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A Two-Parameter Spectral Theorem (*).

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Summary. - *We study the characteristic set*

$$\mathcal{C} = \{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}: \ker(I - \alpha A - \beta B) \neq 0\}$$

of a couple (A, B) *of selfadjoint compact operators on a real Hilbert space* $\boldsymbol{H}$. *We prove that* $\mathcal{C}$ *is the union of a sequence of characteristic curves* $\mathcal{C}_n$ *in the* (α, β) *plane. Each curve is the analytic image of an open interval and it is either closed or it goes to infinity at both ends of the interval. Moreover, it may intersect either itself or other characteristic curves in an at most countable set of points, which may accumulate only at infinity. Finally, to each characteristic curve one can associate an analytic function* E_n, *which gives the eigenprojection onto the eigenspace attached to each point of the characteristic curve, except at the intersection points, where the eigenspace is the direct sum of the projection relevant to each branch passing through the point. The dimension of the eigenprojection is constant along each curve and it is called the multiplicity of the characteristic curve.*

1. - Introduction.

In this paper we want to study the following problem: describe the *characteristic set*

$$\mathcal{C} = \{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R}: \ker(I - \alpha A - \beta B) \neq 0\}, \tag{1}$$

where A and B are selfadjoint compact operators on a real Hilbert space $\boldsymbol{H}$.

This question arises in many problems of Mathematical Physics, for example in multiparameter bifurcation theory. At the end of the paper we illustrate our result with some example.

Ours is by no means the only extension to many parameters of the usual spectrum theory. We refer to [1] for a survey of the vast literature on this subject.

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2. – The spectral theorem.

Before to state our result we give a definition. Consider a possibly finite subset $\mathfrak{J}$ of the natural numbers $\mathbb{N}$ and two sequences $(\alpha_n)_{n\in\mathfrak{J}}$ and $(\beta_n)_{n\in\mathfrak{J}}$ of functions defined on an open interval $J_n\subset\mathbb{R}$. Then for each $n\in\mathfrak{J}$ and $\theta\in J_n$ we define

$$\mathcal{M}_n(\theta)=\{(k,l)\in\mathbb{Z}\times\mathfrak{J}\colon k\in\mathcal{Z}_l\,,\theta+2k\pi\in J_l\,,\alpha_l(\theta+2k\pi)=\alpha_n(\theta)\,,\beta_l(\theta+2k\pi)=\beta_n(\theta)\}$$

where

$$\mathcal{Z}_n=\begin{cases}\{k\in\mathbb{Z}\colon 0\leqslant 2k\pi<\Pi_n\} & \text{if } \alpha_n \text{ and } \beta_n \text{ are both periodic of minimum period } \Pi_n\,,\\ \mathbb{Z} & \text{otherwise}\,.\end{cases}$$

$\mathcal{M}_n(\theta)$ is the set of all branches of curves (α_k,β_k) which intersect the curve (α_n,β_n) at the point $(\alpha_n(\theta),\beta_n(\theta))$. In particular $\mathcal{M}_n(\theta)=\{(0,n)\}$ means that there is only one branch, that is that the curve (α_n,β_n) intersects neither itself nor the other curves.

Now we state our result.

THEOREM. – Consider two compact selfadjoint linear operators A, B on a *real* Hilbert space $\boldsymbol{H}\neq 0$, such that

$$\|A\|+\|B\|\neq 0\,.$$

Then

(i) There exists an at most countable sequence $((\alpha_n,\beta_n))_{n\in\mathfrak{J}}$ of couples of real analytic functions

$$\alpha_n,\beta_n\colon J_n\to\mathbb{R}$$

with $\mathfrak{J}\subset\mathbb{N}$, and $J_n=]\varphi_n,\psi_n[$, such that:

(*a*) the characteristic set of (A,B) is given by:

$$\mathcal{C}=\bigcup_{n\in\mathfrak{J}}\{(\alpha_n(\theta),\beta_n(\theta))\colon\theta\in J_n\}\,;$$

(*b*) $\alpha_n^2(\theta)+\beta_n^2(\theta)\geqslant(\|A\|+\|B\|)^{-2}$ for all $\theta\in J_n$;

(*c*) either we have

$$\lim_{\theta\to\varphi^+}(\alpha_n^2(\theta)+\beta_n^2(\theta))=\lim_{\theta\to\psi^-}(\alpha_n^2(\theta)+\beta_n^2(\theta))=+\infty$$

or the above limits do *not* exist, $\varphi_n=-\infty$, $\psi_n=+\infty$ and α_n, β_n are periodic with period given by a multiple of 2π;

(*d*) the set of «multiple points»

$$J_n'=\{\theta\in J_n\colon\mathcal{M}_n(\theta)\neq\{(0,n)\}\}$$

is at most countable and with no cluster points in J_n.

