Annali di Matematica pura ed applicata
(IV), Vol. CLXI (1992), pp. 113-138

Partial Regularities of Minimizers of Certain Quadratic Functionals with Unbounded Obstacles (*).

HONG MIN-CHUN

Summary. – *In this paper, we study the partial regularity properties of vector valued functions u minimizing certain quadratic functionals with an unbounded obstacle which is defined by*

$$\mu = \{v \in H^{1,2}(\Omega, R^N) \mid v^N \geq f(x, v^1(x), \ldots, v^{N-1}(x)) \text{ a.e. on } \Omega,\ v - u_0 \in H_0^{1,2}(\Omega, \boldsymbol{R}^N)\}.$$

1. – Introduction.

The purpose of this paper is to provide partial regularity results for the problem of vector-valued functions minimizing functionals with an unbounded obstacle.

We study the quadratic functional

$$\mathcal{F}(u;\Omega) = \int_\Omega A_{ij}^{\alpha\beta}(x,u)\, D_\alpha u_\beta^i D u^j\, dx\,, \tag{1.1}$$

where Ω is an open set in $\boldsymbol{R}^n$, and $A_{ij}^{\alpha\beta}(x,u)$, $i, j = 1, \ldots, N$; $\alpha, \beta = 1, \ldots, n$, are continuous functions in $\Omega \times \boldsymbol{R}^N$ satisfying the following:

$$|A_{ij}^{\alpha\beta}(x,u)| \leq L\,, \qquad \text{for some } L > 0\,, \tag{1.2}$$

$$\sum_{\alpha,\beta=1}^{n} \sum_{i,j=1}^{N} A_{ij}^{\alpha\beta}(x,u)\, \xi_\alpha^i \xi_\beta^j \geq |\xi|^2, \qquad \text{for all } \xi \in \boldsymbol{R}^{n\times N}. \tag{1.3}$$

We recall that a function $u\colon \Omega \to \boldsymbol{R}^N$ is a local minimum of $\mathcal{F}$ in Ω if for every φ with a compact support in Ω, we have

$$\mathcal{F}(u; \operatorname{spt}\varphi) \leq \mathcal{F}(u+\varphi; \operatorname{spt}\varphi)\,. \tag{1.4}$$

(*) Entrata in Redazione l'1 dicembre 1988.

Indirizzo dell'A.: Department of Mathematics, Zhejiang University, Hangzhou, P.R. China.

In last few years, the regularity theory of minimizers has been developed a great deal. For example, see [5] and [6]. Here we mention some beautiful works of M. GIAQUINTA and E. GIUSTI [9], [10]:

A) Each minimum $u \in H^{1,2}_{\text{loc}}(\Omega, \boldsymbol{R}^N)$ for the functional (1.1) is Hölder-continuous in the interior Ω up to a set vanishing $\boldsymbol{H}^{n-2}$-measure.

B) If an additional splitting condition is assumed on the coefficients, i.e.,

$$A_{ij}^{\alpha\beta}(x, u) = g_{ij}(x, u) G^{\alpha\beta}(x), \tag{1.5}$$

then the dimension of set Σ of interior singularities does not even exceed $n-3$.

In this paper, we extend Giaquinta-Giusti's results quoted above to the same minimizing problem with an unbounded obstacle.

We define the problem of functions minimizing the functional (1.1) with an obstacle as follows:

We say that a function $u: \Omega \to \boldsymbol{R}^N$ is a minimum of the functional (1.1) with an obstacle μ if there exists a subset μ of $H^{1,2}(\Omega, \boldsymbol{R}^N)$, and $u \in \mu$ such that

$$\mathcal{F}(u; \Omega) \to \min_{v \in \mu} \mathcal{F}(v; \Omega). \tag{1.6}$$

The subset μ is supposed to be given by

$$\mu = \{u \in H^{1,2}(\Omega, \boldsymbol{R}^N) | \, u - u_0 \in H_0^{1,2}(\Omega, \boldsymbol{R}^N), u(x) \in M \text{ on } \Omega\},$$

where M is a given set in $\boldsymbol{R}^N$, and $u_0 \in H^{1,2}(\Omega, \boldsymbol{R}^N)$ defining the prescribed boundary values. In order to have μ nonvoid, we assume that $u_0 \in \mu$, which means that $u_0(x) \in M$ a.e. on Ω. For the sake of simplicity we suppose that u_0 is smooth.

In the scalar case, i.e. $N=1$, the problem (1.6) turns out to be equivalent to the problem of seeking solutions to general variational inequalities. The existence and regularity theory of solutions to the variational inequalities have been developed a great deal by many authors. For results and proofs we see the books [14], [15] and their references.

In this paper, we only consider the regularity theory of the obstacle problem (1.6) in the vector case, i.e. $N \geqslant 2$.

In a straight-forward extension of the obstacle problem from scalar-case to vector-case, having considered the following obstacle:

$$\mu = \{y \in \boldsymbol{R}^N | \, y^i(x) \geqslant \psi^i(x), \quad i = 1, \dots, N, \text{ a.e. on } \Omega, \ \psi^i(x) \text{ are given functions}\}, \tag{1.7}$$

HILDEBRANDT, WIDMAN in [11] and GIAQUINTA in [7] presented the regularity theory of functions minimizing the functional (1.3) with an obstacle (1.7) under a diagonal condition on the coefficients:

$$A_{ij}^{\alpha\beta}(x, u) = \delta_{ij} A_{ij}^{\alpha\beta}(x, u). \tag{1.8}$$

It appears that there are difficulties to extend their results to a more general case without the assumption (1.8).

In a different way, the minimizing problem with an obstacle of the type defined by

$$(1.9)\qquad \mu = \{u \in H^{1,2}(\Omega, R^N) | \, u^N \geq f(x, u^1(x), \dots, u^{N-1}(x)) \text{ a.e. on } \Omega,\ f(x, u^1, \dots, u^{N-1}) \text{ is a given function}\},$$

has sbeen studied by many authors. TOMI [16] first proved the existence of a minimum of the problem (1.6) with the obstacle (1.7) by using a lower-semicontinuity argument. For $n=2$, he proved that each minimum is regular. For $n \geq 3$, FUCHS [3] considered the Dirichlet type minimizing problem with an obstacle as follows:

$$(1.10)\qquad \mathcal{F}_0(u; \Omega) = \int_\Omega |Du|^2 dx \to \min_\mu \mathcal{F}_0(\cdot; \Omega),$$

with $\mu = \{u \in H^{1,2}(\Omega, \boldsymbol{R}^N) | \, u^N \geq f(u^1, \dots, u^{N-1})\}$.

He showed that each showed that each minimum of the problem (1.10) is partial $C^{1,\alpha}$-continuous. For a more general obstacle

$$\mu = \{u(x) \in H^{1,2}(\Omega, \boldsymbol{R}^N) | \, u^N \geq f(x, u^1, \dots, u^{N-1}) \text{ a.e. on } \Omega\},$$

WIEGNER [17] proved that the minimum of the Dirichlet-type problem (1.10) belongs to $C^{1,\alpha}(\Omega, \boldsymbol{R}^N)$.

The aim of this paper is to present some results about partial regularity for the minimums of the problem (1.6) with an obstacle of the form (1.9). More precisely, we extent the Fuchs' ([3]) and Wiegner's ([4]) results to the quadratic functional (1.3) under the following assumption:

$$(1.11)\qquad A_{ij}^{\alpha\beta}(x, y) = g_{ij}(x, y) G^{\alpha\beta}(x).$$

We want to point out that the techniques used in this paper are similar to those used by GIAQUINTA ([8]) in the scalar case in 1981, but they are different from Fuchs' in [3] and Wiegner's in [16]. In some recent papers ([1], [2]), FUCHS and ZUZAAR applied their methods to deal with the regularity of minimizing problem with a bounded obstacle under the condition (1.11). But their methods can not be carried over to our case. It is also pointed out in [3] by Fuchs that his results can only carried over to the case of $A_{ij}^{\alpha\beta}(x, y) = a^{\alpha\beta}(x)\delta_{ij}$, if the obstacle is unbounded.

A brief outline of this paper is as follows. In § 2, we prove a reverse Hölder inequality for the unbounded obstacle problem. In § 3, suppose that the minimum u of the problem (1.6) with the obstacle (1.9) is bounded, we prove the partial regularity of the minimum u. In § 4, by using direct methods, we drop the assumption of boundedness of minimizers to prove the partial regularity. In § 5, we present some extensions and mention a few problem which we have not touched at all.

Acknowledgment. I am very grateful to Professor MARIANO GIAQUINTA for his stimulating discussions, and to Professor GUANGCHANG DONG for his very useful suggestions.

2. – Reverse Hölder inequalities.

In this section, we improve the methods used in [10], [5], and prove the high integrability of the gradient of a minimum of the obstacle problem (1.6).

