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λ-Equidistributed Sequences of Partitions and a Theorem of the De Bruijn-Post Type (*).

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Summary. – *The notion of uniform distribution of a sequence is generalized to sequences of partitions in a separable metric space X. Results concern Riemann integrability with respect to a probability λ on X, and Riemann approximations of Lebesgue integrals.*

Introduction.

A classical theorem by H. WEYL states that, if f is a Riemann integrable real function on the unit interval I, then for every uniformly distributed sequence $\{x_n\}$ in I the averages $1/N \sum_{k=1}^{N} f(x_k)$ converge to $\int_0^1 f(x)\,dx$; the convergence condition was later shown to be sufficient for the Riemann integrability ([1], 1968). The notion of uniform distribution with respect to a given measure has been generalized in several ways: sequences of points in a locally compact space (see [6], 1974 and [10], 1972), sequences of probability measures on a separable compact space ([7], 1970), in particular sequences of discrete measures associated to partitions of a compact interval ([5], 1975). See also [3], 1984.

Given a separable metric space (X, d) and a probability measure λ on X, we define the notion of a λ-equidistributed sequence of finite partitions of X. We give a sufficient condition for the existence of such a sequence of partitions made up of λ-continuity subsets (condition (c) and Theorem 1 below). This method allows to adapt the mentioned results of H. Weyl and De Bruijn-Post to the case of real functions on X: see Corollary 1 and Remark 3. Moreover: if f is bounded, but not λ-Riemann integrable, every number α between its lower and upper Riemann integrals can be approximated by a sequence of Riemann sums associated to the mentioned partitions (Theorem 2). Therefore, if f is bounded and λ-summable, its Lebesgue integral is the

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limit of a particular sequence of Riemann sums (a result of this kind, in a different context, was obtained by P. MORALES: see [9], 1972).

The meaning of condition (c), with the search for non-trivial cases, seems to be an interesting open problem.

Let (X, d) be a separable metric space, λ a probability measure on $\mathcal{B}(X)$. Let Q be a partition, at most countable, of X into Borel sets: $Q = \{E_j\}_{j \in N}$, $E_j \in \mathcal{B}(X)$. For each j such that $\lambda(E_j) > 0$ let z_j be a point in E_j. Define a probability measure μ by

$$\mu := \sum_{j \in N} \lambda(E_j) \cdot \delta_{z_j}, \tag{1}$$

where δ_{z_j} is the Dirac mass at z_j.

LEMMA 1. – If $\{Q_k\}_{k \geq 1}$ is a sequence of partitions as above, verifying

$$\lim_k \left(\sup_{j \in N} \operatorname{diam}(E_{k,j}) \right) = 0, \tag{2}$$

then for any choice of the points $z_{k,j} \in E_{k,j}$ the corresponding sequence of measures $\{\mu_k\}$, associated to $\{Q_k\}$ by the formula (1), converges weakly to λ.

PROOF. – It is enough to check that

$$\lim_k \mu_k(g) = \lambda(g) \tag{3}$$

for all $g \in U(X)$, where $U(X)$ is the space of bounded, uniformly continuous real functions on X (see [11] II, Theor. 6.1). This follows easily from condition (2).

Now let $P = \{E_i\}_{0 \leq i \leq n}$ stand for a *finite* measurable partition of X.

DEFINITION 1. – A sequence $P_k = \{E_{k,i}\colon 0 \leq i \leq n(k)\}$ of finite measurable partitions is called «λ-equidistributed» iff, for any choice of points $z_{k,i} \in E_{k,i}$, for all functions $f \in \mathcal{C}^b(X, \boldsymbol{R})$, the following holds:

$$\lim_k \frac{1}{n(k)+1} \sum_{i=0}^{n(k)} f(z_{k,i}) = \int_X f \, d\lambda. \tag{4}$$

REMARK 1. – Definition 1 generalizes the one used by S. KAKUTANI ([5], p. 370) concerning partitions of the unit interval. In particular: if the sequence $\{P_k\}$ verifies condition (2) and moreover $\lambda(E_{k,i}) = 1/(n(k)+1)$ for all i, then by Lemma 1 it is λ-equidistributed.