(ii) For each $n \in \mathfrak{J}$ there exists a real analytic operator-valued function

$$E_n : J_n \to \mathcal{L}(\boldsymbol{H})$$

such that:

(e) $E_n^2(\theta) = E_n(\theta)$;

(f) $\ker(I - \alpha_n(\theta)A - \beta_n(\theta)B) = \bigoplus_{(k,l) \in \mathfrak{M}_n(\theta)} E_l(\theta + 2k\pi)\boldsymbol{H}$;

(g) $\dim(E_n(\theta)\boldsymbol{H})$ does not depend on θ;

(h) $E_n(\theta + 2k\pi)$ and $E_{n'}(\theta + 2k'\pi)$ are orthogonal for each $(k, n) \neq \neq (k', n')$ such that $\theta + 2k\pi \in J_n$ and $\theta + 2k'\pi \in J_{n'}$. ■

The set

$$\mathcal{C}_n = \{(\alpha_n(\theta), \beta_n(\theta)) : \theta \in J_n\}$$

is called *n-th characteristic curve* of (A, B) with *multiplicity* $m_n = \dim(E_n(\theta)\boldsymbol{H})$.

3. - Proof of the spectral theorem.

Let

$$T(\theta) = \cos(\theta)A + \sin(\theta)B, \tag{2}$$

then

$$T : \mathbb{R} \to \mathcal{L}(\boldsymbol{H})$$

is a real analytic function and $T(\theta)$ is compact selfadjoint for each $\theta \in \mathbb{R}$.

Let

$$\sigma(T(\theta)) = \{\lambda \in \mathbb{R} : \ker(\lambda I - T(\theta)) \neq 0\},$$

be the spectrum of $T(\theta)$.

Consider the complexification of $\boldsymbol{H}$, A and B. Then T extends to a complex analytic function

$$T : \mathbb{C} \to \mathcal{L}(\boldsymbol{H})$$

such that

$$T^*(\zeta) = T(\bar{\zeta}) \quad \text{for all } \zeta \in \mathbb{C}. \tag{3}$$

Given $\eta \in \mathbb{R}$, let

$$\{\lambda_{\eta,m} : m \in \mathfrak{N}_\eta\}, \quad \text{with } \mathfrak{N}_\eta \subset \mathbb{N}, \tag{4}$$

be the set of the *positive* eigenvalues of $T(\eta)$. Let $\varepsilon_{\eta,m} \in \mathbb{R}_+$ be such that

$$\sigma(T(\eta)) \cap [\lambda_{\eta,m} - \varepsilon_{\eta,m}, \lambda_{\eta,m} + \varepsilon_{\eta,m}] = \{\lambda_{\eta,m}\}.$$

Thanks to (3), by standard results on perturbation theory ([4], Theorem VII.6.9 and [5], Theorem II-6.1 and Section II,3.1) there exist $\delta_{\eta,m} \in \mathbb{R}_+$ and $q_{\eta,m}$ real analytic functions

(5) $$\Lambda_{\eta,m,p}\colon]\eta - \delta_{\eta,m}, \eta + \delta_{\eta,m}[\to \mathbb{R}_+, \qquad \text{with } 1 \leq p \leq q_{\eta,m},$$

such that

(6) $$\Lambda_{\eta,m,p}(\eta) = \lambda_{\eta,m},$$

(7) $$\Lambda_{\eta,m,p}(\theta) \neq \Lambda_{\eta,m,p'}(\theta) \quad \text{for } p \neq p' \text{ and } \theta \neq \eta,$$

(8) $$\sigma(T(\theta)) \cap]\lambda_{\eta,m} - \varepsilon_{\eta,m}, \lambda_{\eta,m} + \varepsilon_{\eta,m}[= \{\Lambda_{\eta,m,1}(\theta), \ldots, \Lambda_{\eta,m,q_{\eta,m}}(\theta)\}.$$

Moreover, the projection $E_{\eta,m,p}(\theta)$ onto $\ker(\Lambda_{\eta,m,p}(\theta)I - T(\theta))$ is analytic for $|\theta - \eta| < \delta_{\eta,m}$, if $q_{\eta,\mu} = 1$, whereas it is analytic for $0 < |\theta - \eta| < \delta_{\eta,m}$ and has a removable singularity at $\theta = \eta$, if $q_{\eta,m} > 1$. The extension of $E_{\eta,m,1}, \ldots, E_{\eta,m,q_{\eta,m}}$ to $\theta = \eta$ satisfy the following equality:

(9) $$\ker(\lambda_{\eta,m} I - T(\theta)) = \bigoplus_{1 \leq p \leq q_{\eta,m}} E_{\eta,m,p}(\theta)\boldsymbol{H} \quad \text{for } |\theta - \eta| < \delta_{\eta,m}.$$

Let

$$\Lambda_{\eta,m,p}(\theta)\colon J_{\eta,m,p} \to \mathbb{R}_+$$

be the maximum *positive* real analytic continuation of (5). Of course $J_{\eta,m,p}$ is an open interval.

