Annali di Matematica pura ed applicata
(IV), Vol. CLXI (1992), pp. 315-343

On a Hypergeometric Function of Four Variables with a New Aspect of *SL*-Symmetry (*).

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Summary. – *The integration of the system of partial differential equations associated with the quadruple hypergeometric function K_{16}, first introduced by the author (Exton* (1976), *page* 78), *is undertaken. The interest in this function has been stimulated by the investigation of certain SL-symmetry groups by Hrabowski* (1984), (1985).

1. – Introduction.

In a recent investigation of certain multiple hypergeometric functions of the second order with SL-symmetry from the point of view of parabolic subgroups of a simple Lie algebra, Hrabowski (1984), (1985) has been led to consider, using different notation, the function

$$(1.1)\qquad K_{16}(a_1, a_2, a_3, a_4; b; x, y, z, t) = \sum_{l,m,n,p=0}^{\infty} \frac{(a_1, l+m)(a_2, l+n)(a_3, m+p)(a_4, n+p)}{(b, l+m+n+p)} \frac{x^l\ y^m\ z^n\ t^p}{l!\ m!\ n!\ p!},$$

where

$$(1.2)\qquad (a, n) = a(a+1)(a+2)\ldots(a+n) = \Gamma(a+n)/\Gamma(a), \qquad (a, 0) = 1.$$

This function was originally introduced by Exton (1976) page 78, as part of a systematic study of a certain class of quadruple hypergeometric functions of the second order. Karlsson (1976) determined its region of convergence, namely that where the absolute value of the four variables x, y, z and t should each be less than unity.

In order to highlight certain aspects of the symmetry involved, we modify the no-

(*) Entrato in Redazione il 27 febbraio 1989.

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tation slightly and write, instead of the function K_{16}, the following function:

(1.3) $K^*(a_1, a_2, a_3, a_4; b; x_1, x_2, x_3, x_4) =$

$$= \sum_{m_1,m_2,m_3,m_4=0}^{\infty} \frac{(a_1, m_1+m_2)(a_2, m_2+m_3)(a_3, m_3+m_4)(a_4, m_4+m_1)}{(b, m_1+m_2+m_3+m_4)} \frac{x_1^{m_1} \; x_2^{m_2} \; x_3^{m_3} \; x_4^{m_4}}{m_1! \; m_2! \; m_3! \; m_4!},$$

This function K^* satisfies the partial differential system

$$(1.4)\quad \begin{cases} x_1(1-x_1)\dfrac{\partial^2 F}{\partial x_1^2} + x_2(1-x_1)\dfrac{\partial^2 F}{\partial x_1 \partial x_2} + x_4(1-x_1)\dfrac{\partial^2 F}{\partial x_1 \partial x_4} + x_3\dfrac{\partial^2 F}{\partial x_1 \partial x_3} - \\ -x_2 x_4 \dfrac{\partial^2 F}{\partial x_2 \partial x_4} + [b-(a_1+a_4+1)x_1]\dfrac{\partial F}{\partial x_1} - a_4 x_2 \dfrac{\partial F}{\partial x_2} - a_1 x_4 \dfrac{\partial F}{\partial x_4} - a_1 a_4 F = 0, \\[2ex] x_2(1-x_2)\dfrac{\partial^2 F}{\partial x_2^2} + x_1(1-x_2)\dfrac{\partial^2 F}{\partial x_2 \partial x_1} + x_3(1-x_2)\dfrac{\partial^2 F}{\partial x_2 \partial x_3} + x_4\dfrac{\partial^2 F}{\partial x_2 \partial x_4} - \\ -x_1 x_3 \dfrac{\partial^2 F}{\partial x_1 \partial x_3} + [b-(a_2+a_1+1)x_2]\dfrac{\partial F}{\partial x_2} - a_1 x_3 \dfrac{\partial F}{\partial x_3} - a_2 x_1 \dfrac{\partial F}{\partial x_1} - a_2 a_1 F = 0, \\[2ex] x_3(1-x_3)\dfrac{\partial^2 F}{\partial x_3^2} + x_2(1-x_3)\dfrac{\partial^2 F}{\partial x_3 \partial x_2} + x_4(1-x_3)\dfrac{\partial^2 F}{\partial x_3 \partial x_4} + x_1\dfrac{\partial^2 F}{\partial x_3 \partial x_1} - \\ -x_2 x_4 \dfrac{\partial^2 F}{\partial x_2 \partial x_4} + [b-(a_3+a_2+1)x_3]\dfrac{\partial F}{\partial x_3} - a_2 x_4 \dfrac{\partial F}{\partial x_4} - a_3 x_2 \dfrac{\partial F}{\partial x_2} - a_3 a_2 F = 0 \\ \text{and} \\ x_4(1-x_4)\dfrac{\partial^2 F}{\partial x_4^2} + x_3(1-x_4)\dfrac{\partial^2 F}{\partial x_4 \partial x_3} + x_1(1-x_4)\dfrac{\partial^2 F}{\partial x_4 \partial x_1} + x_2\dfrac{\partial^2 F}{\partial x_3 \partial x_2} - \\ -x_1 x_3 \dfrac{\partial^2 F}{\partial x_1 \partial x_3} + [b-(a_4+a_3+1)x_4]\dfrac{\partial F}{\partial x_4} - a_3 x_1 \dfrac{\partial F}{\partial x_1} - a_4 x_3 \dfrac{\partial F}{\partial x_3} - a_4 a_3 F = 0. \end{cases}$$

As was pointed out by Hrabowski (1984), the general integral of this system depends linearly upon six arbitrary constants and that the Laplace equation of four dimensions is embedded in (1.4).

The purpose of this study is to obtain a sufficient number of independent solutions of the system (1.4) to furnish its general integral with respect to all of its singularities.

2. – The singularities of the system (1.4).

In order to discuss the nature of the singular points of (1.4), we note that the function $K^*(a_1, a_2, a_3, a_4; b; x_1, x_2, x_3, x_4)$ may be written firstly as a double series of Appell functions F_1, namely,

$$\sum_{m_1, m_2=0}^{\infty} \frac{(a_1, m_1+m_2)(a_4, m_1)(a_2, m_2)\, x_1^{m_1} \;\; x_2^{m_2}}{(b, m_1+m_2) \qquad\qquad m_1! \;\; m_2!} \cdot \tag{2.1}$$

$$\cdot F_1(a_3, a_2+m_2, a_4+m_1; b+m_1+m_2; x_3, x_4).$$

Secondly, we have the double series of Appell functions F_3:

$$\sum_{m_1, m_3=0}^{\infty} \frac{(a_1, m_1)(a_2, m_3)(a_4, m_1)(a_3, m_3)\, x_1^{m_1} \;\; x_3^{m_3}}{(b, m_1+m_3) \qquad\qquad m_1! \;\; m_3!} \cdot \tag{2.2}$$

$$\cdot F_3(a_1+m_1, a_3+m_3, a_2+m_3, a_4+m_1; b+m_1+m_3; x_2, x_4).$$

It is evident from the known properties of the Appell functions, that the singular manifolds of the system of partial differential equations under consideration include

$$x_i = 0, 1, \infty, \qquad \text{for } i = 1 \text{ to } 4 \tag{2.3}$$

together with

$$x_1 = x_2, \;\; x_2 = x_3, \;\; x_3 = x_4, \;\; x_4 = x_1, \tag{2.4}$$

$$1/x_1 + 1/x_3 = 1, \qquad 1/x_2 + 1/x_4 = 1, \tag{2.5}$$

and

$$(1-x_1)(1-x_3) = (1-x_2)(1-x_4). \tag{2.6}$$

A contact with a singular manifold (2.5) or (2.6) will be referred to as being of the first or second type respectively.

The singular points of the system (1.4) arise from the intersection and/or contact of two or more of the associated singular manifolds. We now consider the singularities in progressive degrees of complexity.

The simplest type of singularity arises from the intersection of four singular manifolds only, and consist of the points $(1, \infty, 1, \infty)$, $(0, \infty, 0, \infty)$, $(0, 1, 0, 1)$, $(1, 0, 1, \infty)$, $(0, \infty, 1, \infty)$ and those points obtainable from these by the cyclic permutation of the subscripts, which process will be tacitly assumed below to be applied as appropriate.

The singularity $(0, 0, 1, \infty)$ occurs at the intersection of five singular manifolds, while the singularity $(0, 1, 0, \infty)$ is formed from the intersection of four singular manifolds and a contact of the first type. The intersection of six singular manifolds gives rise to the singularities of the type $(0, 0, 0, \infty)$, $(0, \infty, \infty, \infty)$ and $(0, 0, 0, 1)$.

In the case of a contact of the second type with the intersection of five singular manifolds, we obtain $(1, 1, 0, \infty)$ and $(0, 1, \infty, \infty)$. Next in order of complexity, we have those singularities which result from the intersection of six singular manifolds with a contact of the first type, namely $(1, 1, 1, \infty)$ and $(1, \infty, \infty, \infty)$. The intersection of six singular manifolds and a contact of the second type generates the singularities

$(0,0,1,1)$, $(1,1,1,0)$ and $(0,0,\infty,\infty)$. One type of singularity arises from the intersection of six singular manifolds together with a double contact of the first type and a contact of the second type. This is $(1,1,\infty,\infty)$.

Finally, the most complicated class of singularities consists of $(0,0,0,0)$, $(1,1,1,1)$ and $(\infty,\infty,\infty,\infty)$, formed from the intersection of eight singular manifolds and a contact of the second type. In accordance with the general nature of the results expected, complete fundamental systems of solutions in which every member is convergent in the whole neighbourhood of any given singularity is only possible when that singularity is formed from the intersection of no more than four singular manifolds without any contacts. For the system under consideration, this applies only for the singular points $(1,\infty,1,\infty)$, $(0,\infty,0,\infty)$, $(0,1,0,1)$, $(1,0,1,\infty)$ and $(0,\infty,1,\infty)$.

3. - Elementary symmetries and an Euler transformation.