Now let $\mathcal{C}_\lambda$ denote the field of all sets having frontier of null measure, or λ-continuity sets.

LEMMA 2. – For each $k \in \boldsymbol{N}$ there is a partition, at most countable, of X into subsets $\{A_{k,j}\}_{j \in \boldsymbol{N}}$ such that:

(I) for all k and j, $A_{k,j} \in \mathcal{C}_\lambda$,

(II) $\sup_{j \in \boldsymbol{N}} \operatorname{diam}(A_{k,j}) \leq 2/k$.

PROOF. – Let $\{x_j\}_{j \in \boldsymbol{N}}$ be a countable set of points, dense in X. Given k, for each j there is an open ball $B_{k,j}$ with centre x_j and radius $r_{k,j}$ such that $1/2k \leq r_{k,j} \leq 1/k$ and $B_{k,j} \in \mathcal{C}_\lambda$ (the positive function $r \mapsto \lambda(B(x_j, r))$ is increasing, therefore continuous at all but a countable set of points r; see also Remark 2 below). These balls form a covering of X. The sequence of sets defined as follows (where B' denotes $X \setminus B$):

$$A_{k,1} := B_{k,1}; \qquad A_{k,n} = B_{k,n} \cap B'_{k,n-1} \cap \ldots \cap B'_{k,1}, \tag{5}$$

verifies the conditions stated above.

Now, given a point $x_0 \in X$, consider the function

$$f(r;\, x_0) := \lambda(B(x_0, r)), \qquad r \geq 0$$

where $B(x_0, r)$ is the open ball $\{y \in X\colon\ d(x_0, y) < r\}$. As noticed above, f is continuous at all but a countable set of numbers r, where it may have jumps. Let us introduce the following condition relative to the couple (d, λ):

(c) There is a point x_0 such that $f(r, x_0)$ is continuous at all r.

REMARK 2. – f is continuous at r iff $\lambda(B(x_0, r)) = \lambda\left(\bigcap_{s > r} B(x_0, s)\right)$. This equality implies that $B(x_0, r)$ belongs to $\mathcal{C}_\lambda$, but the converse is not true, in general: the frontier of the ball is contained in (but not necessarily equal to) the sphere $\{y\colon\ d(x_0, y) = r\}$ (see [2], p. 204).

Condition (c) is verified by Lebesgue measure in $\boldsymbol{R}^n$ with the usual distance.

LEMMA 3. – If condition (c) holds, there exists a sequence $\{P_k\}$ of finite partitions of X into λ-continuity sets $E_{k,i}$ $(0 \leq i \leq n(k))$ such that: for any choice of points $z_{k,i} \in E_{k,i}$, for every bounded real function f, the following holds:

$$\lim_k \left| \sum_{i=0}^{n(k)} f(z_{k,i}) \cdot \left(\frac{1}{n(k)} - \lambda(E_{k,i}) \right) \right| = 0 . \tag{6}$$

PROOF. – For each $k \in \boldsymbol{N}$ let $\{A_{k,j}\}_{j \in \boldsymbol{N}}$ be a partition of X into λ-continuity sets; let s_k be the smallest integer such that

$$\sum_{j=1}^{s_k} \lambda(A_{k,j}) \geq 1 - \frac{1}{k}; \tag{7}$$

moreover let

$$(8) \qquad l_k := \min_{1 \leqslant j \leqslant s_k} \{\lambda(A_{k,j}) : \lambda(A_{k,j}) > 0\},$$

and A_k be one set having such measure. We will partition A_k into $k+1$ λ-continuity sets: $E_{k,1}, \ldots, E_{k,k}$ each having measure l_k/k, plus one of null measure.