We need the following

LEMMA. – Given $\varphi, \psi \in \mathbb{R}$ and a continuous function

$$\Lambda\colon]\varphi, \psi[\to \mathbb{R}$$

such that $\Lambda(\theta)$ is an eigenvalue of $T(\theta)$ for each $\theta \in]\varphi, \psi[$, then

$$\lim_{\theta \to \varphi^+} \Lambda(\theta) \quad \text{and} \quad \lim_{\theta \to \psi^-} \Lambda(\theta)$$

exist and belong to $\sigma(T(\varphi))$ and $\sigma(T(\psi))$.

PROOF. – We begin by proving that if $(\theta_n)_{n \in \mathbb{N}}$ is a sequence such that:

$$\theta_n \to \varphi^+, \qquad \Lambda(\theta_n) \to \lambda$$

then $\lambda \in \sigma(T(\varphi))$. Indeed, if $\lambda \notin \sigma(T(\varphi))$, because $\sigma(T(\varphi))$ is closed, there exists a neighborhood U of $\sigma(T(\varphi))$ such that $\lambda \notin \overline{U}$. On the other hand by continuity there exists $\delta > 0$ such that $\sigma(T(\theta)) \subset U$ for each $\theta \in [\varphi, \varphi + \delta[$ in contradiction with $\Lambda(\theta_n) \to \lambda \notin \overline{U}$.

Let $(\theta'_n)_{n\in\mathbb{N}}$ and $(\theta''_n)_{n\in\mathbb{N}}$ be two sequences in $]\varphi, \psi[$ such that

$$\lim_{n\to+\infty} \theta'_n = \varphi^+, \qquad \lim_{n\to+\infty} \Lambda(\theta'_n) = \liminf_{\theta\to\varphi^+} \Lambda(\theta) = \lambda',$$

$$\lim_{n\to+\infty} \theta''_n = \varphi^+, \qquad \lim_{n\to+\infty} \Lambda(\theta''_n) = \limsup_{\theta\to\varphi^+} \Lambda(\theta) = \lambda''.$$

In particular $\lambda', \lambda'' \in \sigma(T(\varphi))$. Moreover if $\lambda \in]\lambda', \lambda''[$, then there exists $n_0 \in \mathbb{N}$ such that

$$\lambda \in]\Lambda(\theta'_n), \Lambda(\theta''_n)[\quad \text{for each } n > n_0.$$

Because Λ is continuous there exists

$$\theta_n \in]\min\{\theta'_n, \theta''_n\}, \max\{\theta'_n, \theta''_n\}[,$$

such that $\Lambda(\theta_n) = \lambda$. Therefore

$$\varphi = \lim_{n\to+\infty} \theta_n \quad \text{and} \quad \lambda = \lim_{n\to+\infty} \Lambda(\theta_n) \in \sigma(T(\varphi)).$$

This means that

$$[\lambda', \lambda''] \subset \sigma(T(\varphi)).$$

But $\sigma(T(\varphi))$ is discrete, therefore it cannot contain a non-degenerate interval, so

$$\liminf_{\theta\to\varphi^+} \Lambda(\theta) = \limsup_{\theta\to\varphi^+} \Lambda(\theta).$$

In the same way one treats the other end-point ψ. ∎

Thanks to the previous lemma we have that

$$\Lambda_{\eta,m,p}(\theta) \in \sigma(T(\theta)) \quad \text{for all } \theta \in J_{\eta,m,p}. \tag{10}$$

It follows in particular that also $E_{\eta,m,p}$ extends to an analytic map on $J_{\eta,m,p}$.

