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A Generalization of a Theorem by C. Miranda (*) (**).

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Summary. – *In this paper we study the well-posedness of the Dirichlet problem for an elliptic non divergence form second order equation. The coefficients are not assumed to be continuous but their derivatives are supposed to belong to a suitable Morrey space hence generalizing a classical result by C. Miranda.*

Introduction.

In this paper we consider the uniformly elliptic equation in non divergence form

$$Lu \equiv \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} = f \tag{0}$$

in Ω, a bounded open set of $\boldsymbol{R}^n$.

We look for strong solutions of the Dirichlet problem, i.e. solutions from class $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ (for precise statement see Section 2).

As it is well known (see [8]) the assumption $a_{ij} \in W^{1,n}$ gives the wellposedness of the Dirichlet problem for equation (0) with $f \in L^2$.

This is an optimal result among the L^p spaces because for $a_{ij} \in W^{1,n-\varepsilon}$ $(\varepsilon > 0)$, there are examples showing the non uniqueness of the solution. An improvement of the Miranda's classical result has been given in [1] assuming the $(a_{ij})_{x_s}$, $i, j, s = 1, \ldots, n$ in the weak L^n space with an additional smallness condition for the weak norm of the coefficients. Our result is along the same line but with a different scale of spaces.

More precisely, assuming that $(a_{ij})_{x_s}$, $i, j, s = 1, \ldots, n$, belong to $VL^{2p, n-2p}$, $p > 1$, a convenient subspace of the classical Morrey space $L^{2p, n-2p}$ (for the definition see

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Section 1) we prove an existence and uniqueness result for the Dirichlet problem (Th. 2.1).

The belonging of $(a_{ij})_{x_s}$ to $VL^{2p,n-2p}$ does neither imply the continuity of the coefficients a_{ij}, nor the fact that their derivatives are in the weak L^n (or a fortiori in L^n) as shown in Example 2.1.

Our result rests on some imbedding theorems proved by C. FEFFERMAN in [5] (see also [2], [3], [9]). These are recalled with some consequences and remarks in Section 1. In Section 2 and 4 we study the nondivergence equation (0) proving our main result, while in Section 3 we give some auxiliary results concerning a related divergence form equation.

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1. – Some function spaces.

Let Ω be a bounded open subset of $\boldsymbol{R}^n$ such that

$$|\Omega(x,r)| \equiv |\{y \in \Omega: |x-y|<r\} \geq Ar^n \,(^1)$$

for $r \in]0, \delta[$, where δ is the diameter of Ω and A some positive constant independent of r and x.

For $\lambda \in]0, n[$, $1 \leq p < +\infty$ we set

$$\|u\|_{p,\lambda} = \sup_{\substack{x\in\Omega \\ r\in]0,\delta[}} \left[\frac{1}{r^\lambda}\int_{\Omega(x,r)} |u(y)|^p\,dy\right]^{1/p}. \tag{1.1}$$

The subspace of function in L^p such that (1.1) is finite in the Morrey space $L^{p,\lambda}(\Omega)$.

For $r \in]0, \delta[$ we set

$$\sup_{\substack{x\in\Omega \\ \rho\in]0,r[}} \frac{1}{\rho^\lambda}\int_{\Omega(x,\rho)} |u(y)|^p\,dy = \eta^p(r) \tag{1.2}$$

and we say that $u \in VL^{p,\lambda}(\Omega)$ if $\lim_{r\to 0}\eta(r) = 0$. We will refer to $\eta(r)$ in (1.2) as the $VL^{p,\lambda}$ modulus of u.

Similarly, for $u \in L^p(\Omega)$, we set

$$\sup_{\substack{|E|\leq\sigma \\ E\subseteq\Omega}} \int_E |u(x)|^p\,dx = \omega^p(\sigma).$$

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(1) If $E \subseteq \boldsymbol{R}^n$ is Lebesgue measurable we set $|E|$ for its Lebesgue measure.

Clearly $\omega(\sigma)$ is a decreasing function in $]0, |\Omega|]$ such that $\lim_{\sigma\to 0}\omega(\sigma) = 0$. We will refer to $\omega(\sigma)$ as the AC modulus of $|u|^p$.

We will need for further developments some properties of $VL^{p,\lambda}(\Omega)$ which we state in the following lemmas.

