Annali di Matematica pura ed applicata
(IV), Vol. CLXI (1992), pp. 379-397

A Singular-Degenerate Free Boundary Problem Arising from the Moisture Evaporation in a Partially Saturated Porous Medium (*).

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Summary. – *We consider a moisture evaporation process in a porous medium which is partially saturated by a fluid. The mathematical model is a singular-degenerate nonlinear parabolic free boundary problem. We first transform the problem into a weak form in a fixed domain and then derive some uniform estimates for the proper approximate solution. The existence of a weak solution is established by a compactness argument. Finally, the regularity of the solution and interfaces are investigated.*

1. – Introduction.

Assume that the soil of the earth is a homogeneous, isotropic and rigid porous medium partially saturated by a fluid. Furthermore, we suppose that the surface is dry. Let us examine the evaporation process of the moisture in this soil. Because of the heat of the sun, the water in the soil will be changed into vapor and the vapor will diffuse to the surface and be carried away. So the dry soil will appear not only on the surface of the earth but also below the surface as a region which varies with time. Here, we shall ignore the absorbent, chemical, osmotic and thermal effects of the soil.

It is well-known that the flow of the fluid thorough the porous medium satisfies Darcy's law:

$$\boldsymbol{q} = -K(\psi)\operatorname{grad}\varphi \tag{0.1}$$

and the continuity equation:

$$\frac{\partial\theta}{\partial t} = -\operatorname{div}\boldsymbol{q}, \tag{0.2}$$

(*) Entrata in Redazione il 15 marzo 1989.

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where θ is the volumetric moisture content, $\boldsymbol{q}$ is the macroscopic velocity of the fluid, K is the hydraulic conductivity, and φ is the hydraulic head which may be expressed as the sum of a hydrostatic potential ψ and a gravitational potential x. If we confine our attention to one-dimensional flow and have the x-axis pointing downward, we have the Richard's equation:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[K(\psi)\frac{\partial \psi}{\partial x}\right] - \frac{\partial K(\psi)}{\partial x}. \tag{0.3}$$

Between the quantities θ, ψ and K, there exist experimental relationships $\theta = \theta(\psi)$ and $K = K(\psi)$ if the hysteresis effect is ignored. For most soils, the functions $\theta(\psi)$ and $K(\psi)$ possess some typical properties (cf. [12], [13] and [14]):

(1) there exists a constant ψ_s such that the functions $\theta(\psi)$ and $K(\psi)$ are strictly increasing for $\psi \leqslant \psi_s$;

(2) the functions $\theta(\psi)$ and $K(\psi) \to 0$ as $\psi \to -\infty$ (dry part) and $\theta(\psi)$ and $K(\psi)$ are constants for $\psi \leqslant \psi_s$ (saturated part);

(3) the derivatives $\theta'(\psi)$ and $K'(\psi) \to 0$ as $\psi \to -\infty$ (see [14]),

$$\lim_{\psi \to -\infty} \frac{q'(\psi)}{K(\psi)} = +\infty$$

and $\theta'(\psi) \to 0$ as $\psi \to \psi_s$.

When θ is small we can always transform the equation (0.3) into the θ-equation:

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta)\frac{\partial \theta}{\partial x}\right] - \frac{\partial K(\psi(\theta))}{\partial x}, \tag{0.4}$$

where $D(\theta) = K(\psi(\theta))$ is the diffusivity of the soil which satisfies $D(0) = 0$ and the equation (0.4) degenerates into a hyperbolic-parabolic equation. Otherwise, we transform the equation (0.3) into the ψ-equation:

$$C(\psi)\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial x}\left[K(\psi)\frac{\partial \psi}{\partial x}\right] - \frac{\partial K(\psi)}{\partial x}, \tag{0.5}$$

where $C(\psi) = d\theta/d\psi$ is the capacity of the soil which satisfies $C(\psi) = 0$ when $\psi \geqslant \psi_s$ and $K(0) = 0$. Hence the equation (0.5) degenerates into a hyperbolic-parabolic and elliptic-parabolic equation.

REMARK. – From the physical point of view, it seems to be not meaningful to define the hydraulic conductivity and to use the Darcy's law at very low saturation soil. One can, however, apply the model to the movement of an ideal gas in a homogeneous porous medium in which $u(x, t)$ represents the density of the gas.

Let

$$u(x,t)=\int_{-\infty}^{\psi(x,t)} K(s)\,ds\,,\qquad u_s=\int_{-\infty}^{0} K(s)\,ds\,.$$

We assume $K(s)\equiv K_s>0$ for $s>0$ and strictly increasing for $s\in(-\infty,0)$. Thus, the correspondence of u and ψ is one to one. Furthermore, we see that

$\psi(x,t)=-\infty$ if and only if $u(x,t)=0$ (a dry state) i.e. $\theta=0$;

$\psi(x,t)\in(-\infty,0)$ if and only if $u\in(0,u_s)$ (an unsaturated state) i.e. $0<\theta<1$;

$\psi(x,t)\in[0,+\infty)$ if and only if $u\in[u_s,+\infty)$ (a saturated state) i.e., $\theta=1$.

Expressing the correspondence of u and ψ, we write $\psi(x,t)=\psi(u(x,t))$. We also write $\theta(u(x,t))=\theta(\psi(u(x,t)))$ and $K(u(x,t))=K(\psi(u(x,t)))$ for simplicity. Then we have the following equation from (0.5) for u:

$$\theta'(u)\,u_t=u_{xx}-K(u)_x\,,$$

where

$$\theta'(u)=\frac{1}{K(\psi(u(x,t)))}\,\frac{d\theta}{d\psi}\,,$$

satisfies $\theta'(u)=+\infty$ for $u=0$ and $\theta'(u)=0$ for $u\geqslant u_s$. $K(u)$ is strictly increasing for $u\in(0,u_s)$ with $K(0)=0$ and $K(u)\equiv K_s$ for $u\geqslant u_s$.

In the initial state $u(x,0)=u_0(x)$, we assume that $u_0(0)=0$ and that there exists a constant $s_0\,(0<s_0<1)$ such that $u_0(s_0)=u_s$ (This means that there exist three states initially: dry $(x=0)$, unsaturated $(0<x<s_0)$ and saturated $(s_0<x<1)$). On the boundary $x=1$ we assume no flux, i.e.

$$\left[\frac{\partial u}{\partial x}-K(u)\right]\Bigg|_{x=1}=0\,.$$

On the front of the moisture evaporation, which is denoted by $x=s(t)$, of this moving surface at time t, one has $u(s(t),t)=0$. Moreover, we assume that the evaporation velocity is

$$[u_x-K(u)]|_{x=s(t)}=u_x(s(t),t)=g(s(t),t)\geqslant 0\,,$$

since $K(u(s(t),t))=K(0)=0$.

Now we can state our mathematical problem (P): Let $t>0$.

Find $s(t)\colon[0,T]\to[0,1]$ and $u(x,t)\colon S_T=\{(x,t)\colon s(t)<x<1,\,0<t<T\}\to R^1$ which satisfy

(1.1) $$\beta(u)_t=u_{xx}-K(u)_x\,,\qquad (x,t)\in S_T\,,$$

(1.2) $$u(s(t),t)=0\,,\qquad 0\leqslant t\leqslant T\,,$$

(1.3) $$[u_x-K(u)]_{x=1}=0\,,\qquad 0\leqslant t\leqslant T\,,$$

(1.4) $$\beta(u(x,0)) = \beta(u_0(x)), \qquad 0 \leqslant x \leqslant 1,$$

(1.5) $$u_x(s(t),t) = g(s(t),t), \qquad 0 \leqslant t \leqslant T,$$

where $\beta(u)$ has the same behavior as the function $\theta(u)$ as stated in assumption H(1) below.