Choose $x_0 \in X$ as in condition (*c*); then also the function

$$(9) \qquad \phi(r) := \lambda(A_k \cap B(x_0, r))$$

is continuous at all r, and for r sufficiently large it reaches the value l_k. Let $r_1 > 0$ be such that $\phi(r_1) = l_k/k$; put

$$(10) \qquad E_{k,1} := A_k \cap B(x_0, r_1).$$

In general, let r_p be such that $\phi(r_p) = p \cdot l_k/k$, $1 \leqslant p \leqslant k$; for $p \geqslant 2$ define

$$(11) \qquad E_{k,p} := (A_k \cap B(x_0, r_p)) \setminus E_{k,p-1}.$$

Each $E_{k,p}$ belongs to $\mathcal{C}_\lambda$ and has measure equal to l_k/k.

For any other index j such that $\lambda(A_{k,j}) = l_{k,j} > 0$, so that $l_{k,j} \geqslant l_k$ by definition, we will partition $A_{k,j}$ into a finite number of subsets $E_{k,j,q} \in \mathcal{C}_\lambda$ of equal measure. The ratios

$$(12) \qquad \rho_{k,j} := \frac{\lambda(E_{k,j,q})}{\lambda(E_{k,p})} \qquad (p \geqslant 1,\ q \geqslant 1),$$

will verify, for any k and j, the inequalities

$$(13) \qquad 1 \leqslant \rho_{k,j} < \frac{k+1}{k}.$$

Let $t_{k,j}$ be the largest integer such that

$$(14) \qquad t_{k,j} \cdot \frac{l_k}{k} \leqslant l_{k,j};$$

thus it verifies

$$(15) \qquad t_{k,j} \geqslant k \quad \text{and} \quad (t_{k,j} + 1) \cdot l_k/k > l_{k,j}.$$

By the same method as above, the set $A_{k,j}$ is partitioned into the λ-continuity subsets $E_{k,j,q}$, $1 \leqslant q \leqslant t_{k,j}$, each one of measure l_{kj}/t_{kj}, plus a λ-continuity set of measure zero. By construction

$$(16) \qquad \rho_{k,j} = \frac{l_{kj}}{t_{kj}} \cdot \frac{k}{l_k}.$$

Inequality (14) implies the left-hand side of (13); inequalities (15) imply $\rho_{k,j} < (t_{k,j} + 1)/t_{k,j} \leqslant (k+1)/k$, i.e. the right-hand side of (13).

For each k let us put

$$n_k := \sum_{j=1}^{s_k} t_{k,j}, \tag{17}$$

i.e. the total number of the λ-continuity sets $E_{k,j,q}$ having strictly positive measure (for at least one value of j it is $t_{k,j} = k$); we will find an upper bound and a lower bound for n_k.

Let us renumber as $E_{k,i}$, $1 \leqslant i \leqslant n_k$, all the sets $E_{k,j,q}$. Since $\sum_{i=1}^{n_k} \lambda(E_{k,i}) \leqslant 1$ and each addend is at least l_k/k, then

$$n_k \leqslant \frac{k}{l_k}. \tag{18}$$

On the other hand, by construction

$$\sum_{j=1}^{s_k} t_{k,j} \cdot \rho_{k,j} \cdot \frac{l_k}{k} = \sum_{j=1}^{s_k} \lambda(A_{k,j}) \geqslant 1 - \frac{1}{k}; \tag{19}$$

from this and from (13) follows

$$1 - \frac{1}{k} < \sum_{j=1}^{s_k} t_{k,j} \cdot \frac{k+1}{k^2} \cdot l_k \tag{20}$$

and therefore

$$n_k > \frac{1 - 1/k}{(k+1) \cdot l_k} \cdot k^2; \tag{21}$$

moreover, for all $k \geqslant 2$ we get $1 - 1/k \geqslant (k+1)/(k+4)$.

