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The Local Structure of Trans-Sasakian Manifolds (*).

J. C. MARRERO

Summary. – *In this paper, we completely characterize the local structure of trans-Sasakian manifolds of dimension* $\geqslant 5$ *by giving suitable examples.*

0. – Introduction.

An almost Hermitian manifold V is called locally conformal Kähler (l.c.K.) if its metric is conformally related to a Kähler metric in some neighbourhood of every point of V. Such manifolds have been studied by various authors (see, for instance, [L], [GH], [V1], [V2] and [V3]).

In [O], J. A. OUBIÑA has studied a new class of almost contact metric structure, called trans-Sasakian, which is, in some sense, an analogue of a locally conformal Kähler structure on an almost Hermitian manifold (see definition in § 1).

On the other hand, in [ChG] the authors have introduced two subclasses of trans-Sasakian structures, the $\mathcal{C}_5$ and $\mathcal{C}_6$-structures, which contain the Kenmotsu and Sasakian structures, respectively.

In this paper, we completely characterize the local nature of the trans-Sasakian structures on connected differentiable manifolds of dimension $\geqslant 5$. In section 1, we recall some results on almost contact metric manifolds. In section 2 and 3, we characterize the local nature of $\mathcal{C}_5$ and $\mathcal{C}_6$ structures, respectively (see Theorems 2.1 and 3.1), by using the techniques of [Ol]. In section 4 we prove that the trans-Sasakian structures are of class $\mathcal{C}_5$ or $\mathcal{C}_6$ (see Theorem 4.1). Finally, we obtain some examples of 3-dimensional trans-Sasakian structures which are neither of class $\mathcal{C}_5$ nor $\mathcal{C}_6$.

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Indirizzo dell'A.: Departamento de Matemática Fundamental, Universidad de la Laguna, La Laguna-Tenerife, Canary Island, Spain.
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1. – Preliminaries.

Let V be a C^∞ almost Hermitian manifold with metric g and almost complex structure J. Denote by $\mathfrak{X}(V)$ the Lie algebra of C^∞ vector fields on V. The Kähler form Ω is given by $\Omega(X,Y)=g(X,JY)$; and the Lee form is the 1-form θ defined by $\theta(X)=1/(n-1)\ \delta\Omega(JX)$, where δ denotes the coderivate, $\dim V=2n$ and $X,Y\in\mathfrak{X}(V)$.

Recall that V is said to be Kähler if $d\Omega=0$ and $N_J=0$ and locally conformal Kähler (l.c.K.) if $d\Omega=\theta\wedge\Omega$ and $N_J=0$, where N_J denotes the Nijenhuis tensor of J.

On the other hand, let M be a C^∞ almost contact metric manifold with metric g and almost contact structure (φ,ξ,η). Then we have,

$$\varphi^2=-I+\eta\otimes\xi,\qquad \eta(\xi)=1,$$

$$g(\varphi X,\varphi Y)=g(X,Y)-\eta(X)\,\eta(Y),$$

for $X,Y\in\mathfrak{X}(M)$, where I denotes the identity transformation. The fundamental 2-form Φ of the almost contact metric manifold (M,φ,ξ,η,g) is defined by $\Phi(X,Y)=g(X,\varphi Y)$ for all $X,Y\in\mathfrak{X}(M)$.

An almost contact structure (φ,ξ,η) is said to be normal if the almost complex structure J on $M\times\boldsymbol{R}$ given by

$$J(X,a\,d/dt)=(\varphi X-a\xi,\eta(X)\,d/dt),\tag{1.1}$$

where a is a C^∞ function on $M\times\boldsymbol{R}$, is integrable, which is equivalent to the condition $N_\varphi+2d\eta\otimes\xi=0$, where N_φ denotes the Nijenhuis torsion of φ (see [SH1] and [SH2]).

Now, let (φ,ξ,η,g) be an almost contact metric structure on M. We define an almost Hermitian structure (J,h) on $M\times\boldsymbol{R}$, where the almost complex structure J is given by (1.1) and h is the Riemannian metric following:

$$h((X,a\,d/dt),(Y,b\,d/dt))=g(X,Y)+ab\,.$$

An almost contact metric structure (φ,ξ,η,g) is said to be trans-Sasakian (see [O]) if the almost Hermitian structure (J,h) on $M\times\boldsymbol{R}$ is l.c.K.