Since the first research [10] was accomplished in 1958, considerable attention has been paid to the investigation of the infiltration (0.3) and its generalizations (cf. [2], [5] and [9]). Much research, however, is concentrated on either the completely unsaturated medium or the completely saturated one. A. FASANO and M. PRIMICERO [4] considered the liquid flow in partially saturated porous mediam. They transformed it into a nonlinear free boundary problem in which the free boundary represents the interface between the saturated and unsaturated regions and obtained the existence and uniqueness by the contractive mapping argument. C. J. VAN DUYN and L. P. PELETIER[13] considered the same problem with different approach. They first defined a weak form of the problem and established the existence and uniqueness of the weak solution by compactness method. In [11], C. J. VAN DUYN obtained the continuity of the interface. More recently, S. XIAO et al. [14] studied an initial and boundary value problem with a more complicated situation in which the three physical states are co-existent. By the comparison principle, they derived a number of uniform estimates for the solution of a suitable approximate problem and obtained the existence and uniqueness of the solution under the certain conditions on the data. In this paper, we consider a general equation (0.5) with an evaporation process of the moisture. Unlike the Stefan-like free boundary problems with parabolic equations, our problem (1.1)-(1.5) is much more complicated since the equation is degenerate at both $u=0$ and $u \geqslant u_s$ as well as a free boundary is involved. It seems very hard to apply these previous methods to the present mathematical model. However, we use the idea in [3] and [15] to separate the two different kinds of degeneracy via the Sard's Lemma and the implicit function theorem. With the help of this separation, we can derive an equi-Hölder continuous modulus of the proper approximate solution by means of the maximum principle and the integral estimates. The compactness argument allows us to establish the existence of a solution.

In Section 2, we transform the problem into a weak form and state the main results. Sections 3 and 4 are devoted to the proofs of those results.

Throughout this paper, we shall assume the following conditions are satisfied:

H(1) The function $\beta(s) \in C[0,+\infty) \cap C^{0+1}(0,+\infty)$, $\beta'(0) = +\infty$, and $\beta'(s) > 0$ if $0 < s < u_s$, $\beta(s) \equiv 1$ if $s \geqslant u_s$ and $\beta(s) \in C^4(0,u_s)$.

H(2) The function $K(s) \in C^{0+1}(R^1)$, $K(s) \equiv 0$ for $s \in (-\infty, 0]$, $K(s) \equiv K(u_s)$ for $s \in [u_s, +\infty)$, $0 \leqslant K'(s) \leqslant K_0$ and $K(s) \in C^4(0,u_s)$, where K_0 is a constant.

H(3) (i) $u_0(x) \in C^2[0,1]$, $u_0(0) = 0$, $u_0(s_0) = u_s (0 < s_0 < 1)$, $u_s'(x) > 0$, $u_0''(x) \leqslant 0$, $u_0''(1) < 0$.

(ii) The consistency conditions:

$$-u_0''(0) = K'(0)\,g(0,0)\,,\quad u_0'(0) = g(0,0),\quad u_0'(1) - K(u_0(1)) = 0 \text{ and } u'''_0(1) = 0$$

hold.

H(4) The function $g(x,t) \in C^{2,2}(\overline{Q}_T)$ and $g(x,t) \geq 0$ and $g_t(x,t) \geq 0$.

2. – The weak form of the problem and the main results.

Since the equation (1.1) is degenerate, we cannot expect the existence of a classical solution for the problem (P). This motivates us to look for a weak form of the problem.

Let $T > 0$ and $Q_T = \{(x,t) \in R^2\colon 0 < x < 1, 0 < t < T\}$. We take

$$X = \{\varphi \in C^{2,1}(\overline{Q}_T)\colon \varphi(0,t) = \varphi_x(1,t) = \varphi(x,T) = 0\}\,,$$

as a test function space. Formally, multiplying the equation (1.1) by φ for any $\varphi \in X$, integrating it over the region $S_T = \{(x,t) \in R^2\colon s(t) < x < 1, 0 < t < T\}$ and integrating by parts once or twice depending on the numbers of derivatives in the original equation (1.1), we have

$$\iint_{S_T} [\beta(u)\,\varphi_t + u\varphi_{xx} + K(u)\,\varphi_x]\,dx\,dt - \int_0^T g(s(t),t)\,\varphi(s(t),t)\,dt = -\int_0^1 \beta(u_0(x))\,\varphi(x,0)\,dx\,.$$

Introduce a jump function

$$\chi(u) = \begin{cases} 0\,, & u > 0\,,\\ 1\,, & u \leq 0 \end{cases}$$

and set

$$u^*(x,t) = \begin{cases} u(x,t)\,, & (x,t) \in \overline{S}_T\,,\\ 0\,, & (x,t) \in \overline{Q}_T \setminus \overline{S}_T\,. \end{cases}$$

Thus, by $\beta(0) = 0$ and $\chi(0) = 1$, we have

$$\iint [\beta(u^*)\,\varphi_t + u^*\varphi_{xx} + K(u^*)\,\varphi_x - (g(x,t)\,\varphi(x,t))_x\,\chi(u^*)]\,dx\,dt = -\int_0^1 \beta(u_0(x))\,\varphi(x,0)\,dx\,.$$

We denote by $E(\xi)$ the graph:

$$E(\xi) = \begin{cases} 0\,, & \text{if } \xi > 0\,,\\ [0,1]\,, & \text{if } \xi = 0\,,\\ 1\,, & \text{if } \xi < 0\,. \end{cases}$$

DEFINITION. – A bounded, measurable function $u(x, t)$ defined on $\overline{Q}_T$ is called a weak solution to the problem (P), if

$$(2.1)\quad \iint_{Q_T} [\beta(u)\,\varphi_t + u\varphi_{xx} + K(u)\,\varphi_x - (g(x,t)\,\varphi(x,t))_x E(u)]\,dx\,dt = -\int_0^1 \beta(u_0(x))\,\varphi(x,0)\,dx\,,$$

where $E(u(x, t))$ is a bounded, measurable function whose values are contained in the graph of $E(u)$ and φ is an arbitrary function in the Banach space X.

PROPOSITION. – The classical solution of the problem (P) is a weak solution if we extend $u(x, t)$ as 0 across the front moving boundary $x = s(t)$ up to $x = 0$.

REMARK. – Formally, for the weak solution $u(x, t)$, the sets $\{(x, t)\colon u(x, t) = 0\}$, $\{(x, t)\colon 0 < u < u_s\}$ and $\{(x, t)\colon u(x, t) \geqslant u_s\}$ correspond to the dry, unsaturated and saturated physical regions, respectively. There are two interfaces: the first one is between the sets $\{(x, t)\colon u = 0\}$ and $\{(x, t)\colon 0 < u(x, t) < u_s\}$ corresponding to the value $u(x, t) = 0$ and the other between the sets $\{(x, t)\colon 0 < u < u_s\}$ and $\{(x, t)\colon u \geqslant u_s\}$, corresponding to the value $u(x, t) = u_s$.

The main results in this paper are as follows:

THEOREM 1. – Assume that the hypotheses H(1)-H(4) hold. Then there exists a weak solution $u(x, t)$ in Q_T for some $T > 0$. Let T^* be the largest value of T, then $0 < T^* \leqslant +\infty$. In the case of $T^* < +\infty$, we have $u(x, T^*) \equiv 0$. Furthermore, $\theta(x, t) \stackrel{\text{def}}{=} \beta(u(x, t))$ is continuous.

