $\alpha \in \partial^- f(u)$, it is $|\nabla f|(u) \leq \|\alpha\|$;*

*(e) $\partial^- f(u)$ is (strongly) closed and convex in $X'$.*

In the following result we consider $C^1$ perturbations of convex functions. For such a class a critical point theory has been elaborated in [19].

(2.11) THEOREM. – *Let A be a convex open subset of a Banach space X, let $f_0\colon A \to \mathbb{R} \cup \{+\infty\}$ be a convex and lower semicontinuous function, $f_1\colon A \to \mathbb{R}$ a function of class $C^1$ and let $f = f_0 + f_1$.*

*Then the following facts hold:*

*(a) $\forall u \in \mathcal{D}(f)$: $|df|(u) = |\nabla f|(u)$;*

*(b) for every $u \in \mathcal{D}(f)$ it is $|df|(u) < +\infty$ if and only if $\partial^- f(u) \neq \emptyset$ and in that case*

$$|df|(u) = \min\{\|\alpha\|\colon \alpha \in \partial^- f(u)\}\,.$$

PROOF. – We first observe that by Proposition (2.10) the set $\partial^- f_0(u)$ is weak*-closed. Since

$$\partial^- f(u) = \{\alpha + df_1(u)\colon \alpha \in \partial^- f_0(u)\}\,,$$

$\partial^- f(u)$ is weak*-closed too. It follows that, if $\partial^- f(u) \neq \emptyset$, there exists $\min\{\|a\|\colon \alpha \in \partial^- f(u)\}$.

It is obviously enough to consider the case $df_1(u) = 0$. Then by Proposition (2.7) we can assume $f_1 = 0$.

By Proposition (2.10) it holds

$$\partial^- f(u) \neq \emptyset \Rightarrow |df|(u) \leq |\nabla f|(u) \leq \min\{\|\alpha\|\colon \alpha \in \partial^- f(u)\}\,.$$

Now assume that $0 \notin \partial^- f(u)$ and let $\sigma > 0$ be such that

$$\forall \alpha\colon \alpha \in \partial^- f(u) \Rightarrow \|\alpha\| > \sigma\,.$$

By [19, Lemma 1.3] there exists $w \in A$ such that $f(w) < f(u) - \sigma\|w - u\|$. Since $f$ is lower semicontinuous, there exists $\delta > 0$ such that

$$\forall v \in B(u, \delta): f(w) < f(v) - \sigma\|w - v\|.$$

Unless reducing $\delta$, it can be supposed $w \notin B(u, 2\delta)$. We define $\mathcal{H}: B(u, \delta) \times \times [0, \delta] \to X$ by

$$\mathcal{H}(v, t) = v + t\frac{w - v}{\|w - v\|}.$$

The map $\mathcal{H}$ is evidently continuous and $\|\mathcal{H}(v, t) - v\| = t$. Moreover, since $0 \leq \leq (t/\|w - v\|) \leq 1$, it is

$$f(\mathcal{H}(v, t)) \leq f(v) + \frac{t}{\|w - v\|}(f(w) - f(v)) \leq f(v) - \sigma t.$$

By Proposition (2.5), it is $|df|(u) \geq \sigma$. Therefore $\partial^- f(u) = \emptyset$ implies $|df|(u) = = +\infty$, while $\partial^- f(u) \neq \emptyset$ implies

$$|df|(u) \geq \min\{\|\alpha\|: \alpha \in \partial^- f(u)\}. \quad ■$$

(2.12) COROLLARY. – *Let $X$ be a Finsler manifold of class $C^1$ and let $f: X \to \mathbb{R}$ be a function of class $C^1$.*

*Then we have $|df|(u) = |\nabla f|(u) = \|df(u)\|$ for every $u \in X$.*

PROOF. – If $f$ is defined in a convex open subset of a Banach space, the thesis follows from Theorem (2.11) with $f_0 = 0$.

In general, for every $\varepsilon > 0$ there exists a neighbourhood $U$ of $u$ and a diffeomorphism $\Phi$ from $U$ onto a convex open subset of $T_u X$ such that $\Phi$ and $\Phi^{-1}$ are both Lipschitz continuous of constant $1 + \varepsilon$. By the previous step the thesis follows. ■

We point out that the critical point theory for $C^1$ functions on $C^2$ manifolds is a classical topic [17]. The case of $C^1$ manifolds has been studied in [20].

(2.13) DEFINITION (see [8, 11]). – Let $A$ be an open subset of a Hilbert space, let $f: A \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and let $p \geq 0$. We say that $f$ is a function with *$\varphi$-monotone subdifferential of order $p$*, if there exists a continuous function

$$\chi: (\mathcal{D}(f))^2 \times \mathbb{R}^2 \to \mathbb{R}^+$$

such that

$$\langle \alpha - \beta, u - v \rangle \geq -\chi(u, v, f(u), f(v))(1 + \|\alpha\|^p + \|\beta\|^p)\|u - v\|^2$$

whenever $u, v \in \mathcal{D}(f)$, $\alpha \in \partial^- f(u)$, $\beta \in \partial^- f(v)$.

A critical point theory for functions with $\varphi$-monotone subdifferential of order two has been elaborated in [8, 10, 11] by a suitable evolution theory. However that approach does not allow $C^1$ perturbations.

(2.14) THEOREM. – *Let $A$ be an open subset of a Hilbert space, let $f_0: A \to \mathbb{R} \cup \cup \{+\infty\}$ be a lower semicontinuous function with $\varphi$-monotone subdifferential of order two, $f_1: A \to \mathbb{R}$ a function of class $C^1$ and let $f = f_0 + f_1$.*

*Then the following facts hold:*

(*a*) $\forall u \in \mathcal{D}(f)$: $|df|(u) = |\nabla f|(u)$;

(*b*) *for every $u \in \mathcal{D}(f)$ it is $|df|(u) < +\infty$ if and only if $\partial^- f(u) \neq \emptyset$ and in that case*

$$|df|(u) = \min\{\|\alpha\|: \alpha \in \partial^- f(u)\}.$$

PROOF. – Again by Proposition (2.7) it is sufficient to treat the case $f_1 = 0$.

For every $v \in \mathcal{D}(f)$ let $\mathcal{V}: [0, \mathcal{T}(v)[ \to \mathcal{D}(f)$ be the curve of maximal slope for $f$ such that $\mathcal{V}(0) = v$ defined on its maximal interval (see [11, section 3]).