By inspection of the series representation of the function K^*, we see that the system (1.4) remains unchanged by the cyclic permutation of the subscripts. For example

$$K^*(a_1,a_2,a_3,a_4;b;x_1,x_2,x_3,x_4)=K^*(a_4,a_1,a_2,a_3;b;x_4,x_1,x_2,x_3),\quad \text{etc.} \tag{3.1}$$

Similarly, it is evident that

$$K^*(a_1,a_2,a_3,a_4;b;x_1,x_2,x_3,x_4)=K^*(a_4,a_3,a_2,a_1;b;x_1,x_4,x_3,x_2). \tag{3.2}$$

We now deduce a transformation of Euler type applicable to the function K^*. Within its region of convergence, we recall from (2.1) that the function K^* may be expanded as a double series of the Appell functions F_1 and the inner Appell function may be re-written in the form

$$(1-x_3)^{-a_3}F_1\left(a_3,b-a_2-a_4,a_4+m_1;b+m_1+m_2;\frac{x_3}{x_3-1},\frac{x_4-x_3}{1-x_3}\right), \tag{3.3}$$

so that the right-hand member of (2.1) becomes

$$(1-x_3)^{-a_3}\sum_{m_1,m_2,m_3,m_4=0}^{\infty}\cdot \tag{3.4}$$

$$\cdot\frac{(a_1,m_1+m_2)(a_3,m_3+m_4)(a_4,m_1+m_4)(a_2,m_2)(b-a_2-a_4,m_3)}{(b,m_1+m_2+m_3+m_4)\,m_1!\,m_2!\,m_3!\,m_4!}\cdot$$

$$\cdot x_1^{m_1}x_2^{m_2}\left(\frac{x_3}{x_3-1}\right)^{m_3}\left(\frac{x_4-x_3}{1-x_3}\right)^{m_4}=$$

$$=(1-x_3)^{-a_3}K_{18}\left(a_4,a_3,a_1,a_2,b-a_2-a_4;b;\frac{x_4-x_3}{1-x_3},x_1,x_2,\frac{x_3}{x_3-1}\right),$$

where the quadruple hypergeometric function K_{18} is listed in Exton (1976), page 79. This last expression may be expanded in a second way as a double series of functions F_1, namely

$$(3.5)\qquad (1-x_3)^{-a_3}\sum_{m_3,m_4=0}^{\infty}\frac{(a_3,m_3+m_4)(b-a_2-a_4,m_3)(a_4,m_4)}{(b,m_3+m_4)\,m_3!\,m_4!}\cdot \left(\frac{x_3}{x_3-1}\right)^{m_3}\left(\frac{x_4-x_3}{1-x_3}\right)^{m_4}F_1(a_1,a_4+m_4,a_2;b+m_3+m_4;x_1,x_2).$$

This last Appell function is transformed in a similar fashion to the preceding, so that, after re-arranging the subscripts, we have the required expression which is

$$(3.6)\qquad K^*(a_1,a_2,a_3,a_4;b;x_1,x_2,x_3,x_4)=(1-x_1)^{-a_4}(1-x_2)^{-a_2}K^*\cdot \left(b-a_1-a_3,a_2,a_3,a_4;b;\frac{x_1}{x_1-1},\frac{x_2}{x_2-1},\frac{x_3-x_2}{1-x_2},\frac{x_4-x_1}{1-x_1}\right).$$

This Euler transformation has been obtained previously with different notation. See Karlsson (1976), page 34.

4. – A Pochhammer integral formula.

Consider the integral

$$(4.1)\qquad I=\int_C(-u)^{a_4-1}(u-1)^{a_2-1}F_1(q,a_1,a_3;b;x_1u+x_2(1-u),x_4u+x_3(1-u))\,du,$$

where the contour of integration consists of a Pochhammer double loop slung around the points 0 and 1. Since the series representation of the Appell function of the integrand converges uniformly on the contour for sufficiently small values of the variables x_1, x_2, x_3 and x_4, we may write

$$(4.2)\qquad I=\sum_{m_1,m_2,m_3,m_4=0}^{\infty}\cdot \frac{(q,m_1+m_2+m_3+m_4)(a_1,m_1+m_2)(a_3,m_3+m_4)}{(b,m_1+m_2+m_3+m_4)}\frac{(-x_1)^{m_1}(-x_2)^{m_2}(-x_3)^{m_3}(-x_4)^{m_4}}{m_1!\quad m_2!\quad m_3!\quad m_4!}\cdot \int_C(-u)^{a_4+m_1+m_4-1}(u-1)^{a_2+m_2+m_3-1}du.$$

The inner integral may be evaluated as

$$(4.3)\qquad (2\pi i)^2/[\Gamma(1-a_4-m_1-m_4)\Gamma(1-a_2-m_2-m_3)\Gamma(a_2+a_4+m_1+m_2+m_3+m_4)].$$

See Whittaker and Watson (1952), page 256. Put $q = a_2 + a_4$, when it is clear that

$$(4.4)\quad K^*(a_1, a_2, a_3, a_4; b; x_1, x_2, x_3, x_4) = \Gamma(1-a_2)\Gamma(1-a_4)\Gamma(a_2+a_4)(2\pi i)^{-2}\cdot$$

$$\cdot\int_C (-u)^{a_4-1}(u-1)^{a_2-1} F_1(a_2+a_4, a_1, a_3; b; x_1 u + x_2(1-u), x_4 u + x_3(1-u))\, du\,.$$

If the Appell function in the integrand of (4.4) is replaced by any solution of the system of partial differential equations associated with the function $F_1(a_2 + a_4, a_1, a_3; b; x_1 u + x_2(1-u), x_4 u + x_3(1-u))$, then the integral taken over any contour closed on the Riemann surface of the integrand will be a solution of the partial differential system under consideration, (1.4). Compare Erdélyi (1939) and Exton (1976), page 174, for example. Since the integral (4.4) does not separate all four of the independent variables of (1.4), it would not be of direct use in deducing the complete solution of the system under dscussion. However, the system of partial differential equations associated with the function F_1 has been extensively studied (Erdélyi (1950) and Olsson (1964), for example), so that a considerable amount of information on the solutions of (1.4) is obtainable from (4.4). This, together with the properties of these solutions which will emerge below, enables complete sets of independent integrals of (1.4) relative to all of its singularities to be deduced. The fact that certain of these sets of integrals are only valid for partial neighbourhoods of the associated singular points results in the existence of boundaries between such regions which remain intractable.

5. – Solution of (1.4) of the form K^*.

In what follows, we employ the list of solutions of the partial differential system associated with F_1 as given by Olsson (1964). This is preferred to le Vavasseur's list as given on page 62 of Appell et Kampé de Fériet (1926) because the latter does not include any of the solutions expressed as the Horn function G_2.

Take the solution (19) of Olsson's list of ten distinct F_1 solutions namely,

$$(5.1)\quad F_1(a_2+a_4, a_1, a_3; 1+a_1+a_2+a_3+a_4-b;$$
$$1 - x_1 u - x_2(1-u), 1 - x_4 u - x_3(1-u))$$

and insert this into the integral on the right of (4.4). If we note that

$$(5.2)\qquad 1 - x_1 u - x_2(1-u) = (1-x_1)u + (1-x_2)(1-u), \quad \text{etc.},$$

it is immediately evident that the function

$$(5.3)\quad K^*(a_1, a_2, a_3, a_4; 1+a_1+a_2+a_3+a_4-b; 1-x_1, 1-x_2, 1-x_3, 1-x_4)$$

is a solution of the system (1.4).

Now replace the Appell function in (4.4) by

$$(5.4)\qquad [x_1 u + x_2(1-u)]^{-a_1}[x_4 u + x_3(1-u)]^{-a_3}\cdot$$

$$\cdot F_1\left(1+a_1+a_3-b, a_1, a_3; 1+a_1+a_3-a_2-a_4; \frac{1}{x_1 u + x_2(1-u)}, \frac{1}{x_4 u + x_3(1-u)}\right).$$

We then have

$$(5.5)\qquad \int_C (-u)^{a_4-1}(u-1)^{a_2-1}[x_1 u + x_2(1-u)]^{-a_1}[x_4 u + x_3(1-u)]^{-a_3}\cdot$$

$$\cdot F_1\left(1+a_1+a_3-b, a_1, a_3; 1+a_1+a_3-a_2-a_4; \frac{1}{x_1 u + x_2(1-u)}, \frac{1}{x_4 u + x_3(1-u)}\right) du.$$

The contour of integration is so selected that the following double series is convergent upon it:

$$(5.6)\qquad \sum_{m,n=0}^{\infty} \frac{(1+a_1+a_3-b, m+n)(a_1, m)(a_3, n)}{(1+a_1+a_3-a_2-a_4, m+n)\, m!\, n!}\cdot$$

$$\cdot\int_C (-u)^{a_4-1}(u-1)^{a_2-1}[x_1 u + x_2(1-u)]^{-a_1-m}[x_4 u + x_3(1-u)]^{-a_3-n}\, du.$$

If both of the binomial factors of the integrand are expanded in powers of (1-u)/u, then we obtain the result

$$(5.7)\qquad x_1^{-a_1} x_4^{-a_4} K^*(a_1, a_2, a_3, 1+a_1+a_3-b; 1+a_1+a_3-a_4; 1/x_1, x_2/x_1, x_3/x_4, 1/x_4).$$

No further fundamental forms of the solutions of (1.4) which may be directly represented in terms of the function K^* have been found.

6. – Other solutions of (1.4) directly obtainable from the integral (4.4).

Two further quadruple hypergeometric functions of five parameters arise in the following analysis. These are now introduced.

$$(6.1)\qquad L^*(a, b, c; d, e; x_1, x_2, x_3, x_4) =$$

$$= \sum_{m_1,m_2,m_3,m_4=0}^{\infty} \frac{(a, m_1+m_2)(b, m_1+m_3)(c, m_3+m_4)\, x_1^{m_1}\, x_2^{m_2}\, x_3^{m_3}\, x_4^{m_4}}{(d, m_1+m_2-m_4)(e, m_3+m_4-m_2)\ m_1!\ m_2!\ m_3!\ m_4!},$$

and

(6.2) $M^*(a, b, c; d, e; x_1, x_2, x_3, x_4) =$

$$= \sum_{m_1,m_2,m_3,m_4=0}^{\infty} \frac{(a, m_1+m_2+m_3-m_4)(b, m_1+m_2)(c, m_2+m_3)}{(d, m_1+m_2-m_4)(e, m_2+m_3-m_4)} \frac{x_1^{m_1}\ x_2^{m_2}\ x_3^{m_3}\ x_4^{m_4}}{m_1!\ m_2!\ m_3!\ m_4!},$$

By means of the methods discussed by Karlsson (1976), it may easily be shown that these two series converge for $|x_i| < 1$, $i = 1, 2, 3, 4$.

