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Some Properties of Planar Polynomial Systems of Even Degree (*).

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Summary. – *We study autonomous systems in the plane of polynomial type. We obtain conditions for the existence of unbounded trajectories of such systems. As a consequence we prove that it does not exist a planar polynomial system of even degree with a global center.*

0. – Introduction.

Let us consider a system of ordinary differential equations

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{0.1}$$

where $x, y \in \mathbf{R}$ and $P(x, y), Q(x, y)$ are relatively prime real polynomials.

If $n = \max(\deg P(x, y), \deg Q(x, y))$ we shall denote (0.1) as a planar polynomial system of degree n. The main aim of this paper is to prove that, if n is even, the system (0.1) has at least one unbounded trajectory.

The most natural way to deal with this subject is to use the compactification by Poincaré sphere.

This tool leads us to a result (Theorem 2.7) which will show the existence of unbounded trajectories and will describe their behaviour at infinity, too. As a consequence we will obtain an extension of analogous results concerning homogeneous systems [SC] and systems of degree two [C1]; moreover we will be able to prove a conjecture [ICNO] according to which a polynomial system of even degree cannot have a global center (see definition below). Finally (Theorem 2.9) for a system of odd degree necessary conditions for the existence of a global centre are obtained.

Before starting to prove these results let us recall some fundamental definitions.

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A singular point $S \in \boldsymbol{R}^2$ of (0.1) is a *centre* if there exists a neighbourhood of S entirely filled by closed non singular trajectories except S. The *centre region*, N_S, is the maximal neighbourhood of S with respect to this property. Finally, a centre S of (0.1) is a *global center* if $N_S = \boldsymbol{R}^2$.

1. - Behaviour at infinity of a polynomial planar system.

Let us denote the system (0.1) as $(\boldsymbol{R}^2, (P, Q))$, too. Following the treatment given in [GV] of a classic procedure due to Poincaré, we shall define a dynamical system (S^2, π) on the 2-sphere S^2, and we shall estabilish its topological equivalence with $(\boldsymbol{R}^2, (P, Q))$. Then we shall deal with properties of the field $\pi_\infty : S^1 \mapsto TS^1$, where S^1 is the equator of the sphere S^2 and π_∞ is the restriction of π to S^1. Such properties will be very useful to understand the behaviour at infinity of the system (0.1).

Let $\omega = (\omega_1, \omega_2, \omega_3)$ be a coordinate system in $\boldsymbol{R}^3$, and let us define the submanifolds in $\boldsymbol{R}^3$:

$$\boldsymbol{R}^2: \quad \omega_3 = 1,$$
$$S^2: \quad \omega_1^2 + \omega_2^2 + \omega_3^2 = 1.$$

Let $x = \omega_1$ $y = \omega_2$ be a coordinate system in $\boldsymbol{R}^2$. An analytic manifold structure on S^2 is determined by the local charts (A_i, ϕ_i), (B_i, σ_i), $i = 1, 2, 3$ where:

$$A_i = [\omega \in S^2 : \omega_i > 0] \qquad B_i = [\omega \in S^2 : \omega_i < 0],$$

$$\phi_i : A_i \mapsto \boldsymbol{R}^2 \qquad \phi_i(\omega) = (\omega_j/\omega_i, \omega_k/\omega_i) = (z_1, z_2),$$

$$\sigma_i : B_i \mapsto \boldsymbol{R}^2 \qquad \sigma_i(\omega) = (\omega_j/\omega_i, \omega_k/\omega_i) = (z_1, z_2),$$

where $j < k$ and $j, k \in [1, 2, 3] \setminus [i]$. We shall denote by S^2_+ (north hemisphere) the set A_3 and S^2_- (south hemisphere) the set B_3. Moreover throughout this paper we shall denote the equator of S^2 as: $S^1 = [\omega \in S^2 : \omega_3 = 0]$. Let

$$h^+ : \boldsymbol{R}^2 \mapsto S^2_+ \quad h^+(x, y) = (1/\delta) \cdot (x, y, 1),$$
$$h^- : \boldsymbol{R}^2 \mapsto S^2_- \quad h^-(x, y) = -(1/\delta) \cdot (x, y, 1),$$

where $\delta = \delta(x, y) = [1/(x^2 + y^2 + 1)]^{1/2}$, be the central projection diffeomorphisms onto the two hemispheres. The induced fields:

$$dh^+ \begin{bmatrix} P \\ Q \end{bmatrix} : S^2_+ \mapsto TS^2_+, \qquad dh^- \begin{bmatrix} P \\ Q \end{bmatrix} : S^2_- \mapsto TS^2_-,$$

cannot be extended to S^1. On the contrary such an extension is possible for the fields:

$$\omega_3^{n-1} dh^+ \begin{bmatrix} P \\ Q \end{bmatrix} : S_+^2 \mapsto TS_+^2 , \qquad \omega_3^{n-1} dh^- \begin{bmatrix} P \\ Q \end{bmatrix} : S_-^2 \mapsto TS_-^2 ,$$

where n is the degree of system (0.1). With such a method we obtain a field π: $S^2 \mapsto TS^2$, the *Poincaré field* on S^2 relative to (0.1), with the following analytic expression in local coordinates (z_1, z_2):