We recall that either $\mathcal{T}(v) = +\infty$ or

$$\lim_{t \to \mathcal{T}(v)} f(\mathcal{V}(t)) = -\infty$$

or there exists

$$\lim_{t \to \mathcal{T}(v)} \mathcal{V}(t) \in \partial A.$$

Let $\varepsilon > 0$ and let $\Psi(t) = \int_0^t (\varepsilon + \|V'(\tau)\|)\, d\tau$. For every $v \in \mathcal{D}(f)$ and $s \in \Psi([0, \mathcal{T}(v)[)$ we set $\mathcal{H}(v, s) = \mathcal{V}(\Psi^{-1}(s))$.

Let $u \in \mathcal{D}(f)$, let $\delta > 0$ be such that $\overline{B(u, 2\delta)} \subseteq A$ and $f$ is bounded below on $\overline{B(u, 2\delta)}$ and let $b = f(u) + \delta$.

By the maximality of $[0, \mathcal{T}(v)[$ it follows that $\mathcal{H}$ is defined on $(B(u, \delta) \cap \mathcal{D}(f)) \times \times [0, \delta]$.

Of course $\mathcal{H}$ is Lipschitz continuous of constant 1 with respect to the second variable, so that

$$\|\mathcal{H}(v, s) - v\| = \|\mathcal{H}(v, s) - \mathcal{H}(v, 0)\| \leq s.$$

Furthermore we have

$$(2.15) \quad f(\mathcal{H}(v, s)) = f(\mathcal{V}(\Psi^{-1}(s))) = f(v) - \int_0^{\Psi^{-1}(s)} \|\mathcal{V}'(\tau)\|^2 d\tau =$$

$$= f(v) - \int_0^s \frac{\|\mathcal{V}'(\Psi^{-1}(\xi))\|^2}{\varepsilon + \|\mathcal{V}'(\Psi^{-1}(\xi))\|} d\xi.$$

Let us prove that $\mathcal{H}$: $(B(u, \delta) \cap f^b) \times [0, \delta] \to A$ is continuous. It suffices to consider a sequence $(v_h)$ in $B(u, \delta)$, converging to $v \in B(u, \delta)$, with $f(v_h) \leq b$ and $s \in [0, \delta]$ fixed.

It is $\mathcal{H}(v_h, s) = \mathcal{V}_h(t_h)$, where $\mathcal{V}_h : [0, \mathcal{T}(v_h)[ \to \mathcal{D}(f)$ is the curve of maximal slope for $f$ with $\mathcal{V}_h(0) = v_h$ and

$$s = \int_0^{t_h} (\varepsilon + \|\mathcal{V}_h'(\tau)\|)\, d\tau$$

for a suitable $t_h < \mathcal{T}(v_h)$.

Since $(t_h)$ is bounded, we can suppose $t_h \to t$.

We claim that $t < \mathcal{T}(v)$. If not, there would exist $\tau > 0$ with $0 < t - \tau < \mathcal{T}(v)$ such that $\mathcal{V}(t - \tau) \notin \overline{B(u, 2\delta)}$.

On the other hand we have [11, Theorem 3.7] $\mathcal{V}_h(t_h - \tau) \to \mathcal{V}(t - \tau)$, which is a contradiction because

$$\mathcal{V}_h(t_h - \tau) \in \overline{B(u, 2\delta)}.$$

Since we have also

$$\mathcal{V}_h' \to \mathcal{V}' \quad \text{in } L^1\left(0, \frac{t + \mathcal{T}(u)}{2}\right),$$

we obtain

$$s = \int_0^{t} (\varepsilon + \|\mathcal{V}'(\tau)\|)\, d\tau,$$

hence

$$\mathcal{H}(v_h, s) = \mathcal{V}_h(t_h) \to \mathcal{V}(t) = \mathcal{H}(v, s).$$

Therefore $\mathcal{H}$ is continuous.

Assume now that $0 \notin \partial^- f(u)$ and take $\sigma > 0$ such that

$$\forall \alpha: \alpha \in \partial^- f(u) \Rightarrow \|\alpha\| > \sigma.$$

Unless reducing $\delta$, we have by [11, Theorem 1.18]

$$\forall v \in B(u, 2\delta) \cap f^b, \qquad \forall \alpha: \alpha \in \partial^- f(v) \Rightarrow \|\alpha\| > \sigma.$$

By (2.15) it follows that

$$f(\mathcal{H}(v, s)) \leq f(v) - \frac{\sigma^2}{\varepsilon + \sigma} s,$$

hence by Proposition (2.5)

$$|df|(u) \geq \frac{\sigma^2}{\varepsilon + \sigma}.$$

By the arbitrariness of $\varepsilon$ we get $|df|(u) \geq \sigma$.
Therefore we have $|df|(u) = +\infty$ if $\partial^- f(u) = \emptyset$ and

$$|df|(u) \geq \min\{\|\alpha\|: \alpha \in \partial^- f(u)\}$$

if $\partial^- f(u) \neq \emptyset$. Since $\partial^- f(u) \neq \emptyset$ trivially implies

$$|df|(u) \leq |\nabla f|(u) \leq \min\{\|\alpha\|: \alpha \in \partial^- f(u)\},$$

the thesis follows. ■

Finally, we establish a relation between the weak slope and the Clarke's subdifferential for locally Lipschitz continuous functions. We recall that a critical point theory for such a class of functions has been elaborated in [4].

(2.16) DEFINITION (see [5]). – Let $X$ be a Banach space, $A$ an open subset of $X$, $f: A \to \mathbb{R}$ a locally Lipschitz continuous function and $u \in A$. We set

$$\forall w \in X: f^0(u; w) = \limsup_{\substack{v \to u \\ t \to 0^+}} \frac{f(v + tw) - f(v)}{t},$$

$$\partial f(u) = \{\alpha \in X': f^0(u; w) \geq \langle \alpha, w \rangle \text{ for all } w \text{ in } X\}.$$

It turns out [5] that $\partial f(u)$ is non-empty and weak*-compact.

