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The Infinitesimal Generators of Semigroups of Holomorphic Maps (*).

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Let X, Y be two complex manifolds; we shall denote by $\mathrm{Hol}(X, Y)$ the space of holomorphic maps from X into Y, endowed with the compact-open topology. A *one-parameter semigroup of holomorphic maps* (in short, a *semigroup*) on X is a continuous map $\Phi: \boldsymbol{R}^+ \to \mathrm{Hol}(X, X)$ such that $\Phi_0 = \mathrm{id}_X$, the identity map of X, and

$$\forall s, t \in \boldsymbol{R}^+ \qquad \Phi_s \circ \Phi_t = \Phi_{s+t}. \tag{1}$$

This notation is a continuous analogue of the concept of sequence of iterates of a map $f \in \mathrm{Hol}(X, X)$; indeed, a sequence of iterates can be characterized as a map $\Phi: \boldsymbol{N} \to \mathrm{Hol}(X, X)$ such that $\Phi_0 = \mathrm{id}_X$ and satisfying (1) for every $s, t \in \boldsymbol{N}$.

The first paper concerning one-parameter semigroups of holomorphic maps seems to be [T], where problems somehow regarding the asymptotic behavior of one-parameter semigroups on Δ, the unit disk in $\boldsymbol{C}$, are studied. Later on, the typical approach used to be via the idea of *fractional iteration;* loosely stated, one wants to find a sensible way of defining, at least locally, the r-th iterate of a holomorphic function for any positive real number r. For a recent work on this subject, see [C].

The real break-through in the study of one-parameter semigroups in one complex variable is due to BERKSON and PORTA [BP] and HEINS [H]. Following [W2], they related semigroups and the theory of ordinary differential equations, being able to classify all one-parameter semigroups on Riemann surfaces (for a unified account of their results see [A3]).

Strangely, there seems to be almost no papers on semigroups in several complex variables; as far as we know, they have been studied only in [A1, 2] and [V]. In this paper we want to generalize to arbitrary complex manifolds some of the results of [BP]; in particular, we want to describe at some extent the relationships between semigroups and ODE in several complex variables.

First of all, we fix some notations. Let X and Y be two complex manifolds. A sequence $\{f_\nu\} \subset \mathrm{Hol}(X, Y)$ is said *compactly divergent* if for every pair of compact sets

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$H \subset X$ and $K \subset Y$ we have $f_\nu(H) \cap K = \emptyset$ eventually. A family $\mathcal{F} \subset \mathrm{Hol}(X, Y)$ is *normal* if every sequence in $\mathcal{F}$ has either a converging subsequence or a compactly divergent subsequence. A complex manifold X is *taut* if $\mathrm{Hol}(\Delta, X)$ is normal; it turns out that if X is taut then $\mathrm{Hol}(Y, X)$ is normal for every complex manifold Y (see [Wu] and [K]). Standard examples of taut manifolds are provided by strongly pseudoconvex domains and by complete hermitian manifolds of strictly negative holomorphic sectional curvature (see [Wu]).

If $\Phi\colon \boldsymbol{R}^+ \to \mathrm{Hol}(X, X)$ is a semigroup on a complex manifold X, we shall denote by a prime (like in Φ_t') the derivatives with respect to the complex variables, and by a dot (like in $\dot{\Phi}$) the derivatives with respect to the real variable. $d\Phi_t$ will *always* denote the differential with respect to the complex variables.

We begin proving a general fact about one-parameter semigroups:

PROPOSITION 1. – *Let $\Phi\colon \boldsymbol{R}^+ \to \mathrm{Hol}(X,X)$ be a one-parameter semigroup on a complex manifold X. Then Φ_t is injective for all $t \geq 0$.*

PROOF. – First of all note that, since $\det(d\Phi_t) \to 1$ as $t \to 0$, for t small enough every Φ_t is locally injective.

Assume, by contradiction, that $\Phi_{t_0}(z_1) = \Phi_{t_0}(z_2) = z_0$ for some $t_0 > 0$ and $z_1, z_2 \in X$, with $z_1 \neq z_2$. In particular, if $t > t_0$ we have $\Phi_t(z_1) = \Phi_{t-t_0}(\Phi_{t_0}(z_1)) = \Phi_{t-t_0}(\Phi_{t_0}(z_2)) = \Phi_t(z_2)$; in other words, the two curves $t \mapsto \Phi_t(z_1)$ and $t \mapsto \Phi_t(z_2)$ start at distinct points, meet at $t = t_0$ and coincide thereafter. Let t_0 be the least $t > 0$ such that $\Phi_t(z_1) = \Phi_t(z_2)$, and set $z_0 = \Phi_{t_0}(z_1)$. Then no Φ_t can be injective in a neighbourhood of z_0, and this is a contradiction. q.e.d.