LEMMA 1.1. – *Given $f \in VL^{p,\lambda}(\Omega)$ and $\varepsilon > 0$ there exist two functions f_1, f_2 such that*

$$f = f_1 + f_2; \qquad f_2 \in L^\infty(\Omega); \qquad \|f_1\|_{p,\lambda} < \varepsilon .$$

PROOF. – Let $\bar r$ such that $\eta(\bar r) < (\varepsilon/2)^p$. Letting $A_k = \{x \in \Omega: |f(x)| > k\}$ we have

$$\lim_{K\to+\infty} \int_{A_K} |f(x)|^p\, dx = 0$$

and then we can find $\overline{K} > 0$ such that

$$\int_{A_{\overline{K}}} |f(x)|^p\, dx < \left(\frac{\varepsilon}{2}\right)^p \bar r^{\lambda}.$$

Now let $f_1 = f\chi_{A_{\overline{K}}}$, $f_2 = f - f_1$, where $\chi_{A_{\overline{K}}}$ is the characteristic function of $A_{\overline{K}}$. Obviously $f_2 \in L^\infty(\Omega)$ and $\|f_2\|_\infty \leq \overline{K}$. Moreover

$$\|f_1\|_{p,\lambda} = \sup_{\substack{x\in\Omega\\ \rho\in]0,\delta[}} \left[\frac{1}{\rho^\lambda}\int_{\Omega(x,\rho)} |f_1(y)|^p\, dy\right]^{1/p} \leq \sup_{\substack{x\in\Omega\\ \rho\in]0,\bar r]}} \left[\frac{1}{\rho^\lambda}\int_{\Omega(x,\rho)} |f_1(y)|^p\, dy\right]^{1/p} +$$

$$+ \sup_{\substack{x\in\Omega\\ \rho\in]\bar r,\delta[}} \left[\frac{1}{\rho^\lambda}\int_{\Omega(x,\rho)} |f_1(y)|^p\, dy\right]^{1/p} < \frac{\varepsilon}{2} + \left[\frac{1}{\bar r^\lambda}\int_{A_{\overline{K}}} |f_1(y)|^p\, dy\right]^{1/p} < \varepsilon .$$

LEMMA 1.2. – *Let $f \in VL^{p,\lambda}(\Omega)$*(2)*. Then* $\lim_{y\to 0} \|f(x-y) - f(x)\|_{p,\lambda} = 0$.

PROOF. – We have, for $\bar r \in]0, \delta[$,

$$\|f(x-y) - f(x)\|^p_{p,\lambda} = \sup_{\substack{z\in\Omega\\ r\in]0,\delta[}} \frac{1}{r^\lambda}\int_{\Omega(z,r)} |f(x-y) - f(x)|^p\, dx \leq$$

$$\leq \sup_{\substack{z\in\Omega\\ r\in]0,\bar r[}} \frac{1}{r^\lambda}\int_{\Omega(z,r)} |f(x-y) - f(x)|^p\, dx + \frac{1}{\bar r^\lambda}\int_{\Omega} |f(x-y) - f(x)|^p\, dx .$$

(2) We extend f to $\boldsymbol{R}^n$ setting $f(x) = 0$ for $x \in \boldsymbol{R}^n \setminus \Omega$.

The first term on the right hand side can be made small by the assumption and the second is small, by the continuity of translation in L^p, if we take $|y|$ small.

By this lemma and a known result (see [12]) we have that if f belongs to $VL^{p,\lambda}(\Omega)$, the usual mollifiers converge to f in the $L^{p,\lambda}$ norm. In other words, given any $f \in VL^{p,\lambda}(\Omega)$ with $VL^{p,\lambda}$ modulus $\eta(r)$, it is possible to find a family of C^∞ functions $\{f_h\}_{h>0}$ converging to f in $L^{p,\lambda}$ and with their $VL^{p,\lambda}$ moduli $\eta_h(r) \leq \eta(r)$.

The following theorems firstly proved by C. FEFFERMAN, is well known.

THEOREM 1.1 ([5], [2], [3]). – *Let $v \in L^{p,n-2p}(\Omega)$ for some p such that $n/2 \geq p > 1$. Then, for any $u \in C_0^\infty(\Omega)$,*

$$\int_\Omega u^2(x)\, v(x)\, dx \leq C_0 \|v\|_{p,n-2p} \int_\Omega |Du(x)|^2\, dx\,,$$

where C_0 is a positive constant depending on n and p only (see [3]).