(2.17) THEOREM. – *Let $X$ be a Banach space, $A$ an open subset of $X$ and $f: X \to \mathbb{R}$ a locally Lipschitz continuous function.*
*Then for every $u \in A$ it holds*

$$|df|(u) \geq \min\{\|\alpha\|: \alpha \in \partial f(u)\}.$$

PROOF. – If the right hand side is zero, the fact is trivial. Otherwise, let us take

$$0 < \sigma < \min\{\|\alpha\|: \alpha \in \partial f(u)\}.$$

It is known [5] that the function $f^0(u; \cdot)$ is convex, positively homogeneous and continuous.
By [19, Lemma 1.3] it follows that there exists $w \in X$ such that

$$f^0(u; w) < -\sigma\|w\|.$$

We can suppose that $\|w\| = 1$. Let $\delta > 0$ be such that $B(u, 2\delta) \subseteq A$ and

$$f(v + tw) \leqslant f(v) - \sigma t$$

whenever $v \in B(u, \delta)$ and $t \in [0, \delta]$.

Then, if we define $\mathcal{H}: B(u, \delta) \times [0, \delta] \to A$ by

$$\mathcal{H}(v, t) = v + tw,$$

we find that $|df|(u) \geqslant \sigma$. ■

## 3. – Continuous functionals.

In the following $X$ will still denote a metric space endowed with the metric $d$.

(3.1) DEFINITION. – Let $f: X \to \mathbb{R}$ be a continuous function. A point $u \in X$ is said to be *critical* (*from below*) for $f$, if $|df|(u) = 0$. A real number $c$ is said to be a *critical value* (*from below*) for $f$, if there exists $u \in X$ such that $|df|(u) = 0$ and $f(u) = c$.

(3.2) DEFINITION. – Let $f: X \to \mathbb{R}$ be a continuous function and $c \in \mathbb{R}$. We say that $f$ satisfies *the Palais-Smale condition* at level $c$, if from every sequence $(u_h)$ in $X$ with $|df|(u_h) \to 0$ and $f(u_h) \to c$ it is possible to extract a subsequence $(u_{h_k})$ converging in $X$ (by Proposition (2.6) the limit of $(u_{h_k})$ is necessarily a critical point for $f$).

In this chapter a fundamental tool is constituted by the following Ekeland's variational principle [2, 12].

(3.3) THEOREM. – *Let $X$ be a complete metric space and $f: X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function. Let $r > 0$, $\sigma > 0$ and $u \in X$ be such that*

$$f(u) < \inf_X f + r\sigma.$$

*Then there exists $v \in X$ such that*

$$f(v) \leqslant f(u), \qquad d(v, u) < r, \qquad \forall w \in X: f(w) \geqslant f(v) - \sigma d(w, v).$$

In the rest of the section we will make repeated use of the following consequence of Ekeland's principle.

(3.4) COROLLARY. – *Let $X$ be a complete metric space and $f: X \to \mathbb{R} \cup \{+\infty\}$ a lower semicontinuous function. Let $r > 0$, $\sigma > 0$ and $E \subseteq X$ be such that $E \neq \emptyset$ and*

$$\inf_E f < \inf_X f + r\sigma.$$

*Then there exists* $v \in X$ *such that*

$$f(v) < \inf_X f + r\sigma, \qquad d(v, E) < r, \qquad |df|(v) < \sigma.$$

PROOF. – Let $u \in E$ and $\sigma' \in ]0, \sigma[$ be such that $f(u) < \inf_X f + r\sigma'$. By Ekeland's principle there exists $v \in X$ such that $f(v) \leq f(u)$, $d(v, u) < r$ and

$$\forall w \in X: f(w) \geq f(v) - \sigma' d(w, v).$$

It follows $f(v) < \inf_X f + r\sigma$, $d(v, E) < r$ and

$$|df|(v) \leq |\nabla f|(v) \leq \sigma' < \sigma. \quad \blacksquare$$

(3.5) LEMMA. – *Let* $X$ *be a metric space and* $f: X \to \mathbb{R}$ *a continuous function. Let* $K$ *be a compact subset of* $X$ *and* $\sigma > 0$ *such that*

$$\inf\{|df|(u): u \in K\} > \sigma.$$

*Then there exist a neighbourhood* $U$ *of* $K$ *in* $X$, $\delta > 0$ *and a continuous map* $\mathcal{H}: X \times [0, \delta] \to X$ *such that:*

*a)* $\forall (u, t) \in X \times [0, \delta]$: $d(\mathcal{H}(u, t), u) \leq t$;

*b)* $\forall (u, t) \in X \times [0, \delta]$: $f(\mathcal{H}(u, t)) \leq f(u)$;

*c)* $\forall (u, t) \in U \times [0, \delta]$: $f(\mathcal{H}(u, t)) \leq f(u) - \sigma t$.

PROOF. – For every $u \in K$ let us choose $\delta_u > 0$ and

$$\mathcal{H}_u: B(u, \delta_u) \times [0, \delta_u] \to X$$

according to Definition (2.1). Let $u_1, \dots, u_n \in K$ be such that

$$K \subseteq \bigcup_{j=1}^{n} B\left(u_j, \frac{1}{2}\delta_{u_j}\right).$$

We set $\delta_j = \delta_{u_j}$, $\mathcal{H}_j = \mathcal{H}_{u_j}$ and choose

$$0 < \delta < \min\left\{\frac{1}{2}\delta_1, \dots, \frac{1}{2}\delta_n\right\}.$$

Let us take a neighbourhood $U$ of $K$ in $X$ and continuous functions $\theta_j: X \to [0, 1]$ $(1 \leq j \leq n)$ with

$$\operatorname{supt}\theta_j \subseteq B\left(u_j, \frac{1}{2}\delta_j\right), \qquad \forall v \in X: \sum_{j=1}^{n} \theta_j(v) \leq 1, \qquad \forall v \in U: \sum_{j=1}^{n} \theta_j(v) = 1.$$

We claim that for every $j = 1, \dots, n$ there exists a continuous map

$$\mathcal{K}_j: X \times [0, \delta] \to X$$

such that

$$\forall (u, t) \in X \times [0, \delta]:\ d(\mathcal{K}_j(u, t), u) \leq \left(\sum_{h=1}^{j} \theta_h(u)\right) t,$$

$$\forall (u, t) \in X \times [0, \delta]:\ f(\mathcal{K}_j(u, t)) \leq f(u) - \sigma \left(\sum_{h=1}^{j} \theta_h(u)\right) t.$$

To prove that, in the first place we set

$$\mathcal{K}_1(u, t) = \begin{cases} \mathcal{H}_1(u, \theta_1(u) t) & \text{if } u \in \overline{B\left(u_1, \frac{1}{2}\delta_1\right)}, \\ u & \text{if } u \notin B\left(u_1, \frac{1}{2}\delta_1\right). \end{cases}$$

Evidently $\mathcal{K}_1$ satisfies the requested conditions.