In particular, it may happen that $\Phi_t \in \mathrm{Aut}(X)$, the group of holomorphic automorphisms of X, for all $t \geq 0$. In this case Φ extends to a *one-parameter group*, i.e., to a continuous group homomorphism of $(\boldsymbol{R}, +)$ to $\mathrm{Aut}(X)$. Actually, if X is taut then Φ is a one-parameter group iff Φ_{t_0} is an automorphism for some $t_0 > 0$:

PROPOSITION 2. – *Let $\Phi\colon \boldsymbol{R}^+ \to \mathrm{Hol}(X,X)$ be a one-parameter semigroup on a taut manifold X. Assume $\Phi_{t_0} \in \mathrm{Aut}(X)$ for some $t_0 > 0$; then Φ is a one-parameter semigroup of automorphisms.*

PROOF. – Since $(\Phi_{t_0/n})^n = \Phi_{t_0} \in \mathrm{Aut}(X)$ for all $n \in \boldsymbol{N}^*$, we clearly have $\Phi_{rt_0} \in \mathrm{Aut}(X)$ for all $r \in \boldsymbol{Q}^+$. By continuity, $\Phi_{rt_0} \in \mathrm{Aut}(X)$ for all $r \in \boldsymbol{R}^+$ (for X taut implies $\mathrm{Aut}(X)$ is closed in $\mathrm{Hol}(X, X)$; see [Wu]). q.e.d.

In this paper we shall need a few facts on ordinary differential equations; namely, we shall use the following basic existence theorem:

THEOREM 3. – *Let Ω be an open subset of $\boldsymbol{R}^N$, and $F\colon \Omega \to \boldsymbol{R}^N$ a real analytic map. Then for any compact subset K of Ω there are $\delta > 0$, a neighbourhood $U \subset \Omega$ of K and a* *real analytic map* $u\colon (-\delta,\delta)\times U\to\Omega$ *such that*

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x)=F(u(t,x)), \\ u(0,x)=x. \end{cases} \tag{2}$$

Furthermore, the solution of (2) *is unique in the sense that if there are* $\delta'>0$, *another neighbourhood* $U'\subset\Omega$ *of* K *and another map* $u'\colon(-\delta',\delta')\times U'\to\Omega$ *satisfying* (2), *then* $u\equiv u'$ *on* $[(-\delta,\delta)\times U]\cap[(-\delta',\delta')\times U']$. *Finally, if* Ω *actually is a domain in* $\boldsymbol{C}^n$ *and* $F\colon\Omega\to\boldsymbol{C}^n$ *is holomorphic, then for every* $t\in(-\delta,\delta)$ *the map* $u(t,\cdot)\colon U\to\Omega$ *is holomorphic.*

A proof can be found in [N] or [Hö].

The link between the previous theorem and one-parameter semigroups is provided by the following (well-known) corollary:

COROLLARY 4. – *Let* Ω *be an open subset of* $\boldsymbol{R}^N$, $F\colon\Omega\to\boldsymbol{R}^N$ *a real analytic map, and* K *a compact subset of* Ω. *Choose* $\delta>0$ *and a neighbourhood* $U\subset\Omega$ *of* K *such that there is a real analytic solution* $u\colon(-\delta,\delta)\times U\to\Omega$ *of the Cauchy problem* (2). *Then for every* $s,t\in(-\delta,\delta)$ *and* $x\in K$ *such that* $s+t\in(-\delta,\delta)$ *and* $u(t,x)\in U$ *we have*

$$u(s,u(t,x))=u(s+t,x). \tag{3}$$

PROOF. – Fix $t_0\in(-\delta,\delta)$ and $x_0\in K$ such that $u(t_0,x_0)\in U$, and take $\delta'\leqslant\delta-|t_0|$. Now define $v_1,v_2\colon(-\delta',\delta')\to\Omega$ setting $v_1(s)=u(s,u(t_0,x_0))$ and $v_2(s)=u(s+t_0,x_0)$. Then v_1 and v_2 are two real analytic solutions of

$$\begin{cases} \dfrac{dv}{ds}=F\circ v, \\ v(0)=u(t_0,x_0). \end{cases}$$

By uniqueness, $v_1=v_2$, and (3) is proved. q.e.d.