Let

$$J = \{\theta \in \mathbb{R}: \sigma(T(\theta)) \cap \mathbb{R}_+ \neq \emptyset\},$$

$$\mathcal{T} = \{(\eta, m, p) \in J \times \mathbb{N} \times \mathbb{N}: m \in \mathcal{N}_\eta, 1 \leq p \leq q_{\eta,m}\}$$

and define the following equivalence relation

$(\eta, m, p) \sim (\eta', m', p') \Leftrightarrow$ there exists $k \in \mathbb{Z}$ such that

$$J_{\eta,m,p} = J_{\eta',m',p'} + 2k\pi \text{ and } \Lambda_{\eta,m,p}(\theta + 2k\pi) = \Lambda_{\eta',m',p'}(\theta) \text{ for } \theta \in J_{\eta',m',p'}.$$

Let

$$\mathcal{S} = \mathcal{T}/\sim.$$

By the axiom of choice there exists a function

$$\Gamma: \mathcal{S} \to \mathcal{T},$$

such that $\Gamma(s) \in \mathcal{T}$ is a representative of the equivalence class s. Of course

$$J = \bigcup_{s \in \mathcal{S}} J_{\Gamma(s)}.$$

Because $\Lambda_{\Gamma(s)} \in \sigma(T(\theta))$ for all $\theta \in J_{\Gamma(s)}$, $\sigma(T(\theta))$ is at most countable and by (6), (7) and (8) there is only a finite number of $\Lambda_{\Gamma(s')}$ passing through $\Lambda_{\Gamma(s)}$, we have that

$$\mathcal{S}_\theta = \{s \in \mathcal{S}: \theta \in J_{\Gamma(s)}\}$$

is at most countable for each $\theta \in J$. On the other hand $J \cap \mathbb{Q}$ is dense in J, so

$$\mathcal{S} = \bigcup_{r \in J \cap \mathbb{Q}} \mathcal{S}_r.$$

Therefore $\mathcal{S}$ is at most countable because $J \cap \mathbb{Q}$ is countable.

Let

$$s_n \leftrightarrow n,$$

be a bijection between $\mathcal{S}$ and a suitable subset $\mathfrak{J}$ of $\mathbb{N}$. For each $n \in \mathfrak{J}$ set

$$J_n = J_{\Gamma(s_n)}, \qquad \Lambda_n = \Lambda_{\Gamma(s_n)}, \qquad E_n = E_{\Lambda(s_n)}. \tag{11}$$

Of course

$$n \neq n' \Rightarrow \Lambda_n \neq \Lambda_{n'}.$$

Now we prove that for each $\theta \in J$ we have

$$\sigma(T(\theta)) \cap \mathbb{R}_+ = \{\Lambda_n(\theta + 2k\pi)\text{: there exists } (k, n) \in \mathbb{Z} \times \mathfrak{J} \text{ such that } \theta + 2k\pi \in J_n\}. \tag{12}$$

Consider $\theta + 2k\pi \in J_n$, then by (10) we have

$$\Lambda_n(\theta + 2k\pi) \in \sigma(T(\theta + 2k\pi)) = \sigma(T(\theta)).$$

On the contrary, let $\lambda_{\theta,m}$, with $m \in \mathcal{N}_\theta$, be a positive eigenvalue of $T(\theta)$ and let $n \in \mathfrak{J}$ be such that $\Gamma(s_n) \sim (\theta, m, 1)$. Then there exists $k \in \mathbb{Z}$ such that

$$J_n = J_{\theta,m,1} + 2k\pi \quad \text{ and } \quad \Lambda_n(\theta + 2k\pi) = \Lambda_{\theta,m,1}(\theta) = \lambda_{\theta,m}.$$

From (12) and [4], Lemma VII.3.4 we have that

$$\Lambda_n(\theta) \leq \|T(\theta)\| \leq \|A\| + \|B\|. \tag{13}$$

Set

$$J_n =]\varphi_n, \psi_n[,$$

then we prove that

$$
(14)\qquad \begin{cases} \text{either } \lim\limits_{\theta \to \varphi_n^+} \Lambda_n(\theta) = \lim\limits_{\theta \to \psi_n^-} \Lambda(\Lambda_n(\theta)) = 0\,, \\ \text{or } \varphi_n = -\infty,\ \psi_n = +\infty \text{ and } \Lambda_n \text{ is either constant} \\ \quad \text{or periodic with period given by a multiple } 2\pi. \end{cases}
$$

Consider the behavior of Λ_n at the point φ_n. If $\varphi_n > -\infty$, by our lemma we must have $\lim\limits_{\theta \to \varphi_n^+} \Lambda_n(\theta) = 0$, otherwise Λ_n could be continued outside J_n.