If the second binomial factor of (5.6) is expanded in powers of $u/(1-u)$, then we obtain a solution of (1.4) in the form

(6.3) $x_1^{-a_1} x_1^{-a_3} L^* \cdot$

$$\cdot (a_1, 1+a_1+a_3-b, a_3; 1+a_1-a_4, 1+a_3-a_2; 1/x_1, -x_2/x_1, 1/x_3, -x_4/x_3).$$

Similarly, by using Olsson's F_1 solution (28), we may deduce the solutions

(6.4) $x_1^{-a_1} x_4^{a_1-b+1} L^* \cdot$

$$\cdot (1+a_1+a_3-b, a_1, a_2; a_1-b+2, a_4-b; x_4/x_1, -x_4, x_2/x_1, -x_3/x_4)$$

and

(6.5) $x_1^{-a_1} x_3^{a_1-b+1} M^* \cdot$

$$\cdot (1+a_2+a_4-b, a_1, 1+a_1+a_3-b; 1+a_1-a_4, a_1-b+2; x_2/x_1, -x_3/x_1, x_3, -x_4/x_3).$$

Also, Olsson's G_2 solutions (17) and (29) give respectively

(6.6) $$x_1^{-a_1} L^*(a_1, a_2, a_3; 1+a_1-a_4, b-a_1; x_2/x_1, 1/x_1, -x_3, x_4)$$

and

(6.7) $$x_1^{1-b} M^*(b-1, a_3, a_2; b-a_1, 1+b-a_4; x_4/x_1, -x_3/x_1, x_2/x_1, x_1).$$

Many other types of solutions of the system under consideration, involving a wide variety of quadruple hypergeometric functions, may be obtained in a similar fashion. The above examples are sufficient, however, to provide a basis for the complete integration of (1.4) relative to all its singular points as will be indicated below.

7. – Corresponding solutions.

By considering the solutions of (1.4) which may be expressed directly in terms of the function K^*, it is possible to deduce further solutions as follows:

If $Z(a_1, a_2, a_3, a_4; b; x_1, x_2, x_3, x_4)$ is a solution of (1.4), then by (5.3), so also is

(7.1) $$Z(a_1, a_2, a_3, a_4; 1+a_1+a_2+a_3+a_4-b; 1-x_1, 1-x_2, 1-x_3, 1-x_4).$$

Similarly, by appealing to (5.7),

(7.2) $x_1^{-a_1} x_4^{-a_4} Z\cdot$

$$\cdot(a_1, a_2, a_3, 1+a_1+a_3-b; 1+a_1+a_3-a_4; 1/x_1, x_2/x_1, x_3/x_4, 1/x_4)$$

is a solution of the same system. Furthermore, by (3.1), cyclic permutation of the subscripts leaves (1.4) unchanged and (3.2) gives the solution

(7.3) $$Z(a_4, a_3, a_2, a_1; b; x_1, x_4, x_3, x_2).$$

By means of the Euler transformation (3.7), a further type of solution is seen to be

(7.4) $(1-x_1)^{-a_4}(1-a_2)^{-a_2}Z\cdot$

$$\cdot\left(b-a_1-a_3, a_2, a_3, a_4; b; \frac{x_1}{x_1-1}, \frac{x_2}{x_2-1}, \frac{x_3-x_2}{1-x_2}, \frac{x_4-x_1}{1-x_1}\right).$$

All of the above correspondances may be applied repeatedly or in any successive combination. Olsson (1977) has used this approach in his investigation of the partial differential system associated with the Appell function F_2. If we write (5.7) as

(7.5) $x_1^{-a_1} x_4^{-a_4} K^*(1+a_1+a_3-b, a_1, a_2, a_3; 1+a_1+a_3-a_4; 1/x_4, 1/x_1, x_2/x_1, x_3/x_4)$

and combine (7.5) with (5.7) itself, we obtain the important solution

(7.6) $x_1^{-a_1} x_3^{-a_2} x_4^{1+a_1+a_2-b} K^*\cdot$

$$\cdot\left(1+a_1+a_3-b, a_1, a_2, 1+a_2+a_4-b; 2+a_1+a_2-b; x_4, \frac{x_4}{x_1}, \frac{x_2x_4}{x_1x_3}, \frac{x_4}{x_3}\right).$$

Hence,

(7.7) $x_1^{-a_1} x_3^{-a_2} x_4^{1+a_1+a_2-b}\cdot$

$$\cdot Z\left(1+a_1+a_3-b, a_1, a_2, 1+a_2+a_4-b; 2+a_1+a_2; x_4, \frac{x_4}{x_1}, \frac{x_2x_4}{x_1x_3}, \frac{x_4}{x_3}\right)$$

is a solution of (1.4).

The fundamental sets of solutions are listed in the following appendix and the manner in which each individual solution is obtained is briefly indicated as appropriate.

8. – Appendix. Fundamental sets of solutions of (1.4).

A fundamental set of six linearly independent integrals of the system (1.4) is given relative to each singularty. Any singular point or region not covered below may be dealt with by appropriately interchanging the subscripts.

8.1. THE SINGULARITY (0, 0, 0, 0).

(8.1.1) $K^*(a_1, a_2, a_3, a_4; b; x_1, x_2, x_3, x_4)$,

(8.1.2) $x_1^{-a_1} x_3^{-a_2} x_4^{1+a_1+a_2-b} K^* \cdot$

$$\cdot\left(1+a_1+a_3-b, a_1, a_2, 1+a_2+a_4-b; 2+a_1+a_2-b; x_4, \frac{x_4}{x_1}, \frac{x_2 x_4}{x_1 x_3}, \frac{x_4}{x_3}\right), \quad \text{see (7.6)},$$

(8.1.3) $x_1^{-a_1} x_3^{-a_1-b+1} M^* \cdot$

$$\cdot\left(1+a_2+a_4-b, a_1, 1+a_1+a_3-b; 1+a_1-a_4, a_1-b+2; \frac{x_2}{x_1}, -\frac{x_3}{x_4}, x_3, -\frac{x_4}{x_3}\right), \quad \text{see (5.5)},$$

(8.1.4) $x_1^{1-b} M^*\left(b-1, a_3, a_2; b-a_1, 1+b-a_4; \frac{x_4}{x_1}, -\frac{x_3}{x_1}, \frac{x_2}{x_1}, x_1\right)$, see (6.7),

(8.1.5) $x_1^{-a_4} x_3^{-a_4-b+1} M^* \cdot$

$$\cdot\left(1+a_1+a_3-b, a_4, 1+a_2+a_4; 1+a_4-a_1, a_4-b+2; \frac{x_4}{x_1}, -\frac{x_3}{x_1}, x_3, -\frac{x_2}{x_3}\right)$$

and

(8.1.6) $x_1^{1-b} M^*\left(b-1, a_2, a_3; b-a_4, 1+b-a_1; \frac{x_2}{x_1} - \frac{x_3}{x_1}, x_3, -\frac{x_2}{x_3}\right)$.

The last two solutions are obtained respectively from (8.1.3) and (8.1.4) by interchanging the subscripts.

This set of solutions is valid in that portion of the neighbourhood of the origin for which

$$|x_1| > |x_3| > |x_2| > |x_4|.$$

8.2. THE SINGULARITY (1, 1, 1, 1).

Apply (7.1) to all the members of 8.1.

(8.2.1) $K^*(a_1, a_2, a_3, a_4; 1+a_1+a_2+a_3+a_4-b; 1-x_1, 1-x_2, 1-x_3, 1-x_4)$,

(8.2.2) $(1-x_1)^{-a_1}(1-x_3)^{-a_2}(1-x_4)^{b-a_3-a_4} \cdot K^* \cdot$

$$\cdot\left(b-a_2-a_4, a_1, a_2, b-a_1-a_3; b-a_3-a_4+1; 1-x_4, \frac{1-x_4}{1-x_1} \frac{(1-x_2)(1-x_4)}{(1-x_1)(1-x_3)}, \frac{1-x_4}{1-x_3}\right),$$

(8.2.3) $(1-x_1)^{-a_1}(1-x_3)^{b-a_2-a_3-a_4} \cdot M^* \cdot$

$$\cdot\left(b-a_2-a_3, a_1, b-a_2-a_4; 1+a_1-a_4, b-a_2-a_3-a_4+1; \frac{1-x_2}{1-x_1} \frac{x_3-1}{1-x_1}, 1-x_3, \frac{x_4-1}{1-x_3}\right),$$

$$(8.2.4)\quad (1-x_1)^{b-a_1-a_2-a_3-a_4}\cdot M^*\Big(a_1+a_2+a_3+a_4-b, a_3, a_2; \; 1+a_2+a_3+a_4-b, 2+a_1+a_2+a_3-b; \frac{1-x_4}{1-x_1}, \frac{x_3-1}{1-x_1}, \frac{1-x_2}{1-x_1}, 1-x_1\Big),$$

$$(8.2.5)\quad (1-x_1)^{-a_4}(1-x_3)^{b-a_1-a_2-a_3}M^*\Big(b-a_2-a_4, a_4, b-a_1-a_3; \; 1+a_4-a_1, b-a_1-a_2-a_3+1; \frac{1-x_4}{1-x_1}, \frac{x_3-1}{1-x_1}, 1-x_3, \frac{x_2-1}{1-x_3}\Big)$$

and

$$(8.2.6)\quad (1-x_1)^{b-a_1-a_2-a_3-a_4}M^*\Big(a_1+a_2+a_3+a_4-b, a_2, a_3; \; 1+a_1+a_2+a_3-b, 2+a_2+a_3+a_4-b; \frac{1-x_2}{1-x_1}, \frac{x_3-1}{1-x_1}, \frac{1-x_4}{1-x_1}, 1-x_1\Big).$$

This set of solutions is valid in that part of the neighbourhood of the point $(1,1,1,1)$ for which

$$|1-x_1| > |1-x_2| > |1-x_3| > |1-x_4|.$$

8.3. THE SINGULARITY $(\infty, \infty, \infty, \infty)$.

Apply (7.1) to (5.7) and (6.6) after appropriate inchanges of the subscripts. This gives the following set of solutions:

$$(8.3.1)\quad x_2^{-a_2}x_4^{-a_4}L^*\cdot$$
$$\cdot(a_4, 1+a_2+a_4-b, a_2; 1+a_4-a_3, 1+a_2-a_1; 1/x_4, -x_1/x_4, 1/x_2, -x_3/x_2),$$

$$(8.3.2)\quad x_2^{-a_1}x_4^{-a_3}L^*\cdot$$
$$\cdot(a_1, 1+a_1+a_3-b, a_3; 1+a_1-a_2, 1+a_3-a_4; 1/x_2, -x_1/x_2, 1/x_4, -x_1/x_4),$$

$$(8.3.3)\quad x_1^{a_2-a_1}x_2^{-a_2}x_4^{-a_3}L^*\Big(a_2a_3, 1+a_1+a_3-b; \; 1+a_2-a_4, 1+a_1+a_3-a_2-a_4; \frac{x_1x_3}{x_2x_4}, \frac{x_1}{x_2}, -\frac{1}{x_4}, \frac{1}{x_1}\Big),$$