In [O], the author proves that (φ,ξ,η,g) is a trans-Sasakian structure if and only if it is normal and

$$d\Phi=2\alpha\eta\wedge\Phi\,,\tag{1.2}$$

$$d\eta=\beta\Phi\,,\tag{1.3}$$

where $\alpha=\operatorname{div}\xi/(2n)$ and $\beta=\delta\Phi(\xi)/(2n)$.

An almost contact metric structure (φ,ξ,η,g) is said to be:

$\mathcal{C}_5$ if it is trans-Sasakian with $\beta=0$; Kenmotsu if it is $\mathcal{C}_5$ with $\alpha=1$; $\mathcal{C}_6$ if it is trans-Sasakian with $\alpha=0$; Sasakian if it is $\mathcal{C}_6$ with $\beta=1$; cosymplectic if it is trans-Sasakian with $\alpha=\beta=0$ (see [B2], [ChG] and [K]).

We say that the almost contact structure (φ, ξ, η) has rank r if and only if the 1-form η has rank r. Consequently, (φ, ξ, η) has rank $r = 2s$ if $(d\eta)^s \neq 0$ and $\eta \wedge (d\eta)^s = 0$, and has rank $r = 2s + 1$ if $\eta \wedge (d\eta)^s \neq 0$ and $(d\eta)^{s+1} = 0$.

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold and σ a positive differentiable function on M. We put,

$$g' = \sigma g + (1 - \sigma)\, \eta \otimes \eta .$$

Then, (φ, ξ, η, g') is also an almost contact metric structure on M. Moreover, if (φ, ξ, η, g) is trans-Sasakian and we denote by Φ' the fundamental 2-form of the almost contact metric structure (φ, ξ, η, g'), we have:

$$d\Phi' = (d(\ln \sigma) + (\text{div}\ \xi/n)\, \eta) \wedge \Phi' , \tag{1.4}$$

$$d\eta = (\beta/\sigma)\, \Phi' . \tag{1.5}$$

AGREEMENT. – Through the rest of this paper, M always denotes a $(2n + 1)$-dimensional $(n \geq 2)$ connected manifold unless stated otherwise.

2. – $\mathcal{C}_5$-structures.

In this section, we describe the local structure of manifolds of class $\mathcal{C}_5$. Before, we examine the following example:

EXAMPLE 1. – Let M be the product manifold $L \times V$, where L is the circle S^1 or an open interval (a', b'), $-\infty \leq a' < b' \leq \infty$, and (V, J, G) is a $2n$-dimensional Kählerian manifold. Let E be a nowhere vanishing vector field on L, E^* its dual field of 1-forms and σ a positive function on L. Put

$$\begin{cases} \varphi(aE, X) = (0, JX) , \quad \xi = (E, 0) , \quad \eta = (E^*, 0) , \\ g((aE, X), (bE, Y)) = \sigma G(X, Y) + ab , \end{cases} \tag{2.1}$$

where a and b are differentiable functions on M, and $X, Y \in \mathfrak{X}(V)$. Then it is not difficult to check that (φ, ξ, η, g) is an almost contact metric structure on M of class $\mathcal{C}_5$.

We remark that in the above example the 1-form $(\text{div}\, \xi)\, \eta = d(\ln \sigma^n)$ is closed. We generalize this result for trans-Sasakian manifolds.

Let (φ, ξ, η, g) be a trans-Sasakian structure on M, then

LEMMA 2.1. – *The 1-form* $(\text{div}\, \xi)\, \eta$ *is closed.*

PROOF. – From the definition of trans-Sasakian structure (see (1.2) and (1.3)), we obtain

$$d((\text{div}\ \xi)\, \eta) \wedge \Phi = 0 , \tag{2.2}$$

$$d((\operatorname{div}\xi)\eta)\wedge\eta = (\beta\operatorname{div}\xi)\eta\wedge\Phi. \tag{2.3}$$

Let p be a point of M. We shall prove that $d((\operatorname{div}\xi)\eta)_p = 0$. If $\beta(p) = 0$, we deduce the result from the relations (2.2) and (2.3), since $\dim M \geqslant 5$.