Let us introduce a few notations.

$$Q_R(x_0) := \{x \in \boldsymbol{R}^n : |x^\alpha - x_0^\alpha| \leqslant R,\ \alpha = 1, \dots n\},$$

$$B_R(x_0) := \{x \in \boldsymbol{R}^n : |x - x_0| \leqslant R\}, \qquad u_r := \fint_{B_r(x_0)} u(x)\, dx,$$

$$\tilde{y} := (y^1, \dots, y^{N-1}), \qquad \tilde{u} := (u^1, \dots, u^{N-1}), \qquad f(x, \tilde{u})_R := \fint_{B_R(x_0)} f(x, \tilde{u})\, dx,$$

$$u^{\tilde{\imath}} y^{\tilde{\jmath}} := \sum_{\tilde{\imath}=1}^{N-1} \sum_{\tilde{\jmath}=1}^{N-1} y^{\tilde{\jmath}} u^{\tilde{\imath}},$$

we have

LEMMA 2.1. – Let Q be an n-cube and $f \in L^\nu(Q)$ for some $\nu > q$, and suppose

$$\fint_{Q_R(x_0)} g^q\, dx \leqslant b \left(\fint_{Q_{2R}(x_0)} g\, dx \right)^q + \fint_{Q_{2R}(x_0)} f^q\, dx + \theta \fint_{Q_{2R}(x_0)} g^q\, dx,$$

for each $x_0 \in Q$ and each $R < 1/2 \operatorname{dist}(x_0, Q) \wedge R_0$, where R_0, b, θ b, θ are constants with $b > 1$, $0 \leqslant \theta < 1$. Then we have $g \in L^p_{\text{loc}}(Q)$ for $p \in [q, q + \varepsilon]$, and for $Q_{2R} \subset Q$, $R < R_0$, the following estimate holds

$$\left(\fint_{Q_R} g^p\, dx \right)^{1/p} \leqslant c \left(\fint_{Q_{2R}} g^q\, dx \right)^{1/q} + \left(\fint_{Q_{2R}} f^p\, dx \right)^{1/p},$$

where c and ε are constants depending on b, Q, q, ν, n.

The proof of Lemma 2.1 can be found in [5], [9].

So we have for the obstacle problem.

THEOREM 2.2. – Suppose that (1.2) and (1.3) hold, and let u be a minimum of the obstacle problem (1.6) with the obstacle of type (1.9) in which $f(x, y)$: $\bar{\Omega} \times \boldsymbol{R}^{N-1} \to \boldsymbol{R}$.

And assume that

$$(2.1)\qquad \begin{cases} \text{either (i): } u \in L^\infty_{\mathrm{loc}}(\Omega, \boldsymbol{R}^N),\ f \in C^1(\bar\Omega \times \boldsymbol{R}^{N-1}) \\ \text{or (ii): } \quad \left| \dfrac{\partial f}{\partial y^i}(x, y^1, \dots, y^{N-1}) \right| < L, \qquad i = 1, 2, \dots, N-1, \\ \qquad\qquad \left| \dfrac{\partial f}{\partial x^\alpha}(x, y) \right| \leqslant L, \qquad \alpha = 1, 2, \dots, n \text{ for some constant } L > 0, \end{cases}$$

Then if $u \in H^{1,2}(\Omega, \boldsymbol{R}^N)$, there exists an exponent $p > 2$ such that $u \in H^{1,p}_{\mathrm{loc}}(\Omega, \boldsymbol{R}^N)$. Moreover, for $B_R(x_0) \subset \Omega$ we have

$$(2.2)\qquad \left(\fint_{B_{(1/2)R}(x_0)} |Du|^p\, dx \right)^{1/p} \leqslant c \left[\fint_{B_R(x_0)} (1 + |Du|^2)\, dx \right]^{1/2}$$

provided $R < R_0$, where $c = c(n, \nu, p, L)$.

PROOF. – Without lose of generality, we assume that $x_0 = 0$.
Let

$$(2.3)\qquad \begin{cases} \tilde\varphi = -\eta(\tilde u - \tilde u_R), \\ \varphi^N = -\eta(u^N - u^N_R) + f(x, \tilde u + \tilde\varphi) - (1 - \eta) f(x, \tilde u) - \eta [f(x, \tilde u)]_R, \end{cases}$$

where $\tilde\varphi = (\varphi^1, \dots, \varphi^{N-1})$, $\tilde u = (u^1, \dots, u^{N-1})$, $\eta \in C^\infty_0(B_s)$, $0 \leqslant \eta \leqslant 1$ on B_s, $\eta = 1$ on B_t, $s > t$, $|D\eta| \leqslant c(1/(s-t))$.

It is easy to check that

$$u^N + \varphi^N \geqslant f(x, \tilde u + \tilde\varphi).$$

Thus by the definition of a minimum, we have

$$(2.4)\qquad \int_{B_s} |Du|^2\, dx \leqslant c \int_{B_s} |D(u + \varphi)|^2\, dx.$$

Set

$$\tilde\Phi = (1 - \eta)(\tilde u - \tilde u_R),$$

we get

$$\tilde u - \tilde u_R = -\tilde\varphi + \tilde\Phi.$$

Define Φ^N such that $u^N - u_R^N = -\varphi^N + \Phi^N$ holds. Hence

$$(2.5)\qquad \int_{B_s} |D\varphi|^2\,dx = \int_{B_s} |D\tilde{\varphi}|^2\,dx + \int_{B_s} |D\varphi^N|^2\,dx =$$

$$= \int_{B_s} |D(u^N - \Phi^N)|^2\,dx + \int_{B_s} |D(\tilde{u} - \tilde{\Phi})|^2\,dx \leqslant c\int_{B_s} |Du|^2\,dx + c\int_{B_s} |D\Phi^N|^2\,dx + c\int_{B_s} |D\tilde{\Phi}|^2\,dx \leqslant$$

$$\leqslant c\int_{B_s} |D(u+\varphi)|^2\,dx + c\int_{B_s} |D\Phi^N|^2\,dx + c\int_{B_s} |D\tilde{\Phi}|^2\,dx \leqslant c\int_{B_s} |D\Phi^N|^2\,dx + c\int_{B_s} |D\tilde{\Phi}|^2\,dx\,.$$

Noticing the definition of φ^N and Φ^N, we obtain

$$(2.6)\qquad \int_{B_s} |D\Phi^N|^2\,dx = \int_{B_s} |D(u^N + \varphi^N)|^2\,dx =$$

$$= \int_{B_s} |D[u^N - \eta(u^N - u_R^N) + f(x, \tilde{u} + \tilde{\varphi}) - (1-\eta)\,f(x,\tilde{u}) - \eta f(x,\tilde{u})_R\,]|^2\,dx =$$

$$= \int_{B_s} |D\{(1-\eta)[u^N - u_R^N - f(x,\tilde{u}) + f(x,\tilde{u})_R\,] + f(x, \tilde{u}+\tilde{\varphi})\}|^2\,dx \leqslant$$

$$\leqslant 2\int_{B_s} |D[(1-\eta)(u^N - u_R^N - f(x,\tilde{u}) + f(x,\tilde{u})_R)]|^2\,dx + 2\int_{B_s} |D[f(x,\tilde{u}+\tilde{\varphi})]|^2\,dx \leqslant$$

$$\leqslant c\int_{B_s\setminus B_t} |D[u^N - f(x,\tilde{u})]|^2\,dx + c/(s-t)^2\int_{B_s} |u^N - f(x,\tilde{u}) -$$

$$-(u_R^N - f(x,\tilde{u})_R)|^2\,dx + c\int_{B_s} |D[f(x,\tilde{u}+\tilde{\varphi})]|^2\,dx\,.$$

Under the assumption either (2.1) (i) or (ii), we have

$$(2.7)\qquad \int_{B_s\setminus B_s} |D[f(x,\tilde{u})]|^2\,dx \leqslant c\int_{B_s\setminus B_t} |D\tilde{u}|^2\,dx + L^2|B_s|$$

and

$$(2.8)\qquad \int_{B_s} |D[f(x,\tilde{u}+\tilde{\varphi})]|^2\,dx \leqslant c\int_{B_s} |D(\tilde{u}+\tilde{\varphi})|^2\,dx + L^2|B_s|\,.$$

By (2.6), (2.7), (2.8), we get

$$\int_{B_s} |D\Phi^N|^2\,dx \leqslant c \int_{B_s\setminus B_t} |Du|^2\,dx + c\int_{B_s} |D\tilde{\Phi}|^2\,dx + \tag{2.9}$$

$$+\frac{c}{(s-t)^2}\int_{B_s} |u^N - f(x,u) - [u_R^N - f(x,\tilde{u})_R]|^2\,dx + c|B_s|\,.$$