Assume now that $\varphi_n = -\infty$ and for each $\theta \in J_n$ consider the sequence

$$
(15)\qquad (\Lambda_n(\theta - 2k\pi))_{k \in \mathbb{N}}\,.
$$

Because the elements of (15) are eigenvalues of $T(\theta)$, we have that the sequence (15) is either finite or it converges to 0. Moreover, if one of this possibilities is verified for one θ, it is verified for *all* $\theta \in J_n$. In fact, if the sequence (15) were finite for $\theta = \theta_1$ and infinite for $\theta = \theta_2$, by considering $\Lambda_n(t\theta_1 + (1-t)\theta_2 - 2k\pi)$ with $t \in [0, 1]$ we would have an *infinite* number of eigenvalues flowing together in the same point in contradiction with (6), (7) and (8).

In the case in which all the sequences (15) converge to 0 we have

$$
\lim_{\theta \to -\infty} \Lambda_n(\theta) = 0\,.
$$

In fact, given any $\omega \in J_n$ and $\varepsilon \in \mathbb{R}_+$, because $[\omega - 2\pi, \omega] \subset J_n$ is compact the sequence (15) converges to 0 uniformly on $[\omega - 2\pi, \omega]$. Thus there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$
0 < \Lambda_n(\theta - 2k\pi) < \varepsilon \qquad \text{for each } k \geqslant N(\varepsilon) \text{ and } \theta \in [\omega - 2\pi, \omega]\,.
$$

But this implies that for

$$
\theta < \omega - 2N(\varepsilon)\pi\,,
$$

we have

$$
0 < \Lambda(\theta) = \Lambda\left(\theta + 2\left[\!\!\left[\frac{\omega - \theta}{2\pi}\right]\!\!\right]\pi - 2\left[\!\!\left[\frac{\omega - \theta}{2\pi}\right]\!\!\right]\pi\right) < \varepsilon\,(*)\,,
$$

because

$$
N(\varepsilon) \leqslant \left[\!\!\left[\frac{\omega - \theta}{2\pi}\right]\!\!\right]
$$

(*) $[\![x]\!]$ is the integer part of x.

and

$$\omega - 2\pi < \theta + 2\left[\!\!\left[\frac{\omega - \theta}{2\pi}\right]\!\!\right]\pi \leqslant \omega .$$

Consider now the case in which all the sequences (15) are finite. Let $v(\theta)$ be the number of distinct values taken by (15). Consider $\omega \in J_n$. By (6), (7) and (8) v is upper semi-continuous, thus it has a maximum K in $[\omega - 2\pi, \omega]$. It follows that for each $\theta \in [\omega - 2\pi, \omega]$ at least two of

(16) $$\Lambda_n(\theta - 2K\pi),\ \Lambda_n(\theta - 2(K-1)\pi), \ldots, \Lambda_n(\theta - 2n),\ \Lambda_n(\theta),$$

must coincide. This implies that two of the values of (16) must coincide at an infinite number of $\theta \in [\omega - 2\pi, \omega]$ and so by analyticity $J_n =]-\infty, +\infty[$ and Λ_n must be constant or periodic of period $2K\pi$.

A similar result holds at the other end-point ψ_n and this implies (14).

Now define

$$\alpha_n(\theta) = \cos(\theta)/\Lambda_n(\theta), \qquad \beta_n(\theta) = \sin(\theta)/\Lambda_n(\theta),$$

for $n \in \mathfrak{J}$ and $\theta \in J_n$. Then (a) follows from (2) and (12), while (b) follows from (13). Moreover α_n and β_n are periodic if and only if Λ_n is either constant or periodic, so (c) follows from (14).

Finally (d) follows immediately from the analyticity of the functions Λ_n. This proves part (i) of the theorem.

Now we prove part (ii).

Of course E_n, defined in (11), satisfies (e) because it is a projection.

Consider now $\eta \in J_n$. Then $\Lambda_n(\eta)$ belongs to (4), i.e. there exists $m \in \mathfrak{N}_\eta$ such that

$$\Lambda_n(\eta) = \lambda_{\eta, m}.$$

We know that there exist $\varepsilon_{\eta,m}, \delta_{\eta,m} \in \mathbb{R}_+$ and $q_{\eta,m}$ analytic functions $\Lambda_{\eta,m,p}$ satisfying (6), (7) and (8). Therefore we have that (f) is a consequence of (9), provided that

(17) $$\{\Lambda_{\eta,m,1}(\theta), \ldots, \Lambda_{\eta,m,q_{\eta,m}}(\theta)\} = \{\Lambda_l(\theta + 2k\pi) \colon (k, l) \in \mathfrak{M}_n(\eta)\},$$

for $\theta \in]\eta - \delta_{\eta,m}, \eta + \delta_{\eta,m}[$, and

(18) $$\Lambda_l(\theta + 2k\pi) \neq \Lambda_{l'}(\theta + 2k'\pi),$$

for $(k, l), (k', l') \in \mathfrak{M}_n(\eta)$, $(k, l) \neq (k', l')$ and $0 < |\theta - \eta| < \delta_{\eta,m}$.