Let now $2 \leq j \leq n$ and suppose we have defined $\mathcal{K}_{j-1}$. Since

$$d(\mathcal{K}_{j-1}(u, t), u) \leq \left(\sum_{h=1}^{j-1} \theta_h(u)\right) t \leq \delta < \frac{1}{2}\delta_j,$$

it is

$$\forall u \in \overline{B\left(u_j, \frac{1}{2}\delta_j\right)}: \qquad \mathcal{K}_{j-1}(u, t) \in B(u_j, \delta_j).$$

Then we define

$$\mathcal{K}_j(u, t) = \begin{cases} \mathcal{H}_1(\mathcal{K}_{j-1}(u, t), \theta_j(u) t) & \text{if } u \in \overline{B\left(u_j, \frac{1}{2}\delta_j\right)}, \\ \mathcal{K}_{j-1}(u, t) & \text{if } u \notin B\left(u_j, \frac{1}{2}\delta_j\right). \end{cases}$$

By the inductive hypothesis, it is easy to verify that $\mathcal{K}_j$. satisfies the requested conditions.

To conclude the proof, it is sufficient to set $\mathcal{K} = \mathcal{K}_n$. ∎

In the following we will denote by $\mathfrak{K}$ the family of the compact non-empty subsets of $X$. The set $\mathfrak{K}$ will be endowed with the Hausdorff metric

$$\mathfrak{d}(A, B) = \max \left\{ \max_{a \in A} d(a, B), \max_{b \in B} d(b, A) \right\}.$$

We recall that, if $(X, d)$ is complete, then $(\mathfrak{K}, \mathfrak{d})$ is complete [14].

Given a continuous function $f\colon X \to \mathbb{R}$, we define a function $\mathcal{F}\colon \mathfrak{K} \to \mathbb{R}$ setting

$$\mathcal{F}(K) = \max_K f.$$

It is easily verified that the function $\mathcal{F}$ is continuous with respect to the metric $\mathfrak{d}$.

Now we apply the notion of weak slope to Ljusternik-Schnirelman theory. According to [13], we shall consider the category defined by means of *open* coverings. Therefore every closed subset $C$ of $X$ possesses a neighbourhood $U$ with $\mathrm{cat}_X \bar{U} = \mathrm{cat}_X C$. It follows that for every $h \geq 1$ the set

$$\Gamma_h = \{K \in \mathfrak{K}\colon \mathrm{cat}_X K \geq h\}$$

is closed in $(\mathfrak{K}, \mathfrak{d})$.

As known, if $X$ is an *ANR*, the category in the sense of [13] agrees with the category in the sense of [17], defined by means of closed coverings.

(3.6) LEMMA. – *Let $X$ be a metric space and $f\colon X \to \mathbb{R}$ a continuous function. For every $h \geq 1$ let $\mathcal{F}_h = \mathcal{F}|_{\Gamma_h}$.*

*Let $K \in \Gamma_h$, $\rho > 0$, $\sigma > 0$ be such that*

$$\forall u \in K\colon f(u) \geq \max_K f - \rho \Rightarrow |df|(u) \geq \sigma.$$

*Then* $|d\mathcal{F}_h|(K) \geq \sigma$.

PROOF. – Let $\sigma' \in ]0, \sigma[$ and let $U$ and $\mathcal{H}\colon X \times [0, \delta] \to X$ be obtained applying Lemma (3.5) to the compact set

$$\{u \in K\colon f(u) \geq \max_K f - \rho\}$$

and to $\sigma'$.

Let $\mathcal{V}$ be a neighbourhood of $K$ in $\Gamma_h$ such that

$$\forall A \in \mathcal{V}\colon \max_A f \geq \max_K f - \frac{\rho}{2},$$

$$\forall A \in \mathcal{V}\colon \{u \in A\colon f(u) \geq \max_K f - \rho\} \subseteq U.$$

Let $\delta' = \min\{\rho/(2\sigma'), \delta\}$ and let $\mathcal{K}\colon \mathcal{V} \times [0, \delta'] \to \Gamma_h$ be defined by

$$\mathcal{K}(A, t) = \mathcal{H}(A \times \{t\}).$$

It is easy to verify that $\mathcal{K}$ is continuous and

$$\mathfrak{d}(\mathcal{K}(A, t), A) \leq t.$$

Let now $A \in \mathcal{V}$. If $u \in A$ and $f(u) \leq \max_K f - \rho$, it is for every $t \in [0, \delta']$

$$f(\mathcal{H}(u, t)) \leq f(u) \leq \max_K f - \rho \leq \max_A f - \frac{\rho}{2} \leq \mathcal{F}_h(A) - \sigma' t.$$

Otherwise, if $u \in A$ and $f(u) \geq \max_K f - \rho$, it is for every $t \in [0, \delta']$

$$f(\mathcal{H}(u, t)) \leq f(u) - \sigma' t \leq \mathcal{F}_h(A) - \sigma' t .$$

In any case we have

$$\forall A \in \mathcal{V},\ \forall t \in [0, \delta']:\ \mathcal{F}_h(\mathcal{H}(A, t)) \leq \mathcal{F}_h(A) - \sigma' t ,$$

hence $|d\mathcal{F}_h|(K) \geq \sigma'$.

The assertion follows by the arbitrariness of $\sigma' \in ]0, \sigma[$. ■

Now we can prove the first result concerning Ljusternik-Schnirelman category.

(3.7) THEOREM. – *Let $X$ be a complete metric space and $f: X \to \mathbb{R}$ a continuous function. For $1 \leq h \leq \sup\{\mathrm{cat}_X K$: $K$ is a compact subset of $X\}$ let*

$$c_h = \inf_{\Gamma_h} \mathcal{F} = \inf_{K \in \Gamma_h} (\max_K f) .$$

*Then, if for some $h \geq 1$, $m \geq 1$ it is*

$$-\infty < c_h = \ldots = c_{h+m-1}$$

*and if $f$ verifies the Palais-Smale condition at level $c_h$, it holds*

$$\mathrm{cat}_X \{u \in X:\ |df|(u) = 0,\ f(u) = c_h\} \geq m .$$

*In particular, $c_h$ is a critical value for $f$.*

PROOF. – Let $c = c_h$ and let

$$K_c = \{u \in X:\ |df|(u) = 0,\ f(u) = c\} .$$

By contradiction, assume there exists a neighbourhood $U$ of $K_c$ with $\mathrm{cat}_X \bar{U} \leq m - 1$. For every $\varepsilon > 0$ let

$$\mathcal{N}_\varepsilon(K_c) = \{u \in X:\ d(u, K_c) < \varepsilon\} .$$

Since the Palais-Smale condition at level $c$ holds, $K_c$ is compact. Therefore we can suppose that $U = \mathcal{N}_{2r}(K_c)$ with $r > 0$.