THEOREM 2. – Under the conditions of the Theorem 1, we have the following

(1) there exist two interfaces $s_1(t)$ and $s_2(t)$ such that $\theta(x, t)$ satisfies

$$\begin{aligned} &\theta(x,t) = 0 && \text{for } (x,t) \in P\,,\\ &0 < \theta(x,t) < 1 && \text{for } (x,t) \in R\,,\\ &\theta(x,t) = 1 && \text{for } (x,t) \in D\,, \end{aligned}$$

where

$$\begin{aligned} P &\stackrel{\text{def}}{=} \{(x,t)\colon 0 < x < s_1(t), 0 < t < T^*\}\,,\\ R &\stackrel{\text{def}}{=} \{(x,t)\colon s_1(t) < x < s_2(t), 0 < t < T^{**}\} \cup \{s_1(t) < x < 1, T^{**} < t < T^*\}\,,\\ D &\stackrel{\text{def}}{=} \{(x,t)\colon s_2(t) < x < 1, 0 \leqslant t \leqslant T^{**}\}\,, \end{aligned}$$

while $0 < T^* \leqslant T^*$ and $s_2(T^{**}) = 1$ if $T^{**} < +\infty$. Moreover, $u(x, t)$ satisfies the equation (1.1) in the classical sense in the region R.

(2) The curves $x = s_1(t)$ and $x = s_2(t)$ are monotone increasing. Moreover $s_1(t)$ is continuous on $[0, T^*]$.

3. – The proof of Theorem 1.

Construct function sequences $E_\varepsilon(u) \in C^4(R^1)$, $\beta_\varepsilon(u) \in C^4(R^1)$ and $K_\varepsilon(u) \in C^4(R^1)$ such that the following conditions hold:

(1) $E_\varepsilon(u)$, $\beta_\varepsilon(u)$ and $K_\varepsilon(u)$ converge pointwisely to $E(u)$, $\beta(u)$ and $K(u)$, respectively, when $\varepsilon \to 0$. Moreover for $\beta_\varepsilon(u)$ and $K_\varepsilon(u)$, the convergence is uniform in $[0, +\infty)$.

(2)

$$E_\varepsilon(u) = \begin{cases} 1, & \text{if } u \leqslant 0, \\ \text{strictly decreasing}, & \text{if } 0 \leqslant u \leqslant \varepsilon, \\ 0, & \text{if } u \geqslant \varepsilon \end{cases}$$

and $E_\varepsilon'(0) = E_\varepsilon''(0) = 0$.

$$\beta_\varepsilon(u) = \begin{cases} (\beta_\varepsilon(\varepsilon)/\varepsilon)\, u, & -\infty < u \leqslant \varepsilon, \\ \text{strictly increasing}, & \varepsilon \leqslant u \leqslant 2\varepsilon, \\ \beta(u), & 2\varepsilon \leqslant u \leqslant u_s - \varepsilon, \\ \text{strictly increasing}, & u_s - \varepsilon \leqslant u \leqslant u_s, \\ \beta(u) + \varepsilon u, & u \geqslant u_s. \end{cases}$$

$$K_\varepsilon(u) = \begin{cases} 0, & u \leqslant 0, \\ K(u), & 0 \leqslant u \leqslant u_s - \varepsilon, \\ \text{strictly increasing}, & u_s - \varepsilon \leqslant u \leqslant u_s, \\ K(u_s), & u \geqslant u_s \end{cases}$$

and $0 \leqslant K_\varepsilon'(u) \leqslant K_0$.

Furthermore, at the corner points $(0, 0)$ and $(1, 0)$, the compatibility conditions are satisfied: $u_0(0) = 0$, $u_0'''(1) = 0$, $u_0''(0) = K_\varepsilon'(0)\, u_0'(0)$, $u_0'(1) = K_\varepsilon(u_0(1))$.

Now we consider an approximate problem (P_ε) in a fixed domain Q_T:

$$\beta_\varepsilon(u)_t - u_{xx} + K_\varepsilon(u)_x - g(x, t)\, E_\varepsilon(u)_x = 0 \quad \text{in } Q_T, \tag{3.1}$$

$$u(0, t) = 0, \quad 0 \leqslant t \leqslant T, \tag{3.2}$$

$$[u_x - K_\varepsilon(u)]|_{x=1} = 0, \quad 0 \leqslant t \leqslant T, \tag{3.3}$$

$$u(x, 0) = u_0(x), \quad 0 \leqslant x \leqslant 1. \tag{3.4}$$

By the construction of $E_\varepsilon(u)$, $\beta_\varepsilon(u)$ and $K_\varepsilon(u)$, we can easily see that the compatibility conditions at the corner points are satisfied up to the second order. So the problem (P_ε) has a unique classical solution $u_\varepsilon(x,t) \in C^{4,2}(\overline{Q}_T)$ by [6].

To obtain the existence of a weak solution, we need to have certain uniform estimates with respect to ε. The main difficulty is from the two types of degeneracy and the singularity term $E_\varepsilon(u)_x$ in (3.1).

LEMMA 3.1. – The solution of problem (P_ε) is $u_\varepsilon(x,t) \geq 0$ for all $(x,t) \in \overline{Q}_T$.

PROOF. – This is directly from the strong maximum principle since $u_{\varepsilon x}(1,t) = K(u_\varepsilon) \geq 0$. Q.E.D.

LEMMA 3.2. – The solution $u_\varepsilon(x,t)$ satisfies $u_{\varepsilon x}(x,t) \geq 0$ for all $(x,t) \in \overline{Q}_T$.

PROOF. – Let $v = u_{\varepsilon x}(x,t)$. Since $u_\varepsilon(x,t) \geq 0$ and $u_\varepsilon(0,t)$ which implies that $u_\varepsilon(x,t)$ takes a minimum value 0 at every point on the boundary $x=0$, we have $v(0,t) = u_{\varepsilon x}(0,t) \geq 0$. Note that $v(1,t) = K(u_\varepsilon(1,t)) \geq 0$ and $v(x,0) = u_0'(x) \geq 0$. It follows that $v \geq 0$ on the parabolic boundary $\partial_p Q_T$ of Q_T. It is clear that v satisfies

$$\beta_\varepsilon'(u_\varepsilon)v_t - v_{\varepsilon xx} - [K_\varepsilon'(u_\varepsilon) - g(x,t)E_\varepsilon'(u_\varepsilon)]v_x + \\ + [\beta_\varepsilon''(u_\varepsilon)u_{\varepsilon t} + K_\varepsilon''(u_\varepsilon)v - g(x,t)E_\varepsilon''(u_\varepsilon)v - g'(x,t)E_\varepsilon'(u_\varepsilon)]v = 0 \quad \text{in } Q_T. \tag{3.5}$$

Make the transform

$$v(x,t) = e^{\alpha_\varepsilon t}\overline{v}(x,t),$$

where $\alpha_\varepsilon \geq \|[\beta_\varepsilon''(u_\varepsilon)u_{\varepsilon t} + K_\varepsilon''(u_\varepsilon)v - g(x,t)E_\varepsilon''(u_\varepsilon)v - g'(x,t)E_\varepsilon'(u_\varepsilon)]\|_{L^{+\infty}(Q_T)}$.

Since $\overline{v}(x,t)|_{\partial_p Q_T} = e^{-\alpha_\varepsilon t}v(x,t)|_{\partial_p Q_T} \geq 0$, it follows from the maximum principle for the equation (3.5) that $\overline{v}(x,t) \geq 0$ for all $(x,t) \in \overline{Q}_T$. Therefore, $v(x,t) \geq 0$ for all $(x,t) \in \overline{Q}_T$. Q.E.D.

LEMMA 3.3. – The solution u_ε has the property $u_{\varepsilon t}(x,t) \leq 0$ for any $(x,t) \in \overline{Q}_T$.