Now, we suppose that $\beta(p) \neq 0$. Let U be a neighbourhood of p such that $\beta \neq 0$ on U. We can suppose that $\beta > 0$ on U. Taking in (1.3) the exterior differential and using (1.2), one gets

$$(d\beta + \beta(\operatorname{div}\xi/n)\eta)\wedge\Phi = 0,$$

and since rang $\Phi \geqslant 4$ we obtain $d\beta + \beta(\operatorname{div}\xi/n)\eta = 0$, i.e., $d(\ln\beta) = -(\operatorname{div}\xi/n)\,\eta$ which also proves that $d((\operatorname{div}\xi)\eta)_p = 0$. ∎

Let $(M, \varphi, \xi, \eta, g)$ be an almost contact metric manifold of class $\mathcal{C}_5$. Next, we prove the following structure theorem, which generalizes a similar result obtained by KENMOTSU [K] for Kenmotsu manifolds.

THEOREM 2.1. – *The manifold M is locally the product $(a', b') \times V$, where (a', b') is an open interval and V is a $2n$-dimensional Kählerian manifold, on which the structure (φ, ξ, η, g) is given as in* (2.1).

PROOF. – Fix a point $p \in M$. From Lemma 2.1, there exists a neighbourhood U' of p on which $d(\ln\sigma) = (\operatorname{div}\xi/n)\eta$, for a certain positive function σ. Put, $g' = (1/\sigma)\,g + (1 - 1/\sigma)\,\eta\otimes\eta$. From the definition of structure of class $\mathcal{C}_5$ and using (1.4) and (1.5), (φ, ξ, η, g') is cosymplectic on U'. Therefore the point p has a neighbourhood $U = (a', b') \times V \subseteq U'$ such that (φ, ξ, η, g') is given on U by

$$\varphi(aE, X) = (0, JX), \qquad \xi = (E, 0), \qquad \eta = (E^*, 0),$$

$$g'((aE, X), (bE, Y)) = G(X, Y) + ab,$$

where (J, G) is a Kählerian structure on V, E is a nowhere vanishing vector field on (a', b') and E^* its dual (see, for instance, [B1]). Finally, since $g = \sigma g' + (1 - \sigma)\,\eta\otimes\eta$, we see that the structure (φ, ξ, η, g) on U is given as in (2.1). ∎

REMARK. – In [K], K. KENMOTSU has proved that a Kenmotsu manifold is not compact. However, taking in the example 1, $L = S^1$, V a compact Kähler manifold, η the length element of the circle S^1 and σ a positive function (not constant) on S^1, we obtain an almost contact metric structure of class $\mathcal{C}_5$, which is not Kenmotsu, on the compact manifold $M = S^1 \times V$.

Finally, we suppose that (φ, ξ, η, g) is an almost contact metric structure of class $\mathcal{C}_5$ on a simply connected manifold M. From Lemma 2.1, we have $(\operatorname{div}\xi/n)\,\eta = d(\ln\sigma)$ for a certain positive function σ on M. Put $g' = (1/\sigma)\,g + (1 - 1/\sigma)\,\eta\otimes\eta$. Then, (φ, ξ, η, g') is a cosymplectic structure on M. Consequently, from Proposition 2.3 of [FM], we deduce

PROPOSITION 2.1. – *A compact simply connected manifold can not admit a structure of class $\mathcal{C}_5$.*

3. – $\mathcal{C}_6$-structures.

An almost contact metric structure (φ, ξ, η, g) on M is said to be γ-Sasakian ($\gamma \in \boldsymbol{R}$, $\gamma \neq 0$) if it is normal and $d\eta = \gamma\Phi$, where Φ is the fundamental 2-form (see [JV]). If (φ, ξ, η, g) is Sasakian then it is 1-Sasakian, and if it is γ-Sasakian or cosymplectic then it is of class $\mathcal{C}_6$. Next, we prove the converse.