(8.3.4) $(1-x_2)^{-a_2}(1-x_4)^{-a_4}L^*\cdot$

$$\cdot\left(a_4, b-a_1-a_3, a_2; 1+a_4-a_3, 1+a_2-a_1; \frac{1}{1-x_4}, \frac{x_1-1}{1-x_4}, \frac{1}{1-x_2}, \frac{x_3-1}{1-x_2}\right),$$

(8.3.5) $(1-x_2)^{-a_1}(1-x_4)^{-a_3}L^*\cdot$

$$\cdot\left(a_1, b-a_2-a_4, a_3; 1+a_1-a_2, 1+a_3-a_4; \frac{1}{1-x_2}, \frac{x_1-1}{1-x_2}, \frac{1}{1-x_4}, \frac{x_3-1}{1-x_4}\right),$$

and

(8.3.6) $(1-x_1)^{a_2-a_1}(1-x_2)^{-a_2}(1-x_4)^{-a_3}\propto L^*\Big(a_2, a_3, b-a_2-a_4;$

$$1+a_2-a_4, 1+a_1+a_3-a_2-a_4; \frac{(1-x_1)(1-x_3)}{(1-x_2)(1-x_4)}, \frac{1-x_1}{1-x_2}, \frac{1}{x_4-1}, \frac{1}{1-x_1}\right).$$

This set of solutions is valid in that part of the neighbourhood of the point $(\infty, \infty, \infty, \infty)$ for which

$$|x_4|>|x_1|, \quad |x_2|>|x_3|, \quad |x_2|>|x_1|, \quad |x_2x_4|>|x_1x_3|, \quad |1-x_4|>|1-x_1|,$$
$$|1-x_2|>|1-x_3|, \quad |1-x_2|>|1-x_1| \quad \text{and} \quad |(1-x_2)(1-x_4)|>|(1-x_1)(1-x_3)|.$$

8.4. THE SINGULARITY $(0, 0, 0, \infty)$.

From (6.6), we have

(8.4.1) $$x_4^{-a_3}L^*(a_1, a_2, a_3; b-a_3, 1+a_3-a_4; x_2, -x_1, x_3/x_4, -1/x_4),$$

and

(8.4.2) $$x_4^{-a_4}L^*(a_2, a_1, a_4; b-a_4, 1+a_4-a_3; x_2, -x_3, x_1/x_4, -1/x_4),$$

by the use of (7.2) and (7.3) and re-labelling.
Similarly, from (6.4), we have

(8.4.3) $x_1^{1+a_3-b}x_4^{-a_3}L^*\cdot$

$$\cdot(a_2, a_3, a_1+a_3-b+1; b-a_3, a_4-b+2; x_3/x_4, -x_2/x_1, x_1/x_4, x_1)$$

and

(8.4.4) $x_3^{1+a_4-b}x_4^{-a_4}L^*\cdot$

$$\cdot(a_1, a_4, a_2+a_4-b+1; b-a_3, a_4-b+2; x_1/x_4, -x_2/x_3, x_3/x_4, x_3).$$

Next, apply (7.2) to (8.4.1) and (8.4.3) and obtain respectively

(8.4.5) $$x_2^{a_3-b+1}x_2^{a_4-a_3}x_4^{-a_4}L^*(a_1+a_3-b+1, a_2+a_4-b+1, a_4; a_3-b+1, a_4-a_3+1; x_2, -x_2/x_3, -x_3/x_4, -(x_1x_3)/(x_2x_4))$$

and

(8.4.6) $$x_1^{a_3-a_4}x_2^{a_4-b}x_3^{-a_3}x_4^{-a_4}L^*(a_3, a_1+a_3-b+1, a_2+a_4-b+1; a_3-a_4-1, a_4-b+1; -x_3/x_4, -(x_1x_3)/(x_2x_4), -x_2, -x_2/x_1).$$

We thus have a set of solutions valid near the point $(0, 0, 0, \infty)$ when

$$|x_3|>|x_2|, \quad |x_1|>|x_2| \quad \text{and} \quad |x_2x_4|>|x_1x_3|.$$

8.5. THE SINGULARITY $(1, 1, 1, \infty)$.

Apply (7.1) to the system 8.4 and obtain the following solutions:

(8.5.1) $$(1-x_4)^{-a_3}L^*\cdot\left(a_1, a_2, a_3; 1+a_1+a_2+a_4-b, 1+a_3-a_4; 1-x_2, x_1-1, \frac{1-x_3}{1-x_4}, \frac{1}{x_4-1}\right),$$

(8.5.2) $$(1-x_4)^{-a_4}L^*\cdot\left(a_2, a_1, a_4; 1+a_1+a_2+a_3-b, 1+a_4-a_3; 1-x_2, x_3-1, \frac{1-x_1}{1-x_4}, \frac{1}{x_4-1}\right),$$

(8.5.3) $$(1-x_1)^{b-a_1-a_2-a_4}(1-x_4)^{-a_3}L^*\left(a_2, a_3, b-a_2-a_4; 1+a_1+a_2+a_4-b, 1-a_1-a_2-a_4+b; \frac{1-x_3}{1-x_4}, \frac{x_2-1}{1-x_1}, \frac{1-x_1}{1-x_4}, 1-x_1\right),$$

(8.5.4) $$(1-x_3)^{b-a_1-a_2-a_3}(1-x_4)^{-a_4}L^*\left(a_1, a_4, b-a_1-a_2; 1+a_1+a_2+a_3-b, 1-a_1-a_2-a_3+b; \frac{1-x_1}{1-x_4}, \frac{x_2-1}{1-x_3}, \frac{1-x_3}{1-x_4}, 1-x_3\right),$$

$$(8.5.5)\quad (1-x_2)^{b-a_1-a_2-a_4}(1-x_3)^{-a_4-a_3}(1-x_4)^{-a_4}L^*\Big(b-a_2-a_4,\, b-a_1-a_3,\, a_4;$$

$$b-a_1-a_2-a_4,\, a_4-a_3+1;\, 1-x_2,\, \frac{x_2-1}{1-x_3},\, \frac{x_3-1}{1-x_4},\, \frac{(x_1-1)(1-x_3)}{(1-x_2)(1-x_4)}\Big)$$

and

$$(8.5.6)\quad (1-x_1)^{a_3-a_4}(1-x_2)^{b-a_1-a_2-a_3-1}(1-x_3)^{-a_3}(1-x_4)^{-a_4}L^*\cdot$$

$$\cdot\Big(a_3,\, b-a_2-a_4,\, b-a_1-a_3;\, a_3-a_4-1,\, b-a_1-a_2-a_3;$$

$$\frac{x_3-1}{1-x_4},\, \frac{(x_1-1)(1-x_3)}{(1-x_2)(1-x_4)},\, x_2-1,\, \frac{x_4-1}{1-x_1}\Big).$$

This set of solutions is valid near the point $(1, 1, 1, \infty)$ provided that

$|1-x_3|>|1-x_2|$, $\quad |1-x_1|>|1-x_2|$ $\quad$ and $\quad |(1-x_2)(1-x_4)|>|(1-x_1)(1-x_3)|$.

8.6. The singularity $(0, \infty, \infty, \infty)$.

From (6.3) by interchanging the subscripts, we have the solutions

$$(8.6.1)\quad x_2^{-a_1}x_4^{-a_2}L^*\cdot$$

$$\cdot(a_1,\, 1+a_1+a_3-b,\, a_3;\, 1+a_1-a_2,\, 1+a_3-a_4;\, 1/x_2,\, -x_1/x_2,\, 1/x_4,\, -x_3/x_4),$$

$$(8.6.2)\quad x_2^{-a_2}x_4^{-a_4}L^*\cdot$$

$$\cdot(a_2,\, a_2+a_4-b+1,\, a_4;\, a_2-a_1+1,\, a_4-a_3+1;\, 1/x_4,\, -x_3/x_4,\, 1/x_4,\, -x_1/x_4),$$

$$(8.6.3)\quad x_2^{-a_3}x_4^{-a_4}L^*\cdot$$

$$\cdot(a_4,\, a_2+a_4+1-b,\, a_2;\, a_4-a_3+1,\, a_2-a_1+1;\, 1/x_4,\, -x_1/x_4,\, 1/x_2,\, -x_3/x_2)$$

and

$$(8.6.4)\quad x_2^{-a_1}x_4^{-a_3}L^*\cdot$$

$$\cdot(a_3,\, a_1+a_3-b+1,\, a_1;\, a_3-a_4+1,\, a_1-a_2+1;\, 1/x_2,\, -x_1/x_2,\, 1/x_4,\, -x_3/x_4).$$

If we apply (7.7) to (6.7), we have

$$(8.6.5)\quad x_2^{-a_2}x_4^{-a_3}M^*\cdot$$

$$\cdot(a_2+a_3-b+1,\, a_2,\, a_3;\, a_2-a_1+1,\, a_3-a_4+1;\, -1/x_2,\, -x_3/(x_2x_4),\, -1/x_4,\, x_1),$$

while (7.6) gives

$$(8.6.6)\qquad x_2^{-a_2}x_4^{-a_3}x_1^{1+a_2+a_4-b}K^*\cdot$$

$$\cdot(1+a_2+a_4-b, a_2, a_3, 1+a_1+a_3-b; 2+a_2+a_3-b; x_1, x_1/x_2, (x_1x_3)/(x_2x_4), x_1/x_4).$$