Let us show that, for $k \geqslant 2$ and all i between 1 and n_k,

$$-4\frac{l_k}{k^2} \leqslant \lambda(E_{k,i}) - \frac{1}{n_k} \leqslant \frac{l_k}{k^2}. \tag{22}$$

Inequality (21) above yields

$$\frac{1}{n_k} < \frac{(k+4) \cdot l_k}{k^2} = \frac{l_k}{k} + 4 \cdot \frac{l_k}{k^2}, \tag{23}$$

from which follows

$$\lambda(E_{k,i}) - \frac{1}{k} \geqslant \frac{l_k}{k} - \frac{1}{n_k} > -4\frac{l_k}{k^2}. \tag{24}$$

Moreover, by means of (13) and (18):

$$\lambda(E_{k,i}) - \frac{1}{n_k} \leqslant \frac{l_k}{k} \cdot \frac{k+1}{k} - \frac{l_k}{k} = \frac{l_k}{k^2}; \tag{25}$$

thus (22) is proved. Finally, the λ-continuity set

$$E_{k,0} := X \setminus \bigcup_{i=1}^{n_k} E_{k,i} \tag{26}$$

has the same measure as $X \setminus \bigcup_{j=1}^{s_k} A_{k,j}$, hence $\leq 1/k$.

Now let $M = \sup_{x \in X} |f(x)|$. The absolute value in (6) is less than or equal to

$$M \cdot \left| \frac{1}{n_k} - \lambda(E_{k,0}) \right| + M \cdot \sum_{i=1}^{n_k} \left| \frac{1}{n_k} - \lambda(E_{k,i}) \right| ; \tag{27}$$

from (18) and (22) it follows that

$$\sum_{i=1}^{n_k} \left| \frac{1}{n_k} - \lambda(E_{k,i}) \right| \leq n_k \cdot 4 \cdot \frac{l_k}{k^2} \leq \frac{4}{k}, \tag{28}$$

so that (6) is proved.

THEOREM 1. – If condition (*c*) holds, there exists a sequence $\{P_k\}$ of finite partitions of X, verifying (I) and (II) of Lemma 2 (except, for each k, for one λ-continuity set $E_{k,0}$ of unknown diameter), which is λ-equidistributed.

PROOF. – For each $k \in \boldsymbol{N}$ let $\{A_{k,j}\}_{j \in \boldsymbol{N}}$ be a partition of X into λ-continuity sets, such that $\sup_{j \in \boldsymbol{N}} \operatorname{diam}(A_{k,j}) \leq 2/k$ (Lemma 2); for each k let $P_k = \{E_{k,i} \colon 0 \leq i \leq n_k\}$, the finite partition described in Lemma 3. From Lemma 1 applied to the sequence $\{P_k\}_{k \geq 1}$ and from the fact that $\lambda(E_{k,0}) \leq 1/k$ (the diameter of $E_{k,0}$ has not been estimated), it follows that: for any choice of $z_{k,i} \in E_{k,i}$ and for every $f \in \mathcal{C}^b(X, \boldsymbol{R})$

$$\lim_k \sum_{i=0}^{n_k} f(z_{k,i}) \cdot \lambda(E_{k,i}) = \int_X f \, d\lambda . \tag{29}$$

Thus (6) and (29) imply

$$\lim_k \frac{1}{n_k} \sum_{i=0}^{n_k} f(z_{k,i}) = \int_X f \, d\lambda . \tag{30}$$

COROLLARY 1. – Under the same assumptions, if f is a bounded, λ-Riemann integrable function on X, then for any choice of the points $z_{k,i} \in E_{k,i}$ equality (30) still holds.

PROOF. – The sequence of discrete measure, associated to $\{P_k\}$ by the choice of the $z_{k,i}$'s, converges weakly to λ. By assumption the set of discontinuity points of f is a λ-nullset, therefore (29) still holds.

This Corollary is a new version of the mentioned well-known of H. WEYL ([12], 1016, p. 314).

LEMMA 4. – Suppose condition (c) holds and let $\{P_k\}$ be a sequence of finite partitions of X as in Theorem 1. Then: given a bounded real function f on X, there are two sequences $\{\mu_k\}$ and $\{\nu_k\}$ of discrete probability measures associated to $\{P_k\}$, such that both converge weakly to λ and moreover: the sequence $\mu_k(f)$ converges to the upper Riemann integral of f, the sequence $\nu_k(f)$ converges to the lower Riemann integral of f.