From Theorem 1.1 and Lemma 1.1 we deduce

THEOREM 1.2. – *Let $n/2 \geq p > 1$, $v \in VL^{2p,n-2p}(\Omega)$ with $VL^{2p,n-2p}$ modulus η and AC modulus of $|v|^{2p}\,\omega$. Then, for any $\varepsilon > 0$, it exists $K(\varepsilon) > 0$ such that*

$$\int_\Omega u^2(x)\, v(x)\, dx \leq \varepsilon \|Du\|_2^2 + K(\varepsilon)\|u\|_2^2 \tag{1.3}$$

for all $u \in C_0^\infty(\Omega)$.

Here $K(\varepsilon)$ is a positive constant depending on ε, p, n, η and ω.

PROOF. – We use Lemma 1.1 with $\lambda = n - 2p$, $f = v^2$. Keeping the notation of that lemma we have, for any $\varepsilon > 0$

$$\int_\Omega u^2(x)\, v^2(x)\, dx = \int_\Omega u^2(x)[(v^2)_1 + (v^2)_2]\, dx \leq C\varepsilon \int_\Omega |Du|^2\, dx + \overline{K} \int_\Omega u^2\, dx\,,$$

for any $u \in C_0^\infty(\Omega)$.

REMARK 1.1. – We explicitely observe that the conclusions of Theorems 1.1 and 1.2 remain true for $u \in W_0^{1,2}(\Omega)$. We will now briefly discuss, for later use, the validity of Theorem 1.2 in the case $u \in W^{1,2}(\Omega)$. If $\partial\Omega$ is smooth, say C^1, extending by reflection both v and u, it is possible to produce extensions $\tilde v$ and $\tilde u$ of v and u defined in some open set $\tilde\Omega$, $\Omega \subset\subset \tilde\Omega$, belonging to $VL^{2p,n-2p}(\tilde\Omega)$ and $W_0^{1,2}(\tilde\Omega)$ respectively. In particular it is easy to see that the norms of $\tilde v$ and $\tilde u$ in the relevant spaces are still controlled by the correspondent norms of v and u.

From this facts it clearly follows that

(1.3) *holds true for $u \in W^{1,2}(\Omega)$ if $\partial\Omega$ is C^1.*

Finally we notice that by the remarks following Lemma 1.2 and well known facts about convolutions, given any function v in $L^{2p,n-2p}$, it is possible to find a family of C^∞ functions $\{f_h\}_{h>0}$, converging to f in $L^{2p,n-2p}$, for which (1.3) holds with a constant $K(\varepsilon)$ independent of h.

2. – An elliptic equation in nondivergence form.

In Ω, a bounded open set $\boldsymbol{R}^n$, we consider

$$\begin{cases} a_{ij}(x) \in L^\infty(\Omega), & a_{ij}(x) = a_{ji}(x), \quad i,j = 1, \dots, n, \\ \nu|\xi|^2 \leqslant \sum\limits_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j, & \text{a.e. in } \Omega, \ \forall \xi \in \boldsymbol{R}^n \end{cases} \tag{2.1}$$

where $\nu > 0$ is a positive constant.

We consider in Ω the elliptic equation

$$Lu \equiv -\sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} = f$$

and the associated Dirichlet problem

$$\begin{cases} Lu = f, \\ u \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega), & f \in L^2(\Omega). \end{cases} \tag{2.2}$$

We will prove the following

THEOREM 2.1. – *Let Ω as above and assume its boundary $\partial\Omega$ to be locally the graph of a C^2 function. Assume* (2.1) *and*

$$(a_{ij})_{x_s} \in VL^{2p,n-2p}(\Omega)\,(^3) \tag{2.3}$$

for some $1 < p \leqslant n/2$ and any $s = 1, \dots, n$. Then, for any $f \in L^2(\Omega)$, problem (2.2) *has a unique solution. Moreover*

$$\|u\|_{W^{2,2}(\Omega)} \leqslant K \|f\|_{L^2(\Omega)} \tag{2.4}$$

holds, with a positive constant K independent of f.

Before proving Theorem 2.1 we wish to show by an example how it is related to the existing literature.

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(3) Here and in the following the derivatives are taken in the $D'(\Omega)$ sense.

Precisely we have:

EXAMPLE 2.1. – Let $n \geq 4$, $\Omega = B(0, 1/2)$, $x = (x_1, \ldots, x_n) \in \Omega$

$$a_{ij}(x) = \begin{cases} 1 & \text{if } i = j = 1\,, \\ \delta_{ij}(2 + \sin |\log(x_1^2 + x_2^2 + x_3^2)|^\alpha) & \text{if } (i,j) \neq (1,1)\,, \end{cases}$$

with $\alpha \in]0, 2/3[$. Then we have

$$(a_{ij})_{x_s} \in VL^{3, n-3}(\Omega)\,, \qquad (a_{ji})_{x_s} \notin L^{n-\varepsilon}(\Omega)$$

for $s = 1, \ldots, n$, $j = 1, \ldots, n$, $i = 1, \ldots, n$, $0 < \varepsilon < 1$.