The above solutions are convergent near the singular point in question when

$$|x_4|>|x_3| \quad\text{and}\quad |x_2|>|x_3|.$$

8.7. THE SINGULARITY $(1, \infty, \infty, \infty)$.

If (7.1) is applied to the system of solutions 8.6, we have

$$(8.7.1)\qquad (1-x_2)^{-a_1}(1-x_4)^{-a_2}L^*\Big(a_1, b-a_2-a_4, a_3;$$

$$1+a_1-a_2, 1+a_3-a_4; \frac{1}{1-x_2}, \frac{x_1-1}{1-x_2}, \frac{1}{1-x_4}, \frac{x_3-1}{1-x_4}\Big),$$

$$(8.7.2)\qquad (1-x_2)^{-a_2}(1-x_4)^{-a_4}L^*\Big(a_2, b-a_1-a_3, a_4;$$

$$a_2-a_1+1, a_4-a_3+1; \frac{1}{1-x_2}, \frac{x_3-1}{1-x_4}, \frac{1}{1-x_4}, \frac{x_1-1}{1-x_4}\Big),$$

$$(8.7.3)\qquad (1-x_2)^{-a_3}(1-x_4)^{-a_4}L^*\Big(a_4, b-a_1-a_3, a_2;$$

$$a_4-a_3+1, a_2-a_1+1; \frac{1}{1-x_4}, \frac{x_1-1}{1-x_4}, \frac{1}{1-x_2}, \frac{x_3-1}{1-x_2}\Big),$$

$$(8.7.4)\qquad (1-x_2)^{-a_1}(1-x_4)^{-a_3}L^*\Big(a_3, b-a_2-a_4, a_1;$$

$$a_3-a_4+1, a_1-a_2+1; \frac{1}{1-x_2}, \frac{x_1-1}{1-x_2}, \frac{1}{1-x_4}, \frac{x_3-1}{1-x_4}\Big),$$

$$(8.7.5)\qquad (1-x_2)^{-a_1}(1-x_4)^{-a_3}M^*\Big(b-a_1-a_4, a_2, a_3; a_2-a_1+1, a_3-a_4+1;$$

$$\frac{1}{x_2-1}, \frac{x_3-1}{(1-x_2)(1-x_4)}, \frac{1}{x_4-1}, 1-x_1\Big)$$

and

(8.7.6) $(1-x_1)^{b-a_1-a_4}(1-x_2)^{-a_2}(1-x_4)^{-a_3}K^*\cdot$

$$\cdot\left(b-a_1-a_3\,,a_2\,,a_3\,,b-a_2-a_4\,;b-a_2-a_3+1;1-x_1\,,\frac{1-x_1}{1-x_2}\,,\frac{(1-x_1)(1-x_3)}{(1-x_2)(1-x_4)}\,,\frac{1-x_1}{1-x_4}\right).$$

This set of solutions is valid near the singularity $(1,\infty,\infty,\infty)$ when

$$|1-x_4|>|1-x_3| \quad \text{and} \quad |1-x_2|>|1-x_3|\,.$$

8.8. THE SINGULARITY $(0,0,\infty,\infty)$.

From (5.7) we have

(8.8.1) $x_3^{-a_2}x_4^{-a_4}K^*\cdot$

$$\cdot(a_1\,,a_2\,,1+a_2+a_4-b,a_4\,;1+a_2+a_4-a_3\,;x_1/x_4\,,x_2/x_3\,,1/x_3\,,1/x_4)\,.$$

From (6.6), we have

(8.8.2) $x_4^{-a_3}L^*(a_1\,,a_2\,,a_3\,;b-a_3\,,1+a_3-a_4\,;x_2\,,-x_1\,,x_3/x_4\,,-1/x_4)\,.$

Apply (5.7) twice to (6.5) with the appropriate interchange of the subscripts and obtain the solution

(8.8.3) $x_3^{a_4-a_3}x_4^{-a_4}M^*\cdot$

$$\cdot(a_2+a_4-a_3\,,a_1\,,a_4\,;b-a_3\,,1+a_4-a_3\,;x_2\,,x_1x_3/x_4\,,-x_3/x_4\,,-1/x_3)\,.$$

From (5.4), we have

(8.8.4) $x_1^{1+a_3-b}x_4^{-a_3}L^*\cdot$

$$\cdot(a_2\,,a_3\,,a_1+a_4-b+1;b-a_4\,,a_3-b+2;x_3/x_4\,,-x_2/x_1\,,x_1/x_4\,,x_1)\,.$$

From (6.6) by the two-fold application of (5.7), we have

(8.8.5) $x_2^{a_3-b+1}x_3^{a_4-a_3}x_4^{-a_4}L^*\cdot$

$$\cdot\left(a_1+a_3-b+1,a_2+a_4-b+1,a_4\,;a_3-b+1,a_4-a_3+1;x_2\,,-\frac{x_2}{x_3}\,,-\frac{x_3}{x_4}\,,-\frac{x_1x_3}{x_2x_4}\right).$$

Apply (5.7) twice to (6.4).

(8.8.6) $x_1^{a_3-a_4}x_2^{a_4-b}x_3^{-a_3}x_4^{-a_4}L^*\cdot$

$$\cdot\left(a_3\,,a_1+a_3-b+1,a_2+a_4-b+1;a_3-a_4+1,a_4-b+1;-\frac{x_3}{x_4}\,,-\frac{x_1x_3}{x_2x_4}\,,-x_2\,,-\frac{x_2}{x_1}\right).$$

This set of solutions is valid near the singular point in question when

$$|x_4| > |x_3|, \qquad |x_1| > |x_2| \quad \text{and} \quad |x_2 x_4| > |x_1 x_3|.$$

8.9. THE SINGULARITY $(1, 1, \infty, \infty)$.

Apply (7.1) to the system 8.8 and obtain

$$(8.9.1) \qquad (1-x_3)^{-a_2}(1-x_4)^{-a_4}K^* \cdot \left(a_1, a_2, b-a_1-a_3, a_4; 1+a_2+a_4-a_3; \frac{1-x_1}{1-x_4}, \frac{1-x_2}{1-x_3}, \frac{1}{1-x_3}, \frac{1}{1-x_4}\right),$$

$$(8.9.2) \qquad (1-x_4)^{-a_3}L^* \cdot \left(a_1, a_2, a_4; 1+a_1+a_2-a_4-b, 1+a_3-a_4; 1-x_2, x_1-1, \frac{1-x_3}{1-x_4}, \frac{1}{x_4-1}\right),$$

$$(8.9.3) \qquad (1-x_3)^{a_4-a_3}(1-x_4)^{-a_4}M^*\left(a_2+a_4-a_3, a_1, a_4; 1+a_1+a_2-a_4-b, 1+a_4-a_3; 1-x_2, \frac{(1-x_1)(1-x_3)}{(1-x_2)(1-x_4)}, \frac{1-x_3}{x_4-1}, \frac{1}{x_3-1}\right),$$

$$(8.9.4) \qquad (1-x_1)^{b-a_1-a_2-a_4}(1-x_4)^{-a_3}L^*\left(a_2, a_3, b-a_2-a_4; 1+a_1+a_2+a_3-b, 1+a_1+a_2+a_4-b; \frac{1-x_3}{1-x_4}, \frac{1-x_2}{x_1-1}, \frac{1-x_1}{1-x_4}, 1-x_1\right),$$

$$(8.9.5) \qquad (1-x_2)^{b+a_1+a_2-a_4}(1-x_3)^{a_4-a_3}(1-x_4)^{-a_4}L^*\left(b-a_2-a_4, b-a_1-a_3, a_4; b-a_1-a_2-a_4, a_4-a_3+1; 1-x_2, \frac{1-x_2}{x_3-1}, \frac{1-x_3}{x_4-1}, \frac{(1-x_1)(1-x_3)}{(1-x_2)(1-x_4)}\right)$$

and

$$(8.9.6) \qquad (1-x_1)^{a_3-a_4}(1-x_2)^{b-a_1-a_2-a_3-1}(1-x_3)^{-a_3}(1-x_4)^{-a_4}L^* \cdot \left(a_3, b-a_2-a_4, b-a_1-a_3; a_3-a_4+1, b-a_1-a_2-a_3; \frac{1-x_3}{x_4-1}, \frac{(1-x_1)(1-x_3)}{(1-x_2)(1-x_4)}, x_2-1, \frac{1-x_2}{x_1-1}\right).$$

These solutions are valid near the singular point $(1,1,\infty,\infty)$ when

$$|1-x_4|>|1-x_3|, \quad |1-x_1|>|1-x_2| \quad \text{and} \quad |(1-x_2)(1-x_4)|>|(1-x_1)(1-x_3)|.$$

8.10. THE SINGULARITY $(0,\infty,0,\infty)$.

If (7.4) is applied successively to (6.3), we have

(8.10.1) $$x_2^{-a_1}x_4^{-a_2}L^*\cdot$$
$$\cdot(a_1, a_1+a_3+1-b, a_3; a_1-a_2+1, a_4-a_4+1; 1/x_2, -x_1/x_2, 1/x_4, -x_3/x_4),$$

(8.10.2) $$x_2^{-a_2}x_4^{-a_4}L^*\cdot$$
$$\cdot(a_2, a_2+a_4-b+1, a_4; a_2-a_1+1, a_4-a_3+1; 1/x_2, -x_3/x_2, 1/x_4, -x_1/x_4),$$

(8.10.3) $$x_2^{-a_3}x_4^{-a_4}L^*\cdot$$
$$\cdot(a_4, a_2+a_4-b+1, a_2; a_4-a_3+1, a_2-a_1+1; 1/x_4, -x_1/x_4, 1/x_2, -x_3/x_2)$$

and

(8.10.4) $$x_2^{-a_1}x_4^{-a_3}L^*\cdot$$
$$\cdot(a_3, a_1+a_3-b+1, a_1; a_3-a_4+1, a_1-a_2+1; 1/x_2, -x_1/x_2, 1/x_4, -x_3/x_4).$$

Two further independent solutions may be obtained from (6.5) by the application of (7.2). These are

(8.10.5) $$x_2^{-a_1}x_4^{-a_4}M^*\cdot$$
$$\cdot\left(a_1+a_4-b+1, a_1, a_4; a_1-a_2+1, a_4-a_3+1; -1/x_2, -\frac{x_1}{x_2x_4}, -1/x_4, -x_3\right)$$

and

(8.10.6) $$x_2^{-a_2}x_4^{-a_3}M^*\cdot$$
$$\cdot\left(a_2+a_3-b+1, a_2, a_3; a_2-a_1+1, a_3-a_4+1; -1/x_2, -\frac{x_3}{x_2x_4}, -1/x_4, -x_1\right).$$

This system of solutions is valid throughout the whole neighbourhood of the singular point $(0,\infty,0,\infty)$.

8.11. THE SINGULARITY $(1, \infty, 1, \infty)$.

Apply (7.1) to all of the members of 8.10.