PROOF. – Let $w(x,t) = u_{\varepsilon t}(x,t)$. Then w satisfies

$$\beta_\varepsilon'(u_\varepsilon)w_t - w_{xx} + [K_\varepsilon'(u_\varepsilon) - g(x)E_\varepsilon'(u_\varepsilon)]w_x + \\ + [\beta_\varepsilon''(u_\varepsilon)w + K_\varepsilon''(u_\varepsilon)u_{\varepsilon x} - g(x,t)E_\varepsilon''(u_\varepsilon)u_{\varepsilon x}]w = g_t(x,t)E_\varepsilon'(u_\varepsilon)u_{\varepsilon x} \quad \text{in } Q_T,$$

$$w(0,t) = 0, \qquad 0 \leq t \leq T,$$

$$[w_x(x,t) - K_\varepsilon'(u_\varepsilon(x,t))w(x,t)]|_{x=1} = 0, \qquad 0 \leq t \leq T,$$

$$w(x,0) = \frac{1}{\beta_\varepsilon'(u_\varepsilon)}[u_{\varepsilon xx} - K_\varepsilon'(u_\varepsilon)u_{\varepsilon x} + g(x,t)E_\varepsilon'(u_\varepsilon)u_{\varepsilon x}]\Big|_{t=0}, \qquad 0 \leq x \leq 1.$$

By the condition H(3) and the construction of the $E_\varepsilon(u)$ and $K_\varepsilon(u)$, we know that $u_0'' \leq 0$, $u_0' \geq 0$, $K_\varepsilon'(u) \geq 0$, $E_\varepsilon'(u) \leq 0$ and $u_0''(1) < 0$. So we have $w(x,0) \leq 0$ and $w(1,0) < 0$. Then there exists $T_\varepsilon > 0$ such that $w(1,t) < 0$ for $0 \leq t \leq T_\varepsilon \leq T$.

Let $T_\varepsilon^* = \sup\{\tilde{t}: 0 < \tilde{t} \leq T \text{ and } w(1,t) < 0 \text{ for } 0 \leq t \leq \tilde{t}\}$. Obviously, $T_\varepsilon^* \geq T_\varepsilon > 0$. We assert that $T_\varepsilon^* = T$. Otherwise, assuming the result is not true, we see $w(1, T_\varepsilon^*) = 0$. It is clear that

$$w|_{\partial_p Q_{T_\varepsilon^*}} \leq 0\,,$$

where $Q_{T^*} = \{(x,t) \in Q_T: 0 < x < 1, 0 < t < T_\varepsilon^*\}$. Since $g_t(x,t) E_\varepsilon(u_\varepsilon)_x \leq 0$ by H(4) and $u_{\varepsilon x}(x,t) \geq 0$ by Lemma 3.2, the maximum principle argument similar to that of Lemma 3.2 implies $w(x,t) \leq 0$ for all $(x,t) \in \overline{Q}_{T_\varepsilon^*}$. Since $w(1, T_\varepsilon^*) = 0$, $w(x,t)$ attains a maximum value at the point $(1, T_\varepsilon^*)$ in the region $\overline{Q}_{T_\varepsilon^*}$. Applying strong maximum principle, we have $w_x(1,t)|_{t=T_\varepsilon^*} > 0$. But $w_x(1, T_\varepsilon^*) = K_\varepsilon'(u_\varepsilon(1, T_\varepsilon^*))\, w(1, T_\varepsilon^*) = 0$, this leads a contradiction. Thus, $T_\varepsilon^* = T$ and $w(x,t) \leq 0$ for any $(x,t) \in \overline{Q}_T$. Q.E.D.

COROLLARY 3.4. – There exists a constant M_1 not depending on ε such that

$$0 \leq u_\varepsilon(x,t) \leq M_1 \qquad \text{for any } (x,t) \in \overline{Q}_T\,.$$

PROOF. – The result follows immediately from Lemma 3.1, 3.2 and 3.3. Q.E.D.

LEMMA 3.5. – There exists a constant M_2 not depending on ε such that

$$|u_{\varepsilon x}(x,t)| \leq M_2 \qquad \text{on } \overline{Q}_T\,.$$

PROOF. – We introduce an auxiliary function $w = L_1 x$, where L_1 is a constant to be determined later. Let $Z(x,t) = w(x,t) - u_\varepsilon(x,t)$. Then $Z(0,t) = 0$, $Z_x(1,t) = L_1 - K_\varepsilon(u_\varepsilon(1,t)) > 0$ if we take $L_1 \geq K(u_s) + 1$. This implies that $Z(x,t)$ cannot take a minimum value on the boundary $x = 1$. Moreover, by $u_0(0) = 0$, one has

$$Z(x,0) = L_1 x - u_0(x) \geq 0\,,$$

if $L_1 \geq |u_0'| + 1$.

Now we define operator $\mathcal{F}$ by

$$\mathcal{F}W = \beta_\varepsilon'(u_\varepsilon)\frac{\partial W}{\partial t} - \frac{\partial^2 W}{\partial x^2} + k_\varepsilon'(u_\varepsilon)\frac{\partial W}{\partial x} - g(x,t)E_\varepsilon'(u_\varepsilon)\frac{\partial W}{\partial x}\,.$$

Then,

$$\mathcal{F}Z = K_\varepsilon'(u_\varepsilon)L_1 - g(x,t)E_\varepsilon'(u_\varepsilon)L_1 \geq 0\,,$$

since $E_\varepsilon'(u_\varepsilon) \leq 0$ and $K_\varepsilon'(u_\varepsilon) \geq 0$. It follows by the strong maximum principle that $Z(x,t)$ cannot take negative minimum values in the interior of Q_T. Hence $Z(x,t) \geq 0$ on $\overline{Q}_T$, i.e. $u_\varepsilon(x,t) \leq Lx$ on $\overline{Q}_T$.

Thus, we have

$$\frac{\partial u_\varepsilon(0,t)}{\partial x} \leqslant L_1 .$$

Now

$$\frac{\partial u_\varepsilon(0,t)}{\partial x} = \left.\frac{\partial u_\varepsilon(x,t)}{\partial x}\right|_{x=0} \geqslant 0 ,$$

by Lemma 3.1. Moreover, by the boundary condition (3.3),

$$\frac{\partial u_\varepsilon(1,t)}{\partial x} = K_\varepsilon(u_\varepsilon(1,t))$$

and $u_{\varepsilon x}(x,0) = u_0'(x)$ are uniformly bounded. It follows that

$$\left|\frac{\partial u}{\partial x}\right|_{\partial_p Q_T} < \max\{L_1, C_0+1, K(u_s)\} ,$$

which is independent of ε. Since $u_{\varepsilon t}(x,t) \leqslant 0$ by Lemma 3.3, it follows from equation (3.1) that

$$0 \geqslant \beta_\varepsilon'(u_\varepsilon)\, u_{\varepsilon t} =$$

$$= u_{\varepsilon xx} - K_\varepsilon(u_\varepsilon)_x + g(x,t)\, E_\varepsilon(u_\varepsilon)_x \equiv \frac{\partial}{\partial x}[u_{\varepsilon x} - K_\varepsilon(u_\varepsilon) + g(x,t)\, E_\varepsilon(u_\varepsilon)] - g'(x,t)\, E_\varepsilon(u_\varepsilon) .$$

Integrating the above inequality from 0 to x, we obtain

$$u_{\varepsilon x} - K_\varepsilon(u_\varepsilon) + g(x,t)\, E_\varepsilon(u_\varepsilon) \leqslant \int_0^x g'(x,t)\, E_\varepsilon(u_\varepsilon)\, dx + [u_{\varepsilon x} - K_\varepsilon(u_\varepsilon) + g(x,t)\, E_\varepsilon(u_\varepsilon)]\Big|_{x=0} \leqslant C$$

for all $(x,t) \in \overline{Q}_T$ because of $|E_\varepsilon(u_\varepsilon)| \leqslant 1$, $|K_\varepsilon(u_\varepsilon)| \leqslant K(u_s)+1$ and $|u_{\varepsilon x}(0,t)| \leqslant L_1$, where C is independent of ε.