This in turn implies that $(a_{ij})_{x_s}$ doesn't belong to the space weak-$L^n(\Omega)$. Moreover $a_{ij}(x)$, $i \geq 2$, is not continuous in Ω.

We can conclude that the results in [8] and [1] as well as the classical results concerning continuous coefficients cannot be applied to solve the Dirichlet problem (2.2) with these coefficients. On the contrary the coefficients $a_{ij}(x)$ fall in the scope of Theorem 2.1. We now turn to the preliminaries of the proof of Theorem 2.1.

To begin with we observe that by Lemma 1.2 and the following remarks we can mollify the coefficients a_{ij} as well as the known term f in the equation $Lu = f$ and then assume them smooth, say $C^2(\bar{\Omega})$.

By standard arguments (see [8]) the conclusion of Theorem 2.1 will follow by an existence and uniqueness result for a relected divergence form equation and by bound (2.4) whose proof will be given in Section 4.

3. – A related divergence form equation.

We consider in Ω, bounded open subset of $\boldsymbol{R}^n$ the differential operator

$$Lu \equiv -\sum_{i,j=1}^{n} (a_{ij} u_{x_j})_{x_i} + \sum_{i=1}^{n} b_i u_{x_i}\,,$$

where

$$\begin{cases} a_{ij}(x) \in L^\infty(\Omega)\,, \qquad a_{ij}(x) = a_{ji}(x)\,, \qquad i,j = 1, \ldots, n\,, \\ \nu|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i \xi_j\,, \end{cases} \tag{3.1}$$

a.e. in Ω, for every $\xi \in \boldsymbol{R}^n$, and ν is a positive constant;

$$b_i \in VL^{2p, n-2p}(\Omega) \tag{3.2}$$

for $i = 1, \ldots, n$ and some $1 < p \leq n/2$.

First we want to prove the following lemma which will be useful in the following.

LEMMA 3.1. – *Let f belong to $VL^{p,\lambda}(\Omega)$ with $1 \leqslant p < +\infty$, $0 < \lambda < n$. Then $\forall \varepsilon > 0$ $\exists \sigma < 0$: $E \subset \Omega$, $|E| < \sigma \Rightarrow \|f\chi_E\|_{p,\lambda} < \varepsilon$.*

PROOF. – Let $\bar{r} > 0$ be such that, $\eta(\bar{r}) < (\varepsilon/2)^p$, where η is the $VL^{p,\lambda}$ modulus of f. From the absolute continuity of the integral, we obtain $\sigma > 0$ such that

$$E \subseteq \Omega, \qquad |E| < \sigma \Rightarrow \int_E |f|^p\,dx < \left(\frac{\varepsilon}{2}\right)^p \bar{r}^\lambda .$$

We have

$$\|f\chi_E\|_{p,\lambda} = \sup_{\substack{x\in\Omega\\ r\in]0,\delta[}} \left[\frac{1}{r^\lambda}\int_{\Omega(x,r)} |f\chi_E|^p\,dy\right]^{1/p} \leqslant$$

$$\leqslant \sup_{\substack{0<x\in\Omega\\ r\leqslant\bar{r}\leqslant\delta}} \left[\frac{1}{r^\lambda}\int_{\Omega(x,r)} |f\chi_E|^p\,dy\right]^{1/p} + \sup_{\substack{x\in\Omega\\ \bar{r}<r\leqslant\delta}} \left[\frac{1}{r^\lambda}\int_{\Omega(x,r)} |f\chi_E|^p\,dy\right]^{1/p} \leqslant$$

$$\leqslant \frac{\varepsilon}{2} + \left[\frac{1}{\bar{r}^\lambda}\int_E |f|^p\,dy\right]^{1/p} \leqslant \varepsilon .$$

For any $f \in W^{-1}(\Omega)$, where $W^{-1}(\Omega)$ is the dual space of $W_0^{1,2}(\Omega)$, we consider the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u \in W_0^{1,2}(\Omega). \end{cases} \tag{3.3}$$

For any solution of problem (3.3) we prove the following

THEOREM 3.1. – *Let u be a solution of problem* (3.3) *and assume* (3.1) *and* (3.2). *Then*

$$\|u\|_{W^{1,2}(\Omega)} \leqslant K\|f\|_{W^{-1}(\Omega)},$$

with the constant K depending only on n, ν, p, the $VL^{2p,n-2p}$ modulus of continuity of $|b| = \left(\sum_{i=1}^n b_i^2\right)^{1/2}$ and the AC modulus of $|b|^{2p}$.