Since $D\tilde{\Phi} = (1-\eta)D\tilde{u} - (\tilde{u} - \tilde{u}_R)D\eta$, we have

$$|D\tilde{\Phi}|^2 \leqslant c|D\tilde{u}|^2(1-\eta)^2 + c|\tilde{u} - \tilde{u}_R|^2|D\eta|^2 \tag{2.10}$$

and noticing the definition of η, we obtain

$$\int_{B_s} |Du|^2\,dx \leqslant c_1 \int_{B_s\setminus B_t} |Du|^2\,dx + \frac{c_1}{(s-t)^2}\int_{B_s} |\tilde{u} - \tilde{u}_R|^2\,dx + \tag{2.11}$$

$$+\frac{c_1}{(s-t)^2}\int_{B_s} |u^N - f(x,\tilde{u}) - u_R^N + f(x,\tilde{u})_R|^2\,dx + c_1|B_s|\,.$$

Adding to both sides, c_1 times the quantity on the left, then divided by $c_1 + 1$, we obtain

$$\int_{B_t} |Du|^2\,dx \leqslant \theta\int_{B_s} |Du|^2\,dx + c(s-t)^{-2}\int_{B_s} |\tilde{u} - \tilde{u}_R|^2\,dx + \tag{2.12}$$

$$+ c(s-t)^{-2}\int_{B_s} |u^N - f(x,\tilde{u}) - u_R^N + f(x,\tilde{u})_R|^2\,dx + c|B_s|; \qquad \theta = \frac{c_1}{1+c_1} < 1\,.$$

By the Sobolev-Poincaré inequality, we get

$$\left(\fint_{B_{(1/2)R}} |Du|^2\,dx\right)^{1/2} \leqslant \theta\left(\fint_{B_R} |Du|^2\,dx\right)^{1/2} + c\left(\fint_{B_R} |Du|^{2^*}\,dx\right)^{1/2^*} + c\,,$$

where $2^* = 2n/(2+n) < 2$ and using Lemma 2.1, we can obtain the required results, i.e. for $B_R \subset\subset \Omega$, there exists a constant $\varepsilon > 0$, such that for $p \in [2, 2+\varepsilon)$ we have

$$\left(\fint_{B_{(1/2)R}} |Du|^p\,dx\right)^{1/p} \leqslant c\left(\fint_{B_R} |Du|^2\,dx\right)^{1/2} + c \leqslant c\left[\fint_{B_R}(|Du|^2 + 1)\,dx\right]^{1/2}.$$

REMARK. – The assumption (2.1) is similar to the Fuchs' in [3].
By the Sobolev's theorem we get

COROLLARY 2.3. – Under the assumptions of Theorem 2.2, if $n = 2$, then u is locally Hölder continuous in Ω.

3. – Interior regularities.

In this section, we assume that the coefficients $A_{ij}^{\alpha\beta}$ satisfy

$$(3.1)\qquad |A_{ij(u^k)}^{\alpha\beta}(x,u)| \leq L; \quad \alpha,\beta = 1,\dots,n; \; i,j,k = 1,\dots,N$$

for a constant $L>0$, where $A_{ij(u^k)}^{\alpha\beta}$ denote the coefficients $A_{ij}^{\alpha\beta}$'s partial derivatives with respect to u^k and

$$(3.2)\qquad |A(x,y) - A(x',y')| \leq \omega(|x-x'|^2 + |y-y'|^2),$$

where $\omega(t)$ is a nonnegative bounded function increasing in t, concave continuous in $\omega(0)=0$.

Of course, we assume that (1.11) holds. Moreover, we suppose that the following conditions hold:

There exist constants $\lambda_1, \lambda_2, \lambda_1', \lambda_2' > 0$ such that

$$(3.3)\qquad \begin{cases} \lambda_1'|\xi|^2 \geq \sum\limits_{i,j=1}^{N} g_{ij}(x,y)\,\xi^i\xi^j \geq \lambda_1|\xi|^2; & \forall \xi \in \boldsymbol{R}^N, \\ \lambda_2'|\zeta|^2 \geq \sum\limits_{\alpha,\beta=1}^{n} G^{\alpha\beta}(x,y)\,\zeta_\alpha\zeta_\beta \geq \lambda_2|\zeta|^2; & \forall \zeta \in \boldsymbol{R}^n. \end{cases}$$

Then for the obstacle problem, we have

THEOREM 3.1. – Let (1.2), (1.3), (1.11), (3.1), (3.2), (3.3) hold. And let u be a minimum of the obstacle (1.6) with the obstacle

$$\mu = \{v \in H^{1,2}(\Omega, \boldsymbol{R}^N) |\, v^N \geq f(x,\tilde{v}) \text{ a.e. on } \Omega\},$$

and suppose that f is twice continuous differentiable, then if $u \in H^{1,2} \cap L^\infty(\Omega, \boldsymbol{R}^N)$, there exists an open set $\Omega_0 \subset \Omega$ such that $u \in C^{0,\alpha}(\Omega_0, \boldsymbol{R}^N)$ for all $\alpha < 1$. Moreover $\boldsymbol{H}^{n-q}(\Omega \setminus \Omega_0) = 0$ for some $q > 2$. Here $\boldsymbol{H}^{n-q}$ denotes the $(n-q)$-dimensional Hassdorff measure.

PROOF. – By the definition the obstacle problem (1.6), we set

$$\tilde{v} = \tilde{u} + t\tilde{\phi}, \qquad v^N = u^N + f(x, \tilde{u} + t\tilde{\phi}).$$

Since

$$v^N \geq f(x,\tilde{u})$$

we have

$$(3.4)\qquad \int_\Omega A_{ij}^{\alpha\beta}(x,u)\,D_\beta u^j D_\alpha \tilde{\phi}^{\tilde{i}}\,dx + \int_\Omega A_{Nj}^{\alpha\beta}(x,u)\,D_\beta u^j D_\alpha[(D_y f)(x,\tilde{u})\cdot\tilde{\phi}]\,dx =$$

$$\int_\Omega b_{\tilde{i}}(x,u,Du)\,\tilde{\phi}^{\tilde{i}}\,dx + \int_\Omega b_N(x,u,Du)[(D_y f)(x,\tilde{u})\cdot\tilde{\phi}]\,dx;$$

for all $\tilde{\phi} \in H_0^{1,2} \cap L^\infty(\Omega, \boldsymbol{R}^{N-1})$, where $b_k = \sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{n} A_{ij(u^k)}^{\alpha\beta} D_\alpha u^i D_\beta u^j$.

Let us define a set

$$\mathcal{L}_M(u) := \{\phi^N \in H_0^{1,2}(B_R, R) | (0, \phi^N) \in H_0^{1,2} \cap L^\infty(B_R, \boldsymbol{R}^N) \text{ such that } u^N + t\phi^N \geqslant f(x, \tilde{u}) \text{ for all } 0 \leqslant t \leqslant 1\}.$$

For $\phi^N \in \mathcal{L}_M(u)$, we obtain

$$\int_{B_R} A_{Nj}^{\alpha\beta}(x, u) D_\beta u^j D_\alpha \phi^N dx \geqslant \int_{B_R} b_N(x, u, Du) \phi^N dx; \qquad \forall \phi^N \in \mathcal{L}_M(u). \tag{3.5}$$

By the methods of freezing the coefficients, we get from (3.4)

$$\int_{B_R} A_{\tilde{i}j}^{\alpha\beta}(x_0, u_R) D_\beta u^j D_\alpha \tilde{\phi}^{\tilde{i}} dx = \tag{3.6}$$

$$= \int_{B_R} [A_{\tilde{i}j}^{\alpha\beta}(x_0, u_R) - A_{\tilde{i}j}^{\alpha\beta}(x, u)] D_\beta u^j D_\alpha \tilde{\phi}^{\tilde{i}} dx - \int_{B_R} A_{Nj}^{\alpha\beta}(x, u) D_\beta u^j D_\alpha [(D_y f)(x, \tilde{u}) \cdot \tilde{\phi}] dx +$$

$$+ \int_{B_R} b_{\tilde{i}}(x, u, Du) \tilde{\phi}^{\tilde{i}} dx + \int_{B_R} b_N(x, u, Du)[(D_y f)(x, \tilde{u}) \cdot \tilde{\phi}] dx; \quad \forall \tilde{\phi} \in H_0^{1,2} \cap L^\infty(B_R, \boldsymbol{R}^{N-1})$$

and we have from (3.5)

$$\int_{B_R} A_{Nj}^{\alpha\beta}(x_0, u_R) D_\beta u^j D_\alpha \phi^N dx \geqslant \tag{3.7}$$

$$\geqslant \int_{B_R} [A_{Nj}^{\alpha\beta}(x_0, u_R) - A_{Nj}^{\alpha\beta}(x, u)] D_\beta u^j D_\alpha \phi^N dx + \int_{B_R} b_N(x, u, Du) \phi^N dx,$$

$$\text{for all } \phi^N \in \mathcal{L}_M(u).$$

For the sake of simplicity, we introduce some matrix notations.