PROOF. – For each $k \in N$ and i between 0 and n_k there are two points $w_{k,i} \in \overset{0}{E}_{k,i}$ and $v_{k,i} \in \overset{0}{E}_{k,i}$ such that:

$$f(w_{k,i}) > \sup_{\overset{0}{E}_{k,i}} f - \frac{1}{k}; \qquad f(v_{k,i}) < \inf_{\overset{0}{E}_{k,i}} f + \frac{1}{k} \tag{31}$$

(for $i \geqslant 1$ the interior of $E_{k,i}$ is not empty, since $E_{k,i} \in \mathcal{C}_\lambda$ and $\lambda(E_{k,i}) > 0$; for $i = 0$ it might be $\lambda(E_{k,0}) = 0$, but nothing would change in the sums below).

Define

$$\mu_k := \sum_{i=0}^{n_k} \lambda(E_{k,i}) \cdot \delta_{w_{k,i}} \tag{32}$$

and

$$\nu_k := \sum_{i=0}^{n_k} \lambda(E_{k,i}) \cdot \delta_{v_{k,i}}. \tag{33}$$

By Lemma 1 and by the inequality $\lambda(E_{k,0}) \leqslant 1/k$ both $\{\mu_k\}$ and $\{\nu_k\}$ converge weakly to λ. Let f^* be the smallest u.s.c. function $\geqslant f$, and f_* be the greatest l.s.c. function $\leqslant f$. Trivially, f is summable relative to each of the measures μ_k and ν_k; f^* and f_* are summable relative to λ, and

$$\int_X f^* d\lambda = \overline{\int_X} f d\lambda, \qquad \int_X f_* d\lambda = \underline{\int_X} f d\lambda \tag{34}$$

(by definition or by approximation, according to the point of view: see [8] p. 375 and [4], I.9.1).

We will prove the following: there exist

$$\lim_k \int_X f d\mu_k = \int_X f^* d\lambda \quad \text{and} \quad \lim_k \int_X f d\nu_k = \int_X f_* d\lambda . \tag{35}$$

The equality $f^*(x) = \limsup_{y \to x} f(y)$ implies that, for each couple (k, i)

$$\left(\sup_{\overset{0}{E}_{k,i}} f\right) \cdot 1_{\overset{0}{E}_{k,i}} \geq f^* \cdot 1_{\overset{0}{E}_{k,i}} \tag{36}$$

and therefore

$$\int_X f\, d\mu_k = \sum_i \lambda(E_{k,i}) \cdot f(w_{k,i}) > \sum_i \int_{\overset{0}{E}_{k,i}} f^*\, d\lambda - \frac{1}{k} \sum_i \lambda(E_{k,i}) = \int_X f^*\, d\lambda - \frac{1}{k} \tag{37}$$

(recall that $\lambda(\overset{0}{E}_{k,i}) = \lambda(E_{k,i})$). Thus:

$$\liminf_{k \to \infty} \int_X f\, d\mu_k \geq \int_X f^*\, d\lambda . \tag{38}$$

On the other hand

$$\int_X f^*\, d\lambda \geq \limsup_{k \to \infty} \int_X f^*\, d\mu_k \geq \limsup_{k \to \infty} \int_X f\, d\mu_k , \tag{39}$$

because of the weak convergence of $\{\mu_k\}$ to λ. Inequalities (38) and (39) imply the first of (35). The second of (35) is proved in the same way, using the inequality

$$\left(\inf_{\overset{0}{E}_{k,i}} f\right) \cdot 1_{\overset{0}{E}_{k,i}} \leq f_* \cdot 1_{\overset{0}{E}_{k,i}} \tag{40}$$

and the weak convergence of $\{\nu_k\}$ to λ.