PROOF. – For any nonnegative real number K_1, we set $\Omega_{K_1} = \{x \in \Omega: u \geqslant K_1, |Du| > 0\}$, $M_1(K_1) = c_0^{1/2}\||b|\chi_{\Omega_{K_1}}\|_{2p,n-2p}^{1/2}\nu^{-1}$, where c_0 is from Theorem 1.1.

The function $M_1(K_1)$ is nonincreasing for $K_1 \geqslant 0$ and tends to zero at the infinity. Let us prove that it is also continuous. In fact we prove that

$$\lim_{h\to 0} \left|\||b|\chi_{\Omega_{K+h}}\| - \||b|\chi_{\Omega_K}\|\right| = 0,$$

for any nonnegative real number K. We note that

$$\left| \| |b|\chi_{\Omega_{K+h}} \| - \| |b|\chi_{\Omega_K} \| \right| \leq \| |b|(\chi_{\Omega_{K+h}} - \chi_{\Omega_K}) \|_{2p,\, n-2p}$$

and for any $\varepsilon > 0$, by Lemma 3.1, there exists $\delta > 0$ such that

$$|\Omega_{K+h} - \Omega_K| < \delta, \qquad |\Omega_K - \Omega_{K+h}| < \delta,$$

implies

$$\| |b| \, |\chi_{\Omega_{K+h}} - \chi_{\Omega_K}| \, \|_{2p,\, n-2p} < \varepsilon .$$

If $M_1(0) \leq 1/2$ we set $K_1 = 0$, otherwise we select K_1 such that $M_1(K_1) = 1/2$. If $K_1 > 0$, for any $K_2 \in [0, K_1[$ we set $\Omega_{K_2} = \{x \in \Omega : K_2 \leq u < K_1\}$, $M_2(K_2) = c_0^{1/2} \| |b|\chi_{\Omega_{K_2}} \|_{2p,\, n-2p}^{1/2} \nu^{-1}$. Again, if $M_2(0) \leq 1/2$ we take $K_2 = 0$, on the contrary we select K_2 such that $M_2(K_2) = 1/2$. Repeating this process we show that there exists

$$t \leq 1 + \frac{4 c_0^{1/2} \nu^{-1}}{\bar{r}^{n/4p - 1/2}} \left(\int_\Omega |b|^{2p} \, dx \right)^{1/4p}$$

such that $K_t = 0$, where $\bar{r}$: $\eta(\bar{r}) < (\nu^2/16 c_0)^{1/2p}$ and η is the $VL^{2p,\, n-2p}$ modulus of continuity of $|b|$.

In fact if $M_1(0) \leq 1/2$ we have $t = 1$, otherwise $\exists m > 1$, $m \in N$: $M_i(K_i) = 1/2$ $\forall i \in \{1, \dots, m\}$ and $m/2 = c_0^{1/2} \nu^{-1} \sum_{i=1}^{m} \| |b|\chi_{\Omega_{K_i}} \|_{2p,\, n-2p}^{1/2}$. Let us fix then $\bar{r}$: $(\eta(\bar{r}))^{1/2p} < \nu / 4 c_0^{1/2}$. We have

$$\| |b|\chi_{\Omega_{K_i}} \|_{2p,\, n-2p}^{1/2} \leq \sup_{\substack{x \in \Omega \\ 0 < r < \bar{r}}} \left[\frac{1}{r^{n-2p}} \int_{\Omega(x,r)} |b|^{2p} \chi_{\Omega_{K_i}} \, dy \right]^{1/4p} + \left[\frac{1}{\bar{r}^{n-2p}} \int_{\Omega_{k_i}} |b|^{2p} \, dy \right]^{1/4p}$$

and then

$$\sum_{i=1}^{n} \| |b|\chi_{\Omega_{K_i}} \|_{2p,\, n-2p}^{1/2} \leq m(\eta(\bar{r}))^{1/2p} + \frac{1}{\bar{r}^{(n-2p)/4p}} \sum_{i=1}^{m} \left(\int_{\Omega_{k_i}} |b|^{2p} \, dx \right)^{1/4p} \leq$$

$$\leq m \frac{\nu}{4 c_0^{1/2}} + \frac{1}{\bar{r}^{n/4p - 1/2}} \left(\int_\Omega |b|^{2p} \, dx \right)^{1/4p} .$$