$$\pi(z_1, z_2) = \begin{cases} \dfrac{z_2^n}{\nabla} \begin{bmatrix} Q\left(\dfrac{1}{z_2}, \dfrac{z_1}{z_2}\right) - z_1 P\left(\dfrac{1}{z_2}, \dfrac{z_1}{z_2}\right) \\ - z_2 P\left(\dfrac{1}{z_2}, \dfrac{z_1}{z_2}\right) \end{bmatrix}, & (z_1, z_2) \in A_1 , \\[2ex] \dfrac{z_2^n}{\nabla} \begin{bmatrix} P\left(\dfrac{1}{z_2}, \dfrac{z_1}{z_2}\right) - Q\left(\dfrac{z_1}{z_2}, \dfrac{1}{z_2}\right) \\ - z_2 Q\left(\dfrac{1}{z_2}, \dfrac{z_1}{z_2}\right) \end{bmatrix}, & (z_1, z_2) \in A_2 , \\[2ex] \dfrac{1}{\nabla} \begin{bmatrix} P(z_1, z_2) \\ Q(z_1, z_2) \end{bmatrix}, & (z_1, z_2) \in A_3 , \end{cases}$$

where $\nabla = (1 + z_1^2 + z_2^2)^{(n-1)/2}$.

REMARK 1.1. – The expression of $\pi(z_1, z_2)$ when $(z_1, z_2) \in B_i$ is obtained from the corresponding one in A_i except for a factor $(-1)^{n-1}$.

From the analytic expression of $\pi(z_1, z_2)$ it easily follows:

THEOREM 1.1 [GV]. – (*a*) Systems $\left(\boldsymbol{R}^2, \begin{bmatrix} P \\ Q \end{bmatrix}\right)$ and (S_+^2, π_+), where the field π_+ is the restriction of π to S_+^2, are topologically equivalent.

(*b*) The equator is an invariant set with respect to the flow induced on S^2 by π. ∎

From the property (*B*) of Theorem 1.1 we define

$$\pi_\infty(s) = \pi(s, 0)$$

and so the analytic expression of the field $\pi_\infty : S^1 \mapsto TS^1$ is the following:

$$\pi_\infty(s) = \begin{cases} \dfrac{1}{(1+s^2)^{1/2}} F(s) , & (s, 0) \in S^1 \cap A_1 , \\[2ex] \dfrac{1}{(1+s^2)^{1/2}} G(s) , & (s, 0) \in S^1 \cap A_2 , \end{cases}$$

where:

$$F(s) = Q_n(1, s) - sP_n(1, s), \qquad G(s) = P_n(s, 1) - sQ_n(s, 1).$$

Here, and throughout this paper, we denote by $Q_k(x, y)$ and $P_k(x, y)$, $0 \leq k \leq n$, the homogeneous part of degree k of polynomials $Q(x, y)$ and $P(x, y)$. Moreover, let us recall that, as a consequence of Remark 1.1, the definition of $\pi_\infty(s)$, when $(s, 0) \in S^1 \cap \cap B_i$, $i = 1, 2$, is obtained from the definition of $\pi_\infty(s)$ when $(s, 0) \in S^1 \cap A_i$ and by multiplication by the factor $(-1)^{n-1}$. This implies:

LEMMA 1.2. – Let $(s, 0)$, $(\bar{s}, 0) \in S^1$ be two antipodal points. Then:

(*a*) if n is odd: $\pi_\infty(s) \cdot \pi_\infty(\bar{s}) \geq 0$,

(*b*) if n is even: $\pi_\infty(s) \cdot \pi_\infty(\bar{s}) \leq 0$.

PROOF. – The proof is a straightforward consequence of the equality

$$\pi_\infty(\bar{s}) = (-1)^{n-1} \pi_\infty(s). \quad \blacksquare$$

Moreover the polynomial nature of the field π_∞ implies:

LEMMA 1.3 [GV] [SC]. – (*a*) If $(s, 0), (\bar{s}, 0) \in S^1$ are antipodal points and $\pi_\infty(s) = 0$ then $\pi_\infty(\bar{s}) = 0$.

(*b*) Only one of the following situations can happen:

(*b*.1) π_∞ is identically zero,

(*b*.2) the set $S = [(s, 0) \in S^1 \colon \pi_\infty(s) = 0]$ has k elements, $1 \leq k \leq 2(n+1)$.

(*c*) If n is even, π_∞ has at least two singular points.

(*d*) The following five properties are equivalent:

(*d*.1) π_∞ is identically zero,

(*d*.2) $F(s)$ is identically zero,

(*d*.3) $G(s)$ is identically zero,

(*d*.4) $xQ_n(x, y) - yP_n(x, y)$ is identically zero,

(*d*.5) if

$$P_n(x, y) = \sum_{k=0}^{n} a_{kn} x^{n-k} y^k \qquad Q_n(x, y) = \sum_{k=0}^{n} b_{kn} x^{n-k} y^k$$

then:

$$a_{nn} = b_{0n} = 0 \quad \text{and} \quad a_{kn} = b_{k+1, n} \text{ where } 0 \leq k \leq n-1. \quad \blacksquare$$

DEF. 1.1. – Let $(\alpha, 0) \in S^1$ be an isolated singular point of π_∞, and let W_α^s and W_α^u be respectively the stable and unstable manifold of $(\alpha, 0)$ in S^1 with respect to π_∞. We say that