(8.11.1) $(1-x_2)^{-a_1}(1-x_4)^{-a_2}L^* \cdot$

$$\cdot\left(a_1, b-a_2-a_4, a_3; a_1-a_2+1, a_3-a_4+1; \frac{1}{1-x_2}, \frac{1-x_1}{x_2-1}, \frac{1}{1-x_4}, \frac{1-x_3}{x_4-1}\right),$$

(8.11.2) $(1-x_2)^{-a_2}(1-x_4)^{-a_4}L^* \cdot$

$$\cdot\left(a_2, b-a_1-a_3, a_4; a_2-a_1+1, a_4-a_3+1; \frac{1}{1-x_2}, \frac{1-x_3}{x_2-1}, \frac{1}{1-x_4}, \frac{1-x_1}{x_4-1}\right),$$

(8.11.3) $(1-x_2)^{-a_3}(1-x_4)^{-a_4}L^* \cdot$

$$\cdot\left(a_4, b-a_1-a_3, a_2; a_4-a_3+1, a_2-a_1+1; \frac{1}{1-x_4}, \frac{1-x_1}{x_4-1}, \frac{1}{1-x_2}, \frac{1-x_3}{x_2-1}\right),$$

(8.11.4) $(1-x_2)^{-a_2}(1-x_4)^{-a_3}L^* \cdot$

$$\cdot\left(a_3, b-a_2-a_4, a_1; a_3-a_4+1, a_1-a_2+1; \frac{1}{1-x_2}, \frac{1-x_1}{x_2-1}, \frac{1}{1-x_4}, \frac{1-x_3}{x_2-1}\right),$$

(8.11.5) $(1-x_2)^{-a_1}(1-x_4)^{-a_4}M^* \cdot$

$$\cdot\left(b-a_2-a_3, a_2, a_4; a_1-a_2+1, a_4-a_3+1; \frac{1}{x_2-1}, \frac{1-x_1}{(1-x_2)(x_4-1)}, \frac{1}{x_4-1}, x_3-1\right)$$

and

(8.11.6) $(1-x_2)^{-a_2}(1-x_4)^{-a_3}M^* \cdot$

$$\cdot\left(b-a_1-a_3, a_1, a_3; a_2-a_1+1, a_3-a_4+1; \frac{1}{x_2-1}, \frac{1-x_3}{(1-x_2)(x_4-1)}, \frac{1}{x_4-1}, x_1-1\right).$$

As in the previous case, this family of solutions converges throughout the whole of the neighbourhood of the associated singular point with the obvious proviso that the radii of convergence are not exceeded.

8.12. THE SINGULARITY $(0, 0, 1, 1)$.

From (6.3), by means of (7.5) and interchanging the subscripts as require, we have

(8.12.1) $x_1^{-a_2} x_2^{a_3+a_2-b+1} x_3^{-a_3} L^* \cdot$

$$\cdot\left(1+a_2+a_4-b, 1+a_1+a_3-b, a_3; 2+a_2-b, a_2-a_3+1; x_1, -\frac{x_1}{x_2}, \frac{x_2}{x_3}, -\frac{x_2 x_4}{x_1 x_3}\right).$$

Similarly, by the twofold application of (7.5) to (6.3) and (6.4), we obtain

(8.12.2) $$x_1^{a_2+1-b}x_2^{a_3-a_2}x_3^{-a_3}L^*\cdot$$
$$\cdot\left(a_3\,,1+a_1+a_3-b,1+a_2+a_4-b;1+a_3-a_1\,,b-a_2-2;\frac{x_2}{x_3}\,,-\frac{x_2x_4}{x_1x_3}\,,x_1\,,\frac{x_1}{x_2}\right)$$

and

(8.12.3) $$x_1^{a_2+1-b}x_2^{a_2-a_3}x_3^{-a_3}L^*\cdot$$
$$\cdot\left(1+a_2+a_4-b,1+a_1+a_3-b,a_3;2+a_2-b,1+a_3-a_2;x_1\,,-\frac{x_1}{x_2}\,,\frac{x_2}{x_3}\,,-\frac{x_2x_4}{x_1x_3}\right).$$

The following three solutions may be obtained from the above by the application of (7.1) with appropriate interchanging of the subscripts:

(8.12.4) $$(1-x_1)^{-a_1}(1-x_3)^{-a_3}(1-x_4)^{b-a_2-a_3}L^*\Big(b-a_1-a_3\,,b-a_2-a_4\,,a_1;$$
$$b-a_1-a_2-a_3+1,a_4-a_1+1;1-x_3\,,\frac{1-x_3}{x_4-1}\,,\frac{1-x_4}{1-x_1}\,,\frac{(1-x_2)(1-x_4)}{(1-x_1)(1-x_3)}\Big),$$

(8.12.5) $$(1-x_1)^{-a_1}(1-x_3)^{b-a_1-a_2-a_3}(1-x_4)^{a_1-a_4}L^*\Big(a_1\,,b-a_2-a_4\,,b-a_1-a_3;$$
$$1+a_1-a_3\,,a_1+a_2+a_3+1-b;\frac{1-x_4}{1-x_1}\,,\frac{(1-x_2)(1-x_4)}{(1-x_1)(1-x_3)}\,,1-x_3\,,\frac{1-x_3}{1-x_4}\Big)$$

and

(8.12.6) $$(1-x_1)^{-a_1}(1-x_3)^{b-a_1-a_2-a_3}(1-x_4)^{a_4-a_1}L^*\Big(b-a_1-a_3\,,b-a_2-a_4\,,a_1;$$
$$b-a_1-a_2-a_3+1,a_1-a_4+1;1-x_3\,,\frac{1-x_3}{x_4-1}\,,\frac{1-x_4}{1-x_1}\,,\frac{(1-x_2)(1-x_4)}{(1-x_1)(1-x_3)}\Big).$$

This set of solutions is valid in that part of the neighbourhood of the singular point $(0,0,1,1)$ for which

$|x_2|>|x_1|$, $|x_1x_3|>|x_2x_4|$, $|1-x_4|>|1-x_3|$ and $|(1-x_1)(1-x_3)|>|(1-x_2)(1-x_4)|$.

8.13. THE SINGULARITY $(0,0,0,1)$.

A fundamental set of integrals associated with this singular point may be constructed by taking (8.1.2), (8.1.3), (8.4.4), (8.12.1), (8.12.2) and (8.12.3) with appropriate changes of the subscripts. We then have

(8.13.1) $$x_1^{1+a_2+a_3-b}x_2^{-a_2}x_4^{-a_3}K^*\cdot$$
$$\cdot\left(1+a_1+a_3-b, a_3, a_2, 1+a_2+a_4-b; 2+a_2+a_3-b; x_1, \frac{x_1}{x_4}, \frac{x_1x_3}{x_2x_4}, \frac{x_1}{x_2}\right),$$

(8.13.2) $$x_2^{a_3-b+1}x_4^{-a_3}M^*\cdot$$
$$\cdot\left(1+a_2+a_4-b, a_3, 1+a_1+a_3-b; 1+a_3-a_4, a_3-b+2; \frac{x_3}{x_4}, -\frac{x_2}{x_4}, x_2, -\frac{x_1}{x_2}\right),$$

(8.13.3) $$x_3^{1+a_4-b}x_4^{-a_4}L^*\cdot$$
$$\cdot(a_1, a_4, a_2+a_4-b+1; b-a_3, a_4-b+2; x_1/x_4, -x_2/x_3, x_3/x_4, x_2, -x_1/x_2),$$

(8.13.4) $$x_2^{-a_3}x_3^{a_4+a_3-b+1}x_4^{-a_4}L^*\cdot$$
$$\cdot\left(1+a_1+a_3-b, 1+a_2+a_4-b, a_4; 2+a_3-b, a_3-a_4-1; x_2, -\frac{x_2}{x_3}, \frac{x_1}{x_4}, -\frac{x_1x_3}{x_2x_4}\right),$$

(8.13.5) $$x_2^{a_3+1-b}x_3^{a_4-a_3}x_4^{-a_4}L^*\cdot$$
$$\cdot\left(a_4, 1+a_2+a_4-b, 1+a_1+a_3-b; 1+a_4-a_2, b-a_3-2; \frac{x_3}{x_4}, -\frac{x_1x_3}{x_2x_4}, x_2, \frac{x_2}{x_3}\right)$$

and

(8.13.6) $$x_2^{a_3+1-b}x_3^{a_3-a_4}x_4^{-a_4}L^*\cdot$$
$$\cdot\left(1+a_1+a_3-b, 1+a_2+a_4-b, a_4; 2+a_3-b, 1+a_4-a_3; x_2, -\frac{x_2}{x_3}, \frac{x_3}{x_4}, -\frac{x_1x_3}{x_2x_4}\right).$$

This system of integrals is valid in that portion of the neighbourhood of the singular point in question for which

$$|x_3|>|x_2|>|x_1| \quad\text{and}\quad |x_2x_4|>|x_1x_3|.$$

8.14. THE SINGULARITY (1, 1, 1, 0).

Apply (7.1) to the members of 8.13.

(8.14.1) $$(1-x_1)^{b-a_1-a_4}(1-x_2)^{-a_2}(1-x_4)^{-a_3}K^*\Big(b-a_2-a_4, a_3, a_2, b-a_1-a_3;$$
$$b-a_1-a_4+1; 1-x_1, \frac{1-x_1}{1-x_4}, \frac{(1-x_1)(1-x_3)}{(1-x_2)(1-x_4)}, \frac{1-x_1}{1-x_2}\Big),$$

(8.14.2) $$(1-x_2)^{b-a_1-a_2-a_4}(1-x_4)^{-a_3}M^*\Big(b-a_1-a_3, a_3, b-a_2-a_4;$$
$$1+a_3-a_4, b-a_1-a_2-a_4+1; \frac{1-x_3}{1-x_4}, \frac{1-x_2}{x_4-1}, 1-x_2, \frac{1-x_1}{x_2-1}\Big),$$

(8.14.3) $$(1-x_3)^{b-a_1-a_2-a_3}(1-x_4)^{-a_4}L^*\Big(a_1, a_4, b-a_1-a_3;$$
$$1+a_1+a_2+a_4-b, b-a_1-a_2-a_3+1; \frac{1-x_1}{1-x_4}, \frac{1-x_2}{x_3-1}, \frac{1-x_3}{1-x_4}, 1-x_3\Big),$$

(8.14.4) $$(1-x_3)^{-a_3}(1-x_4)^{b-a_1-a_2}(1-x_4)^{-a_4}L^*\Big(b-a_2-a_4, b-a_1-a_3, a_4;$$
$$b-a_1-a_2-a_4+1, a_3-a_4-1; 1-x_2, \frac{1-x_2}{x_3-1}, \frac{1-x_1}{1-x_4}, \frac{(1-x_1)(1-x_3)}{(1-x_2)(1-x_4)}\Big),$$