There exists $\sigma > 0$ such that

$$u \notin \mathcal{N}_r(K_c) \quad \text{and} \quad c - \sigma \leq f(u) \leq c + \sigma \Rightarrow |df|(u) \geq \sigma .$$

For every $c' > c$ there exists $A_1 \in \Gamma_{h+m-1}$ such that $\mathcal{F}(A_1) < c'$. Let

$$A_2 = \overline{A_1 \setminus \overline{\mathcal{N}_{2r}(K_c)}} .$$

Then $\mathrm{cat}_X A_2 \geq h$, $\mathcal{F}(A_2) < c'$ and $A_2 \cap \mathcal{N}_{2r}(K_c) = \emptyset$.

It follows that, setting

$$E = \{A \in \Gamma_h : A \cap \mathcal{N}_{2r}(K_c) = \emptyset\},$$

it is

$$\inf_E \mathcal{F}_h = \inf_{\Gamma_h} \mathcal{F}_h .$$

Being $\mathcal{F}_h$ bounded from below, by Corollary (3.4) there exists $A \in \Gamma_h$ such that

$$\mathcal{F}_h(A) < c + \sigma , \qquad \mathfrak{d}(A, E) < r , \qquad |d\mathcal{F}_h|(A) < \sigma .$$

It is $A \cap \mathcal{N}_r(K_c) = \emptyset$ and of course $\max_A f \geq c$, so that

$$\forall u \in A \colon f(u) \geq \max_A f - \sigma \Rightarrow |df|(u) \geq \sigma .$$

By Lemma (3.6) it follows

$$|d\mathcal{F}_h|(A) \geq \sigma ,$$

which is a contradiction. ■

For the two next results concerning Ljusternik-Schnirelman category, some regularity on the metric space is required.

Instead of imposing $X$ to be an *ANR*, we prefer to consider a weaker condition, which has the advantage to be homotopically invariant.

(3.8) DEFINITION. – A metric space $X$ is said to be *weakly locally contractible*, if for every $u \in X$ there exists a neighbourhood $U$ of $u$ contractible in $X$.

This means that $\mathrm{cat}_X\{x\} = 1$ for every $x \in X$.

(3.9) THEOREM. – *Let $X$ be a complete and weakly locally contractible metric space and $f\colon X \to \mathbb{R}$ a continuous function. Suppose that $f$ is bounded from below and that for every $b \in f(X)$ and for every $c \leq b$ the Palais-Smale condition at level $c$ holds.*

*Then $f$ has at least*

$$\sup\{\mathrm{cat}_X K \colon K \text{ is a compact subset of } X\}$$

*points which are critical from below.*

PROOF. – If

$$1 \leq h \leq \sup\{\mathrm{cat}_X K \colon K \text{ is a compact subset of } X\},$$

the Palais-Smale condition at level $c_h$ holds.

Furthermore, every singleton in $X$ has category 1. By Theorem (3.7) the thesis follows in a standard way. ■

(3.10) THEOREM. – *Let $X$ be a complete and weakly locally contractible metric space and $f: X \to \mathbb{R}$ a continuous function such that*

(*a*) $\inf_X f > -\infty$;

(*b*) *for every $b \in f(X)$ and for every $c \leq b$ the Palais-Smale condition at level $c$ holds*;

(*c*) $\sup\{\mathrm{cat}_X K$: $K$ is a compact subset of $X\} = +\infty$.

*Then the supremum of $f$ is not achieved and*

$$\sup_h c_h = \sup_X f,$$

*where $(c_h)$ is the sequence defined in Theorem* (3.7).

*In particular, there exists a sequence $(u_h)$ in $X$ with $|df|(u_h) = 0$ and $f(u_h) \to \sup_X f$.*

PROOF. – Let $c = \sup_h c_h$. Let us suppose that the assertion is false. In that case, it is $c < +\infty$ and the Palais-Smale condition at level $c$ holds.

Furthermore the set

$$K = \{u \in X:\ |df|(u) = 0 \text{ and } f(u) \leq c\}$$

is compact.

By the weak local contractibility there exists $r > 0$ such that $\mathrm{cat}_X(\overline{\mathcal{N}_{2r}(K)}) = k < +\infty$. Let $\sigma > 0$ be such that

$$u \notin \mathcal{N}_r(K) \quad \text{and} \quad c - r\sigma \leq f(u) \leq c + r\sigma \Rightarrow |df|(u) \geq \sigma$$

and let $h$ be such that $c < c_h + r\sigma$.

If $c' > c_{h+k}$, there exists $A_1 \in \Gamma_{h+k}$ with $\mathcal{F}(A_1) < c'$. Setting

$$A_2 = \overline{A_1 \setminus \overline{\mathcal{N}_{2r}(K)}},$$

it is $A_2 \in \Gamma_h$, $\mathcal{F}(A_2) < c'$ and $A_2 \cap \mathcal{N}_{2r}(K) = \emptyset$.

Setting

$$E = \{A \in \Gamma_h : A \cap \mathcal{N}_{2r}(K) = \emptyset\},$$

it results

$$\inf_E \mathcal{F}_h \leq c_{h+k} \leq c < c_h + r\sigma = \inf_{\Gamma_h} \mathcal{F}_h + r\sigma.$$

By Corollary (3.4) there exists $A \in \Gamma_h$ such that

$$\mathcal{F}_h(A) < c_h + r\sigma, \qquad \mathfrak{d}(A, E) < r, \qquad |d\mathcal{F}_h|(A) < \sigma.$$

It follows that $A \cap \mathcal{N}_r(K) = \emptyset$ and then

$$\forall u \in A:\ f(u) \geq c - r\sigma \Rightarrow |df|(u) \geq \sigma.$$

Since $c - r\sigma < c_h \leq \mathcal{F}_h(A)$, by Lemma (3.6) we deduce that

$$|d\mathcal{F}_h|(A) \geq \sigma,$$

which is a contradiction. ∎

Now we want to prove a saddle point theorem in the spirit of [1, 18] for continuous functions.