In other words, the solution of the Cauchy problem (2) is locally a one-parameter group. In particular, if X is a complex manifold and F is a holomorphic vector field on X such that the Cauchy problem

$$\begin{cases} \dfrac{\partial\Phi}{\partial t}=F\circ\Phi, \\ \Phi(0,z)=z, \end{cases}$$

has a global solution $\Phi\colon\boldsymbol{R}^+\times X\to X$, then Φ automatically is a one-parameter semigroup, holomorphic in z and real analytic in t. In this case, F is called the *infinitesimal generator* of Φ. Note that, by the uniqueness statement of Theorem 3, Φ is completely determined by its infinitesimal generator.

Following [BP], we now show that every one-parameter semigroup is obtained in this way:

THEOREM 5. – *Let $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(X,X)$ be a one-parameter semigroup on a complex manifold X. Then there is a holomorphic vector field F on X such that*

$$\frac{\partial \Phi}{\partial t} = F \circ \Phi . \tag{4}$$

In particular, Φ is analytic in t.

PROOF. – Since Φ is a semigroup, it suffices to show that $\partial\Phi/\partial t$ exists at $t=0$. Then we can fix a coordinate neighborhood D_0, a domain $D \subset\subset D_0$ and $\alpha_0 \in (0,1)$ such that $\Phi([0,\alpha_0]\times D) \subset\subset D_0$, reducing ourselves to the case of domains in $\mathbf{C}^n$, where n is the dimension of X.

Let K be a compact convex subset of D. We can choose $\alpha \in (0, \alpha_0/2)$ such that the convex hull $\hat{K}$ of $\Phi([0,\alpha]\times K)$ is still contained in D. In particular we can choose $\delta \in (0,\alpha]$ such that

$$\sup_{z\in\hat{K}} \|\Phi_t'(z) - I_n\| \leq 1/10$$

for all $t \leq \delta$, where Φ_t' is the Jacobian matrix of Φ_t, I_n is the $n\times n$ identity matrix and $\|\cdot\|$ here denotes the usual operator norm. Hence for all $t\in[0,\delta]$ and $z \in K$

$$\|\Phi_{2t}(z) - 2\Phi_t(z) + z\| = \left\| \int_z^{\Phi_t(z)} d[\Phi_t - \mathrm{id}_D] \right\| \leq \frac{1}{10}\|\Phi_t(z) - z\| , \tag{5}$$

where the integration path is the segment from z to $\Phi_t(z)$, and thus is contained in $\hat{K}$.

Therefore for every $t\in[0,\delta]$ and $z\in K$

$$\|\Phi_t(z) - z\| \leq \frac{10}{19}\|\Phi_{2t}(z) - z\| \leq 2^{-2/3}\|\Phi_{2t}(z) - z\| . \tag{6}$$

Let $k\in\mathbf{N}$ be such that $2^k\delta \geq 1$, and put

$$M = 2^{2k/3}\sup\{\|\Phi_t(z) - z\| \mid z\in K,\ t\in[2^{-k},\alpha_0]\} .$$

Then (6) implies

$$\forall t\in[0,\alpha_0]\ \forall z\in K \qquad \|\Phi_t(z) - z\| \leq Mt^{2/3} . \tag{7}$$

Now repeat the same argument on a compact convex subset K_1 of D containing properly $\hat{K}$, coming up with a constant $M_1 > 0$ such that

$$\forall t\in[0,\alpha_0]\ \forall z\in K_1 \qquad \|\Phi_t(z) - z\| \leq M_1 t^{2/3} .$$

Then the Cauchy inequalities produce a constant $\tilde{M} > 0$ such that

$$\forall t \in [0, \alpha_0]\ \forall z \in \hat{K} \quad \|\Phi_t'(z) - I_n\| \leq \tilde{M} t^{2/3}. \tag{8}$$

If we plug (7) and (8) in (5), we find that for all $t \in [0, \alpha]$ and $z \in K$

$$\|\Phi_{2t}(z) - 2\Phi_t(z) + z\| \leq \tilde{M} t^{2/3} \|\Phi_t(z) - z\| \leq M\tilde{M} t^{4/3}.$$

Thus

$$\left\| \frac{\Phi_{2t}(z) - z}{2t} - \frac{\Phi_t(z) - z}{t} \right\| \leq \frac{M\tilde{M}}{2} t^{1/3},$$

for $z \in K$ and $t \in (0, \alpha]$. Hence

$$\lim_{n \to \infty} \frac{\Phi(2^{-n}, z) - z}{2^{-n}} = F(z)$$

exists uniformly on compact subsets of D, defining a holomorphic function $F\colon D \to \mathbf{C}^n$, i.e., a holomorphic vector field on D.