LEMMA 5. – Suppose (c) holds and let $\{P_k\}$ be as in Theorem 1. For every bounded real function f, for every $t \in [0,1]$ there is a subsequence $\{\phi_{k_n, t}\}_n$ of discrete probability measures associated to $\{P_k\}$ such that: there exists the limit

$$\lim_n \phi_{k_n, t}(f) = \Phi(t) \tag{41}$$

and Φ is a Lipschitz-continuous function of t, such that

$$\Phi(0) = \int_X f_*\, d\lambda , \qquad \Phi(1) = \int_X f^*\, d\lambda . \tag{42}$$

PROOF. – For each $\mathcal{C}_\lambda$-set $E_{k,i}$ let $w_{k,i}$, $v_{k,i}$ be the interior points specified in Lemma 4. Given t as above and any $\varepsilon > 0$, there exist $r \geq 0$ and $\delta = \delta(\varepsilon) > 0$ such that

$$\lambda(B(x_0, r)) = t \quad \text{and} \quad \lambda(B(x_0, r + \delta) = t + \varepsilon , \tag{43}$$

by assumption (c). Let $k = k(\varepsilon)$ be an integer such that $2/k < \delta$, and P_k the corresponding finite partition as in Theorem 1; then

$$\max_{1 \leq i \leq n_k} \operatorname{diam}(E_{k,i}) < \delta \quad \text{while } \lambda(\mathrm{E}_{\mathrm{k},0}) \leq \frac{1}{\mathrm{k}}. \tag{44}$$

Define the discrete measure $\phi_{k,t}$ as in (1), with the following choice of the points: for each $E_{k,i}$ intersecting the ball $B(x_0, r)$ the point $w_{k,i}$; for each $E_{k,i}$ intersecting the complement of $B(x_0, r+\delta)$ choose $v_{k,i}$; for all the others and for $E_{k,0}$ take any interior point $z_{k,i}$. Let us consider the function

$$f_t(x) := \begin{cases} f^*(x) & \text{if } x \in B(x_0, r), \qquad r = r(t), \\ f_*(x) & \text{otherwise}, \end{cases} \tag{45}$$

which is bounded and Borel measurable. Let

$$\Phi(t) := \oint_X f_t(x)\, d\lambda \tag{46}$$

defined on $[0,1]$ and Lipschitz continuous with a constant $M_f = 2 \cdot \sup_x |f(x)|$.

Equalities (42) are obvious. We will prove that, for every $t \in [0,1]$, there is a sequence of integers k_n such that (41) holds. Given $t \in [0,1]$ and $\varepsilon > 0$, let k be the integer $k(\varepsilon)$ and $\phi_{k,t}$ the discrete measure associated to P_k, both described above. Then, writing B_r instead of $B(x_0, r)$ and $\overline{B}_\delta$ instead of $\overline{B}(x_0, r+\delta)$:

$$\phi_{k,t}(f) = \oint_{B_r} f\, d\phi_{k,t} + \oint_{\overline{B}_\delta \setminus B_r} f\, d\phi_{k,t} + \oint_{X \setminus \overline{B}_\delta} f\, d\phi_{k,t}. \tag{47}$$

At each point $x \in B_r$: $(f \cdot 1_{B_r})^*(x) = f^*(x)$, and at each $z \in X \setminus \overline{B}_\delta$: $(f \cdot 1_{X \setminus \overline{B}_\delta})_*(z) = f_*(z)$. Therefore, by the same reasons as in Lemma 4, there exist

$$\lim_k \oint_{B_r} f\, d\phi_{k,t} = \oint_{B_r} f^*\, d\lambda, \qquad \lim_k \oint_{X \setminus \overline{B}_\delta} f\, d\phi_{k,t} = \oint_{X \setminus \overline{B}_\delta} f_*\, d\lambda. \tag{48}$$

Moreover

$$\left| \oint_{\overline{B}_\delta \setminus B_r} f\, d\phi_{k,t} - \oint_{\overline{B}_\delta \setminus B_r} f_t\, d\lambda \right| \le M_f \cdot \varepsilon. \tag{49}$$

Now: take $\varepsilon = 1/n$ and a corresponding $\delta = \delta(1/n)$ as above; let k_n be an integer such that $2/k_n < \delta(1/n)$. Then the subsequence $\phi_{k_n,t}$ of discrete measures just defined verifies equality (41), because of (47), (48) and (49).