(*a*) α is *semistable* if $W_\alpha^s \neq \emptyset$ and $W_\alpha^u \neq \emptyset$:

(*b*) α is *stable* if $W_\alpha^s \neq \emptyset$ and $W_\alpha^u = \emptyset$:

(*c*) α is *unstable* if $W_\alpha^s = \emptyset$ and $W_\alpha^u \neq \emptyset$:

LEMMA 1.4. – Let (0.1) be a system of even degree n, and let the relative field π_∞ be non identically zero. Then the number h of non semistable singular points of π_∞ verifies the equality:

$$h = 2 + 4\nu \quad \text{where } \nu = 0, 1, 2, \ldots .$$

PROOF. – In view of the polynomial nature of the field $\pi_\infty(s)$ it is possible to define the multiplicity of the couple of antipodal points $(s, 0), (\bar{s}, 0) \in S^1$ as singular points of $\pi_\infty(s)$ by their multiplicity as solutions of polynomials $F(s)$ and/or $G(s)$. Moreover it is simple to see that a singular point of $\pi_\infty(s)$ is not semistable if and only if it is a singular point of multiplicity of π_∞.

To each couple of antipodal singular points $(s, 0), (\bar{s}, 0) \in S^1$ of $\pi_\infty(s)$, with multiplicity ν, is associated a straight line $ax + by = 0$ $a, b \in \boldsymbol{R}$, which is a solution of multiplicity ν relatively to the homogeneous equation

$$H(x, y) = xQ_n(x, y) - yP_n(x, y) = 0 .$$

It is well known [W] that $H(x, y)$, being an homogeneous $(n+1)$-form, can be expressed as

$$H(x, y) = \left[\prod_{i=1}^{p} (a_i x + b_i y)^{\nu_i}\right] \cdot \left[\prod_{j=1}^{q} (c_j x^2 + d_j xy + l_j y^2)^{\lambda_j}\right]$$

where:

$$n + 1 = \sum_{i=1}^{p} \nu_i + 2 \sum_{j=1}^{q} \lambda_j .$$

If n is even the number k of the real solutions of $H(x, y) = 0$ with odd multiplicity is odd. From the equality $h = 2k$ the assertion of the lemma follows. ∎

2. – Behaviour at infinity of the trajectories and system degree.

In this section we will prove a conjecture in [ICNO]. Let us define π_+ and π_- respectively as the restriction of the Poincaré field to S^2_+ and to S^2_-.

LEMMA 2.1. – Let $(\alpha, 0) \in S^1$ be an isolated and non semistable singular point of π_∞. Then there exists a trajectory of (S^2_+, π_+) and a trajectory of (S^2_-, π_-) both having $[(\alpha, 0)]$ as a limit set.

PROOF. – With no loss of generality we can suppose that $(\alpha,0)$ is a stable singular point of π_∞. Let U be a neighbourhood of $(\alpha, 0)$ in S^2 such that $(\alpha, 0)$ is the only singular point of π in U. Let Φ: $U \mapsto \boldsymbol{R}^2$ be a local coordinate such that $\Phi(z_1, z_2) = (u, v)$ and:

$$\Phi(\alpha, 0) = (0, 0), \qquad \chi(u, v) :\equiv \pi[\Phi^{-1}(u, v)], \qquad \Phi(S^1 \cap U) \subset [(u, v)\colon v = 0].$$

Let us consider a rectangle R, $R \subset \Phi(U)$, which is centered at $(0, 0)$ and with half sides of length 3a and b $(a, b > 0)$. Let ϕ: $R \mapsto \boldsymbol{R}$ be a $C^\infty(R)$ function such that:

$$\begin{aligned} &\phi(u, v) = 1 && \text{if } |u| \geq 2a, \\ &\phi(u, v) = 0 && \text{if } |u| \leq a, \\ &0 \leq \phi(u, v) \leq 1 && \text{if } (u, v) \in R, \\ &\frac{\partial \phi}{\partial v}(u, v) = 0 && \text{if } (u, v) \in R. \end{aligned}$$

Finally let us consider two segments L_1 and L_2 in R with endpoints $(-2a, -b)$, $(-2a, b)$ and $(2a, -b)$, $(2a, b)$ respectively. Let U_1 and U_2 be two neighbourhoods of L_1 and L_2 respectively, and σ_1: $U_1 \mapsto TU_1$, σ_2: $U_2 \mapsto TU_2$ the two constant vector fields:

$$\sigma_1(u, v) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \sigma_2(u, v) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