For $z_0 \in D$ and $t_0 > 0$ small enough, $\Phi([0, t_0] \times \{z_0\})$ is a compact subset of D. Hence $2^n[\Phi(t + 2^{-n}, z_0) - \Phi(t, z_0)]$ tends uniformly to $F(\Phi(t, z_0))$ for $t \in [0, t_0]$. This implies

$$\Phi_t(z) = z + \int_0^t F(\Phi_s(z))\, ds,$$

and so

$$\left.\frac{\partial \Phi}{\partial t}\right|_{t=0} = F$$

on D. But D was an arbitrary coordinate neighborhood, and thus (4) is proved. q.e.d.

So every one-parameter semigroup is the solution of a Cauchy problem, and there is a one-to-one correspondance between infinitesimal generators and one-parameters semigroups.

A first application of this observation is the following. A point z_0 in a complex manifold X is a *fixed point* of the semigroup Φ if $\Phi_t(z_0) = z_0$ for all $t \geq 0$. We can tell fixed points using the infinitesimal generator:

PROPOSITION 6. – *Let* $\Phi\colon \mathbf{R}^+ \to \mathrm{Hol}(X, X)$ *be a one-parameter semigroup on a complex manifold* X*, and let* F *be its infinitesimal generator. Then* $z_0 \in X$ *is a fixed point of* Φ *iff* $F(z_0) = 0$.

PROOF. – If $z_0 \in X$ is a fixed point of Φ, then (4) immediately yields $F(z_0) = 0$. Conversely, assume $F(z_0) = 0$, and set $\varphi(t) = \Phi(t, z_0)$. Then φ solves the Cauchy problem

$$\begin{cases} \dfrac{d\psi}{dt} = F \circ \psi, \\ \psi(0) = z_0. \end{cases}$$

Since $F(z_0) = 0$, $\psi \equiv z_0$ is a solution and, since F is holomorphic, it is the only solution. Hence $\varphi \equiv z_0$, and z_0 is a fixed point of Φ. q.e.d.

Our main goal now is the characterization of the holomorphic vector fields arising as infinitesimal generators of one-parameter semigroups; we shall eventually describe a characterization valid in a large class of taut manifolds.

We need a digression in distribution theory. Let $C_c^\infty(X)$ denote the space of smooth real-valued functions with compact support in a manifold X, and let $\mathcal{D}'(X)$ denote the space of distributions on X. We shall say that a distribution $T \in \mathcal{D}'(X)$ is *non-positive*, and we shall write $T \leqslant 0$, if $\langle T, \varphi \rangle \leqslant 0$ for any $\varphi \in C_c^\infty(X)$ such that $\varphi \geqslant 0$ everywhere.

Let $u \in C^0(\boldsymbol{R})$, and denote by $u' \in \mathcal{D}'(\boldsymbol{R})$ its distributional derivative. If $u \in C^1(\boldsymbol{R})$, we know that $u' \leqslant 0$ iff u is not increasing; we claim that this is true in general.

PROPOSITION 7. – *Let $u \in C^0(\boldsymbol{R})$. Thus $u' \leqslant 0$ iff u is not increasing.*

PROOF. – Let $\{\rho_k\} \subset C_c^\infty(\boldsymbol{R})$ be a sequence of functions such that for all $k \in \boldsymbol{N}$ we have

(i) $\rho_k \geqslant 0$;

(ii) $\rho_k(t) = 0$ if $|t| \geqslant 1/k$;

(iii) $\int_{\boldsymbol{R}} \rho_k \, dt = 1$;

(iv) $\rho_k(-t) = \rho_k(t)$ for all $t \in \boldsymbol{R}$.

Set

$$u_k(t) = u * \rho_k(t) = \int_{\boldsymbol{R}} u(s)\rho_k(t-s)\, ds = \int_{\boldsymbol{R}} u(t+s)\rho_k(s)\, ds. \tag{9}$$

It is well known (see [Ru, pp. 155-161] and [Br, pp. 66-72]) that

(*a*) $u_k \in C^\infty(\boldsymbol{R})$ for all $k \in \boldsymbol{N}$;

(*b*) $u_k \to u$ as $k \to +\infty$, uniformly on compact subsets;

(*c*) $u_k'(t) = \langle u', \tau_t \rho_k \rangle$ for all $t \in \boldsymbol{R}$ and $k \in \boldsymbol{N}$, where $\tau_t \rho_k(s) = \rho_k(t-s)$;

(*d*) $u_k' \to u'$ as $k \to +\infty$ in $\mathcal{D}'(\boldsymbol{R})$.

Assume $u' \leqslant 0$. Then, by (*c*), $u_k' \leqslant 0$; it follows that every u_k is not increasing and, by (*b*), u is not increasing.