(3.11) LEMMA. – *Let $X$ be a metric space, $f\colon X \to \mathbb{R}$ a continuous function, $(D, S)$ a compact pair and $\psi\colon S \to X$ a continuous map. Let us consider*

$$\Phi = \{\varphi \in C(D; X)\colon \varphi|_S = \psi\}$$

*endowed with the uniform metric $\mathfrak{d}$ and let us define a continuous function $\mathcal{F}\colon \Phi \to \mathbb{R}$ by*

$$\mathcal{F}(\varphi) = \max_D (f \circ \varphi).$$

*Let $\varphi \in \Phi$, $\rho > 0$, $\sigma > 0$ be such that*

$$\max_S (f \circ \psi) < \max_D (f \circ \varphi),$$

$$\forall \xi \in D\colon\ f(\varphi(\xi)) \geq \max_D (f \circ \varphi) - \rho \Rightarrow |df|(\varphi(\xi)) \geq \sigma.$$

*Then $|d\mathcal{F}|(\varphi) \geq \sigma$.*

PROOF. – Without loss of generality we can assume

$$\max_S (f \circ \psi) \leq \max_D (f \circ \varphi) - 3\rho.$$

Let $\sigma' \in ]0, \sigma[$ and let $U$ and $\mathcal{H}\colon X \times [0, \delta] \to X$ be obtained applying Lemma (3.5) to the compact set

$$\{\varphi(\xi)\colon\ f(\varphi(\xi)) \geq \max_D (f \circ \varphi) - \rho\}$$

and to $\sigma'$. Of course we can assume $f(u) > \max_D (f \circ \varphi) - 2\rho$ for every $u \in U$.

We can also suppose that $\mathcal{H}(u, t) = u$ whenever $f(u) \leq \max_D (f \circ \varphi) - 3\rho$. Otherwise we substitute $\mathcal{H}(u, t)$ with $\mathcal{H}(u, t\lambda(u))$, where $\lambda\colon X \to [0, 1]$ is a continuous function such that $\lambda(u) = 0$ for $f(u) \leq \max_D (f \circ \varphi) - 3\rho$ and $\lambda(u) = 1$ for $f(u) \geq \max_D (f \circ \varphi) - 2\rho$.

Let $\mathcal{V}$ be a neighbourhood of $\varphi$ in $\Phi$ such that

$$\forall \eta \in \mathcal{V}\colon \max_D (f \circ \eta) \geq \max_D (f \circ \varphi) - \frac{\rho}{2},$$

$$\forall \eta \in \mathcal{V}\colon\ \{\eta(\xi)\colon\ f(\eta(\xi)) \geq \max_D (f \circ \varphi) - \rho\} \subseteq U.$$

Let $\delta' = \min\{\rho/(2\sigma'), \delta\}$ and let $\mathcal{K}: \mathcal{V} \times [0, \delta'] \to \Phi$ be defined by

$$\forall \xi \in D: \ \mathcal{K}(\eta, t)(\xi) = \mathcal{H}(\eta(\xi), t).$$

It is easy to verify that $\mathcal{K}$ is continuous and

$$\mathfrak{d}(\mathcal{K}(\eta, t), \eta) \leq t.$$

Let now $\eta \in \mathcal{V}$ and $\xi \in D$. If $f(\eta(\xi)) \leq \max_D (f \circ \varphi) - \rho$, then it is for every $t \in [0, \delta']$

$$f(\mathcal{K}(\eta, t)(\xi)) = f(\mathcal{H}(\eta(\xi), t)) \leq f(\eta(\xi)) \leq \max_D (f \circ \varphi) - \rho \leq \max_D (f \circ \eta) - \frac{\rho}{2} < \mathcal{F}(\eta) - \sigma' t.$$

Instead, if $f(\eta(\xi)) \geq \max_D (f \circ \varphi) - \rho$, then it is for every $t \in [0, \delta']$

$$f(\mathcal{K}(\eta, t)(\xi)) = f(\mathcal{H}(\eta(\xi), t)) \leq f(\eta(\xi)) - \sigma' t \leq \mathcal{F}(\eta) - \sigma' t.$$

Then,

$$\forall t \in [0, \delta']: \ \mathcal{F}(\mathcal{K}(\eta, t)) \leq \mathcal{F}(\eta) - \sigma' t$$

and therefore $|d\mathcal{F}|(\varphi) \geq \sigma'$.

The assertion follows by the arbitrariness of $\sigma' \in ]0, \sigma[$. ■

(3.12) THEOREM. – *Let $X$ be a complete metric space, $f: X \to \mathbb{R}$ a continuous function, $(D, S)$ a compact pair, $\psi: S \to X$ a continuous map and*

$$\Phi = \{\varphi \in C(D; X): \varphi|_S = \psi\}.$$

*Let us suppose that $\Phi \neq \emptyset$ and*

$$\forall \varphi \in \Phi: \ \max_S (f \circ \psi) < \max_D (f \circ \varphi).$$

*Then, if $f$ verifies the Palais-Smale condition at level*

$$c = \inf_{\varphi \in \Phi} (\max_D (f \circ \varphi)),$$

*it follows that $c$ is a critical value for $f$.*

PROOF. – Let us suppose that $c$ is not a critical value for $f$. Since the Palais-Smale condition at level $c$ holds, there exists $\sigma > 0$ such that

$$u \in \mathcal{D}(f) \quad \text{and} \quad c - \sigma \leq f(u) \leq c + \sigma \Rightarrow |df|(u) \geq \sigma.$$

Let us define $\mathcal{F}: \Phi \to \mathbb{R}$ as in Lemma (3.11). Being $\mathcal{F}$ bounded from below, by Corollary (3.4) there exists $\varphi \in \Phi$ such that

$$\mathcal{F}(\varphi) < c + \sigma, \qquad |d\mathcal{F}|(\varphi) < \sigma.$$

It is

$$f(\varphi(\xi)) \geq \max_D (f \circ \varphi) - \sigma \Rightarrow |df|(\varphi(\xi)) \geq \sigma .$$

By Lemma (3.11) it follows $|d\mathcal{F}|(\varphi) \geq \sigma$, which is a contradiction. ∎

So far we have treated the critical point theory for continuous functions. If $f: X \to \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous, it is possible to consider the continuous function $\mathcal{G}_f$: $\mathrm{epi}(f) \to \mathbb{R}$. By Definition (2.4) we have $|df|(u) = 0$ if and only if $|d\mathcal{G}_f|(u, f(u)) = 0$ and also $(u_h)$ is a Palais-Smale sequence for $f$ if and only if $(u_h, f(u_h))$ is a Palais-Smale sequence for $\mathcal{G}_f$.