Thus, we have

$$u_{\varepsilon x}(x,t) \leqslant K_\varepsilon(u_\varepsilon) + |g(x,t)\, E_\varepsilon(u_\varepsilon)| + C \leqslant K_0 + |g(x,t)| + C$$

which is uniformly bounded for all $(x,t) \in \overline{Q}_T$. One combines the above inequality with Lemma 3.2 to obtain

$$|u_{\varepsilon x}(x,t)| \leqslant M_2 \qquad \text{on } \overline{Q}_T ,$$

where M_2 is a constant which is independent of ε. Q.E.D.

In order to obtain the equi-continuity of $u_\varepsilon(x,t)$ or $\beta_\varepsilon(u_\varepsilon(x,t))$, we cannot use the usual arguments for the problem (3.1)-(3.4) because of the terms $\beta_\varepsilon(u)$ and $H_\varepsilon(u)$. We overcome this difficulty by separating the two different kinds of degeneracy via Sard's Lemma and the implicit function theorem.

SARD'S LEMMA. – Suppose $f: \Omega \in R^2 \to R^1$ is a C^2-function, where Ω is a bounded region. Then the set $\{c \in R(f)$: for all $(x, t) \in \Omega$ with $f(x, t) = c$ and $\nabla f(x, t) = \{f_x(x, t), f_t(x, t)\} = 0\}$, which is all the critical values in the range of $f(x, t)$, is a null-measure set in the sense of R^1-Lebesgue.

LEMMA 3.6. – For a.e. α $(0 < \alpha < u_s)$, there exists a curve $x = h_\varepsilon^\alpha(t)$ which is a monotonic increasing function of t such that

$$A_\alpha^\varepsilon = \{(x, t): h_\varepsilon^\alpha(t) \leqslant x \leqslant 1, 0 \leqslant t \leqslant T(\varepsilon)\} \equiv \{(x, t): u_\varepsilon(x, t) \geqslant \alpha\},$$

$$B_\alpha^\varepsilon = \{(x, t) \in Q_T: 0 \leqslant u_\varepsilon(x, t) < \alpha\} = \overline{Q}_T \setminus A_\alpha^\varepsilon,$$

where $0 < T(\varepsilon) \leqslant T$ and $h_\varepsilon^\alpha(t) = 1$ if $T(\varepsilon) \leqslant t \leqslant T$.

PROOF. – By Sard's lemma, we know that for a.e. non-critical value α of $u_\varepsilon(x, t)$ with $0 < \alpha < u_s$ and any $(x, t) \in \Gamma_\alpha^\varepsilon = \{(x, t) \in Q_T: u_\varepsilon(x, t) = \alpha\}$, $u_\varepsilon(x, t)$ has the property that

$$\nabla u_\varepsilon(x, t) = \{u_{\varepsilon x}(x, t), u_{\varepsilon t}(x, t)\} \neq 0. \tag{3.6}$$

According to the existence and uniqueness theorem of the implicit function, we know that $\Gamma_\alpha^\varepsilon$ consists of the graphs of finitely many smooth curves which do not intersect each other. We assert that $\Gamma_\alpha^\varepsilon$ is composed of the graph of one smooth curve $x = h_\varepsilon^\alpha(t)$. Assume the result is not true. First of all, no segment I in $\Gamma_\alpha^\varepsilon$ (that means the graph of the segment I is contained in $\Gamma_\alpha^\varepsilon$) exists such that I is parallel to x-axis. Otherwise, on such a segment I, because of $u_\varepsilon(x, t) \equiv \alpha$, one has $u_{\varepsilon x} = u_{\varepsilon xx} = 0$. Hence $u_{\varepsilon t} = 0$ from the equation (3.1). This contradicts with (3.6). Now assume that at $t = t_0 \in (0, T]$, there exist points (x_0, t_0) and $(x_1, t_0) \in \Gamma_\alpha^\varepsilon$ with $x_0 \neq x_1$. Without loss of generality, let $x_0 < x_1$. Then $u_\varepsilon(x_0, t_0) = \alpha$ and $u_\varepsilon(x_1, t_0) = \alpha$. Since $u_{\varepsilon x}(x, t) \geqslant 0$ by Lemma 3.2, it follows that $u_\varepsilon(x, t_0) \equiv \alpha$ for $x \in [x_0, x_1]$. Hence the interval $J = [x_0, x_1]$ is contained in $\Gamma_\alpha^\varepsilon$ (in the sense of graph) and J is parallel to the x-axis. This is a contradiction. So $\Gamma_\alpha^\varepsilon$ comprises a graph of one smooth curve. Next we show that $\Gamma_\alpha^\varepsilon$ can be represented by the graph of a function $x = h_\varepsilon^\alpha(t)$ which enters at a point $(x_1, 0)$ and exits at (x_2, T) or $(1, T(\varepsilon))$ with $0 < T(\varepsilon) < T$ and $h_\varepsilon^\alpha(T(\varepsilon)) = 1$ $(0 < x_1 < 1, 0 < x_2 < 1)$. Indeed, since $u_0(0) = 0$ and $u_0(x_0) = u_s$, there exists a x_1 $(0 < x_1 < x_0)$ such that $u_0(x_1) = \alpha$ by $0 < \alpha < u_s$. The inequality $u_0'(x) > 0$ implies that $u_{\varepsilon x}(x, t) > 0$ in a neighborhood $N_r = \{(x, t): |x - x_1| < r, 0 < t < r\}$. Let $T(\varepsilon) = \sup\{t^*: u_{\varepsilon x} > 0$ for $0 < t < t^*, x \in [0, 1]\}$. By the implicit theorem, there exists a function, denoted by $h_\varepsilon^\alpha(t)$, such that on $[0, T(\varepsilon)]$

$$\Gamma_\alpha^\varepsilon = \{(h_\varepsilon^\alpha(t), t): 0 \leqslant t \leqslant T(\varepsilon)\}.$$

If $T(\varepsilon) = T$ or $x = h_\varepsilon^\alpha(T(\varepsilon)) = 1$, we have the desired result. Otherwise, assume $T(\varepsilon) < T$ and $0 \leqslant h_\varepsilon^\alpha(t) < 1$. It is clear that $x = h_\varepsilon^\alpha(T(\varepsilon)) > 0$ since $u_\varepsilon(h_\varepsilon^\alpha(T(\varepsilon)), T(\varepsilon)) =$

$= \alpha > 0$. Furthermore,

$$u_{\varepsilon x}(h_\varepsilon^\alpha(T(\varepsilon)), T(\varepsilon)) = 0. \tag{3.7}$$

Let

$$Q_1 = \{(x, t) \in Q_T : 0 < x < h_\varepsilon^\alpha(t),\ 0 < t < T(\varepsilon)\},$$

and

$$Q_2 = \{(x, t) \in Q_T : h_\varepsilon^\alpha(t) < x < 1,\ 0 < t < T(\varepsilon)\}.$$

Since $h_\varepsilon^\alpha(t)$ is smooth, the interior cone property at point $(h_\varepsilon^\alpha(T(\varepsilon)), T(\varepsilon))$ is satisfied in at least one of the regions Q_1 and Q_2. By the maximum principle, $u_\varepsilon(x, t)$ attains a strictly maximum value at $(h_\varepsilon^\alpha(T(\varepsilon)), T(\varepsilon))$ on $\overline{Q}_1$ and $u_\varepsilon(x, t)$ attains a strictly minimum value at $(h_\varepsilon^\alpha(T(\varepsilon)), T(\varepsilon))$ on $\overline{Q}_2$. If the interior cone property holds for Q_1, the strong maximum principle (cf. [15]) implies that $u_{\varepsilon x}(h_\varepsilon^\alpha(T(\varepsilon)), T(\varepsilon)) < 0$ which contradicts with (3.7). Similarly, we can obtain a contradiction of (3.7) when the interior cone property holds for Q_2. Therefore, $T(\varepsilon) = T$ and

$$\Gamma_\alpha^\varepsilon = \{(h_\varepsilon^\alpha(t), t) : 0 \le t < T\}.$$

The rest of our Lemma follows from the maximum principle.