Conversely, assume u not increasing. Then, by (9), every u_k is not increasing; therefore $u_k' \leqslant 0$ in the classical sense. By (*d*), this implies $u' \leqslant 0$, and we are done. q.e.d.

Now we come back to complex manifolds. A *Finsler metric* $H\colon TX \to \boldsymbol{R}^+$ on a complex manifold X is an upper semicontinuous real-valued nonnegative function defined on the tangent space TX of X such that

(i) for every $z \in X$, $v \in T_zX$ and $\lambda \in \boldsymbol{C}$ we have

$$H(z; \lambda v) = |\lambda| H(z; v);$$

(ii) for every compact subset of K of X there is a constant $c_K > 0$ such that

$$\forall z \in K \ \forall v \in T_zX \qquad H(z; v) \geqslant c_K \|v\|,$$

where $\|v\|$ is the length of v computed with respect to any given hermitian metric on X. If H is continuous, we shall speak of a *continuous Finsler metric*.

The *integrated form* of a Finsler metric H is the function $d_H\colon X \times X \to \boldsymbol{R}^+$ defined by

$$\forall z, w \in X \qquad d_H(z, w) = \inf\left\{ \int_0^1 H(\gamma(t); \dot\gamma(t))\, dt \right\},$$

where the infimum is taken with respect to the set of piece-wise differentiable curves $\gamma\colon [0, 1] \to X$ such that $\gamma(0) = z$ and $\gamma(1) = w$.

The integrated form d_H of a Finsler metric H is a distance inducing the original topology on the manifold X, by (ii). We shall say that H is *complete* if d_H is a complete distance. We remark that, by the Hopf-Rinow theorem, H is complete iff the closed d_H-balls are compact. For more informations on Finsler metrics, consult [R].

In this paper we shall be mainly concerned with two examples of Finsler metrics. First of all, if h is a hermitian metric on a complex manifold X, then the function $H\colon TX \to \boldsymbol{R}^+$ given by

$$\forall v \in TX \qquad H(v) = [h(v, v)]^{1/2}$$

is clearly a continuous (even smooth) Finsler metric; H is complete iff h is complete.

The second example is given by the Kobayashi metric. Let X be a complex manifold; the *Kobayashi (pseudo)metric* $\varkappa_X\colon TX \to \boldsymbol{R}^+$ is defined by

$$\varkappa_X(z; v) = \inf\{|\xi| \mid \exists \varphi \in \mathrm{Hol}(\Delta, X)\colon \varphi(0) = z,\ d\varphi_0(\xi) = v\},$$

for any $z \in X$ and $v \in T_z X$. Its integrated form is the *Kobayashi (pseudo)distance* k_X.

Clearly, the Kobayashi metric satisfies (i) in the definition of a Finsler metric; furthermore, it is upper semicontinuous. A complex manifold X is said *hyperbolic* if $\varkappa_X$ is a Finsler metric, that is if $\varkappa_X$ satisfies (ii) in the definition of a Finsler metric: it turns out that X is hyperbolic iff k_X is a true distance on X. The Kobayashi metric of a taut manifold is a continuous Finsler metric. A complex manifold X is said *complete hyperbolic* if $\varkappa_X$ is a complete Finsler metric; it turns out that every complete hyperbolic manifold is taut. Examples of complete hyperbolic manifolds are provided by strongly pseudoconvex domains, complete hermitian manifolds with negative holomorphic sectional curvature, and by manifolds covered by complete hyperbolic manifolds. For proofs and more informations see, for instance, [Ro], [Ko1, 2] and [A3].

If H is a Finsler metric on X, we shall say that a map $f \in \mathrm{Hol}(X, X)$ is a *H-contraction* if

$$\forall v \in TX \qquad H(df(v)) \leq H(v);$$

if the equality holds for every $v \in TX$ we shall say that f is a *H-isometry*. For instance, by definition every holomorphic map is a contraction, and every automorphism of X an isometry, for the Kobayashi metric.

A one-parameter semigroup Φ on X is a *semigroup of H-contractions* if Φ_t is a H-contraction for every $t \geq 0$; analogously, Φ is a *semigroup of H-isometries* if Φ_t is a H-isometry for every $t \geq 0$.

Now let H be a continuous Finsler metric on a complex manifold X, and F_1, F_2 two holomorphic vector fields on X. Then $d(H \circ F_1)$ is a current on X (i.e., a differential form with distributional coefficients), and $d(H \circ F_1) \cdot F_2$ is a distribution on X. For instance, if H comes from a hermitian metric, then H is smooth out of the zero section, and so $d(H \circ F_1)$ is the usual differential out of the zero set of F_1.