The essential difficulty is that we do not know in general the behaviour of $|d\mathcal{G}_f|(u, \xi)$ when $f(u) < \xi$. However, as we shall see in the next result, we can calculate $|d\mathcal{G}_f|(u, \xi)$ when $f$ is a $C^1$ perturbation of a convex function. Therefore it is possible to obtain in our setting the results of [19].

(3.13) THEOREM. – *Let $A$ be a convex open subset of a Banach space $X$, let $f_0: A \to \mathbb{R} \cup \{+\infty\}$ be a convex and lower semicontinuous function, $f_1: A \to \mathbb{R}$ a function of class $C^1$ and let $f = f_0 + f_1$.*

*Then for every $u \in \mathcal{D}(f)$, $\xi > f(u)$ it holds*

$$|d\mathcal{G}_f|(u, \xi) = 1 .$$

PROOF. – Let $u \in \mathcal{D}(f)$ and $\xi > f(u)$. It is obviously enough to consider the case $df_1(u) = 0$. Then, by Proposition (2.7) we can suppose $f_1 = 0$.

Let $\mathcal{H}$: $B((u, \xi), \delta) \times [0, \delta] \to \mathrm{epi}(f)$ be defined by

$$\mathcal{H}((v, \mu), t) = \left( v + \frac{t(u - v)}{\sqrt{\|v - u\|^2 + |\mu - f(u)|^2}}, \mu - (\mu - f(u)) \frac{t}{\sqrt{\|v - u\|^2 + |\mu - f(u)|^2}} \right)$$

where $\delta > 0$ is such that $f(u) < \xi - 2\delta$.

Since

$$f\left( v + \frac{s(u - v)}{\sqrt{\|v - u\|^2 + |\mu - f(u)|^2}} \right) \leq f(v) + \frac{s}{\sqrt{\|v - u\|^2 + |\mu - f(u)|^2}} (f(u) - f(v)) \leq$$

$$\leq \mu - (\mu - f(u)) \frac{s}{\sqrt{\|v - u\|^2 + |\mu - f(u)|^2}},$$

we actually have $\mathcal{H}((v, \mu), s) \in \mathrm{epi}(f)$. It is easily verified that $\mathcal{H}$ is continuous and

$$d(\mathcal{H}((v, \mu), t), (v, \mu)) = t .$$

On the other hand we have

$$\mathcal{G}_f(\mathcal{H}((v, \mu), s)) = \mu - (\mu - f(u))\frac{s}{\sqrt{\|v-u\|^2 + |\mu - f(u)|^2}} =$$

$$= \mathcal{G}_f(v, \mu) - \frac{(\mu - f(u))}{\sqrt{\|v-u\|^2 + |\mu - f(u)|^2}}s \leq \mathcal{G}_f(v, \mu) - \frac{\xi - \delta - f(u)}{\sqrt{\delta^2 + (\xi + \delta - f(u))^2}}s,$$

which implies

$$|d\mathcal{G}_f|(u, \xi) \geq \frac{\xi - \delta - f(u)}{\sqrt{\delta^2 + (\xi + \delta - f(u))^2}}.$$

Since $\delta$ can be made arbitrarily small, we conclude that

$$|d\mathcal{G}_f|(u, \xi) \geq 1. \quad \blacksquare$$

## 4. - Eigenvalue problems for variational inequalities.

In this section we give an application of the theory developed in the previous sections. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$, $n \geq 3$, let $g: \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with $g(x, -s) = -g(x, s)$ and let $\varphi \in H^1(\Omega)$, $\varphi \geq 0$. We want to consider the eigenvalue problem

$$(4.1) \qquad \begin{cases} (\lambda, u) \in \mathbb{R} \times \mathbb{K}, \\ \lambda \int_\Omega (DuD(v-u) + g(x, u)(v-u))\,dx \geq \int_\Omega u(v-u)\,dx \qquad \forall v \in \mathbb{K}, \end{cases}$$

where

$$\mathbb{K} = \{u \in H_0^1(\Omega) : -\varphi \leq u \leq \varphi \text{ a.e.}\}.$$

We assume that there exist $a \in L^{2n/(n+2)}(\Omega)$, $b \in \mathbb{R}$ and $p < 2n/(n-2)$ such that

$$(4.2) \qquad -a(x)|s| - bs^2 \leq sg(x, s) \leq a(x)|s| + b|s|^p.$$

Problem (4.1) has been already treated in [3, 6, 7, 15], provided that the difference quotient

$$\frac{g(x, s) - g(x, t)}{s - t}$$

is subjected to a suitable lower estimate. On the other hand only a one-sided version of (4.2) is assumed.

However, in the mentioned papers an essential tool is constituted by the evolution theory of [8, 11], which guarantees existence and uniqueness for the evolution problem associated with (4.1).

For this reason, the case in which only an estimate of $|g(x, s)|$ is assumed cannot be treated by that approach.

We set

$$G(x, s) = \int_0^s g(x, t)\,dt$$

and we define $f_0: H_0^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ and $f_1: H_0^1(\Omega) \to \mathbb{R}$ by

$$f_0(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |Du|^2\,dx & \text{if } u \in \mathbb{K}, \\ +\infty & \text{if } u \in H_0^1(\Omega)\setminus\mathbb{K}, \end{cases}$$

$$f_1(u) = \int_\Omega G(x, u)\,dx.$$

Let $\rho > 0$ and let $S = \left\{u \in H_0^1(\Omega): \int_\Omega u^2\,dx = \rho^2\right\}$. We define $I_S: H_0^1(\Omega) \to \mathbb{R} \cup \cup\{+\infty\}$ by

$$I_S(u) = \begin{cases} 0 & \text{if } u \in S, \\ +\infty & \text{if } u \in H_0^1(\Omega)\setminus S, \end{cases}$$

and $f: \mathbb{K} \cap S \to \mathbb{R}$ by $f = f_0 + f_1$.