To obtain the monotonicity of $h_\varepsilon^\alpha(t)$, let $t_1 < t_2$ and $u_\varepsilon(h_\varepsilon^\alpha(t_1), t_1) = u_\varepsilon(h_\varepsilon^\alpha(t_2), t_2) = \alpha$. It follows by the mean value theorem that

$$0 = u_\varepsilon(h_\varepsilon^\alpha(t_1), t_1) - u_\varepsilon(h_\varepsilon^\alpha(t_2), t_2) = u_{\varepsilon x}(\xi_1, t_1)[h_\varepsilon^\alpha(t_1) - h_\varepsilon^\alpha(t_2)] + u_{\varepsilon t}(h_\varepsilon^\alpha(t_2), \xi_2)(t_1 - t_2),$$

where ξ_1 is between $h_\varepsilon^\alpha(t_1)$ and $h_\varepsilon^\alpha(t_2)$ and ξ_2 between t_1 and t_2.

Since $u_{\varepsilon x}(x, t) \ge 0$ and $u_{\varepsilon t}(x, t) \le 0$ on $\overline{Q}$, we have

$$h_\varepsilon^\alpha(t_1) \le h_\varepsilon^\alpha(t_2),$$

i.e. $h_\varepsilon^\alpha(t)$ is a monotonic increasing function. Q.E.D.

The reader is referred to paper [15] for a more detailed analysis of the structure of the level sets of solutions of parabolic equations.

In what follows we need the following result.

LEMMA 3.7. – Suppose $f(x) \in C^1[0, 1]$ and (1)$|f'(x)| \le A$, (2)$\left|\int_a^b f(x)\,dx\right| \le B$ for any $a, b \in [0, 1]$. Then $|f(x)| \le \max\{2B, (2AB)^{1/2}$ for all $x \in [0, 1]\}$.

PROOF. – See C. J. VAN DUYN [13].

Now we fix $\alpha (0 < \alpha < u_s)$ which is not a critical value of $u_\varepsilon(x, t)$.

LEMMA 3.8. – On the region A_α^ε, the function $\beta_\varepsilon(u_\varepsilon)$ is equi-Hölder continuous in x and t with exponents 1 and 1/2, respectively.

PROOF. – On the region A_α^ε, $u_\varepsilon(x,t) \geq \alpha$ implies $E_\varepsilon(u_\varepsilon) \equiv 0$ and then $E'_\varepsilon(u_\varepsilon) \equiv 0$. Moreover, $\beta'_\varepsilon(u_\varepsilon)$ is uniformly bounded with respect to ε by the hypothesis H(1), provided that ε is small enough such that $\varepsilon < \alpha/2$. The bound of $\beta'_\varepsilon(u_\varepsilon)$ may depend on α, but not on ε.

Set $\theta_\varepsilon(x,t) = \beta_\varepsilon(u_\varepsilon(x,t))$. We have $|\theta_{\varepsilon x}(x,t)| \leq |\beta'_\varepsilon(u_\varepsilon)|\,|u_{\varepsilon x}| \leq M_2 C(\alpha) = M_3(\alpha)$, where $M_3(\alpha)$ is independent of ε.

For any rectangle $[a,b]\times[t_1,t_2] \subset A_\alpha^\varepsilon$, we utilize the equation (3.1) and Lemma 3.5 to obtain

$$\left|\int_a^b [\theta_\varepsilon(x,t_2)-\theta_\varepsilon(x,t_1)]\,dx\right| = \left|\int_a^b\int_{t_1}^{t_2} \beta_\varepsilon(u_\varepsilon)_t\,dx\,dt\right| = \left|\int_{t_1}^{t_2} [u_{\varepsilon x} - K(u_\varepsilon)]\Big|_{x=a}^{x=b}\right| \leq M_4|t_1-t_2|\,.$$

By Lemma 3.7, we have

$$|\theta_\varepsilon(x,t_1)-\theta_\varepsilon(x,t_2)| \leq M_5|t_1-t_2|^{1/2},$$

for any $(x,t_i) \in A_\alpha^\varepsilon$ $(i=1,2)$, where M_5 depends only on α and known data. Q.E.D.

LEMMA 3.9. – On the region B_α^ε, $u_\varepsilon(x,t)$ is equi-Hölder continuous in x and t with exponents 1 and 1/2, respectively.

PROOF. – On the region B_α^ε, $\beta'_\varepsilon(u_\varepsilon) \geq a(\alpha) > 0$, where $a(\alpha)$ is a constant not depending on ε. For any $(x_0,t_0) \in B_\alpha^\varepsilon$, we take $\Delta t > 0$ such that $[x_0, x_0+\sqrt{\Delta t}]\times[t_0,t_0+\Delta t] \subset B_\alpha^\varepsilon$. Then

$$\left|\int_{x_0}^{x_0+\sqrt{\Delta t}} [\beta_\varepsilon(u_\varepsilon(x,t_0+\Delta t)) - \beta_\varepsilon(u_\varepsilon(x,t_0))]\,dx\right| = \left|\int_{x_0}^{x_0+\sqrt{\Delta t}}\int_{t_0}^{t_0+\Delta t} \beta_\varepsilon(u_\varepsilon)_t\,dx\,dt\right| =$$

$$= \left|\int_{x_0}^{x_0+\sqrt{\Delta t}}\int_{t_0}^{t_0+\Delta t} \left\{\frac{\partial}{\partial x}[u_{\varepsilon x} - K_\varepsilon(u_\varepsilon) + g(x,t)E_\varepsilon(u_\varepsilon)] - g'(x,t)H_\varepsilon(u_\varepsilon)\right\}dx\,dt\right| \leq M_6|\Delta t|,$$

by the Lemma 3.5.

We use the mean value theorem for integrals to obtain

$$\left|\int_{x_0}^{x_0+\sqrt{\Delta t}} [\beta_\varepsilon(u_\varepsilon(x,t_0+\Delta t)) - \beta_\varepsilon(u_\varepsilon(x,t_0))]\,dx\right| =$$

$$= |[\beta_\varepsilon(u_\varepsilon(x^*,t_0+\Delta t)) - \beta_\varepsilon(u_\varepsilon(x^*,t_0))]|\sqrt{\Delta t} \geq a(\alpha)|u_\varepsilon(x^*,t_0+\Delta t) - u_\varepsilon(x^*,t_0)|\sqrt{\Delta t},$$

where $x^* \in (x_0, x_0+\sqrt{\Delta t})$.