First we prove (17). Given $1 \leqslant p \leqslant q_{\eta,m}$, there exists $l \in \mathfrak{J}$ such that $\Gamma(s_l) \sim \sim (\eta, m, p)$. Therefore there exists $k \in \mathbb{Z}$ such that

$$J_l = J_{\eta,m,p} + 2k\pi \quad \text{and} \quad \Lambda_l(\theta + 2k\pi) = \Lambda_{\eta,m,p}(\theta) \quad \text{for } \theta \in J_{\eta,m,p}.$$

Without loss of generality we may assume that $k \in \mathbb{Z}_l$. Of course

$$\Lambda_l(\eta + 2k\pi) = \Lambda_{\eta, m, p}(\eta) = \lambda_{\eta, m} = \Lambda_n(\eta),$$

thus $(k, l) \in \mathcal{M}_n(\eta)$ by the definition of $\mathcal{M}_n$, α_n and β_n.

Consider now $(k, l) \in \mathcal{M}_n(\eta)$. Then

$$\eta + 2k\pi \in J_l \quad \text{and} \quad \Lambda_l(\eta + 2k\pi) = \Lambda_n(\eta).$$

This means that

$$\Lambda_l(\eta + 2k\pi) = \lambda_{\eta, m}.$$

Then by (12) and continuity we have that there exist $\delta, \varepsilon \in \mathbb{R}_+$ such that

$$\Lambda_l(\theta + 2k\pi) \in \sigma(T(\theta)) \cap]\lambda_{\eta, m} - \varepsilon, \lambda_{\eta, m} + \varepsilon[$$

for $|\theta - \eta| < \delta$. Then (8) implies that $\Lambda_l(\theta + 2k\pi)$ must coincide on $]\eta - \delta_{\eta, m}, \eta + \delta_{\eta, m}[$ with one of $\Lambda_{\eta, m, 1}, \ldots, \Lambda_{\eta, m, q_{\eta, m}}$. This proves (17).

Assume now that (18) is not satisfied, then by (17) and (11) it follows that $l = l'$. This implies that Λ_l is either constant or periodic. If Λ_l is constant, then $k = k' = 0$ because α_l and β_l have period 2π and $k, k' \in \mathbb{Z}_l$, so that $0 \leqslant k < 1$ and $0 \leqslant k' < 1$. If Λ_l is periodic of minimum period $2p_l\pi$, then $k - k'$ must be a multiple of p_l and this is in contradiction with $k, k' \in \mathbb{Z}_l$ (that is $0 \leqslant k < p_l$, $0 \leqslant k' < p_l$), unless $k = k'$. This proves (18).

It is clear that $\dim(E_n(\theta)\boldsymbol{H})$ is a continuous function with integer values, thus it is constant.

Finally orthogonality follows from the fact that A and B are selfadjoint.

The proof of the theorem is therefore complete.

4. – Examples.

First we give a finite dimensional example. Let A and B be two symmetric 2×2 real matrices

$$A = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_3 \end{bmatrix}, \qquad B = \begin{bmatrix} b_1 & b_2 \\ b_2 & b_3 \end{bmatrix},$$

such that

$$\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \neq 0. \tag{19}$$

The characteristic curves of $I - \alpha A - bB$ are given by the characteristic equation

$$\begin{vmatrix} 1 - \alpha a_1 - \beta b_1 & -\alpha a_2 - \beta b_2 \\ -\alpha a_2 - \beta b_2 & 1 - \alpha a_3 - \beta b_3 \end{vmatrix} = 0,$$

that is

$$(a_1 a_3 - a_2^2)\alpha^2 + (a_1 b_3 + a_3 b_1 - 2a_2 b_2)\alpha\beta + (b_1 b_3 - b_2^2)\beta^2 - (a_1 + a_3)\alpha - (b_1 + b_3)\beta + 1 = 0,$$

which is the equation of a conic.

Indeed

$$a_1 a_3 - a_2^2 = a_1 b_3 + a_3 b_1 - 2a_2 b_2 = b_1 b_3 - b_2^2 = 0,$$

implies

$$a_1 b_3 - a_3 b_1 = 0$$

in contradiction with (19).