THEOREM 2. – Let (X, d) be a separable metric space, λ a probability on X verifying condition (c). There is a λ-equidistributed sequence $\{P_k\}$ of finite partitions into λ-continuity sets such that: for every bounded real function f on X, for every number α between the lower Riemann integral and the upper Riemann integral of f, there is a subsequence of discrete measures $\{\phi_{k_n}\}_{n \in N}$ associated to $\{P_k\}$ such that

$$\lim_n \oint_X f\, d\phi_{k_n} = \alpha. \tag{50}$$

PROOF. – By Lemma 5, there is a $t \in [0,1]$ s.t. $\Phi(t) = \alpha$; take $\phi_{k_n} := \phi_{k_n, t}$ defined above.

COROLLARY 2. – If f is bounded and λ-summable, there exists a sequence of Riemann sums, associated to the sequence of finite, $\mathcal{C}_\lambda$, λ-equidistributed partitions $\{P_k\}_{k \in N}$, which converges to $\lambda(f)$.

REMARK 3. – Let $f: X \to \boldsymbol{R}$ be bounded but not Riemann-integrable; define the sequences $\{\mu_k\}$ and $\{\nu_k\}$ as in Lemma 4. Then the sequence

$$\mu_1(f), \quad \nu_1(f), \ldots, \mu_k(f), \quad \nu_k(f), \ldots \tag{51}$$

does not converge. If f is not bounded on the support of λ, for each $k \in N$ there is at least one of the sets $E_{k,i}$ (with strictly positive mesure) such that f is unbounded on it. Therefore for each k there is a choice of the points $z_{k,i} \in E_{k,i}$ such that

$$\left| \frac{1}{n_k} \sum_{i=0}^{n_k} f(z_{k,i}) \right| > k \tag{52}$$

and thus the sequence of averages in (30) does not converge.

This is a new version of the De Brujin-Post theorem (see [1], 1968).

REFERENCES

[1] N. G. DE BRUJIN - K. A. POST, *A remark on uniformly distributed sequences and Riemann integrability*, Indag. Math., **30** (1968), pp. 149-150.

[2] J. DUGUNDJI, *Topology*, Allyn and Bacon Inc. (1966).

[3] E. HLAWKA, *The Theory of Uniform Distribution*, AB Academic Publishers, Berkhamsted (1984).

[4] K. JACOBS, *Measure and Integral*, Academic Press (1978).

[5] S. KAKUTANI, *A problem of equidistribution on the unit interval*, in: *Measure Theory*, Oberwolfach (1975), pp. 369-375; Lect. Notes in Math., **541**, Springer-Verlag (1976).

[6] L. KUIPERS - H. NIEDERREITER, *Uniform Distribution of Sequences*, J. Wiley (1974).

[7] K. SCHMIDT, *Über einen Zusammenhang zwischen gleichverteilten Punkt- and Massfolgen*, J. Reine Angew. Math., **244** (1970), pp. 94-96.

[8] L. SCHWARTZ, *Radon Measure on Arbitrary Topological Spaces and Cylindrical Measures*, Tata Institute of Fundamental Research, Oxford Univ. Press (1973).

[9] P. MORALES, *Mean value theorem for the m-integral of Dinculeanu*, Canad. Math. Bull., **15** (2) (1972), pp. 243-251.

[10] H. NIEDERREITER, *On the existence of uniformly distributed sequences in compact spaces*, Compositio Math., **25** (1972), pp. 93-99.

[11] L. R. PARTHASARATHY, *Probability Measures on Metric Spaces*, Academic Press (1967).

[12] H. WEYL, *Über die Gleichverteilung von Zahlen mod. Eins*, Math. Ann., **77** (1916, reprint 1964), pp. 313-352.