LEMMA 3.1. – *If (φ, ξ, η, g) is an almost contact metric structure on M of class $\mathcal{C}_6$, then it is γ-Sasakian or cosymplectic.*

PROOF. – Taking in (1.3) the exterior differential, we get that $d\beta \wedge \Phi = 0$ and, since M is connected and dim $M \geqslant 5$, we obtain $\beta = \gamma =$ constant. Thus, if $\gamma \neq 0$, the structure (φ, ξ, η, g) is γ-Sasakian and if $\gamma = 0$, it is cosymplectic. ∎

Therefore, a not cosymplectic $\mathcal{C}_6$ manifold is essentially a Sasakian manifold. In fact, if the structure (φ, ξ, η, g) is γ-Sasakian then the structure $(\varphi, (1/\gamma)\xi, \gamma\eta, \gamma^2 g)$ is Sasakian.

Now, let M be the product manifold $L \times V$, where L is the circle S^1 or an open interval (a', b'), $-\infty \leqslant a' < b' \leqslant \infty$, and (V, J, G) is an almost Hermitian manifold of dimension $2n$. Let E be a nowhere vanishing vector field on L, E^* its dual field of 1-forms and ω a 1-form on V.

Put,

$$\begin{cases} \varphi(aE, X) = (-\omega(JX)E, JX), \qquad \xi = (E, 0), \qquad \eta = (E^*, \omega), \\ g((aE, X), (bE, Y)) = G(X, Y) + ab + \omega(X)\omega(Y) + \omega(X)b + \omega(Y)a, \end{cases} \tag{3.1}$$

where a, b are differentiable functions on M and $X, Y \in \mathfrak{X}(V)$. By straightforward verification we can see that (φ, ξ, η, g) is an almost contact metric structure on M. Moreover, if we denote by N_J and N_φ the Nijenhuis tensors of J and φ, respectively, and by Ω the Kähler form of (V, J, G), then it is not difficult to check the following:

PROPOSITION 3.1.

$$N_\varphi((aE, X), (bE, Y)) + 2d\eta((aE, X), (bE, Y))(E, 0) = \tag{3.2}$$
$$= ((-\omega(N_J(X, Y)) - 2d\omega(JX, JY) + 2d\omega(X, Y))E, N_J(X, Y)),$$

$$\Phi((aE, X), (bE, Y)) = \Omega(X, Y), \tag{3.3}$$

for $X, Y \in \mathfrak{X}(V)$ and a, b differentiable functions on M.

Next, we describe the local structure of manifolds of class $\mathcal{C}_6$. Previously, we examine the following example.

EXAMPLE 2. – Let M, V, L, J, E and E^* be as in example 1 and ω a 1-form on V, such that $d\omega = \beta\Omega$ where β is constant and Ω the Kähler form of (V, J, G). We define φ, ξ, η and g as in (3.1). Then, from (3.2) and (3.3) we deduce

a) If $\beta = 0$, (φ, ξ, η, g) is a cosymplectic structure.

b) If $\beta \neq 0$, (φ, ξ, η, g) is a β-Sasakian structure.

Now, we prove that the converse holds locally. Let (φ, ξ, η, g) be an almost contact metric structure of class $\mathcal{C}_6$ on M, then

THEOREM 3.1. – *The manifold M is locally the product $(a', b') \times V$, where (a', b') is an open interval and V is a $2n$-dimensional Kählerian manifold, on which the structure (φ, ξ, η, g) is given as in Example 2.*

PROOF. – Fix a point $p \in M$. Let U be a coordinate neighbourhood of p, with coordinates $(x^0, x^1, \ldots, x^{2n})$ such that $U = (-a, a) \times V$, x^0 is the coordinate on $(-a, a)$, $(x^1, \ldots, x^{2n})$ are the coordinates on V and $\xi = \partial/\partial x^0$ on U. Let g_{ij}, η_i, φ^i_j be the components of g, η and φ in the coordinates $(x^0, x^1, \ldots, x^{2n})$. From Lemma 3.1 we obtain the relations

$$\mathcal{L}_\xi g = \mathcal{L}_\xi \eta = \mathcal{L}_\xi \varphi = 0$$

where $\mathcal{L}_\xi$ denotes the Lie derivate with respect to ξ. Using the above relations we deduce that the components g_{ij}, η_i, φ^i_j are independent of the coordinate x^0. Therefore they can be used to a description of an almost Hermitian structure on V. Thus, define

$$J(\partial/\partial x^j) = \sum_{i=1}^{2n} \varphi^i_j (\partial/\partial x^i) \qquad j = 1, \ldots, 2n,$$

$$G(\partial/\partial x^i, \partial/\partial x^j) = g_{ij} - \eta_i \eta_j, \qquad i, j = 1, \ldots, 2n.$$