Let

$$g = (g_{ij}(x_0, u))_{N \times N}, \qquad \tilde{g} = (g_{ij}(x_0, u_R))_{(N-1) \times (N-1)},$$

$$\tilde{h} = (g_{N1}(x_0, u_R), \ldots, g_{N \times N-1}(x_0, u_R))^T,$$

$$\tilde{\phi} = (\phi^1, \ldots, \phi^{N-1})^T,$$

then we get

$$g = \begin{bmatrix} \tilde{g} & \tilde{h} \\ \tilde{h}^T & g_{NN} \end{bmatrix}, \qquad D_\alpha \tilde{u} = (D_\alpha u^1, \dots D_\alpha u^{N-1})^T$$

and $D_\alpha \tilde{\phi} = (D_\alpha \phi^1, \dots, D_\alpha \phi^{N-1})^T$.

For any $\phi^N \in \mathcal{L}_M(u)$, we can choose $\tilde{\phi}$ such that

$$\tilde{g}\tilde{\phi} = \phi^N \tilde{h}.$$

Then we have

$$\tilde{\phi} = \phi^N (\tilde{g})^{-1} \tilde{h}, \qquad D_\alpha \tilde{\phi} = D_\alpha \phi^N (\tilde{g})^{-1} \tilde{h}. \tag{3.8}$$

Substituting (3.8) in (3.6) gives

$$\int_{B_R} G^{\alpha\beta}(x_0, u_R)(D_\beta \tilde{u})^T \tilde{g}(D_\alpha \tilde{\phi})\, dx + \int_{B_R} G^{\alpha\beta}(x_0, u_R) D_\beta u^N \tilde{h}^T \cdot D_\alpha \tilde{\phi}\, dx = \tag{3.9}$$

$$= \int_{B_R} G^{\alpha\beta}(x_0, u_R)(D_\beta \tilde{u})^T \tilde{h}(D_\alpha \phi^N)\, dx + \int_{B_R} G^{\alpha\beta}(x_0, u_R) D_\beta u^N \tilde{h}^T (\tilde{g})^{-1} \tilde{h} D_\alpha \tilde{\phi}^N\, dx =$$

$$= \int_{B_R} [A_{\tilde{i}j}^{\alpha\beta}(x_0, u_R) - A_{\tilde{i}j}^{\alpha\beta}(x, u)] D_\beta u^j D_\alpha \tilde{\phi}^{\tilde{i}}\, dx - \int_{B_R} A_{Nj}^{\alpha\beta}(x, u) D_\alpha [(D_y f)(x, u) \cdot \tilde{\phi}] D_\beta u^j\, dx +$$

$$+ \int_{B_R} b_{\tilde{i}}(x, u, Du) \tilde{\phi}^{\tilde{i}}\, dx + \int_{B_R} b_N(x, u, Du)[(D_y f)(x, \tilde{u}) \cdot \tilde{\phi}]\, dx$$

$$\forall \phi^N \in \mathcal{L}_M(u), \qquad \tilde{\phi} = \phi^N (\tilde{g})^{-1} h.$$

Substracting (3.7) by (3.9) would yield

$$\int_{B_R} G^{\alpha\beta}(x_0, u_R) D_\beta u^N [g_{NN}(x_0, u_R) - \tilde{h}^T (\tilde{g})^{-1} \tilde{h}] D_\alpha \phi^N\, dx \geqslant \tag{3.10}$$

$$\geqslant \int_{B_R} A_{Nj}^{\alpha\beta}(x, u) D_\beta u^j D_\alpha [(D_y f)(x, \tilde{u}) \cdot \tilde{g}^{-1} \tilde{h} \phi^N]\, dx -$$

$$- \int_{B_R} [A_{\tilde{i}j}^{\alpha\beta}(x_0, u_R) - A_{\tilde{i}j}^{\alpha\beta}(x, u)] D_\beta u^j D_\alpha (\phi^N \tilde{g}^{-1} \tilde{h})^{\tilde{i}}\, dx +$$

$$+ \int_{B_R} [A_{Nj}^{\alpha\beta}(x_0, u_R) - A_{Nj}^{\alpha\beta}(x, u)] D_\beta u^j D_\alpha \phi^N\, dx +$$

$$+\int_{B_R} b_N(x,u,Du)\,\phi^N\,dx-\int_{B_R} b_{\tilde i}(x,u,Du)(\phi^N\tilde g^{-1}\tilde h)^{\tilde i}\,dx-$$

$$-\int_{B_R} b_N(x,u,Du)[\phi^N(D_y f)(x,\tilde u)(\tilde g)^{-1}\tilde h]\,dx:=I_1+I_2+I_3+I_4+I_5+I_6.$$

Let

$$g^*_{NN}=g_{NN}-\tilde h^T(g)^{-1}\tilde h.$$

Since

$$\begin{bmatrix} I & 0\\ -\tilde h^T\tilde g^{-1} & 1\end{bmatrix}\begin{bmatrix}\tilde g & \tilde h\\ \tilde h^T & g_{NN}\end{bmatrix}=\begin{bmatrix}\tilde g & \tilde h\\ 0 & g^*_{NN}\end{bmatrix},$$

we have

$$g^*_{NN}=\det(\tilde g^{-1}\det g>\tau,\qquad \text{with some constant } \tau>0.$$

Then we assume that v is a solution of the following Dirichlet problem:

$$\text{(3.11)}\qquad \begin{aligned}&\sum_{\alpha,\beta,j} D_\alpha[A^{\alpha\beta}_{\tilde i j}(x_0,u_R)D_\beta v^j]=0; \qquad 1\le\tilde i\le N-1,\\ &\sum_{\alpha,\beta} D_\alpha[g^*_{NN}G^{\alpha\beta}(x_0,u_R)D_\beta v^N]=0\end{aligned}$$

with $v-u\in H^{1,2}_0(B_R,\boldsymbol{R}^N)$.

By the standard theory of systems of linear partial differential equation (see [5]), we have

$$\text{(3.12)}\qquad \int_{B_\rho}|Dv|^2\,dx\le c\left(\frac{\rho}{R}\right)^n\int_{B_R}|Dv|^2\,dx\le c\left(\frac{\rho}{R}\right)^n\int_{B_R}|Du|^2\,dx$$

for all $\rho\le R$, $B_R\subset\Omega$.

By (3.11), (3.6), and setting $w=u-v$, we get

$$\text{(3.13)}\qquad \int_{B_R} A^{\alpha\beta}_{\tilde i j}(x_0,u_R)D_\beta w^i D_\alpha\tilde\phi^{\tilde i}\,dx=\int_{B_R}[A^{\alpha\beta}_{\tilde i j}(x_0,u_R)-A^{\alpha\beta}_{\tilde i j}(x,u)]D_\beta u^j D_\alpha\tilde\phi^{\tilde i}\,dx+$$

$$+\int_{B_R} b_{\tilde i}(x,u,Du)\,\tilde\phi^{\tilde i}\,dx+\int_{B_R} b_N(x,u,Du)[(D_y f)(x,\tilde u)\cdot\tilde\phi]\,dx-$$

$$-\int_{B_R} A^{\alpha\beta}_{Nj}(x,u)D_\beta u^i D_\alpha[(D_y f(x,\tilde u)\cdot\tilde\phi]\,dx:=I_7+I_8+I_9+I_{10},$$

$$\forall\tilde\phi\in H^{1,2}_0\cap L^\infty(B_R,R^{N-1}).$$

From (3.11) and (3.10), we have

$$\int_{B_R} G^{\alpha\beta}(x_0, u_R) g^*_{NN} D_\beta w^N D_\alpha \phi^N \, dx \geq I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \qquad \forall \phi^N \in \mathcal{L}_M(u). \tag{3.14}$$

Choosing $\phi^N = -[u^N - v^N \vee f(x, u^1, \ldots, u^{N-1})]$ in (3.14) gives

$$u^N + t\phi^N \geq f(x, u^1, \ldots, u^{N-1}) \quad \text{for all } t \in [0, 1]$$

and

$$\int_{B_R} |Dw^N|^2 \, dx \leq |I_1| + \ldots + |I_6| + c \int_{B_R} |D[v^N - v^N \vee f(x, \tilde{u})]|^2 \, dx \tag{3.15}$$

where $\tilde{\phi} = \phi^N \tilde{g}^{-1} \tilde{h}$, $\phi^N = -[u^N - v^N \vee f(x, \tilde{u})]$.