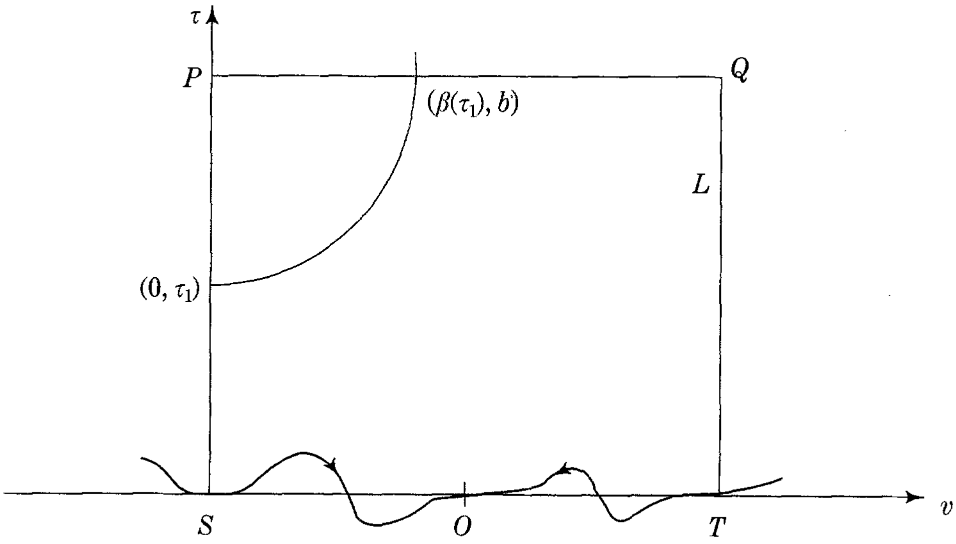
By the Rectification Theorem [A] it is possible to choose U_1 and U_2 sufficiently small such that σ_1 and σ_2 are locally diffeomorphic to the vector field $\chi(u, v)$.

If $\sigma: R \mapsto TR$ is any differentiable extension of σ_1 and σ_2 on the whole rectangle R, we can define the vector field on R:

$$\Theta(u, v) = \phi(u, v)\,\sigma(u, v) + [1 - \phi(u, v)]\,\chi(u, v),$$

which is topologically equivalent in R to $\chi(u, v)$. Thus it is enough to prove the assertion of the lemma relatively to the field $\Theta(u, v)$. Let us consider the points in R: $P = (-2a, b)$, $Q = (2a, b)$, $O = (0,0)$, $S = (-2a, 0)$, $T = (2a, 0)$; and let us define L as the inner region of the closed curve formed by the segments PS, PQ, QT, by the arcs of trajectories through S and T and by the point O.

Let (ν, τ) be new coordinates in the region L, with origin S and such that, for example, $P = (0, b)$ and $Q = (4a, b)$. If $g(t, (\nu, \tau))$ is the flow induced by $\Theta(u, v)$ on R, we define the functions:



$\tau \mapsto \beta(\tau)$ $\quad \tau \mapsto \gamma(\tau)$

where $\tau \in [0, b]$, as:

$\beta(\tau) = \nu$-coordinate of the first intersection point between the segment PQ and the trajectory $t \mapsto g(t, (0, \tau))$;

$\gamma(\tau) = \nu$-coordinate of the first intersection point between the segment PQ and the trajectory $t \mapsto g(t, (4a, \tau))$.

Both $\beta(\tau)$ and $\gamma(\tau)$ are monotone functions, and so we can define:

$$\beta_0 = \lim_{\tau \mapsto 0} \beta(\tau), \qquad \gamma_0 = \lim_{\tau \mapsto 0} \gamma(\tau).$$

Let us suppose that the trajectory $t \mapsto g(t, (\beta_0, b))$ has not the set [O] as its limit set. Thus by the Poincaré-Bendixson Theorem there exists $t_0 > 0$ such that:

$$g(t_0, (\beta_0, b)) = (\bar{\nu}, b + \varepsilon),$$

where $\varepsilon > 0$. Moreover we can suppose that $0 \leqslant \bar{\nu} \leqslant 4a$. Indeed if $\bar{\nu} < 0$ it follows that:

$$\beta_0 = g(t', (0, \tau_1)) \qquad t' > 0 \qquad 0 \leqslant \tau_1 \leqslant b$$

and if $\bar{\nu} > 4a$ it follows that:

$$\beta_0 = g(t'', (4a, \tau_2)) \qquad t'' > 0 \qquad 0 \leqslant \tau_2 \leqslant b$$

and both these conditions contradict the definition of β_0. Now, from the continuous dependence of the solutions of ordinary differential equations on the initial data of a

Cauchy problem, there exists a neighbourhood $U_{t_0,\varepsilon}$ of the point (β_0, b) such that, if $(\nu, \tau) \in U_{t_0,\varepsilon}$ then:

$$\|g(t,(\nu,\tau)) - g(t,(\beta_0,b))\| < \frac{\varepsilon}{2},$$

for every $t \in \boldsymbol{R}$ such that $t_0 \leqslant t \leqslant 0$.

This inequality implies that every trajectory $t \mapsto g(t,(\nu,\tau))$, where $(\nu,\tau) \in U_{t_0,\varepsilon}$, must intersect backwards the segment PQ, and this is a contradiction with the definition of β_0. This conclusion proves that there exists at least one trajectory of the field $\Theta(u,v)$ in R which has [O] as its limit set and, finally, this proves the lemma by virtue of the definition of the field $\Theta(u,v)$. ∎

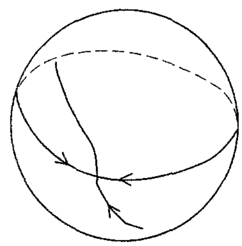
COROLLARY 2.2. – If $\alpha \in \boldsymbol{R}$ is a root of odd multiplicity of $F(s)=0$ and/or $G(s)=0$, there exist a trajectory of (S^2_+, π_+) and a trajectory of (S^2_-, π_-) both having $[(\alpha,0)]$ as their limit set.