(8.14.5) $$(1-x_2)^{b-a_1-a_2-a_4}(1-x_3)^{a_4-a_3}(1-x_4)^{-a_4}L^*\Big(a_4, b-a_1-a_3, b-a_2-a_4;$$
$$1+a_4-a_2, a_1+a_2+a_4-b+1; \frac{1-x_3}{1-x_4}, \frac{(1-x_1)(1-x_3)}{(1-x_2)(1-x_4)}, 1-x_2, \frac{1-x_2}{1-x_3}\Big)$$

and

(8.14.6) $$(1-x_2)^{b-a_1-a_2-a_4}(1-x_3)^{a_3-a_4}(1-x_4)^{-a_4}L^*\Big(b-a_2-a_4, b-a_1-a_3, a_4;$$
$$b-a_1-a_2-a_4+1, 1+a_4-a_3; 1-x_2, \frac{1-x_2}{x_3-1}, \frac{1-x_3}{1-x_4}, \frac{(1-x_1)(1-x_3)}{(1-x_2)(1-x_4)}\Big).$$

This system of solutions is valid near the point $(1,1,1,0)$ when

$$|1-x_3| > |1-x_2| > |1-x_1| \quad \text{and} \quad |(1-x_2)(1-x_4)| > |(1-x_1)(1-x_3)|.$$

8.15. THE SINGULARITY $(0,1,0,1)$.

By appropriate interchanging of the subscripts in (8.6.6), we have

(8.15.1) $$x_2^{-a_2}x_4^{-a_3}x_1^{1+a_2+a_4-b}K^*\cdot$$
$$\cdot\Big(1+a_2+a_4-b, a_2, a_3, 1+a_1+a_3-b; 2+a_2+a_3-b; x_1, \frac{x_1}{x_2}, \frac{x_1x_3}{x_2x_4}, \frac{x_1}{x_4}\Big),$$

(8.15.2) $x_4^{-a_4} x_2^{-a_1} x_3^{1+a_2+a_4-b} K^* \cdot$

$$\cdot\left(1+a_2+a_4-b, a_4, a_1, 1+a_1+a_3-b; 2+a_2+a_1-b; x_3, \frac{x_3}{x_4}, \frac{x_1 x_3}{x_2 x_4}, \frac{x_3}{x_2}\right),$$

(8.15.3) $x_4^{-a_3} x_2^{-a_2} x_1^{1+a_1+a_3-b} K^* \cdot$

$$\cdot\left(1+a_1+a_3-b, a_3, a_2, 1+a_2+a_4-b; 2+a_1+a_2-b; x_1, \frac{x_1}{x_4}, \frac{x_1 x_3}{x_2 x_4}, \frac{x_1}{x_2}\right)$$

and

(8.15.4) $x_2^{-a_1} x_4^{-a_4} x_3^{1+a_1+a_3-b} K^* \cdot$

$$\cdot\left(1+a_1+a_3-b, a_1, a_4, 1+a_2+a_4-b; 2+a_1+a_4-b; x_3, \frac{x_3}{x_2}, \frac{x_1 x_3}{x_2 x_4}, \frac{x_1}{x_4}\right).$$

Apply (7.1) to (8.13.1) and interchange the subscripts as required.

(8.15.5) $(1-x_1)^{-a_4}(1-x_2)^{b-a_2-a_4}(1-x_3)^{-a_3} K^* \cdot$

$$\cdot\left(b-a_2-a_4, a_3, a_4, b-a_1-a_3; b-a_2-a_4+1; 1-x_2, \frac{1-x_2}{1-x_3}, \frac{(1-x_2)(1-x_4)}{(1-x_1)(1-x_3)}, \frac{1-x_2}{1-x_1}\right)$$

and

(8.15.6) $(1-x_1)^{-a_1}(1-x_3)^{-a_2}(1-x_4)^{b-a_2-a_4} K^* \cdot$

$$\cdot\left(b-a_2-a_4, a_1, a_2, b-a_1-a_3; b-a_2-a_4+1; 1-x_4, \frac{1-x_4}{1-x_1}, \frac{(1-x_2)(1-x_4)}{(1-x_1)(1-x_3)}, \frac{1-x_4}{1-x_3}\right).$$

The above set of solutions is valid for the whole neighbourhood of the singular point $(0, 1, 0, 1)$.

8.16. THE SINGULARITY $(0, 1, 0, \infty)$.

From (8.1.2) by appropriately interchanging the subscripts, we have

(8.16.1) $x_2^{-a_2} x_4^{-a_3} x_1^{1+a_2+a_4-b} K^* \cdot$

$$\cdot\left(1+a_2+a_4-b, a_2, a_3, 1+a_1+a_3-b; 2+a_1+a_4-b; x_1, \frac{x_1}{x_2}, \frac{x_1 x_3}{x_2 x_4}, \frac{x_1}{x_4}\right),$$

(8.16.2) $x_4^{-a_4} x_2^{-a_1} x_3^{1+a_2+a_4-b} K^* \cdot$

$$\cdot\left(1+a_2+a_4-b, a_4, a_1, 1+a_1+a_3-b; 2+a_1+a_4-b; x_3, \frac{x_3}{x_4}, \frac{x_1 x_3}{x_2 x_4}, \frac{x_3}{x_2}\right),$$

(8.16.3) $$x_4^{-a_3} x_2^{-a_2} x_1^{1+a_1+a_3-b} K^* \cdot$$

$$\cdot\left(1+a_1+a_3-b, a_3, a_2, 1+a_2+a_4-b; 2+a_2+a_3-b; x_1, \frac{x_1}{x_4}, \frac{x_1 x_3}{x_2 x_4}, \frac{x_1}{x_2}\right)$$

and

(8.16.4) $$x_2^{-a_1} x_4^{-a_4} x_3^{1+a_1+a_3-b} K^* \cdot$$

$$\cdot\left(1+a_1+a_3-b, a_1, a_4, 1+a_2+a_4-b; 2+a_2+a_3-b; x_3, \frac{x_3}{x_2}, \frac{x_1 x_3}{x_2 x_4}, \frac{x_1}{x_4}\right).$$

Now apply (7.1) to (8.8.5) and (8.8.6) and obtain respectively

(8.16.5) $$(1-x_2)^{b-a_1-a_2-a_4}(1-x_3)^{a_4-a_3}(1-x_4)^{-a_4} L^*\Big(b-a_2-a_4, b-a_1-a_3, a_4;$$

$$b-a_1-a_2-a_4, a_4-a_3+1; 1-x_2, \frac{1-x_2}{x_3-1}, \frac{1-x_3}{x_4-1}, \frac{(1-x_1)(1-x_3)}{(1-x_2)(1-x_4)}\Big)$$

and

(8.16.6) $$(1-x_1)^{a_3-a_4}(1-x_2)^{b-a_1-a_2-a_3-1}(1-x_3)^{-a_3}(1-x_4)^{-a_4} L^* \cdot$$

$$\cdot\Big(a_3, b-a_2-a_4, b-a_1-a_3; a_3-a_4-1, b-a_1-a_3-a_4;$$

$$\frac{1-x_3}{x_4-1}, \frac{(1-x_1)(1-x_3)}{(1-x_2)(1-x_4)}, x_2-1, \frac{1-x_2}{x_1-1}\Big).$$

This system of integrals is valid in that part of the neighbourhood of the singular point $(0, 1, 0, \infty)$ for which

$$|(1-x_2)(1-x_4)| > |(1-x_1)(1-x_3)|.$$

8.17. THE SINGULARITY $(1, 0, 1, \infty)$.

Apply (7.1) to all the members of 8.16.

(8.17.1) $$(1-x_1)^{b-a_1-a_3}(1-x_2)^{-a_2}(1-x_4)^{-a_3} K^*\Big(b-a_1-a_3, a_2, a_3, b-a_2-a_4;$$

$$b-a_2-a_3+1; 1-x_1, \frac{1-x_1}{1-x_2}, \frac{(1-x_1)(1-x_3)}{(1-x_2)(1-x_4)}, \frac{1-x_1}{1-x_4}\Big),$$

(8.17.2) $$(1-x_2)^{a_2}(1-x_3)^{b-a_1-a_3}(1-x_4)^{-a_4}K^*\Big(b-a_1-a_3, a_4, a_1, b-a_2-a_4; b-a_2-a_3+1; 1-x_3, \frac{1-x_3}{1-x_4}, \frac{(1-x_1)(1-x_3)}{(1-x_2)(1-x_4)}, \frac{1-x_3}{1-x_2}\Big),$$

(8.17.3) $$(1-x_1)^{b-a_2-a_4}(1-x_2)^{-a_2}(1-x_4)^{-a_3}K^*\Big(b-a_2-a_4, a_3, a_2, b-a_1-a_3; b-a_1-a_4+1; 1-x_1, \frac{1-x_1}{1-x_4}, \frac{(1-x_1)(1-x_3)}{(1-x_2)(1-x_4)}, \frac{1-x_1}{1-x_2}\Big),$$

(8.17.4) $$(1-x_2)^{a_1}(1-x_3)^{b-a_2-a_4}(1-x_4)^{-a_4}K^*\Big(b-a_2-a_4, a_1, a_4, b-a_1-a_3; b-a_1-a_4+1; 1-x_3, \frac{1-x_3}{1-x_2}, \frac{(1-x_1)(1-x_3)}{(1-x_2)(1-x_4)}, \frac{1-x_1}{1-x_4}\Big),$$

(8.17.5) $$x_2^{a_3+1-b}x_3^{a_4-a_3}x_4^{-a_4}L^*\cdot$$
$$\cdot\Big(1+a_1+a_3-b, 1+a_2+a_4-b, a_4; 1+a_3-b, a_4-a_3+1; x_2, -\frac{x_2}{x_3}, -\frac{x_3}{x_4}, -\frac{x_1x_3}{x_2x_4}\Big)$$

and

(8.17.6) $$x_1^{a_3-a_4}x_2^{a_4-b}x_3^{-a_3}x_4^{-a_4}L^*\cdot$$
$$\cdot\Big(a_3, 1+a_1+a_3-b, 1+a_2+a_4-b; a_3-a_4-1, 1+a_2-b; -\frac{x_3}{x_4}, -\frac{x_1x_3}{x_2x_4}, -x_2, -\frac{x_2}{x_1}\Big).$$

This system is valid near the point $(1, 0, 1, \infty)$ for which
$$|x_2x_4| > |x_1x_3|.$$

8.18. THE SINGULARITY $(0, \infty, 1, \infty)$.

From (8.10.1) to (8.10.4) and (8.10.6), we have

(8.18.1) $$x_2^{-a_1}x_4^{-a_2}L^*\cdot$$
$$\cdot(a_1, a_1+a_3+1-b, a_3; a_1-a_2+1, a_3-a_4+1; 1/x_2, -x_1/x_2, 1/x_4, -x_3/x_4),$$

(8.18.2) $$x_2^{-a_2}x_4^{-a_4}L^*\cdot$$
$$\cdot(a_2, a_2+a_4+1-b, a_4; a_2-a_1+1, a_4-a_3+1; 1/x_2, -x_3/x_2, 1/x_4, -x_1/x_4),$$

(8.18.3) $x_2^{-a_3} x_4^{-a_4} L^* \cdot$

$$\cdot (a_4, a_2 + a_4 + 1 - b, a_2; a_4 - a_3 + 1, a_2 - a_1 + 1; 1/x_4, -x_1/x_4, 1/x_2, -x_3/x_4),$$

(8.18.4) $x_2^{-a_1} x_4^{-a_3} L^* \cdot$

$$\cdot (a_3, a_1 + a_3 + 1 - b, a_1; a_3 - a_4 + 1, a_1 - a_2 + 1; 1/x_2, -x_1/x_2, 1/x_4, -x_3/x_4)$$

and

(8.18.5) $x_2^{-a_2} x_4^{-a_3} M^* \cdot$

$$\cdot (a_2 + a_3 + 1 - b, a_2, a_3; a_2 - a_1 + 1, a_3 - a_4 + 1; -1/x_2, -x_3/(x_2 x_4), -1/x_4, -x_1).$$

This set of integrals is completed by applying (7.1) to (8.10.5).