From this we obtain

$$m \leq 4 \frac{c_0^{1/2}}{\nu} \frac{1}{\bar{r}^{n/4p - 1/2}} \left(\int_\Omega |b|^{2p} \, dx \right)^{1/4p} .$$

We consider $K_1, \dots, K_t$ and set $u_1 = (u - K_1)^+ \equiv \max(u - K_1, 0)$. Since u is a solution of problem (3.3) we have

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij} u_{x_i} (u_1)_{x_j} - \sum_{i=1}^n b_i u_{x_i} u_1 \right) dx = \langle f, u_1 \rangle ,$$

from which using (3.1), (3.2) and Theorem 1.1

$$\nu \int_\Omega |Du_1|^2 dx \leq \|f\|_{W^{-1}(\Omega)} \|u_1\|_{W^{1,2}(\Omega)} + \left(\int_\Omega |b|^2 u_1^2 dx \right)^{1/2} \left(\int_\Omega |Du_1|^2 dx \right)^{1/2} \leq$$

$$\leq \|f\|_{W^{-1}(\Omega)} \|u_1\|_{W^{1,2}(\Omega)} + c_0^{1/2} \| |b| \chi_{\Omega_{K_1}} \|_{2p,n-2p}^{1/2} \int_\Omega |Du_1|^2 dx .$$

Remembering that $M_1(K_1) = c_0^{1/2} \| |b| \chi_{\Omega_{K_1}} \|_{2p,n-2p}^{1/2} \nu^{-1} = 1/2$, we deduce

$$\frac{\nu}{2} \|u_1\|_{W^{1,2}(\Omega)} \leq \|f\|_{W^{-1}(\Omega)} .$$

Now for any $q \in \{2, \dots, t\}$ we set in Ω

$$\begin{aligned} u_q &= 0 && \text{if } u < K_q , \\ u_q &= u - K_q && \text{if } K_q \leq u < K_{q-1} , \\ u_q &= K_{q-1} - K_q && \text{if } u \geq K_{q-1} . \end{aligned}$$

For any $q \in \{2, \dots, t\}$ we have

$$\int_\Omega \left(\sum_{i,j=1}^n a_{ij} u_{x_i} (u_q)_{x_i} - \sum_{i=1}^n b_i u_{x_i} u_q \right) dx = \langle f, u_q \rangle$$

$$\int_\Omega b_i u_{x_i} u_q \, dx = \sum_{h=1}^q \int_{\Omega_{K_h}} b_i (u_h)_{x_i} u_q \, dx$$

and then

$$\left| \int_\Omega \left(\sum_{i,j=1}^n a_{ij} u_{x_i} (u_q)_{x_i} dx \right) \right| \leq \langle f, u_q \rangle + \sum_{h=1}^q \sum_{i=1}^n \int_{\Omega_{K_h}} |b_i (u_h)_{x_i} u_q| \, dx$$

from which

$$\nu \int_\Omega |Du_q|^2 dx \leq \|f\|_{W^{-1}(\Omega)} \|u_q\|_{W^{1,2}(\Omega)} + c_0^{1/2} \| |b| \chi_{\Omega_{K_q}} \|_{2p,n-2p}^{1/2} \int_\Omega |Du_q|^2 dx +$$

$$+ c_0^{1/2} \sum_{h=1}^{q-1} \| |b| \chi_{\Omega_{K_q}} \|_{2p,n-2p}^{1/2} \left(\int_\Omega |Du_h|^2 dx \right)^{1/2} \left(\int_\Omega |Du_q|^2 dx \right)^{1/2} .$$

Then

$$\frac{\nu}{2}\|u_q\|_{W^{1,2}(\Omega)} \leqslant \|f\|_{W^{-1}(\Omega)} + \frac{\nu}{2}\sum_{h=1}^{q-1}\|u_h\|_{W^{1,2}(\Omega)},$$

and

$$\|u_q\|_{W^{1,2}(\Omega)} \leqslant \frac{2^q}{\nu}\|f\|_{W^{-1}(\Omega)},$$

for any $q=1,2,\dots,t$. The theorem is then proved if we notice that $\|u^+\|_{W^{1,2}(\Omega)} \leqslant \leqslant \sum_{h=1}^{t}\|u_h\|_{W^{1,2}(\Omega)}$ and use a similar argument for $u^-=\min(u,0)$.