Now we are finally able to prove our main theorem, characterizing the infinitesimal generators of semigroups of H-contractions:

THEOREM 8. – *Let H be a complete continuous Finsler metric on a complex manifold X. Then a holomorphic vector field F on X is the infinitesimal generator of a one-parameter semigroup of H-contractions iff*

$$d(H \circ F) \cdot F \leq 0. \tag{10}$$

PROOF. – Assume first F is the infinitesimal generator of a semigroup $\Phi \colon \boldsymbol{R}^+ \to \mathrm{Hol}(X, X)$ of H-contractions. Choose $z_0 \in X$; then for every $t_1 > t_2 > 0$ and every $v \in T_{z_0} X$ we have

$$\begin{aligned} H(\Phi_{t_1}(z_0);\, d(\Phi_{t_1})(z_0) \cdot v) &= \\ &= H(\Phi_{t_1 - t_2}(\Phi_{t_2}(z_0));\, d\Phi_{t_1 - t_2}(\Phi_{t_2}(z_0)) \cdot (d\Phi_{t_2}(z_0) \cdot v)) \leq H(\Phi_{t_2}(z_0);\, d(\Phi_{t_2})(z_0) \cdot v). \end{aligned}$$

Therefore for every $z_0 \in X$ and $v \in T_{z_0}X$ the continuous function

$$t \mapsto H(\Phi_t(z_0);\ d\Phi_t(z_0)\cdot v) = H \circ d\Phi_t(v)$$

is not increasing. Thus Proposition 7 yields

$$(11) \qquad 0 \geqslant \frac{d}{dt}[H \circ d\Phi_t(v)]\bigg|_{t=0} = dH \circ d\dot{\Phi}(0, v) = dH \circ dF \circ d\Phi(0, v) = $$
$$= dH \circ dF(v) = d(H \circ F)(v).$$

In particular, we can take $v = F(z_0)$ obtaining

$$0 \geqslant d(H \circ F)(z_0) \cdot F(z_0),$$

and (10) is proved.

Conversely, assume (10) holds. Fix $z_0 \in X$, and let $\phi_{z_0}\colon [0, \delta_{z_0}) \to X$ be the unique maximal solution of the Cauchy problem

$$(12) \qquad \begin{cases} \dfrac{d\phi}{dt} = F \circ \phi, \\ \phi(0) = z_0. \end{cases}$$

To show that F is an infinitesimal generator, it suffices to prove that $\delta_{z_0} = +\infty$ for all $z_0 \in X$.

If $F(z_0) = 0$, then $\phi_{z_0} \equiv z_0$, and so there is nothing to prove. If $F(z_0) \neq 0$, then ϕ_{z_0} is a non-constant real analytic map; therefore it cannot be eventually constant, and thus $F(\phi_{z_0}(t)) \neq 0$ for all $t \in [0, \delta_{z_0})$.

Assume, by contradiction, $\delta_{z_0} < +\infty$; we claim that $\phi_{z_0}([0, \delta_{z_0}))$ is contained in a compact subset of X. Indeed, we have (setting $\phi = \phi_{z_0}$)

$$\frac{d}{dt}H(\phi(t);\ \dot{\phi}(t)) = d(H \circ \dot{\phi})(t, 1) = d(H \circ F \circ \phi)(t, 1) = d(H \circ F) \cdot \dot{\phi}(t) = $$
$$= d(H \circ F)(\phi(t)) \cdot F(\phi(t)) \leqslant 0,$$

by (10). Hence, again by Proposition 7, the function $t \mapsto H(\phi_{z_0}(t);\ \dot{\phi}_{z_0}(t))$ is not increasing; therefore for all $t \in [0, \delta_{z_0})$ we have

$$d_H(z_0, \varphi_{z_0}(t)) \leqslant \int_0^t H(\varphi_{z_0}(s);\ \dot{\varphi}_{z_0}(s))\, ds \leqslant \delta_{z_0} H(z_0;\ F(z_0)),$$

and so $\phi_{z_0}([0, \delta_{z_0}))$ is contained in a closed d_H-ball K, which is compact because H is a complete Finsler metric.

Let $\delta_1 > 0$ and $u\colon (-\delta_1, \delta_1) \times K \to X$ be given by Theorem 3 applied to K, and choose $t_0 \in [0, \delta_{z_0})$ such that $\delta - t_0 < \delta_1$. Then the uniqueness statement of Theorem 3 shows that the function $\psi\colon [0, \delta_1 + t_0) \to X$ given by

$$\psi(t) = \begin{cases} \phi_{z_0}(t) & \text{if } t < \delta_{z_0}, \\ u(t - t_0, \phi_{z_0}(t_0)) & \text{if } t \geqslant t_0, \end{cases}$$

is still a solution of (12), against the maximality of δ_{z_0}, q.e.d.