Then the problem comes in estimating the following term:

$$\int_{B_R} |D[v^N - v^N \vee f(x, \tilde{u})]|^2 \, dx .$$

By (3.11), we notice that

$$\int_{B_R} g^*_{NN} G^{\alpha\beta}(x_0, u_R) D_\beta [v^N - v^N \vee f(x, \tilde{u})] D_\alpha \phi^N \, dx = \tag{3.16}$$

$$= -\int_{B_R} g^*_{NN} G^{\alpha\beta}(x_0, u_R) D_\beta [v^N \vee f(x, \tilde{u})] D_\alpha \phi^N \, dx \qquad \forall \phi^N \in H_0^{1,2}(B_R, \mathbf{R}).$$

Therefore, choosing $\phi^N = v^N - v^N \vee f(x, \tilde{u})$, and noticing $v^N = v^N \vee f(x, \tilde{u})$ if $v^N \geq f(x, \tilde{u})$ would yield

$$\int_{B_R} |D[v^N - v^N \vee f(x, \tilde{u})]|^2 \, dx \leq c \int_{B_R} |D_x [f(x, \tilde{u})]|^2 \, dx . \tag{3.17}$$

Using (3.13) and setting $\tilde{\phi}^{\tilde{i}} = -(u^{\tilde{i}} - v^{\tilde{i}})$, $1 \leq \tilde{i} \leq N-1$, give

$$\int_{B_R} |Dw|^2 \, dx \leq |I_1| + |I_2| + \ldots + |I_{10}| + c \int_{B_R} |D_x [f(x, u)]|^2 \, dx . \tag{3.18}$$

Next we divide the proof of Theorem 3.1 into two different cases.

Case (i):

$$D_x f(x_0, \tilde{u}_R) = 0, \qquad D_y f(x_0, \tilde{u}_R) = 0 . \tag{$*$}$$

For the estimation of (3.18), we know that the difficulty comes from estimating I_1 and

$$C \int_{B_R} |D_x[f(x, \tilde{u})]|^2 \, dx .$$

Because $f(x, u)$ belongs to the space C^2, and $u \in L^\infty(\Omega, \boldsymbol{R}^N)$, we get the following facts:

There exists a bounded nonnegative function $\omega_2(t)$ increasing in t, concave continuous with $\omega_2(0) = 0$ such that

$$|D_y f(x, \tilde{u}) - D_y f(x_0, \tilde{u}_R)| \leq \omega_2(|x - x_0|^2 + |u - u_R|^2)$$

and

$$|D_x f(x, \tilde{u}) - D_x f(x_0, \tilde{u}_R)| \leq \omega_2(|x - x_0|^2 + |u - u_R|^2) .$$

Thus we have

$$(3.19) \qquad |I_1| = \left| \int_{B_R} A_{Nj}^{\alpha\beta}(x, u) D_\beta u^j D_\alpha [(D_y f)(x, \tilde{u}) \tilde{g}^{-1} \tilde{h} \phi^N] \, dx \right| \leq$$

$$\leq \left| \int_{B_R} A_{Nj}^{\alpha\beta}(x, u) D_\beta u^j D_\alpha D_y f(x, \tilde{u}) \tilde{g}^{-1} \tilde{h} \phi^N \, dx \right| +$$

$$+ \left| \int_{B_R} A_{Nj}^{\alpha\beta}(x, u) D_\beta u^j (D_y^2 f)(x, \tilde{u}) D_\alpha \tilde{u} \cdot \tilde{g}^{-1} \tilde{h} \phi^N \, dx \right| +$$

$$+ \left| \int_{B_R} A_{Nj}^{\alpha\beta}(x, u) D_\beta u^j (D_y f)(x, \tilde{u}) \tilde{g}^{-1} \tilde{h} D_\alpha \phi^N \, dx \right| +$$

$$\leq c \int_{B_R} |Du|^2 |\phi^N| \, dx + c \int_{B_R} |Du| \, |\phi^N| \, dx +$$

$$+ \left| \int_{B_R} A_{Nj}^{\alpha\beta}(x, u) D_\beta u^j [(D_y f)(x, \tilde{u}) - (D_y f)(x_0, \tilde{u}_R)] \tilde{g}^{-1} \tilde{h} D_\alpha \phi^N \, dx \right|$$

$$\leq c \int_{B_R} (1 + |Du|^2)(|w^N| + |v^N| + |v^N - v^N \vee f(x, \tilde{u})|) \, dx +$$

$$+ c \int_{B_R} \omega_2^2(|x - x_0|^2 + |u - u_R|^2) |Du|^2 \, dx + \varepsilon \int_{B_R} |Dw|^2 \, dx +$$

$$+ \varepsilon \int_{B_R} |D[v^N - v^N \vee f(x, \tilde{u})]|^2 \, dx .$$

The condition $(*)$ gives

$$(3.20)\qquad \int_{B_R} |D_x[f(x,\tilde{u})]|^2\,dx \leqslant \int_{B_R} |(D_x f)(x,\tilde{u})|^2\,dx + \int_{B_R} |D_y f(x,\tilde{u})|^2 |Du|^2\,dx \leqslant$$

$$\leqslant \int_{B_R} |D_x f(x,\tilde{u}) - D_x f(x_0,\tilde{u}_R)|^2\,dx + \int_{B_R} |D_y f(x,\tilde{u}) - D_y f(x_0,\tilde{u}_R)|^2 |Du|^2\,dx \leqslant$$

$$\leqslant c\int_{B_R} (1+|Du|^2)\,\omega_2(|x-x_0|^2 + |u-u_R|^2)\,dx\,.$$

We can estimate I_{10} similarly to I_1

$$(3.21)\qquad |I_{10}| \leqslant c\int_{B_R} (1+|Du|^2)|w|\,dx + \varepsilon\int_{B_R} |Dw|^2\,dx +$$

$$+ c\int_{B_R} (1+|Du|^2)\,\omega_2^2(|x-x_0|^2 + |u-u_R|^2)\,dx\,.$$

Estimating I_2, I_3, I_7 would yield

$$(3.22)\qquad |I_2| + |I_3| + |I_7| \leqslant \int_{B_R} \omega^2(|x-x_0|^2 + |u-u_R|^2)|Du|^2\,dx +$$

$$+ \varepsilon\int_{B_R} |Dw|^2\,dx + \varepsilon\int_{B_R} |D(v^N - v^N \vee f(x,u))|^2\,dx\,.$$

Estimating I_4, I_5, I_6, I_8 and I_9 would give

$$(3.23)\qquad |I_4| + |I_5| + |I_6| + |I_8| + |I_9| \leqslant c\int_{B_R} (|Du|^2 + 1)|\phi|\,dx \leqslant$$

$$\leqslant c\int_{B_R} (|Du|^2 + 1)(|w| + |v^N - v^N \vee f(x,\tilde{u})|)\,dx\,.$$

Hence by (3.18), ..., (3.23), we have

$$(3.24)\qquad \int_{B_R} |Dw|^2\,dx \leqslant c\int_{B_R} \omega^2(|x-x_0|^2 + |u-u_R|^2) +$$

$$+ \omega_2^2(|x-x_0|^2 + |u-u_R|^2) + \omega_2(|x-x_0|^2 + |u-u_R|^2)(1+|Du|^2)\,dx +$$

$$+ c\int_{B_R} (1+|Du|^2)(|w| + |v^N - v^N \vee f(x,\tilde{u})|)\,dx\,.$$

We can use the boundednesses of u, ω and ω_2 to get

$$\int_{B_R} |Dw|^2\,dx \leq c\int_{B_R} (1+|Du|^2)\,dx \tag{3.25}$$

and

$$\int_{B_R} |D[v^N - v^N \vee f(x,\tilde{u})]|^2\,dx \leq c\int_{B_R} (1+|Du|^2)\,dx\,. \tag{3.26}$$

By the Sobolev inequality, we have

$$\int_{B_R} |w|^2\,dx \leq cR^2\int_{B_R} (1+|Du|^2)\,dx \tag{3.27}$$

and

$$\int_{B_R} |v^N - v^N \vee f(x,\tilde{u})|^2\,dx \leq cR^2\int_{B_R} (1+|Du|^2)\,dx\,. \tag{3.28}$$

On the other hand, using the L^p-estimates, and the boundednesses of u and v, we obtain

$$\begin{aligned}\int_{B_R} (|w| + |v^N - v^N \vee f(x,\tilde{u})|)|Du|^2\,dx &\leq \\ &\leq c\int_{B_{2R}} (1+|Du|^2)\,dx\left[\int_{B_R} (|w|^2 + |v^N - v^N \vee f(x,\tilde{u})|^2)\,dx\right]^{p-2/2p} \leq \\ &\leq c\int_{B_{2R}} (1+|Du|^2)\,dx\left[R\,⨍_{B_R}(1+|Du|^2)\,dx\right]^{p-2/2p}; \qquad p>2\,.\end{aligned} \tag{3.29}$$

Then from (3.24), (3.29), and the concavites of ω, ω_2, and using the L^p-estimates again, we get

$$\begin{aligned}\int_{B_R} |Dw|^2\,dx \leq c\int_{B_R} (1+|Du|^2)\,dx\left[\omega\left(R^2 + R^{2-n}\int_{B_R} |Du|^2\,dx\right)\right]^{1-2/p} + \\ + \omega_2\left(R^2 + R^{2-n}\int_{B_R} |Du|^2\,dx\right)^{1-2/p} + c\int_{B_{2R}} (1+|Du|^2)\,dx\left[R^2\,⨍_{B_R}(1+|Du|^2)\,dx\right]^{(p-2)/2p}\end{aligned} \tag{3.30}$$

and notice that

$$\int_{B_\rho} |Du|^2\,dx \leq c\left(\frac{\rho}{R}\right)^n\int_{B_R} |Du|^2\,dx + c\int_{B_R} |Dw|^2\,dx$$

for $\rho \leqslant (1/2)R$, then we can finished the proof of Theorem 3.1 in the Case (i) by the standard methods (see chapter VI of [5]).