PROOF. – The assertion is a consequence of Lemma 2.1 and of the equivalence between the semistability condition of $(\alpha,0)$ as a singular point of π_∞ and the odd multiplicity of α as a root of $F(s)=0$ and/or $G(s)=0$. ∎

COROLLARY 2.3. – Let (0.1) be a system of even degree n, such that π_∞ is not identically zero. Then there exist a trajectory of (S^2_+, π_+) and a trajectory of (S^2_-, π_-), both having $[(\alpha,0)]$ as their limit set.

PROOF. – This corollary is still a consequence of Lemma 2.1, and of Lemma 1.3, which implies the existence, for even degree systems, of two singular and not semistable points of π_∞, at least. ∎

REMARK 2.1. – By using the symmetry of the phase portrait on the Poincaré sphere we can suggest a qualitative description of the Poincaré field of systems verifying the hypotheses of Corollary 2.2 and Corollary 2.3:



We shall now consider the case where π_∞ is identically zero: we shall prove that in such situation, independently from the degree n of the system (0.1), there exist many points on S^1 (actually they form an infinite subset of the equator of the Poincaré sphere) which constitute the limit sets of trajectories of (S^2_+, π_+) or of (S^2_-, π_-).

Let us introduce some notations. Let $\boldsymbol{RP}^2$ be the projective plane and let $\boldsymbol{RP}$ be the projective line, which we always consider as an embedded submanifold of $\boldsymbol{RP}^2$. The sphere S^2 is a double covering of $\boldsymbol{RP}^2$ and the equator S^1 of the sphere is a double covering of $\boldsymbol{RP}$. Therefore if $(\xi_1, \xi_2, 0) \in \boldsymbol{RP} \subset \boldsymbol{RP}^2$, with (x, y, z) homogeneous coordinates in $\boldsymbol{RP}^2$, we have correspondingly two points in S^1 (the points of the fiber of the double covering): by using the coordinate system we chose on S^1, such points are, for instance, $(\alpha, 0) = (\xi_2/\xi_1, 0)$ and the diametrically opposed one.

If $(\xi_1, \xi_2, 0) \in \boldsymbol{RP}$, we can consider the vector $\xi = (\xi_1, \xi_2)$ and its orthogonal vector $\xi^\perp = (-\xi_2, \xi_1)$. Let us define:

$$G_\xi(x, y) = \left\langle \begin{bmatrix} P \\ Q \end{bmatrix}, \xi^\perp \right\rangle = -\xi_2 P(x, y) + \xi_1 Q(x, y)$$

(here $\langle\ ,\ \rangle$ denotes the scalar product in $\boldsymbol{R}^2$).

The algebraic curve:

$$\phi_\xi \colon \xi_2 P(x, y) - \xi_1 Q(x, y) = 0\,,$$

is the ξ-isocline of (0.1). The map $\xi \mapsto \phi_\xi$ defines a bundle of algebraic curves of degree n in $\boldsymbol{RP}^2$: the isoclines of system (0.1). As an elementary consequene of the definition of polynomials $F(s)$ and $G(s)$, and by simple properties of a bundle of algebraic curves, we state:

LEMMA 2.4. – (*a*) Let $(\alpha, 0) \in S^1$, with $\alpha = \xi_2/\xi_1$ or $\alpha = \xi_1/\xi_2$. Then $(\alpha, 0)$ is a singular point of π_∞ if and only if $(\xi_1, \xi_2, 0)$ is an intersection in $\boldsymbol{RP}^2$ between $\boldsymbol{RP}$ and ϕ_ξ (or, as we will use to say, if $(\xi_1, \xi_2, 0)$ is a point at infinity of ϕ_ξ).

(*b*) The point $(\xi_1, \xi_2, 0) \in \boldsymbol{RP}$ is a base-point of the bundle of isoclines of system (0.1) (*i.e.* a point which belongs to every isocline of the bundle) if and only if $P_n^2(\xi_1, \xi_2) + Q_n^2(\xi_1, \xi_2) = 0$. ■

Now we give another condition of existence of unbounded trajectories for system of type (0.1):

LEMMA 2.5. – Let $(\xi_1, \xi_2, 0) \in \boldsymbol{RP}$ such that:

(*a*) $(\xi_1, \xi_2, 0)$ is a point of r-multiplicity, $r \geq 1$, of the isocline ϕ_ξ,

(*b*) the projective line is a component of l-multiplicity, $0 \leq l < r$, of the isocline ϕ_ξ,

(*c*) $(\xi_1, \xi_2, 0)$ is not a base-point of the bundle of isoclines of system (0.1),

(*d*) $r \equiv (l+1) \bmod 2$.

Then, if $\alpha = \xi_2/\xi_1$ or $\alpha = \xi_1/\xi_2$, at least one of the two antipodal points, $(\alpha, 0)$ and $(\bar{\alpha}, 0)$, constitutes the limit set of a trajectory of one of the systems (S^2_+, π_+), (S^2_-, π_-).