Now we define for $u,v\in W_0^{1,2}(\Omega)$ the bilinear form

$$B(u,v)=\int_\Omega\left(\sum_{i,j=1}^{n}a_{ij}u_{x_i}v_{x_j}-\sum_{i=1}^{n}b_iu_{x_i}v\right)dx.$$

Using Theorem 1.1 we deduce immediately that this form is continuous. Moreover we have

LEMMA 3.2. – *There exists a positive constant* γ such that

$$B(u,u)\geqslant\frac{\nu}{2}\|u\|^2_{W^{1,2}(\Omega)}-\gamma\|u\|_2^2,$$

for any $u\in W_0^{1,2}(\Omega)$.

PROOF. – For any $u\in W_0^{1,2}(\Omega)$ we have

$$\int_\Omega\sum_{i,j=1}^{n}a_{ij}u_{x_i}u_{x_j}dx-\int_\Omega\left|\sum_{i=1}^{n}b_iu_{x_i}u\right|dx\leqslant B(u,u).$$

Then, by (3.1), for any $\varepsilon>0$

$$\nu\|u\|^2_{W^{1,2}(\Omega)}\leqslant\frac{\varepsilon}{2}\|u\|^2_{W^{1,2}(\Omega)}+\frac{1}{2\varepsilon}\int_\Omega|b|^2u^2dx+B(u,u),$$

and using Theorem 1.2 the conclusion follows choosing ε properly.

By Fredholm alternative, using Theorem 3.1 and Lemma 3.2 the next theorem follows.

THEOREM 3.2. – *If* (3.1) *and* (3.2) *hold, then there exists an unique solution of problem* (3.3). *Moreover the following bound holds*

$$\|u\|_{W^{1,2}(\Omega)}\leqslant K\|f\|_{W^{-1}(\Omega)},$$

with K a positive constant depending only on ν, $\eta(r)$, $\omega(\sigma)$, c_0, n, p *where* $\eta(r)$ *is the* $VL^{2p,n-2p}$ *modulus of continuity of* $|b|=\left(\sum_{i=1}^{n}b_i^2\right)^{1/2}$ *and* $\omega(\sigma)$ *is the AC modulus of* $|b|^{2p}$.

4. – Proof of Theorem 2.1.

Let Ω as in the statement of Theorem 2.1 and assume (2.1) and (2.3). We can rewrite equation $Lu=f$ in the form

$$Lu = -\sum_{i,j=1}^{n}(a_{ij}u_{x_j})_{x_i} + \sum_{i,j=1}^{n}(a_{ij})_{x_i}u_{x_j} = f. \tag{4.1}$$

We assume that $a_{ij}(x)$ and f belong to $C^2(\overline{\Omega})$. We will prove (2.4) for solutions $u \in C^2(\overline{\Omega}) \cap C^3(\Omega)$ of the Dirichlet problem

$$Lu=f, \qquad u|_{\partial\Omega}=0,$$

with K depending only on ν, n, p, $\eta(r)$ and $\omega(\sigma)$, where $\eta(r)$ and $\omega(\sigma)$ are respectively the $VL^{2p,n-2p}$ modulus of continuity of $A \equiv \left[\sum_{i,j,s=1}^{n}((a_{ij})_{x_s})^2\right]^{1/2}$ and the AC modulus of A^{2p}.

Let Ω' be an open subset of Ω with $\Omega' \subset\subset \Omega$, $h \in C_0^1(\Omega)$ such that $h \equiv 1$ in Ω', $0 \leq h \leq 1$ in Ω. Let $1 \leq s \leq n$ and consider $v(x) = h(x)\,u_{x_s x_s}$ as a test function. From (4.1) we get

$$-\int_\Omega \sum_{i,j=1}^{n}(a_{ij}u_{x_j})_{x_i} h u_{x_s x_s}\,dx + \int_\Omega \sum_{i,j=1}^{n}(a_{ij})_{x_i}u_{x_j} h u_{x_s x_s}\,dx = \int_\Omega f h u_{x_s x_s}\,dx.$$