A first corollary of Theorem 8 is:

COROLLARY 9. – *Let H be a complete continuous Finsler metric on a complex manifold X. Then the set of infinitesimal generators of one-parameter semigroups of H-contractions on X is a cone in the space of holomorphic vector fields on X with vertex at the zero section.*

PROOF. – In the proof of Theorem 8 we saw that $F \in \mathrm{Hol}(D, \boldsymbol{C}^n)$ is an infinitesimal generator iff (11) holds for every $v \in TX$. The assertion is then clear. q.e.d.

We are also able to characterize the infinitesimal generators of one-parameter groups:

COROLLARY 10. – *Let H be a complete continuous Finsler metric on a complex manifold X. Then a holomorphic vector field F is the infinitesimal generator of a one-parameter group of H-isometries on X iff*

$$d(X \circ F) \cdot F \equiv 0. \tag{13}$$

PROOF. – Assume F is the infinitesimal generator of a one-parameter group $\Phi\colon \boldsymbol{R} \to \mathrm{Aut}(X)$ of H-isometries, and set $\Phi_t^+ = \Phi_t$ and $\Phi_t^- = \Phi_{-t}$ for $t \geqslant 0$. Then Φ^+, $\Phi^-\colon \boldsymbol{R}^+ \to \mathrm{Hol}(X, X)$ are one-parameter semigroups on X of H-contractions with infinitesimal generators F, respectively $-F$, and Theorem 8 implies (13).

Conversely, assume (13) holds; then both F and $-F$ are infinitesimal generators of one-parameter semigroups of H-contractions Φ, respectively Ψ. The uniqueness of the solution of the Cauchy problem

$$\begin{cases} \dfrac{d\phi}{dt} = F \circ \phi, \\ \phi(0) = z_0, \end{cases}$$

in a neighbourhood of 0 for every $z_0 \in X$ then implies $\Phi_t \circ \Psi_t \equiv \mathrm{id}_D$ for t small enough, and hence always. q.e.d.

Recalling our examples of Finsler metrics, we get the following:

COROLLARY 11. – *Let X be a complete hyperbolic manifold. Then a holomorphic vector field F on X is the infinitesimal generator of a one-parameter semigroup on X iff $d(\varkappa_X \circ F)\cdot F \leq 0$.*

PROOF. – Indeed, every one-parameter semigroup on X is a semigroup of $\varkappa_X$-contractions and, being X complete hyperbolic, we can apply Theorem 8. q.e.d.

We should remark that if $D \subset\subset \boldsymbol{C}^n$ is a strongly convex smooth domain, then $\varkappa_D$ is smooth out of the zero section; see [L].

Another standard metric to consider is the Bergmann metric; see [Ko2] for definition and properties. Since the Bergmann metric is invariant under holomorphic automorphisms, we get

COROLLARY 12. – *Let X be a complex manifold such that its Bergmann metric b_X is complete, and denote by B_X the complete continuous Finsler metric associated to b_X. Then a holomorphic vector field F on X is the infinitesimal generator of a one-parameter group on X iff $d(B_X \circ F)\cdot F \equiv 0$.*

We can compute explicitly (10) in a particular case. Let B^n be the unit ball for the standard euclidean norm $\|\cdot\|$ on $\boldsymbol{C}^n$. On B^n the Kobayashi and the Bergmann metrics coincide, and they are given (cf. [FV]) by

$$\varkappa_{B^n}(z; v) = \frac{1}{1-\|z\|^2}[|(z, v)|^2 + (1-\|z\|^2)\|v\|^2]^{1/2},$$

where $(\cdot, \cdot)$ is the standard hermitian product on $\boldsymbol{C}^n$. Therefore

$$\frac{\partial \varkappa_{B^n}}{\partial z_j}(z; v) = \frac{1}{2(1-\|z\|^2)}\left[2\bar{z}_j \varkappa_{B^n}(z; v) + \frac{(v, z)\bar{v}_j - \|v\|^2 \bar{z}_j}{\varkappa_{B^n}(z; v)(1-\|z\|^2)}\right]$$

and

$$\frac{\partial \varkappa_{B^n}}{\partial v_j}(z; v) = \frac{1}{2\varkappa_{B^n}(z; v)(1-\|z\|^2)^2}[(z, v)\bar{z}_j + (1-\|z\|^2)\bar{v}_j].$$