In order that our example be consistent with our theorem, we have to check that the origin of the (α, β)-plane lies inside the conic, otherwise (c) is not satisfied. This means that for each $\theta \in \mathbb{R}$ the line of equations

$$\alpha = \cos(\theta)\, t, \qquad \beta = \sin(\theta)\, t,$$

intersects the conic at least once, that is that the following equation has at least one solution

$$\text{(20)} \quad [(a_1 a_3 - a_2^2)\cos^2\theta + (a_3 b_3 + a_1 b_1 - 2a_2 b_2)\cos\theta\sin\theta + \\ +(b_1 b_3 - b_2^2)\sin^2\theta]\, t^2 - [(a_1 + a_3)\cos\theta + (b_1 + b_3)\sin\theta]\, t + 1 = 0.$$

Now the discriminant of this equation is

$$[(a_1 + a_3)\cos\theta + (b_1 + b_3)\sin\theta]^2 - 4[(a_1 a_3 - a_2^2)\cos^2\theta + \\ + (a_1 b_3 + a_3 b_1 - 2a_2 b_2)\cos\theta\sin\theta + (b_1 b_3 - b_2^2)\sin^2\theta] = \\ = [(a_1 - a_3)\cos\theta + (b_1 - b_3)\sin\theta]^2 + 4[a_2\cos\theta + b_2\sin\theta]^2 \geq 0.$$

This implies that (20) has at least one real solution. In fact the discriminant cannot vanish together with the coefficient of t^2, because this would imply that also the coefficient of t would vanish yielding

$$\begin{cases} (a_1 - a_3)\cos\theta + (b_1 - b_3)\sin\theta = 0, \\ (a_1 + a_3)\cos\theta + (b_1 + b_3)\sin\theta = 0, \end{cases}$$

which is inconsistent with (19).

Of course one can consider similar examples in $\mathbb{R}^n$. In this case the conic becomes a real plane algebraic curve of order n. Real algebraic curves are made of ovals, that is of closed connected curves which are homeomorphic to the circle in the projective plane. According to our theorem these ovals should contain the origin inside.

Now we give two examples from the theory of elastic stability.

The first one is the following boundary value problem

$$(21)\qquad \begin{cases} x_1'''' + \tau x_2''' + \lambda x_1'' = 0\,, \\ x_2'''' - \tau x_1''' + \lambda x_2'' = 0\,, \end{cases} \qquad \begin{cases} x_1(\pm 1) = x_2(\pm 1) = 0\,, \\ x_1'(\pm 1) = x_2'(\pm 1) = 0\,, \end{cases}$$

which arises in the study of the buckling of a rod subjected to terminal couple τ and thrust λ (see [2]).

In [2], Theorem 1 the following result is proved. There exists a sequence $(\lambda_n)_{n \in \mathbb{N}}$ of real analytic functions

$$\lambda_n : \mathbb{R} \to \mathbb{R}$$

such that (21) has non-trivial solutions if and only if

$$\lambda = \lambda_n(\tau)\,.$$

Moreover for each $n \in \mathbb{N}$ we have

$$\lambda_n(-\tau) = \lambda_n(\tau)\,,$$

$$\frac{1}{4}(\tau_n^2 - \tau^2) < \lambda_n(\tau) \leqslant \frac{1}{4}((n+1)^2\pi^2 - \tau^2) \qquad \text{for } \tau^2 < \tau_n^2\,,$$

$$\lambda_n(\tau_n) = 0\,,$$

$$\frac{1}{4}(n^2\pi^2 - \tau^2) \leqslant \lambda_n(\tau) < \frac{1}{4}(\tau_n^2 - \tau^2) \qquad \text{for } \tau^2 > \tau_n^2\,,$$

where $\tau_n \in \mathbb{R}_+$ is such that

$$\tan(\tau_n) = \tau_n \qquad \text{and} \qquad n\pi < \tau_n < \left(n + \frac{1}{2}\right)\pi\,.$$

For each $n \in \mathbb{N}$ there exists a $\mathcal{C}^\infty$ function

$$\Phi_n : \mathbb{R} \to H_0^2(-1,1) \times H_0^2(-1,1)\,,$$

such that

$$\Phi_n(\tau) = (\varphi_{n,1}(\tau), \varphi_{n,2}(\tau)) \qquad \text{and} \qquad \Phi_n^\perp(\tau) = (-\varphi_{n,2}(\tau), \varphi_{n,1}(\tau))\,,$$

are an orthonormal basis for the eigenfunctions of (21) with $\lambda = \lambda_n(\tau)$.

This result is in agreement with ours. In [2], Theorem 1 we have proved that Φ_n is only $\mathcal{C}^\infty$ instead than analytic, but on the other hand we have obtained more, in the sense that we have shown that not only the eigenprojections but also the eigenfunctions depend smoothly on τ.