(8.18.6) $(1 - x_2)^{-a_1} (1 - x_4)^{-a_4} M^* \cdot$

$$\cdot \left(b - a_2 - a_3, a_1, a_4; a_1 - a_2 + 1, a_4 - a_3 + 1; \frac{1}{x_2 - 1}, \frac{x_1 - 1}{(1 - x_2)(1 - x_4)}, \frac{1}{x_4 - 1}, x_3 - 1 \right).$$

This system of integrals is valid throughout the whole of the neighbourhood of the singular point in question.

8.19. THE SINGULARITY $(0, 0, 1, \infty)$.

From (8.8.2) and (8.8.4), we have

(8.19.1) $x_4^{-a_3} L^* (a_1, a_2, a_3; b - a_3, 1 + a_3 - a_4; x_2, -x_1, x_3/x_4, -1/x_4)$

and

(8.19.2) $x_1^{1 + a_3 - b} x_4^{-a_3} L^* \cdot$

$$\cdot (a_2, a_3, a_1 + a_3 - b + 1; b - a_4, a_3 - b + 2; x_3/x_4, -x_2/x_1, x_1/x_4, x_1).$$

From (8.8.5) and (8.8.6) with the appropriate interchanging of the subscripts, we obtain

(8.19.3) $x_2^{1 + a_3 - b} x_3^{a_4 - a_3} x_4^{-a_4} L^* (1 + a_1 + a_3 - b, 1 + a_2 + a_4 - b, a_4;$

$$1 + a_3 - b, a_4 - a_3 + 1; x_2, -x_2/x_3, -x_3/x_4, -x_1 x_3/(x_2 x_4)),$$

(8.19.4) $x_1^{a_3 - a_4} x_2^{1 + a_4 - b} x_3^{-a_3} x_4^{-a_4} L^* (a_3, 1 + a_1 + a_3 - b, 1 + a_2 + a_4 - b;$

$$a_3 - a_4 + 1, 1 + a_2 - b; -x_3/x_4, -x_1 x_3/(x_2 x_4), -x_2, -x_2/x_1),$$

(8.19.5) $x_1^{a_3 - a_4} x_2^{1 + a_4 - b} x_4^{-a_3} L^* (1 + a_1 + a_3 - b, 1 + a_2 + a_4 - b, a_3;$

$$1 + a_4 - b, a_4 - a_3 + 1; x_2, -x_2/x_1, -x_1/x_4, -x_1 x_3/(x_2 x_4))$$

and

$$(8.19.6)\qquad x_1^{-a_4}x_2^{b-a_3}x_3^{a_4-a_3}x_4^{-a_3}L^*(a_4, 1+a_2+a_4-b, 1+a_1+a_3-b;$$
$$a_4-a_3+1, 1+a_1-b; -x_1/x_4, -x_1x_3/(x_2x_4), -x_2, -x_2/x_3).$$

This set of integrals is valid in the neighbourhood of the point $(0, 0, 1, \infty)$ when $|x_1|>|x_2|$ and $|x_2x_4|>|x_1x_3|$.

8.20. THE SINGULARITY $(1, 1, 0, \infty)$.

Apply (7.1) to the members of 9.19.

$$(8.20.1)\qquad (1-x_4)^{-a_3}L^*\cdot$$
$$\cdot\left(a_1, a_2, a_3; 1+a_1+a_2+a_4-b, 1+a_3-a_4; 1-x_2, x_1-1, \frac{1-x_3}{1-x_4}, \frac{1}{x_4-1}\right),$$

$$(8.20.2)\qquad (1-x_1)^{b-a_1-a_2-a_4}(1-x_4)^{-a_3}L^*\Bigg(a_2, a_3, b-a_2-a_4;$$
$$1+a_1+a_2+a_3-b, b-a_1-a_2-a_4+1; \frac{1-x_3}{1-x_4}, \frac{1-x_2}{x_1-1}, \frac{1-x_1}{1-x_4}, 1-x_1\Bigg),$$

$$(8.20.3)\qquad (1-x_2)^{b-a_1-a_2-a_4}(1-x_3)^{a_4-a_3}(1-x_4)^{-a_4}L^*\Bigg(b-a_2-a_4, b-a_1-a_3, a_4;$$
$$b-a_1-a_2-a_4, a_4-a_3+1; 1-x_2, \frac{1-x_2}{x_3-1}, \frac{1-x_3}{x_4-1}, \frac{(1-x_1)(1-x_3)}{(1-x_2)(1-x_4)}\Bigg),$$

$$(8.20.4)\qquad (1-x_1)^{a_3-a_4}(1-x_2)^{b-a_1-a_2-a_3}(1-x_3)^{-a_3}(1-x_4)^{-a_4}L^*\cdot$$
$$\cdot\Bigg(a_3, b-a_2-a_4, b-a_1-a_3; a_3-a_4+1, b-a_1-a_3-a_4;$$
$$\frac{1-x_3}{x_4-1}, \frac{(1-x_1)(1-x_3)}{(1-x_2)(1-x_4)}, x_2-1, \frac{1-x_2}{x_1-1}\Bigg),$$

$$(8.20.5)\qquad (1-x_1)^{a_3-a_4}(1-x_2)^{b-a_1-a_2-a_3}(1-x_4)^{-a_3}L^*\Bigg(b-a_2-a_4, b-a_1-a_3, a_3;$$
$$b-a_1-a_2-a_3, a_4-a_3+1; 1-x_2, \frac{1-x_2}{x_1-1}, \frac{1-x_1}{x_4-1}, \frac{(1-x_1)(1-x_3)}{(1-x_2)(1-x_4)}\Bigg)$$

and

(8.20.6) $$(1-x_1)^{a_4}(1-x_2)^{1+a_1+a_2+a_4-b}(1-x_3)^{-a_4-a_3}(1-x_4)^{-a_3}L^*\cdot$$
$$\cdot\left(a_4, b-a_1-a_3, b-a_2-a_4; a_4-a_3+1, b-a_2-a_3-a_4;\right.$$
$$\left.\frac{1-x_1}{x_4-1}, \frac{(1-x_1)(1-x_3)}{(1-x_2)(1-x_4)}, x_2-1, \frac{1-x_2}{x_3-1}\right).$$

This set of integrals is valid near the singular point $(1, 1, 0, \infty)$ provided that

$$|1-x_1|>|1-x_2| \quad \text{and} \quad |(1-x_2)(1-x_4)|>|(1-x_1)(1-x_3)|.$$

8.21. THE SINGULARITY $(0, 1, \infty, \infty)$.

From (8.6.6) we have

(8.21.1) $$x_2^{-a_2}x_4^{-a_3}x_1^{1+a_2+a_4-b}K^*\cdot$$
$$\cdot\left(1+a_2+a_4-b, a_2, a_3, 1+a_1+a_3-b; 2+a_2+a_3-b; x_1, \frac{x_1}{x_2}, \frac{x_1x_3}{x_2x_4}, \frac{x_1}{x_4}\right).$$

From (8.7.6) by interchanging the subscripts, we obtain

(8.21.2) $$(1-x_2)^{b-a_1-a_2}(1-x_1)^{-a_4}(1-x_3)^{-a_3}K^*\left(b-a_1-a_3, a_4, a_3, b-a_2-a_4;\right.$$
$$\left.b-a_1-a_2+1; 1-x_2, \frac{1-x_2}{1-x_1}, \frac{(1-x_2)(1-x_4)}{(1-x_1)(1-x_3)}, \frac{1-x_2}{1-x_3}\right).$$

From (8.8.1), (8.9.1), (8.15.3) and (8.15.5), we have

(8.21.3) $$x_3^{-a_2}x_4^{-a_4}K^*\left(a_1, a_2, 1+a_2+a_4-b, a_4; 1+a_2+a_4-a_3; \frac{x_1}{x_4}, \frac{x_2}{x_3}, \frac{1}{x_3}, \frac{1}{x_4}\right),$$

(8.21.4) $$(1-x_3)^{-a_2}(1-x_4)^{-a_4}K^*\cdot$$
$$\cdot\left(a_1, a_2, b-a_1-a_3, a_4; 1+a_2+a_4-a_3; \frac{1-x_1}{1-x_4}, \frac{1-x_2}{1-x_3}, \frac{1}{1-x_3}, \frac{1}{1-x_4}\right),$$

(8.21.5) $$x_2^{-a_2}x_4^{-a_3}x_1^{1+a_1+a_3-b}K^*\cdot$$
$$\cdot\left(1+a_1+a_3-b, a_3, a_2, 1+a_2+a_4-b; 2+a_2+a_3-b; x_1, \frac{x_1}{x_4}, \frac{x_1x_3}{x_2x_4}, \frac{x_1}{x_2}\right)$$

and

$$(8.21.6)\qquad (1-x_1)^{-a_4}(1-x_2)^{b-a_2}(1-x_3)^{-a_3}K^*\Big(b-a_2-a_4, a_3, a_4, b-a_1-a_3;$$

$$b-a_2-a_3+1; 1-x_2, \frac{1-x_2}{1-x_3}, \frac{(1-x_2)(1-x_4)}{(1-x_1)(1-x_3)}, \frac{1-x_2}{1-x_1}\Big).$$

This system of integrals is valid in that part of the neighbourhood of the singular point $(0, 1, \infty, \infty)$ for which

$$|x_2 x_4| > |x_1 x_3| \quad \text{and} \quad |(1-x_1)(1-x_3)| > |(1-x_2)(1-x_4)|.$$

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