It follows that

$$|u_\varepsilon(x_0, t_0+\Delta t) - u_\varepsilon(x_0, t_0)| \leqslant$$

$$\leqslant |u_\varepsilon(x_0, t_0+\Delta t) - u_\varepsilon(x^*, t_0+\Delta t)| + |u_\varepsilon(x^*, t_0+\Delta t) - u_\varepsilon(x^*, t_0)| +$$

$$+ |u_\varepsilon(x^*, t_0) - u_\varepsilon(x_0, t_0)| \leqslant 2M_5 |x_0 - x^*| + C_1 |\Delta t|^{1/2} \leqslant M_6 |\Delta t|^{1/2},$$

where M_6 depends only on known data and α. Q.E.D.

With the above results in hand we can establish the following important corollary.

COROLLARY 3.9. – The sequence $\{\beta_\varepsilon(u_\varepsilon(x,t))\}$ is equi-continuous on region $\overline{Q}_T$ if ε is small enough.

PROOF. – Let (x_0, t_0) be an arbitrary fixed point on $\overline{Q}_T$. We want to show that for any $r>0$, there exists a $\delta = \delta(r) > 0$ such that

$$|\beta_\varepsilon(u_\varepsilon(x,t)) - \beta_\varepsilon(u_\varepsilon(x_0, t_0))| < r$$

if $(x,t) \in U_\delta = \{(x,t) \in \overline{Q}_T : |x - x_0| + |t - t_0|^{1/2} \leqslant \delta\}$ for a small ε.

Firstly, since $\beta_\varepsilon(u)$ converges to $\beta(u)$ uniformly, hence, for any $r>0$ there exists a small ε_0 such that

$$|\beta_\varepsilon(u) - \beta(u)| < r/6$$

for any $u \in [0, M_1]$ if $\varepsilon < \varepsilon_0$.

Secondly, since $\beta(u)$ is uniformly continuous on $[0, M_1]$, we have $\delta_0 = \delta_0(r) > 0$ such that

$$|\beta(u) - \beta(u')| < r/6$$

if $|u - u'| < \delta_0$.

Case 1. If both (x,t) and $(x_0, t_0) \in A_\alpha^\varepsilon$, then, from Lemma 3.8,

$$|\beta_\varepsilon(u_\varepsilon(x,t)) - \beta_\varepsilon(u_\varepsilon(x_0, t_0))| \leqslant \max\{M_2, M_5\}[|x - x_0| + |t - t_0|^{1/2}] \leqslant r/2,$$

provided that we choose $\delta < \delta_1 \equiv r/(2\max\{M_2, M_5\})$.

Case 2. If both (x,t) and $(x_0, t_0) \in B_\alpha^\varepsilon$, then there exists $\delta_2 = \delta_0/(\max\{M_5, M_6\}) >$ > 0 such that

$$|u_\varepsilon(x,t) - u_\varepsilon(x_0, t_0)| < \max\{M_5, M_6\}[|x - x_0| + |t - t_0|]^{1/2} < \delta_0.$$

Then, if $\varepsilon < \varepsilon_0$,

$$|\beta_\varepsilon(u_\varepsilon(x,t)) - \beta_\varepsilon(u_\varepsilon(x_0,t_0))| \leq |\beta_\varepsilon(u_\varepsilon(x,t)) - \beta(u_\varepsilon(x,t))| + |\beta(u_\varepsilon(x,t)) - \beta(u_\varepsilon(x_0,t_0))| +$$
$$+ |\beta(u_\varepsilon(x_0,t_0)) - \beta_\varepsilon(u_\varepsilon(x_0,t_0))| \leq r/6 + r/6 + r/6 = r/2,$$

provided that $\delta < \delta_2$.

Case 3. Without loss of generality, let $(x,t) \in A^\varepsilon_\alpha$ and $(x_0,t_0) \in B^\varepsilon_\alpha$. Then there exists a point $(x^*,t^*) \in A^\varepsilon_\alpha \cap \overline{B}^\varepsilon_\alpha$. Hence

$$|\beta_\varepsilon(u_\varepsilon(x,t)) - \beta_\varepsilon(u_\varepsilon(x_0,t_0))| \leq |\beta_\varepsilon(u_\varepsilon(x,t)) - \beta(u_\varepsilon(x^*,t^*))| +$$
$$+ |\beta_\varepsilon(u_\varepsilon(x^*,t^*)) - \beta_\varepsilon(u_\varepsilon(x_0,t_0))| < r/2 + r/2 = r,$$

provided we choose that $\delta = \min\{\delta_1, \delta_2\}$ for $\varepsilon < \varepsilon_0$.

This concludes our result. Q.E.D.

By the Arzela-Ascoli compactness Lemma, there exists a subsequence (still denoted by $\beta_\varepsilon(u_\varepsilon(x,t))$ for convenience) of the $\{\beta_\varepsilon(u_\varepsilon(x,t))\}$ such that $\beta_\varepsilon(u_\varepsilon)$ converges uniformly on $\overline{Q}_T$. We denote the limit function by $\theta(x,t)$. It is clear that $\theta(x,t)$ is a continuous function because the convergence of the sequence is uniform. Now we define

$$D = \{(x,t)\colon (x,t) \in \overline{Q}_T,\ \theta(x,t)\},$$

$$P = \{(x,t)\colon (x,t) \in \overline{Q}_T,\ 0 < \theta(x,t) < 1\},$$

$$R = \{(x,t)\colon (x,t) \in \overline{Q}_T,\ \theta(x,t) = 1\}.$$

Then $D \cup P \cup R = \overline{Q}_T$ and P is a open set because of the continuity of the funtion $\theta(x,t)$.

Using weak compactness of the L_2-space, we have subsequences (still denoted by the original symbols) $u_\varepsilon(x,t) \rightharpoonup u(x,t)$, $\beta_\varepsilon(u_\varepsilon) \rightharpoonup \tilde{\beta}(x,t)$, $K_\varepsilon(u_\varepsilon) \rightharpoonup \tilde{K}(x,t)$ and $E_\varepsilon(u_\varepsilon) \rightharpoonup \tilde{E}$(x,t) when $\varepsilon \to 0$, where «$\rightharpoonup$» means that the sequences converge weakly in the sense of Banach space $L^2(Q_T)$.

In order to obtain a weak solution, we need to prove that $\tilde{\beta}(x,t) = \beta(u(x,t))$, $\tilde{K}(x,t) = K(u(x,t))$ and $\tilde{E}(x,t) = E(u(x,t))$ for a.e. $(x,t) \in Q_T$.

LEMMA 3.10. – For almost all $(x,t) \in Q_T$, we have $\tilde{\beta}(x,t) = \beta(u(x,t))$, $\tilde{K}(x,t) = K(u(x,t))$ and $\tilde{E}(x,t) = E(u(x,t))$, where the last equality holds in the sense of graph of the $E(u)$.

PROOF. – The equalities $\tilde{\beta}(x,t) = \beta(u(x,t))$ and $\tilde{K}(x,t) = K(u(x,t))$ can be demonstrated similarly to those of C. J. VAN DUYN [13], and XIAO SHUTIE et al. [14]. In fact, for any point (x_0,t_0) in $D \cup P$, the continuity of the function $\theta(x,t)$ implies that we have a neighborhood $N(x_0, t_0)$ such that

$$0 \leq \beta_\varepsilon(u_\varepsilon(x,t)) < \frac{1+\theta(x_0,t_0)}{2} < 1, \qquad (x,t) \in N(x_0,t_0).$$

It follows by the definition of $\beta_\varepsilon(u)$ and $\beta(u_s) = 1$ that

$$0 \leq u_\varepsilon(x,t) \leq \beta^{-1}\left(\frac{1+\theta(x_0,t_0)}{2}\right) < u_s.$$