It is clear that the pair (J, G) is an almost Hermitian structure on V. Moreover, if we put

$$\omega(\partial/\partial x^i) = \eta_i \qquad (i = 1, \ldots, 2n), \qquad E = \partial/\partial x^0, \qquad E^* = dx^0$$

then the almost contact metric structure (φ, ξ, η, g) on U and the almost Hermitian structure (J, G) on V are related by (3.1). Consequently, from relations (3.2) and (3.3), we deduce that (J, G) is a Kähler structure on V. Finally, from the definition of structure of class $\mathcal{C}_6$, we obtain $d\omega = \beta\Omega$ with β constant, where Ω is the Kähler form of (V, J, G). ■

4. – Trans-Sasakian manifolds.

First, we study the rank of a trans-Sasakian structure

PROPOSITION 4.1. – *Let $(M, \varphi, \xi, \eta, g)$ be a trans-Sasakian manifold and r the rank of (φ, ξ, η). Then r cannot be even. Moreover, if $r = 2s + 1$, then $s = 0$ or $s = n$ and we have*

a) (φ, ξ, η, g) *is of class* $\mathcal{C}_5$ *if and only if* $s = 0$.

b) (φ, ξ, η, g) *is of class* $\mathcal{C}_6$ *not cosymplectic if and only if* $s = n$.

PROOF. – If $r = 2s$, from (1.3), we deduce that $\beta \neq 0$ at every point. On the other hand, since $\eta \wedge (d\eta)^s = 0$ and $\eta \wedge \Phi^n \neq 0$ we obtain $\beta = 0$, which is a contradiction.

The assertion *a)* is evident.

Now, we suppose that $r = 2s + 1$, $s \neq 0$. Then, it is clear that $\beta \neq 0$ at every point and thus $r = 2n + 1$. From Lemma 2.1, the 1-form $\alpha\eta = -(\mathrm{div}\,\xi/2n)\ \eta$ is closed. Therefore, by using 1.3, we obtain

$$d\alpha \wedge \eta + \alpha\beta\Phi = 0\,, \tag{4.1}$$

and since $\eta \wedge \Phi \neq 0$, we deduce $\alpha\beta = 0$, *i.e.*, $\alpha = 0$. Consequently, the structure (φ, ξ, η, g) is of class $\mathcal{C}_6$ not cosymplectic.

Conversely, if $r = 2s + 1$ and (φ, ξ, η, g) is of class $\mathcal{C}_6$ and it is not cosymplectic then $\beta \neq 0$ and thus $s = n$. ■

Next, we prove that a trans-Sasakian structure is of class $\mathcal{C}_5$ or $\mathcal{C}_6$.

THEOREM 4.1. – *If (φ, ξ, η, g) is a trans-Sasakian structure, it is of class $\mathcal{C}_5$ or $\mathcal{C}_6$.*

PROOF. – Denote by A the following set

$$A = \{x \in M / \beta(x) = 0\}\,,$$

where $d\eta = \beta\Phi$.

Let x_0 be a point of A. From Lemma 2.1, $2\alpha\eta = (\mathrm{div}\,\xi/n)\ \eta$ is a closed 1-form. Then, there exists an open neighbourhood U of x_0 on which $d(\ln\sigma) = 2\alpha\eta$, for a certain positive function σ. Put,

$$g' = (1/\sigma)\,g + (1 - 1/\sigma)\,\eta \otimes \eta\,. \tag{4.2}$$

From relations (1.4), (1.5) and by using the Lemma 3.1 we obtain that the almost contact metric structure (φ, ξ, η, g') is of class $\mathcal{C}_6$ and $\beta/\sigma = c$ (c constant). Thus, since $\beta(x_0) = 0$, $c = 0$ and therefore $U \subseteq A$.