(4.3) DEFINITION (see [6]). – Let $K$ be a convex subset of a Hilbert space $X$, $M$ a $C^1$ hypersurface in $H$ and $u \in K \cap M$. The sets $K$ and $M$ are said to be *tangential* at $u$, if there exists $\mu \in \mathbb{R}\setminus\{0\}$ such that

$$\forall w \in K: \qquad \mu(\nu(u), w - u) \geq 0,$$

where $\nu(u)$ is a normal unit vector to $\boldsymbol{M}$ at $u$.

In our particular situation, since $0 \in \mathbb{K}$ we have that $S$ and $\mathbb{K}$ are tangential at $u \in S \cap \mathbb{K}$ if and only if

$$\forall v \in \mathbb{K}: \int_\Omega u(v - u)\,dx \leq 0.$$

(4.4) THEOREM. – *Let us assume that* $\rho^2 < \int_\Omega \varphi^2\,dx$. *Then there exists a sequence* $(\lambda_h)$

*in* $\mathbb{R}$ *and a sequence* $(u_h)$ *in* $\mathbb{K} \cap S$ *such that*

$$\lambda_h \int_\Omega (Du_h D(v - u_h) + g(x, u_h)(v - u_h))\,dx \geq \int_\Omega u_h (v - u_h)\,dx \qquad \forall v \in \mathbb{K},$$

$$\lim_h \lambda_h = 0 .$$

PROOF. – If $\mathbb{K}$ and $S$ are tangential at some $u \in \mathbb{K} \cap S$, it is sufficient to set $\lambda_h = 0$ and $u_h = u$ for every $h$.

Therefore let us assume that $\mathbb{K}$ and $S$ are not tangential at any point of $\mathbb{K} \cap S$. Let $X$ be the quotient space obtained from $\mathbb{K} \cap S$ by identifying $u$ with $-u$, endowed with the metric

$$\hat{d}([u],[v]) = \min\{\|u - v\|, \|u + v\|\} .$$

It is readily seen that $X$ is complete. Moreover, in [7, Lemma 1.14] it is proved that $X$ contains compact subsets of arbitrarily large category. Since $f$ is even, there is an induced functional $\hat{f}: X \to \mathbb{R}$, which is of course continuous. By (4.2) $\hat{f}$ is bounded below. Since $\mathbb{K}$ and $S$ are not tangential at any point, by [6, Theorem 1.10] $(f_0 + I_S)$ has $\varphi$-monotone subdifferential of order two. From [10, Theorem 3.14] it follows that $\mathbb{K} \cap S = \mathcal{D}(f_0 + I_S)$ is an *ANR*, hence $X$ is an *ANR*.

Let us show that $\hat{f}$ verifies the Palais-Smale condition at level $c$ for every $c \in \mathbb{R}$.

Let $([u_h])$ be a sequence in $X$ with $|d\hat{f}|([u_h]) \to 0$, $\hat{f}([u_h]) \to c$. Then we have $|df|(u_h) \to 0$ and $f(u_h) \to c$. By (4.2) it follows that $(u_h)$ is bounded in $H_0^1(\Omega)$. Up to a subsequence, we can assume that $(u_h)$ is weakly convergent to $u \in \mathbb{K} \cap S$.

Again by [6, Theorem 1.10] we have $\alpha \in \partial^-(f_0 + I_S)(u_h)$ if and only if

$$\alpha = \beta - \mu u_h \qquad \text{in } \mathrm{H}^{-1}(\Omega)$$

with $\beta \in \partial^- f_0(u_h)$ and $\mu \in \mathbb{R}$. Moreover, from

$$\forall v \in \mathbb{K}: \qquad \mu \int_\Omega u_h (v - u_h)\,dx \leq f_0(v) - f_0(u_h) - \langle \alpha, v - u_h \rangle$$

we deduce $|\mu| \leq C(1 + \|\alpha\|)$, because $\mathbb{K}$ and $S$ are not tangential at $u$.

On the other hand, by Proposition (2.10) and Theorem (2.14) we have $|df|(u_h) = \|\alpha_h + df_1(u_h)\|$ for some $\alpha_h \in \partial^-(f_0 + I_S)(u_h)$. Since by (4.2) $(df_1(u_h))$ is strongly convergent in $H^{-1}(\Omega)$, it follows that $(\alpha_h)$ is also strongly convergent in $H^{-1}(\Omega)$. Therefore $\alpha_h = \beta_h - \mu_h u_h$ with $\mu_h$ bounded in $\mathbb{R}$. Since $(u_h)$ is strongly convergent in $H^{-1}(\Omega)$, we have that $(\beta_h)$ is strongly convergent, up to a subsequence, to some $\beta \in H^{-1}(\Omega)$. By the convexity of $f_0$ it follows that

$$f_0(u) = \lim_h f_0(u_h),$$

i.e. $(u_h)$ is strongly convergent in $H_0^1(\Omega)$ to $u$.

By Theorem (3.10) there exists a sequence $(u_h)$ in $\mathbb{K} \cap S$ with $|df|(u_h) = 0$ and $f(u_h) \to +\infty$. By (4.2) it follows that $\int_\Omega |Du_h|^2 dx \to +\infty$. Let $\mu_h \in \mathbb{R}$ be such that $-g(x, u_h) + \mu_h u_h \in \partial^- f_0(u_h)$, i.e.

$$\int_\Omega (Du_h D(v - u_h) + g(x, u_h)(v - u_h))\, dx \geq \mu_h \int_\Omega u_h(v - u_h)\, dx \qquad \forall v \in \mathbb{K}.$$

The choice $v = 0$ gives

$$\mu_h \rho^2 \geq \int_\Omega |Du_h|^2 dx + \int_\Omega u_h g(x, u_h)\, dx \geq \int_\Omega |Du_h|^2 dx - C\|a\|_{2n/(n+2)} \|Du_h\|_2 - b\rho^2,$$

so that $\mu_h \to +\infty$.

Then it is sufficient to set $\lambda_h = \mu_h^{-1}$ for $\mu_h > 0$. ■

(4.5) REMARK. – Of course it is interesting to known when we can state that $\lambda_h \neq 0$ for every $h$, i.e. when $\mathbb{K}$ and $S$ are not tangential at any point. This question has been solved in [7], where a characterization of the $\rho$'s for which $\mathbb{K}$ and $S$ are not tangential is given. For instance, if $\varphi \in C(\Omega) \cap H^1(\Omega)$ and $\{x \in \Omega : \varphi(x) > 0\}$ is connected, then $\mathbb{K}$ and $S$ are not tangential at any point, provided that $\rho^2 < \int_\Omega \varphi^2 dx$.

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