Therefore, by Lemma 3.9, $u_\varepsilon(x,t)$ must be convergent to $u(x,t)$ uniformly as $\varepsilon \to$ in $N(x_0,t_0)$. This implies

$$\beta_\varepsilon(u_\varepsilon(x,t)) \to \beta(u(x,t))$$

and

$$K_\varepsilon(u_\varepsilon(x,t)) \to K(u(x,t)),$$

uniformly in $N(x_0, t_0)$ as $\varepsilon \to 0$. The proof of the equalities in region R is exactly same as those in [13]. Next we need to show that $\tilde{E}(x,t) = E(u(x,t))$. For any $(x_0,t_0) \in P \cup R$, we find $0 < \theta(x_0,t_0) \leq 1$. Since θ is continuous on $\overline{Q}_T$ we can choose a neighborhood $N(x_0,t_0) \subset Q_T$ and a constant c_0 such that $0 < c_0 \leq \theta(x,t) \leq 1$ for all $(x,t) \in N(x_0,t_0)$. Note that $\beta_\varepsilon(u_\varepsilon(x,t))$ converges to $\theta(x,t)$ uniformly on $\overline{Q}_T$, we have $0 < c_0/2 < \beta_\varepsilon(u_\varepsilon(x,t)) \leq 1$ when ε is small enough. Since $\beta_\varepsilon(u)$ converges to $\beta(u)$ uniformly on $[0, +\infty)$, one has a positive constant c^* independent of ε such that $u_\varepsilon(x,t) > c^* > 0$ on $N(x_0,t_0)$. It follows that $E_\varepsilon(u_\varepsilon(x,t)) = 0$ in $N(x_0,t)$ by the definition of $E_\varepsilon(u)$. Since (x_0,t_0) is arbitrary in $P \cup R$ we see that $E_\varepsilon(u_\varepsilon(x,t))$ converges to 0 as ε goes to 0 for a.e. $(x,t) \in P \cup R$. It implies that $\tilde{E}(x,t) = 0$ for a.e. $(x,t) \in P \cup R$. Note that $u(x,t) > 0$ for a.e. $(x,t) \in P \cup R$, we have that $E(u(x,t)) = 0$ for a.e. $(x,t) \in P \cup R$ by the definition of $E(u)$. Hence $\tilde{E}(x,t) = E(u(x,t)) = 0$ for a.e. $(x,t) \in P \cup R$. On the region D, $u(x,t) = 0$, for a.e. $(x,t) \in D$. Then $0 \leq E(u(x,t)) \leq 1$ on D in the sense of almost everywhere. Observe that $0 \leq \tilde{E}(x,t) \leq 1$. This implies that $\tilde{E}(x,t) = E(u)$ in the sense of the graph in D. Q.E.D.

Now we prove that $u(x,t)$ is a weak solution.
Let $T_0 = \sup\{t: (x,t) \in P \cup R\} > 0$.

Case 1: $T = T_0$. For any $0 < \tilde{T} < T_0$, $u_\varepsilon(1,t) > \varepsilon$ if $0 \leq t \leq \tilde{T} < T_0$ and ε is small enough. Therefore $H_\varepsilon(u_\varepsilon(1,t)) = 0$ when $0 \leq t \leq \tilde{T}$ and ε is sufficiently small. For any $\varphi(x,t) \in X$, $0 \leq t \leq \tilde{T}$, multiplying the equation (3.3) by φ and integrating it over $Q_{\tilde{T}}$ and letting $\varepsilon \to 0$, we find

$$\iint_{Q_{\tilde{T}}} [\beta(u)\varphi_t + u\varphi_{xx} - (g(x,t)\varphi)_x E(u)]\,dx\,dt + \int_0^1 \beta(u_0(x))\varphi(x,0)\,dx = 0.$$

Finally, let $\tilde{T} \to T_0$.

Case 2: $T_0 < T$,. Then $u(x, T_0) = 0$ and $u(x, t)$ satisfies the integral equality (2.1) on $[0, T_0]$.

Since T is arbitrary, the proof of Theorem 1 is done.

REMARK. – Once one has the equi-continuous estimate for $u_\varepsilon(x, t)$ one can establish the existence of a weak solution. Unfortunately, in general the weak solution $u(x, t)$ is not continuous in $\overline{Q}_T$ (cf. [13] and [14]).

COROLLARY 1. – The weak solution $u(x, t)$ satisfies equation (1.1) in the classical sense in the region P.

PROOF. – In fact, for any $(x_0, t_0) \in P$, the set $P_r = \{(x, t): (x, t) \in \overline{Q}_T,\ |x - x_0| + |t - t_0| < r\} \subset P$ if r is small enough. By virtue of Lemma 3.6, we know that $u_\varepsilon(x, t)$ converges uniformly on the set P_r. By the hypotheses H(1), H(2) and the standard regularity theory of weak solution of nondegenerate parabolic equations, we immediately obtain the result.

COROLLARY 2. – The weak solution $u(x, t)$ has the following properties: $\theta(x, t) = \beta(u(x, t))$ is increasing monotonically with respect to x and decreasing monotonically with respect to t.

PROOF. – This result follows directly from Lemma 3.2 and Lemma 3.3. Q.E.D.

4. – The proof of the Theorem 2.

In this section, we shall study additional properties of the weak solution. The sets D, P and R correspond to the dry, unsaturated and saturated regions, respectively. Of course, we are interested in the shapes of D, P and R. We also ask how much smoothness the interfaces possess. For each $t \in [0, T]$, define

$$s_1(t) = \sup\{x \in [0, 1]: \theta(x', t) = 0,\ \text{if } 0 < x' < x,\ (x, t) \in \overline{Q}_T\},$$

$$s_2(t) = \sup\{x \in [0, 1]: \theta(x, t) < 1 \ \text{for}\ (x, t) \in \overline{Q}_T\}.$$

Since $\theta(x, t)$ is increasing monotonically for each fixed t and decreasing monotonically for each fixed x, we know that $s_i(t)$ $(i = 1, 2)$ is well-defined and $s_i(t_1) < s_i(t_2)$ if $t_1 < t_2$ $(i = 1, 2)$. Moreover, $s_1(t) < s_2(t)$ on $[0, T]$. We assert that the curve $x = s_1(t)$ is continuous in the $[0, T^*)$.

In fact, if this is not true, then there exists $t_0 \in (0, T^*)$ such that $s_1(t_0 - 0) < s_1(t_0 + 0)$ since $s_1(t)$ is monotone increasing. Therefore, we can take a small rectangle contained in P with top side $\overline{s_1(t_0 - 0)\, s_1(t_0 + 0)}$. Now $u(x, t)$ is strictly positive in that small open rectangle and satisfies equation (3.3) by Corollary

3.1. But $u(x, t) = 0$ on the top side of that rectangle, this contradicts the maximum principle.

From the definition, we know that

$$D = \{(x, t): 0 \leqslant x \leqslant s_1(t),\ 0 < t < T^*\},$$
$$P = \{(x, t): s_1(t) < x < s_2(t),\ 0 < t < T^{**}\} \cup \{s_1(t) < x < 1,\ T^{**} < t < T^*\},$$

and

$$R = \{(x, t): s_2(t) < x < 1,\ 0 < t < T^{**}\},$$

where $0 < T^{**} < T^* \leqslant +\infty$, $s_2(T^{**}) = 1$ if $T^* < +\infty$ and $s_2(t) \equiv 1$ for $t \in [T^{**}, T^*)$.

REMARKS. – The uniqueness of the problem (1.1)-(1.5) is an open question. It would be interesting to know more about the regularity of the interfaces $s_1(t)$ and $s_2(t)$.

Acknowledgments. – The author wishes to thank Professor C. J. VAN DUYN, Professor S. XIAO and Professor LI-SHANG JIANG for the discussion of the physical and mathematical model. The author is also indebted to Professor JOHN R. CANNON for the careful reading of this manuscript and his many helpful suggestions.

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