Consequently, A is an open subset of M. On the other hand, it is clear that A is closed in M. Therefore, from the connectedness of M, we deduce that $A = M$ or $A = \phi$. If $A = M$, (φ, ξ, η, g) is of class $\mathcal{C}_5$ (in this case M may also be cosymplectic) and if $A=\phi$ the rank of the structure (φ,ξ,η) is $2n+1$ and hence, using the Proposition 4.1, (φ,ξ,η,g) is of class $\mathcal{C}_6$. ∎

The Theorem 4.1 is not true for dim $M=3$. In fact, if (M,φ,ξ,η,g) is a 3-dimensional Sasakian manifold, and

$$g'=\sigma g+(1-\sigma)\,\eta\otimes\eta\,,$$

where σ is a positive function on M, then

PROPOSITION 4.2. – *(φ,ξ,η,g') is a trans-Sasakian structure on M. Moreover, if $\xi(\sigma)\neq 0$, then (φ,ξ,η,g') is neither of class $\mathcal{C}_5$ nor $\mathcal{C}_6$.*

PROOF. – It is clear that (φ,ξ,η,g') is a normal structure on M. Moreover, if Φ' is the fundamental 2-form of the structure (φ,ξ,η,g'), we deduce, from (1.4) and (1.5), that

$$d\Phi'=d(\ln\sigma)\wedge\Phi'\,, \tag{4.3}$$

$$d\eta'=(1/\sigma)\,\Phi'\,. \tag{4.4}$$

Now take a φ-bassis $\{E_0,E_1,E_2\}$ for the structure (φ,ξ,η,g') and its dual bassis of 1-forms $\{E_0^*=\eta,E_1^*,E_2^*\}$. Then, $\Phi'=2E_2^*\wedge E_1^*$ and $d(\ln\sigma)=\xi(\ln\sigma)\,\eta+ +E_1(\ln\sigma)E_1^*+E_2(\ln\sigma)E_2^*$. Thus,

$$d\Phi'=\xi(\ln\sigma)\,\eta\wedge\Phi' \tag{4.5}$$

and (φ,ξ,η,g') is a trans-Sasakian structure. Moreover, by using (4.4), we deduce that (φ,ξ,η,g') is not of class $\mathcal{C}_5$.

On the other hand, from (4.4) and (4.5), (φ,ξ,η,g') is of class $\mathcal{C}_6$ if and only if $\xi(\ln\sigma)=0$, *i.e.*, $\xi(\sigma)=0$.

This ends the proof of the proposition. ∎

Next, by using the Proposition 4.2, we give an example of trans-Sasakian structure which is neither of class $\mathcal{C}_5$ nor $\mathcal{C}_6$.

Let $H(1,1)$ be the group of matrices of real numbers of the form

$$\begin{bmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{bmatrix},$$

where x, y, $z\in\boldsymbol{R}$. $H(1,1)$ is a connected simply connected nilpotent Lie group of dimension 3, which is called Heisenberg group (for an extensive study of Heisenberg groups see [H], [CFL]).

A bassis for the left invariant 1-forms on $H(1,1)$ is given by

$$\alpha=dx\,,\qquad \beta=dy\,,\qquad \gamma=dz-x\,dy$$

and its dual bassis of left invariant vector fields on $H(1,1)$ is given by

$$X = \partial/\partial x\,, \qquad Y = \partial/\partial y + x\partial/\partial z\,, \qquad Z = \partial/\partial z\,.$$

Define an almost contact metric structure (φ, ξ, η, g) on $H(1,1)$ by

$$\varphi X = Y\,, \qquad \varphi Y = -X\,, \qquad \xi = Z\,, \qquad \eta = \gamma\,,$$

$$g = 1/2\,(\alpha \otimes \alpha + \beta \otimes \beta + \gamma \otimes \gamma)\,.$$

Then, (φ, ξ, η, g) is a Sasakian structure on $H(1,1)$.
Now, put

$$g' = e^z g + (1 - e^z)\,\eta \otimes \eta\,.$$

Then, by using the Proposition 4.2 (in this case $\sigma = e^z$), (φ, ξ, η, g') is a trans-Sasakian structure on $H(1,1)$ which is neither of class $\mathcal{C}_5$ nor $\mathcal{C}_6$.

Finally, since the unit sphere S^3 carries an induced Sasakian structure as orientable hypersurface of $\boldsymbol{R}^4$ (see, for instance, [B2]), we also can obtain, by using the Proposition 4.2, a trans-Sasakian structure on S^3, which is neither of class $\mathcal{C}_5$ nor $\mathcal{C}_6$.

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