Case (ii): either $D_x f(x_0, \tilde{u}_R) \neq 0$ or $D_y f(x_0, \tilde{u}_R) \neq 0$.
We transform v into v^*:

$$v^{*i} = v^i, \qquad i = 1, 2, \ldots N-1,$$

$$v^{*N} = v^N - D_x f(x_0, \tilde{u}_R)\cdot x - D_y f(x_0, \tilde{u}_R)\cdot \tilde{v}.$$

Thus the obstacle problem (1.6) turns out to be equivalent to the following obstacle problem:

$$\mathcal{F}^*(u^*; B_R) \to \min_{v^* \in \mu^*} \mathcal{F}(v^*; B_R),$$

where $\mathcal{F}(v^*; B_R)$ is a new functional defined by

$$\text{(3.31)} \qquad \mathcal{F}^*(v^*; B_R) = \int_{B_R} A_{ij}^{\alpha\beta}(x, v) D_\alpha v^i D_\beta v^j dx :=$$

$$:= \int_{B_R} A_{ij}^{*\alpha\beta}(x, v^*) D_\alpha v^{*i} D_\beta v^{*j} dx + 2\int_{B_R} A_{Nj}^{\alpha\beta}(x, v)(D_\alpha f)(x_0, \tilde{u}_R)[D_\beta v^{*N} +$$

$$+ D_y f(x_0, \tilde{u}_R)\cdot D_\beta \tilde{v}^*]dx + \int_{B_R} A_{NN}^{\alpha\beta}(x, v)(D_\alpha f)(x_0, \tilde{u}_R)(D_\beta f)(x_0, \tilde{u}_R)\, dx :=$$

$$:= \int_{B_R} A_{ij}^{*\alpha\beta}(x, v) D_\alpha v^{*i} D_\beta v^{*j} dx + \int_{B_R} b(x, v^*)\cdot Dv^* dx + \int_{B_R} c(x, v^*)\, dx$$

and

$$\mu^* = \{v^* \in H^{1,2}(B_R, R^N) |\, v^{*N} \geqslant f^*(x, \tilde{v}^*),\ v^* - u^* \in H_0^{1,2}(B_R, \boldsymbol{R}^N)\}.$$

Here

$$f^*(x, \tilde{v}^*) = f(x, \tilde{v}^*) - (D_x f)(x_0, \tilde{u}_R)\cdot x - (D_y f)(x_0, \tilde{u}_R)\cdot \tilde{v}^*,$$

$$A_{ij}^{*\alpha\beta}(x, v^*) = G^{\alpha\beta}(x, v^*) g_{ij}^*(x, v^*)$$

with

$$(g_{ij}^*(x, v^*)) = Z^T(x_0, \tilde{u}_R)(g_{ij}(x, v^*)) Z(x_0, \tilde{u}_R),$$

$$Z(x_0, \tilde{u}_R) = \begin{bmatrix} I & 0 \\ \theta_1^T & 1 \end{bmatrix}, \qquad \theta_1 = (D_{y^1} f(x_0, u_R), \ldots, D_{y^{N-1}} f(x_0, u_R))^T.$$

It is easy to check that

$$(D_x\, d^*)(x_0, \tilde{u}_R) = 0\,, \qquad (D_y f^*)(x_0, \tilde{u}_R) = 0$$

and that the coefficients $A_{ij}^{*\alpha\beta}$ also satisfy the condition (3.3), and f^* is twice continuous differentiable.

Comparing $\mathcal{F}^*$ with $\mathcal{F}$, we have two additional terms

$$\int_{B_R} b(x, v^*)\cdot Dv^*\, dx + \int_{B_R} c(x, v^*)\, dx\,,$$

of the form.

Then from the definition of the minimum in B_R, we get a few additional terms besides (3.4) and (3.5)

$$\int_{B_R} b(x, u^*)\, D_\alpha \phi\, dx + \int_{B_R} b_{u^k}(x, u^*)\, \phi^k Du^*\, dx + \int_{B_R} c_{u^i}(x, u^*)\, \phi^i\, dx\,.$$

By standard arguments (see [5]), we can treat above terms easily. The other terms in (3.4) and (3.5) have been treated in the Case (i). Finally we have shown Theorem 3.1. q.e.d.

From the proof of Theorem 3.1 we get

COROLLARY 3.2. – Suppose that the coefficients $A_{ij}^{\alpha\beta}$ are constants, and assume that (1.11) holds, and let the obstacle function $f = 0$ in (1.9), and let condition (3.3) hold, then if $u \in H^{1,2}_{\mathrm{loc}}(\Omega, \mathbf{R}^N)$ is an minimum of the problem (1.6) with an obstacle (1.9), the conclusion of Theorem 3.1 is still true. Moreover the singular set Σ is empty.

4. – The direct approach to regularity.

In Section 3, we have shown that each minimum $u \in H^{1,2} \cap L^\infty(\Omega, \mathbf{R}^N)$ of the obstacle problem (1.6) is partial regular, but usually we can only show the existence of a minimum u belonging to the space $H^{1,2}(\Omega, \mathbf{R}^N)$ (see [16]). In this section we want to drop the assumption of boundedness of the minimizer, i.e. $u \in L^\infty(\Omega, \mathbf{R}^N)$. Instead we assume that the first derivatives of the function f in (1.9) are uniformly continuous and uniformly bounded, i.e. there exists a constant $L > 0$ such that

$$(4.1) \qquad \left|\frac{\partial f}{\partial x^\alpha}(x, y)\right| \le L\,, \quad \alpha = 1, \dots n; \qquad \left|\frac{\partial f}{\partial y^i}(x, y)\right| \le L\,, \quad i = 1, \dots, N-1;$$

for all $(x, y) \in \Omega \times \boldsymbol{R}^{N-1}$ and

$$\frac{\partial f}{\partial x^\alpha}(x, y) \quad \text{and} \quad \frac{\partial f}{\partial y^i}(x, y) \tag{4.2}$$

are uniformly continuous and uniformly bounded.

Then we can show the partial regularity of the minimizer of the obstacle problem (1.6) too.

THEOREM 4.1. – Assume that the coefficients $A_{ij}^{\alpha\beta}$ are uniformly continuous and uniformly bounded, and the splitting condition (1.11) and condition (3.3) hold. Let $u \in H^{1,2}_{\text{loc}}(\Omega, \boldsymbol{R}^N)$ be a local minimum of the problem (1.6) with an obstacle

$$\mu = \{v \in H^{1,2}(\Omega, \boldsymbol{R}^N) |\, u^N \geqslant f(x, \tilde{v}) \text{ a.e. on } \Omega\}$$

and suppose that f satisfies the assumptions (4.1) and (4.2). Then there exists an open set $\Omega_0 \subset \Omega$ such that $u \in C^{0,\alpha}(\Omega_0, \boldsymbol{R}^N)$ for every $\alpha < 1$. Moreover we have

$$\Omega \setminus \Omega_0 = \left\{x_0 \in \Omega \colon \liminf_{R \to 0^+} R^{2-n} \int_{B_R(x_0)} |Du|^2 dx > \varepsilon_0 \right\},$$

where ε_0 is a positive constant independent of u. Finally $\boldsymbol{H}^{n-q}(\Omega \setminus \Omega_0) = 0$ for some $q > 2$.