REMARK 2.2. – The «geometric» hypotheses (a)-(c) have the following algebraic interpretation. Let:

$$P(x,y)=\sum_{k=0}^{n}P_k(x,y)=\sum_{k=0}^{n}\sum_{h=0}^{k}a_{hk}x^{k-h}y^h,$$

$$Q(x,y)=\sum_{k=0}^{n}Q_k(x,y)=\sum_{k=0}^{n}\sum_{h=0}^{k}b_{hk}x^{k-h}y^h.$$

Then hypotheses (a)-(c) are equivalent to hypotheses $(a)'$-$(c)'$, where:

$(a)'$ let $0\leqslant\lambda\leqslant r-1$ and $0\leqslant k\leqslant n$. For every $(l_1,l_2)\in\boldsymbol{N}_0\times\boldsymbol{N}_0$ such that $\lambda=l_1+l_2+n-k$ it results that:

$$\sum_{k=0}^{n}\sum_{h=0}^{k}\binom{k-h}{l_1}\cdot\binom{h}{l_2}l_1!\,l_2!\,(\xi_2a_{hk}-\xi_1b_{hk})\,\xi_1^{k-h-l_1}\cdot\xi_2^{h-l_2}=0$$

and moreover there exists $((\bar l_1,\bar l_2)\in\boldsymbol{N}_0\times\boldsymbol{N}_0$ such that $\bar l_1+\bar l_2+n-k=r$ and

$$\sum_{k=0}^{n}\sum_{h=0}^{k}\binom{k-h}{\bar l_1}\cdot\binom{h}{\bar l_2}\bar l_1!\,\bar l_2!\,(\xi_2a_{hk}-\xi_1b_{hk})\,\xi_1^{k-h-\bar l_1}\cdot\xi_2^{h-\bar l_2}=0\,;$$

$(b)'$ it results that: $\xi_2a_{hk}-\xi_1b_{hk}=0$ when $0\leqslant h\leqslant k$, $n-l+1\leqslant k\leqslant n$ and there exists $\bar h\in\boldsymbol{N}_0$, such that: $\xi_2a_{\bar h,n-l}-\xi_1b_{\bar h,n-l}\neq0$;

$(c)'$ (a consequence of Lemma 2.4): $P_n^2(\xi_1\xi_2)+Q_n^2(\xi_1,\xi_2)>0$.

PROOF OF LEMMA 2.5. – From Lemma 2.4 each of the points $(\alpha,0)$ and $(\bar\alpha,0)$ in the statement of the lemma is a singular point of the Poincaré field π. From hypotheses (b) the isocline ϕ_ξ has the following equation in projective coordinates:

$$z^l\sum_{k=0}^{n-l}(\xi_2a_{hk}-\xi_1b_{hk})\,x^{k-h}y^hz^{n-k-l}=0\,.$$

Let us denote ϕ'_ξ the algebraic curve of order $n-l$ with equation

$$\sum_{k=0}^{n-l}(\xi_2a_{hk}-\xi_1b_{hk})\,x^{k-h}y^hz^{n-k-l}=0\,.$$

By hypothesis (a), $(\xi_1,\xi_2,0)\in\phi'_\xi$. It is well known that a real point of an algebraic curve in $\boldsymbol{RP}^2$ is topologically isolated only if it has an even multiplicity: so by hypothesis (d) there exists a real branch R_ξ of ϕ'_ξ, and so a real branch of ϕ_ξ, too, with the following properties:

(α) $(\xi_1,\xi_2,0)\in R_\xi$,

(β) the points of R_ξ are points of odd multiplicity of ϕ_ξ.

Let r^+ and $\bar r^+$ be two parallel straight lines in $\boldsymbol{RP}^2$, with $(\xi_1,\xi_2,0)\in r^+$. We suppose to choose r^+ and $\bar r^+$ in such a way that R_ξ is the only branch of ϕ_ξ inside the half strip included between r^+ and $\bar r^+$. Of course we can neglect the case $r^+\subset R_\xi$ or $\bar r^+\subset R_\xi$

because in this situation r^+ or $\bar{r}^+$ should contain an unbounded trajectory and the assertion of the lemma would be immediate.

By hypothesis (*c*) it is also possible to choose $r^+, \bar{r}^+$ and a segment l, orthogonal to r^+, such that the half strip L, with boundary $\partial L = r^+ \cup \bar{r}^+ \cup l$, contains R_ξ and no other real points of ϕ_ξ, and does not contain any point of the isocline $\phi_{\xi^\perp}$. By the definition of the isocline ϕ_ξ:

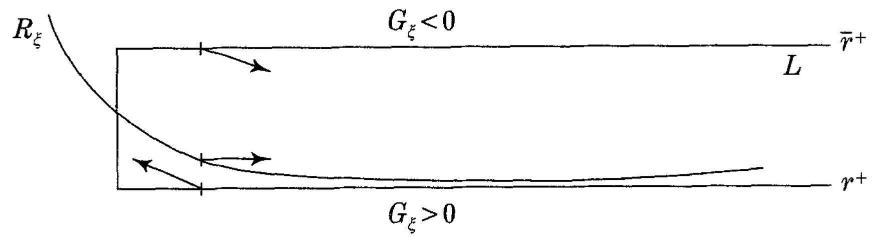
$$\xi_2 P(x,y) - \xi_1 Q(x,y) = G_\xi(x,y) = \left\langle \begin{pmatrix} P \\ Q \end{pmatrix}, \xi^\perp \right\rangle = 0$$

and by virtue of property (β) it follows that if $(x,y) \in r^+$ and $(x',y') \in \bar{r}^+$ then:

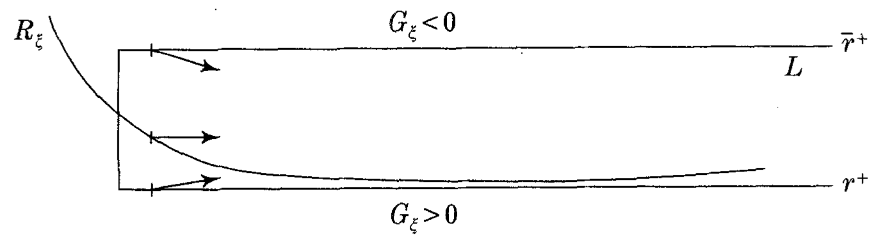
$$G_\xi(x,y)\, G_\xi(x',y') < 0\,.$$

According to this condition, and except from the inversion $t \mapsto -t$, there are only the following three possible situations:

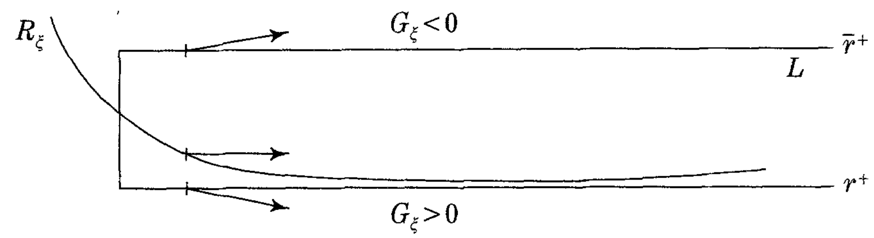
(I)



(II)



(III)



By hypothesis (*c*) and the definition of L it is easy to prove that the situation in fig. (I) is not possible. In the situation of fig. (II) it is clear (see, for instance, [L], pag. 213) that every trajectory through a point in L has an image in (S^2_+, π_+) or in (S^2_-, π_-) having $[(\alpha, 0)]$ or $[(\bar{\alpha}, 0)]$ as limit set. The situation sketched by fig. (III) implies (see still [L], pag. 215) the existence of at least a trajectory of (S^2_+, π_+) or of (S^2_-, π_-), having $[(\alpha, 0)]$ or $[(\bar{\alpha}, 0)]$ as limit set. So the proof of the lemma is complete. ■

We can now study the case of systems of type (0.1), satisfying $\pi_\infty \equiv 0$.

LEMMA 2.6. – Let us consider a system of type (0.1), and let the relative field π_∞ be identically zero. Let us denote $\mathfrak{Z} = [(\xi_1, \xi_2, 0) \in \boldsymbol{RP}\colon P^2_n(x, y) + Q^2_n(x, y) = 0]$. Then:

(*a*) the set $\mathfrak{Z}$ has k elements, $0 \leq k \leq n-1$,

(*b*) for every $\xi \in \boldsymbol{RP} \setminus \mathfrak{Z}$, $\xi = (\xi_1, \xi_2, 0)$, one of the two antipodal points $(\alpha, 0)$, $(\bar{\alpha}, 0) \in S^1$, $\alpha = \xi_2/\xi_1$ or $\alpha = \xi_1/\xi_2$, forms the limit set of at least a trajectory of (S^2_+, π^+) or of (S^2_-, π_-).

PROOF. – The proof consists in verifying that if $\xi \in \boldsymbol{RP} \setminus \mathfrak{Z}$ the hypotheses of Lemma 2.5 are satisfied. Let $\xi = (\xi_1, \xi_2, 0) \in \boldsymbol{RP}$ be a fixed point. Another point $\eta = (\eta_1, \eta_2, 0) \in \boldsymbol{RP}$ belongs to the isocline ϕ_ξ if and only if (η_1, η_2) is a solution of the homogeneous equation:

$$\xi_2 P_n(x, y) - \xi_1 Q_n(x, y) = 0 .$$

By virtue of Lemma 1.2 and by the hypothesis that $\pi_\infty \equiv 0$, the above equation is equivalent to:

$$\xi_2 \sum_{k=0}^{n-1} a_{kn} x^{n-k} y^k - \xi_1 \sum_{k=1}^{n-1} a_{k-1,n} x^{n-k} y^k = 0 ,$$

or:

$$(2.1) \qquad (\xi_2 x - \xi_1 y) \sum_{k=0}^{n-1} a_{kn} x^{n-k} y^k = 0 .$$

By equation (2.1) the intersection points between ϕ_ξ and $\boldsymbol{RP}$ are, in addition to $(\xi_1, \xi_2, 0)$, the solutions of:

$$(2.2) \qquad \sum_{k=0}^{n-1} a_{kn} x^{n-k-1} y^k = 0 .$$

This is an $(n-1)$-form which cannot by identically zero (in such case $P_n(x, y) \equiv \equiv Q_n(x, y) \equiv 0$) so, by (2.1) it follows that the points in $\boldsymbol{RP}$ belonging to every isocline ϕ_ξ are the points corresponding to the k solutions of (2.2), with $0 \leq k \leq n-1$.

This conclusion, together with Lemma 2.4, proves the statement (*a*).