Then

$$\int_{\Omega'}\left|\sum_{i,j=1}^{n} a_{ij}u_{x_i x_s}u_{x_j x_s}\right| dx \leq \int_\Omega \left|\sum_{i,j=1}^{n} a_{ij}u_{x_j}h_{x_i}u_{x_s x_s}\right| dx +$$
$$+\int_\Omega \left|\sum_{i,j=1}^{n} h_{x_s}a_{ij}u_{x_j}u_{x_i x_s}\right| dx + \int_\Omega \left|\sum_{i,j=1}^{n} h(a_{ij})_{x_s}u_{x_j}u_{x_i x_s}\right| dx +$$
$$+\int_\Omega \left|\sum_{i,j=1}^{n}(a_{ij})_{x_i}u_{x_j}h u_{x_s x_s}\right| dx + \int_\Omega |f h u_{x_s x_s}|\,dx = I_1 + \ldots + I_5.$$

Using (3.1) we have

$$\nu\int_{\Omega'}\sum_{i=1}^{n}(u_{x_i x_s})^2\,dx \leq I_1 + \ldots + I_5. \tag{4.2}$$

Also, for any $\varepsilon > 0$

$$\begin{cases} I_1 \leq K\dfrac{\varepsilon}{2}\|u_{x_s x_s}\|_2^2 + \dfrac{K}{2\varepsilon}\|Du\|_2^2, \\ I_2 \leq K\dfrac{\varepsilon}{2}\|(|Du|)_{x_s}\|_2^2 + \dfrac{K}{2\varepsilon}\|Du\|_2^2, \end{cases} \tag{4.3}$$

where K is a positive constant depending on n, h and the L^∞ norms of the coefficients a_{ij}.

We now set $A_s^2 = \sum_{i,j=1}^{n} ((a_{ij})_{x_s})^2$. Then

$$I_3 \leqslant \int_\Omega \left| \sum_{i,j=1}^{n} (a_{ij})_{x_s} u_{x_j} u_{x_i x_s} \right| dx \leqslant \left(\int_\Omega ((|Du|)_{x_s})^2 dx \right)^{1/2} \left(\int_\Omega (A_s |Du|)^2 dx \right)^{1/2}.$$

Using Theorem 1.2 (4) we obtain for any $\varepsilon, \sigma > 0$

$$I_3 \leqslant \frac{\sigma}{2} \int_\Omega ((|Du|)_{x_s})^2 dx + \frac{1}{2\sigma} \int_\Omega (A_s |Du|)^2 dx \leqslant$$

$$\leqslant K \left[\frac{\sigma}{2} \int_\Omega ((|Du|)_{x_s})^2 dx + \frac{\varepsilon}{4\sigma} \int_\Omega |D^2 u|^2 dx + \frac{1}{4\varepsilon\sigma} \int_\Omega |Du|^2 dx \right].$$

Letting $\varepsilon = 2\sigma^2$ we get

$$I_3 \leqslant K \left[\sigma \int_\Omega |D^2 u|^2 dx + \frac{1}{8\sigma^3} \int_\Omega |Du|^2 dx \right]. \tag{4.4}$$

A similar bound holds for I_4.

Finally we consider I_5. We have

$$I_5 \leqslant \frac{\sigma}{2} \int_\Omega |D^2 u|^2 dx + \frac{1}{2\sigma} \int_\Omega f^2 dx. \tag{4.5}$$

From (4.2), (4.3), (4.4), (4.5) we obtain

$$\nu \|D^2 u\|^2_{L^2(\Omega')} \leqslant K(\varepsilon + \sigma) \int_\Omega |D^2 u|^2 dx + K\left(\frac{1}{\varepsilon} + \frac{1}{8\sigma^3} \right) \int_\Omega |Du|^2 dx + \frac{1}{2\sigma} \int_\Omega f^2 dx,$$

where K depends also on $\sup_\Omega |Dh|$.

By Theorem 3.1, fixing $\varepsilon + \sigma < \nu/2K$ we deduce

$$\nu \|D^2 u\|_{L^2(\Omega')} \leqslant \frac{\nu}{2} \|D^2 u\|_2^2 + K \|f\|_2^2, \tag{4.6}$$

where K depends on n, p, ν, $\eta(r)$, $\omega(\sigma)$ and $\sup |Dh|$.

Then with standard techniques (see e.g. [6], p. 187) we obtain a majorization

(4) See Remark 1.1

formula like (4.6) near the boundary that added to (4.6) gives

$$\|D^2 u\|_2^2 \leq K\|f\|_2^2,$$

where K depends on n, p, ν, $\eta(r)$, $\omega(\sigma)$ and Ω.

This concludes the proof of Theorem 2.1.

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