Hence Theorem 8 and a computation yield

COROLLARY 13. – *A holomorphic map $F\colon B^n \to \boldsymbol{C}^n$ is the infinitesimal generator of a one-parameter semigroup on B^n iff for all $z \in B^n$ we have*

$$(14) \qquad 2[\|G(z)\|^2 - |(G(z), z)|^2]\operatorname{Re}(G(z), z) + (1-\|z\|^2)^2 \operatorname{Re}(dF\cdot F(z), G(z)) \leq 0,$$

where

$$G(z) = (1-\|z\|^2)F(z) + (F(z), z)z.$$

In particular, if $n=1$ we have $G\equiv F$ and (14) becomes

$$\forall z\in\Delta \qquad \mathrm{Re}\,[2\bar{z}F+(1-|z|^2)F']\leqslant 0\,. \tag{15}$$

In [BP], BERKSON and PORTA described a different characterization of infinitesimal generators on Δ. We end this paper by showing how to deduce their characterization from ours, proving:

PROPOSITION 14. – *A holomorphic function $F\colon \Delta\to \boldsymbol{C}$ is the infinitesimal generator of a one-parameter semigroup on Δ iff*

$$F(z)=(z-\tau)(\bar{\tau}z-1)\,f(z) \tag{16}$$

for some $\tau\in\bar{\Delta}$ and some holomorphic function $f\colon\Delta\to\boldsymbol{C}$ with $\mathrm{Re}f\geqslant 0$ *everywhere.*

PROOF. – Assume first $F\in\mathrm{Hol}(\Delta,\boldsymbol{C})$ satisfies (15). Then F is the infinitesimal generator of a one-parameter subgroup $\Phi\colon\boldsymbol{R}^+\to\mathrm{Hol}(\Delta,\Delta)$. Without loss of generality we can suppose Φ not trivial, i.e., $F\not\equiv 0$. Then there exists a unique point $\tau\in\bar{\Delta}$, called the *Wolff point* of Φ, such that either $\tau\in\Delta$ is the unique fixed point of Φ, or $\tau\in\partial\Delta$ and $\Phi_t\to\tau$ as $t\to+\infty$, uniformly on compact sets (this is shown in [BP]; a simpler proof is in [A3]). Now, Schwarz's and Wolff's lemmas (for the latter, see [W1] and [Bu]) imply that for every $z\in\Delta$ the function

$$t\mapsto\frac{|1-\bar{\tau}\Phi_t(z)|^2}{1-|\Phi_t(z)|^2}$$

is not increasing. Differentiating at $t=0$ we get

$$\mathrm{Re}\,[(1-\tau\bar{z})(\bar{z}-\bar{\tau})\,F(z)]\leqslant 0\,.$$

Set $f(z)=(\bar{\tau}z-1)^{-1}(z-\tau)^{-1}F(z)$; f is well-defined for if $\tau\in\Delta$ then $F(\tau)=0$ by Proposition 6. Then $\mathrm{Re}f\geqslant 0$ and

$$F(z)=(z-\tau)(\bar{\tau}z-1)\,f(z)\,,$$

as claimed.

Conversely, suppose F is given by (16) for suitable τ and f. Assume first $\mathrm{Re}\,f(z_0)=0$ for some $z_0\in\Delta$. Then the minimum principle for harmonic functions yields $f\equiv ia$ for some $a\in\boldsymbol{R}$, and (15) is easily verified.

Assume then $\mathrm{Re}f(z)>0$ for all $z\in\Delta$. Then the Schwarz-Pick lemma applied to $(f-1)/(f+1)$ yields

$$\forall z\in\Delta \qquad \frac{|f'(z)|}{2\,\mathrm{Re}\,f(z)}\leqslant\frac{1}{1-|z|^2}\,. \tag{17}$$

Using (16) we find

$$2\bar{z}F + (1-|z|^2)F' = (1-|z|^2)(\bar{\tau}z-1)(z-\tau)f' - [|z-\tau|^2 + |\bar{\tau}z-1|^2]f.$$

Hence (17) yields

$$\operatorname{Re}[2\bar{z}F + (1-|z|^2)F'] = (1-|z|^2)\operatorname{Re}[(z-\tau)(\bar{\tau}z-1)f'] - [|z-\tau|^2 + |\bar{\tau}z-1|^2]\operatorname{Re}f \leqslant$$

$$\leqslant (1-|z|^2)|z-\tau||\bar{\tau}z-1||f'| - [|z-\tau|^2 + |\bar{\tau}z-1|^2]\operatorname{Re}f \leqslant -(|z-\tau| - |\bar{\tau}z-1|)^2\operatorname{Re}f \leqslant 0,$$

and we are done. q.e.d.

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