PROOF. – For the obstacle problem (1.6), we make a transformation:

$$v^{*i} = v^i, \qquad i = 1, \ldots, N-1; \qquad v^{*N} = v^N - f(x, \tilde{v}).$$

Then we have

$$D_\alpha v^{*i} = D_\alpha v^i, \qquad i = 1, 2, \ldots, N-1,$$

$$D_\alpha v^N = D_\alpha v^{*N} + D_\alpha [f(x, \tilde{v}^*)] = D_\alpha v^{*N} + D_\alpha f(x, \tilde{v}) + D_y f(x, \tilde{v}) \cdot D_\alpha \tilde{v},$$

we define the new functional

$$\mathcal{F}^*(v^*; B_R) := \mathcal{F}(v; B_R) = \int_{B_R} G^{\alpha\beta}(x, v) g_{ij}(x, v) D_\alpha v^i D_\beta v^j dx = \tag{4.3}$$

$$= \int_{B_R} G^{\alpha\beta}(x, v^*) g_{\tilde{i}\tilde{j}}(x, v^*) D_\alpha v^{*\tilde{i}} D v^{*\tilde{j}}\, dx +$$

$$+ 2 \int_{B_R} G^{\alpha\beta}(x, v^*) g_{N\tilde{j}}(x, v^*) D_\beta v^{*\tilde{j}} [D_\alpha v^{*N} + D_y f(x, \tilde{v}^*) D_\alpha \tilde{v}^* + D_\alpha f(x, \tilde{v}^*)]\, dx +$$

$$+ \int_{B_R} G^{\alpha\beta}(x, v^*) g_{NN}(x, v^*) [D_\alpha v^{*N} + D_y f(x, \tilde{v}^*) D_\alpha v^* + D_\alpha f(x, \tilde{v}^*)][D_\beta v^{*N} +$$

$$+D_y f(x,\tilde{v}^*)D_\beta \tilde{v}^* + D_\beta f(x,\tilde{v}^*)]\,dx = \int_{B_R} G^{\alpha\beta}(x,v^*)g^*_{ij}(x,v^*)D_\alpha v^{*i}D_\beta v^{*j}\,dx +$$

$$+2\int_{B_R} G^{\alpha\beta}(x,v^*)\{g_{Nj}(x,v^*)D_\alpha f(x,\tilde{v}^*)D_\beta v^{*j} + g_{NN}(x,v^*)\cdot D_\alpha f(x,\tilde{v}^*)[D_\alpha v^{*N} +$$

$$+D_y f(x,\tilde{v}^*)\cdot D_\beta \tilde{v}^*]\}\,dx + \int_{B_R} G^{\alpha\beta}(x,v^*)g_{NN}(x,v^*)D_\alpha f(x,\tilde{v}^*)D_\beta f(x,\tilde{v}^*)\,dx :=$$

$$:= \int_{B_R} G^{\alpha\beta}(x,v^*)g^*_{ij}(x,v^*)D_\alpha v^{*i}D_\beta v^{*j}\,dx + \int_{B_R} b(x,v^*)\cdot Dv^*\,dx + \int_{B_R} c(x,v^*)\,dx\,,$$

where $(g^*_{ij}(x,y))_{NN} = Z^T(x,\tilde{y})(g_{ij}(x,y))Z(x,\tilde{y})$,

$$Z(x,y) = \begin{bmatrix} I & 0 \\ \theta_2 & 1 \end{bmatrix}, \qquad \theta_2 = (D_{y^1}f(x,\tilde{y}),\dots,D_{y^{N-1}}f(x,\tilde{y}))\,.$$

Noticing that (4.1) holds, we can show that $g^*_{ij}(x,y)$ $(i,j=1,\dots,N)$ also satisfy (3.3). Moreover we have

$$|b(x,v^*)| \leq L\,, \qquad |c(x,v^*)| \leq L \quad \text{for some } L>0\,.$$

And there exist two nonnegative bounded functions $\omega_3(t)$ and $\omega_4(t)$ increasing in t, concave continuous in $\omega_3(0)=0$ and $\omega_4(0)=0$ such that for x, $x' \in \omega$ and y, $y' \in \boldsymbol{R}^N$

$$|b(x,y)-b(x',y')| \leq \omega_3(|x-x'|^2+|y-y'|^2)\,,$$

$$|c(x,y)-c(x',y')| \leq \omega_4(|x-x'|^2+|y-y'|^2)\,.$$

The obstacle problem (1.6) turns out to be equivalent to the following obstacle problem:

$$\mathcal{F}^*(u^*;B_R) \to \min_{v^{*N}\geq 0} \mathcal{F}^*(v^*;B_R)\,, \qquad v^*-u^* \in H^{1,2}_0(B_R,R^N)\,, \tag{4.4}$$

where $\mathcal{F}^*$ is defined by (4.3).

For the sake of simplicity, we still denote u*, $g^*_{ij}(x,u^*)$, and $\mathcal{F}^*$ by u, $g_{ij}(x,u)$ and $\mathcal{F}$.

By the results of existence about the obstacle problem (see [16]), we assume that U is a minimum of the following obstacle problem:

$$\mathcal{F}_0(U;B_R) \to \min_{v^N\geq 0} \mathcal{F}_0(v;B_R)\,, \qquad v-u \in H^{1,2}_0(B_R,R^N)\,, \tag{4.5}$$

where

$$\mathcal{F}_0(v;B_R) := \int_{B_R} G^{\alpha\beta}(x_0,u_R)g_{ij}(x_0,u_R)D_\alpha v^i D_\beta v^j dx\,.$$

By arguments similar to Section 3, we get

$$\int_{B_R} G^{\alpha\beta}(x_0, u_R)\, g_{i\tilde{j}}(x_0, u_R)\, D_\alpha U^i D_\beta \phi^{\tilde{j}}\, dx = 0, \tag{4.6}$$

$$\forall \phi^{\tilde{j}} \in H_0^{1,2}(B_R, \mathbf{R}), \qquad \tilde{j} = 1, \dots, N-1.$$

and

$$\int_{B_R} G^{\alpha\beta}(x_0, u_R)\, g_{iN}(x_0, u_R)\, D_\alpha U^i D_\beta \phi^N\, dx \geqslant 0, \tag{4.7}$$

$$\forall \phi^N \in H_0^{1,2}(B_R, R)\colon U^N + t\phi^N \geqslant 0,\ \forall t \in [0,1].$$

Let v be a minimum of the following Dirichlet problem:

$$\mathcal{F}_0(v; B_R) \to \min_{\substack{w \in H^{1,2}(B_R) \\ w - u \in H_0^{1,2}(B_R)}} \mathcal{F}_0(w; B_R). \tag{4.8}$$

We have from the standard theory of systems of linear elliptic equations (see [5])

$$\int_{B_\rho} |Dv|^2\, dx \leqslant c\left(\frac{\rho}{R}\right)^n \int_{B_R} |Du|^2\, dx$$

for all $\rho \leqslant (1/2)R$.

From the processes of proofs in Section 3 and Corollary 3.2, or see [13], we can obtain

$$\int_{B_R} |D(v - U)|^2\, dx = 0 \ \Rightarrow\ v = U \text{ a.e. on } B_R.$$

Therefore

$$\int_{B_\rho} |DU|^2\, dx \leqslant c\left(\frac{\rho}{R}\right)^n \int_{B_R} |Du|^2\, dx, \qquad \forall \rho \leqslant (1/2)R.$$

Setting $w = u - U$ would yield

$$\int_{B_\rho} |Du|^2\, dx \leqslant \int_{B_\rho} |DU|^2\, dx + \int_{B_\rho} |D(u - U)|^2\, dx \leqslant c\left(\frac{\rho}{R}\right)^n \int_{B_R} |Du|^2\, dx + \int_{B_R} |Dw|^2\, dx. \tag{4.9}$$

Using the assumption (3.3), and inequalities (4.6), (4.7), we have

$$(4.10)\qquad \int_{B_R} |Dw|^2\,dx \leqslant \int_{B_R} g_{ij}(x_0, u_R)\,G^{\alpha\beta}(x_0, u_R)\,D_\alpha w^i D_\beta w^j dx \leqslant$$

$$\leqslant \int_{B_R} g_{ij}(x_0, u_R)\,G^{\alpha\beta}(x_0, u_R)\,D_\alpha u^i D_\beta u^j dx - \int_{B_R} g_{ij}(x_0, u_R)\,G^{\alpha\beta}(x_0, u_R)\,D_\alpha U^i D_\beta U^j\,dx \leqslant$$

$$\leqslant \int_{B_R} g_{ij}(x, u)\,G^{\alpha\beta}(x, u)\,D_\alpha u^i D_\beta u^j dx + \int_{B_R} b(x, u)\cdot Du\,dx +$$

$$+ \int_{B_R} c(x, u)\,dx - \int_{B_R} g_{ij}(x, U)\,G^{\alpha\beta}(x, U)\,D_\alpha U^i D_\beta U^j\,dx - \int_{B_R} b(x, U)\cdot DU\,dx -$$

$$- \int_{B_R} c(x, U)\,dx + c\int_{B_R} [\omega(R^2 + |u - u_R|^2) + \omega(R^2 + |U - U_R|^2)][|Du|^2 + |DU|^2]\,dx +$$