Our last example is from buckling theory of cylindrical shells. Consider a cylindrical shell with radius R, length L and thickness h, subjected to a compressive force λ and to a uniform lateral pressure p (positive inward). Take cylindrical coordinates ρ,

θ, ζ such that the unstressed shell is described by

$$\rho = R\,, \qquad 0 \leqslant \theta \leqslant 2\pi\,, \qquad 0 \leqslant \zeta \leqslant L$$

and let

$$x = \zeta\,, \qquad y = \theta R\,.$$

Then, in the framework of Donnell theory ([6], Section 1.2), the buckling equations are given by ([6], equations (2.2.23) and (2.2.26))

$$\text{(22}a\text{)} \qquad \begin{cases} \Delta^2 F + \dfrac{Eh}{R} w_{xx} = 0 \\ D\Delta^2 w - \dfrac{1}{R} F_{xx} + \dfrac{\lambda}{2\pi R} w_{xx} + pRw_{yy} = 0\,, \end{cases}$$

where E and D are positive constants: the Young modulus and the flexural rigidity. F is the Airy stress function and w is the normal displacement (positive inward).

Of course F and w must be periodic with respect to y:

$$\text{(22}b\text{)} \qquad F(x, y + 2\pi R) = F(x, y)\,, \qquad w(x, y + 2\pi R) = w(x, y)\,,$$

for all $0 \leqslant x \leqslant L$ and $y \in \mathbb{R}$. Moreover we assume simple support boundary conditions at $x = 0$, L:

$$\text{(22}c\text{)} \qquad \begin{cases} F(0, y) = F(L, y) = 0\,, & F_{xx}(0, y) = F_{xx}(L, y) = 0\,, \\ w(0, y) = w(L, y) = 0\,, & w_{xx}(0, y) = w_{xx}(L, y) = 0\,, \end{cases}$$

for each $y \in \mathbb{R}$.

Using Fourier analysis it is easy to solve this boundary value problem. We have that

$$\sin(mx)\cos(ny)\,, \qquad \sin(mx)\sin(ny)\,, \qquad \text{with } m,\ n \in \mathbb{Z},\ m \geqslant 1,\ n \geqslant 0$$

form a complete orthogonal set in the following Hilbert space. Let

$$\Omega =]0, L[\times]0, 2\pi R[$$

and let

$$\boldsymbol{H} = \{u\colon]0, L[\times \mathbb{R} \to \mathbb{R}\colon u|_\Omega \in H^2(\Omega)\,,\ u(x, y + 2\pi R) = u(x, y)\,,\ \forall (x, y) \in]0, L[\times \mathbb{R},$$
$$u(0, y) = u(L, y) = 0,\ \ \forall y \in \mathbb{R}\}$$

with scalar product

$$\langle u, v\rangle = \int\limits_\Omega (u_{xx} v_{xx} + 2u_{xy} v_{xy} + u_{yy} v_{yy})\, dx\, dy\,.$$

Let

$$F(x, y) = \sum_{m \geqslant 1, n \geqslant 0} [\alpha_{m,n} \sin(mx) \cos(ny) + \beta_{m,n} \sin(mx) \sin(ny)],$$

$$w(x, y) = \sum_{m \geqslant 1, n \geqslant 0} [a_{m,n} \sin(mx) \cos(ny) + b_{m,n} \sin(mx) \sin(ny)],$$

and substitute into (22*a*). We obtain the following equations for $\alpha_{m,n}$, $a_{m,n}$, and $\beta_{m,n}$, $b_{m,n}$

$$\begin{cases} (m^2+n^2)^2 \alpha_{m,n} - \dfrac{Eh}{R} m^2 a_{m,n} = 0 \\ \dfrac{1}{R} m^2 \alpha_{m,n} + \left[D(m^2+n^2)^2 - \dfrac{\lambda}{2\pi R} m^2 - pRn^2\right] a_{m,n} = 0, \end{cases}$$

$$\begin{cases} (m^2+n^2)^2 \beta_{m,n} - \dfrac{Eh}{R} m^2 b_{m,n} = 0 \\ \dfrac{1}{R} m^2 \beta_{m,n} + \left[D(m^2+n^2)^2 - \dfrac{\lambda}{2\pi R} m^2 - pRn^2\right] b_{m,n} = 0. \end{cases}$$

These two systems have the same determinant, which yields the characteristic equation:

$$(m^2+n^2)^2\left[D(m^2+n^2)^2 - \frac{\lambda}{2\pi R} m^2 - pRn^2\right] + \frac{Eh}{R} m^4 = 0. \tag{23}$$

Thus the boundary value problem (22) has non-trivial solutions if and only if λ and p lie on the straight lines of equation (23). Of course these straight lines, which are the characteristic curves of our problem, may either coincide or intersect according to the values of the constants D, E, R, h.

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