We can now observe that every $\xi \in \boldsymbol{RP} \setminus \mathfrak{Z}$ is a simple point (*i.e.* with multiplicity one) of the isocline ϕ_ξ. Moreover no isocline ϕ_ξ can have the projective line $\boldsymbol{RP}$ as its component. Indeed if $\boldsymbol{RP}$ were a component of ϕ_ξ, $\xi = (\xi_1, \xi_2, 0)$, it ought to be satisfied the identity:

$$\xi_2 P_n(x, y) \equiv \xi_1 Q_n(x, y),$$

or

$$\xi_2 a_{kn} = \xi_1 b_{kn}, \qquad 0 \leqslant k \leqslant n$$

and by Lemma 1.3 and the hypothesis $\pi_\infty \equiv 0$ this implies that $P_n(x, y) \equiv Q_n(x, y) \equiv 0$, impossible.

Finally, every $\xi \in \boldsymbol{RP} \setminus \mathfrak{Z}$ verifies hypotheses (*a*)-(*d*) of Lemma 2.5, and the assertion follows. ■

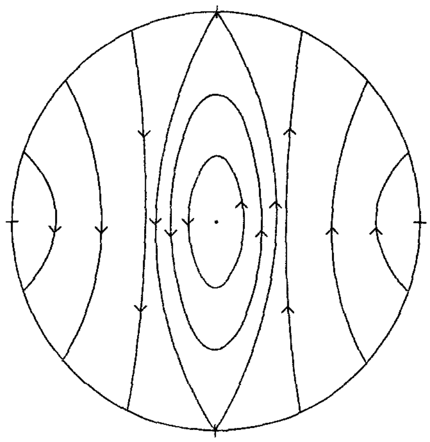
REMARK 2.3. – In general it is not possible to extend Lemma 2.6 and to claim that the same results are true for every $\xi \in \boldsymbol{RP}$. An example is the system [C2]:

$$\dot{x} = -y + x^2 y, \qquad \dot{y} = x + xy^2.$$

The trajectories of the system are (contained in) the conics

$$x^2 + (1 - r^2) y^2 = r^2 \qquad r \in \boldsymbol{R}^2.$$

This system satisfies the condition $\pi_\infty \equiv 0$. Moreover the points $(0, 1, 0)$, $(1, 0, 0) \in \boldsymbol{RP}$ verify $P_3^2(x, y) + Q_3^2(x, y) = 0$. The points in S^1 corresponding to $(0, 1, 0)$ are limit sets of trajectories of (S_+^2, π_+) and of (S_-^2, π_-); on the contrary no one of the two antipodal points in S^1 which correspond to $(1, 0, 0)$ is the limit set of a trajectory of (S_+^2, π_+) or of (S_-^2, π_-) (see the following figure):



By using Corollary 2.3 and Lemma 2.6 we can state our main result:

THEOREM 2.7. – Every system of type (0.1), with even degree n, and with $\pi: S^2 \mapsto TS^2$ as the associated Poincaré field on S^2, has at least one trajectory of (S_+^2, π_+) and one trajectory of (S_-^2, π_-) having a point of the equator of S^2 as a limit set. ■

By using this theorem it is easy to prove the following statement, which represents a conjecture in [ICNO]:

COROLLARY 2.8. – A system of type (0.1) and even degree is such that the equator S^1 of the Poincaré sphere cannot be the limit set of any trajectory of $(S^2_+ \cup S^2_-, \pi_{+-})$ (here π_{+-} is the restriction of the Poincaré field on $S^2_+ \cup S^2_-$), nor can be the accumulation set of closed trajectories. In particular, even degree polynomial planar systems cannot have a global centre. ∎

REMARK 2.4. – For every odd integer n there exist polynomial planar systems of degree n with a global centre. For instance [C2]:

$$\dot{x} = y^n, \qquad \dot{y} = -x^n \qquad n \text{ any odd integer},$$

has a global centre at the origin.

REMARK 2.5. – It is well known [GV] that the equator S^1 of the Poincaré sphere cannot be a closed trajectory of the system (S^2, π) associated to a system (0.1) with even degree. It might seem natural trying to prove Corollary 2.8 using this property. But, for instance the proposition:

«If (0.1) has a global centre, the equator of the Poincaré sphere is a closed trajectory for the associated Poincaré field»

is false, as shown by ([SC], pag. 87):

$$\dot{x} = y, \qquad \dot{y} = -2x^3,$$

which has a global center at the origin and two antipodal singular points on the equator of the Poincaré sphere.

Finally, we state some necessary conditions for the existence of global centers of polynomial systems:

THEOREM 2.9. – Necessary conditions for the existence of a global center of a system of type (0.1) and degree n are:

(*a*) n is odd;

(*b*) the coefficients of $P_n(x, y)$ and $Q_n(x, y)$ do not verify:

$$a_{nn} = b_{0n} = 0, \qquad a_{kn} = b_{k+1,n} \qquad 0 \leq k \leq n-1;$$

(*c*) polynomials $F(s)$ and $G(s)$ have not real roots of odd multiplicity;

(*d*) there does not exist any point $\xi \in \boldsymbol{RP}$ which verifies hypotheses

(*a*)-(*d*) of Lemma 2.5 (or, in alternative, the hypotheses (*a*)′-(*d*)′ of Remark 2.2).

PROOF. – Statement (*a*) follows from Corollary 2.8, and statement (*b*) is a consequence of Lemma 2.6. The statement (*c*) is essentially the Corollary 2.2, and the statement (*d*) follows from Lemma 2.5. ■

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