$$+ c\int_{B_R} [|Du| + |DU|][\omega_3(R^2 + |u - u_R|^2) + \omega_3(R^2 + |U - U_R|^2)\,dx +$$

$$+ c\int_{B_R} [\omega_4(|x - x_0|^2 + |u - u_R|^2) + \omega_4(|x - x_0|^2 + |U - U_R|^2)]\,dx +$$

$$+ \int_{B_R} b(x_0, u_R)\cdot D(U - u)\,dx\,.$$

Noticing that $u|_{\partial B_R} = U|_{\partial B_R}$, and u is a minimum of the obstacle problem (4.5), we have

$$(4.11)\qquad \int_{B_R} |Dw|^2\,dx \leqslant c\int_{B_R} (|Du|^2 + |DU|^2)[\omega(R^2 + |u - u_R|^2) + \omega(R^2 + |U - U_R|^2)]\,dx +$$

$$+ c\int_{B_R} (|Du| + |DU|)[\omega_3(R^2 + u - u_R^2) + \omega_3(R^2 + |U - U_R|^2)\,dx +$$

$$+ c\int_{B_R} [\omega_4(R^2 + |u - u_R|^2) + \omega_4(R^2 + |U - U_R|^2)]\,dx\,.$$

By the Sobolev-Poincaré inequality and L^p-estimate, we get

$$\int_{B_R} |Du|^2 \omega(R^2 + |u - u_R|^2)\, dx \leq \tag{4.12}$$

$$\leq \int_{B_{2R}} (1 + |Du|^2)\, dx\, \omega \left(R^2 + cR^{2-n} \int_{B_R} |Du|^2\, dx\right)^{1-2/p}, \qquad p > 2\,.$$

Seeing [5], [9], we have

$$\int_{B_R} |DU|^p\, dx = \int_{B_R} |Dv|^p\, dx \leq c \int_{B_R} |Du|^p\, dx\,, \qquad \text{for all } p > 2\,.$$

We estimate the other terms of (4.11) to obtain that

$$\int_{B_R} |Du|^2\, dx \leq c\left(\frac{\rho}{R}\right)^n + \omega\left(R^2 + cR^{2-n} \int_{B_R} |Du|^2\, dx\right)^{1-2/p} + \tag{4.13}$$

$$+ \omega_3\left(R^2 + cR^{2-n} \int_{B_R} |Du|^2\, dx\right)^{1-2/p} + \omega_4\left(R^2 + cR^{2-n} \int_{B_R} |Du|^2\, dx\right)^{1-2/p} \int_{B_{2R}} (1 + |Du|^2)\, dx$$

for all $\rho \leq (1/2)R$.

By the standard steps (see [5]), we have finished the proof of Theorem 4.1. q.e.d.

From the results of [5], and the proof of Theorem 4.1, we have.

COROLLARY 4.2. – Assume that the coefficients $A_{ij}^{\alpha\beta}$ are continuous (not necessarily uniformly continuous), the first derivatives of f are continuous, and the other assumptions of Theorem 4.1 hold, then the conclusion of Theorem 4.1 also holds.

COROLLARY 4.3. – Assume that the coefficients $A_{ij}^{\alpha\beta}$ are Hölder continuous, the first derivatives of f are Hölder continuous and bounded, and the assumptions of Theorem 4.1 hold, then the first derivatives of the minimum u of the obstacle problem (1.6) are locally Hölder continuous in $\Omega_0 \subset \Omega$.

5. – A few remarks.

In this section, we want to show that for a special class of quadratic multiple integrals and bounded minimum points of the obstacle problem we can improve the estimate of the Hausdorff dimension of the singular set. The methods used here is due to GIAQUINTA and GIUSTI [10].

More precisely we shall restrict ourselves to the special form of the coefficients given by

$$A_{ij}^{\alpha\beta}(x,y) = G^{\alpha\beta}(x)\, g_{ij}(x,y)$$

and suppose that f is a function depending only on u. Moreover, we assume that there exists a constant $L > 0$ such that

$$|Df| \leqslant L\,, \qquad |D^2 f| \leqslant L\,. \tag{5.1}$$

Thus we have

LEMMA 5.1. – Let $A^{(\nu)}(x,y) = g_{ij}^{(\nu)}(x,y)\, G^{\alpha\beta(\nu)}$ be a sequence of continuous functions in $B \times \boldsymbol{R}^N$ (B is the unit ball in $\boldsymbol{R}^n$) converging uniformly to $A(x,y)$ and satisfying the following assumption:

$$|A^{(\nu)}(x,y)| \leqslant M\,, \qquad \text{for some constant } M > 0\,, \tag{5.2}$$

$$A^{(\nu)}\xi\cdot\xi = A_{ij}^{\alpha\beta(\nu)}(x,y)\,\xi_\alpha^i \xi_\beta^j \geqslant |\xi|^2\,, \qquad \forall \xi \in \boldsymbol{R}^{nN}\,, \tag{5.3}$$

$$|A^{(\nu)}(x,y) - A^{(\nu)}(x',y')| \leqslant \omega(|x-x'|^2 + |y-y'|^2) \quad \text{for } (x,y), (x',y') \in \Omega \times \boldsymbol{R}^N \tag{5.4}$$

where $\omega(t)$ is a bounded continuous concave function with $\omega(0) = 0$. For each $\nu = 1, 2, \dots$, let $u^{(\nu)}$ be a solution of the following obstacle problem:

$$\mathcal{F}^{(\nu)}(u^{(\nu)}; B) \to \min_{\substack{v \in \mu \\ v - u_0 \in H_0^{1,2}(B, \boldsymbol{R}^N)}} \mathcal{F}^{(\nu)}(v; B)\,,$$

where

$$\mathcal{F}^{(\nu)}(u^{(\nu)}; B) = \int_B A^{(\nu)}(x, u^{(\nu)})\, Du^{(\nu)} Du^{(\nu)}\, dx$$

and

$$\mu = \{u \in H^{1,2}(\Omega, \boldsymbol{R}^N) |\, u^N \geqslant f(u^1, \dots, u^{N-1})\}\,.$$

And assume that (5.1), (5.2) and (5.3) hold, and suppose that $u^{(\nu)}$ converge to u weakly in $L^2(B; R^N)$. Then u is a minimum of the following obstacle problem:

$$\mathcal{F}(u; B) = \int_B A(x,u)\, Du\, Du\, dx \to \min_{\substack{v \in \mu \\ v - u_0 \in H_0^{1,2}(B)}} \mathcal{F}(v)\,.$$

Moreover, if x_ν is a singular point for $u^{(\nu)}$, and $x_\nu \to x_0$ then x_0 is a singular point for u.

PROOF. – Similar to the proof of Lemma 1 in [10], it follows from Theorem 2.2 that for $R<1$ we have

$$\int_{B_R} |Du^{(\nu)}|^q \, dx \leq c(R), \qquad q>2, \tag{5.5}$$

where $c(R)$ is a constant independent of ν.

It implies that for every $R<1$ we have

$$\begin{cases} u^{(\nu)} \to u & \text{strongly in } L^2(B_R), \\ Du^{(\nu)} \to Du & \text{weakly in } L^q(B_R). \end{cases} \tag{5.6}$$

Passing possibly to a subsequence we may suppose that $u^{(\nu)} \to u$ a.e. in B.

We can show that (see [10])

$$\mathcal{F}(u; B_R) \leq \liminf_{\nu\to\infty} \mathcal{F}^{(\nu)}(u^{(\nu)}; B_R). \tag{5.7}$$

Let $\eta(x)$ be a C^1 function in B, with $0 \leq \eta \leq 1$, $\eta = 0$ in B_ρ ($\rho < R$) and $\eta = 1$ outside B_R. Then for any $w \in \mu$, $w|_{\partial B} = u|_{\partial B}$, we set $v^{(\nu)^i} = w^i + (u^{(\nu)^i} - u^i)$, $i = 1, \ldots, N-1$, and $v^{(\nu)^N} = w^N + u^{(\nu)^N} - u^N + (\tilde{v}^{(\nu)}) - f(\tilde{w})$. Since $w^N \geq f(\tilde{w})$, we obtain that $v^{(\nu)^N} \geq f(\tilde{v}^{(\nu)})$ and therefore

$$\mathcal{F}^{(\nu)}(u^{(\nu)}; B_R) \leq \mathcal{F}^{(\nu)}(v^{(\nu)}; B_R). \tag{5.8}$$

Taking (5.1), (5.6) and (5.8) into account, we can get

$$\mathcal{F}(u; B_R) \leq \liminf_{\nu\to\infty} \mathcal{F}^{(\nu)}(v^{(\nu)}; B_R) \leq \mathcal{F}(w; B_R) + c\|\eta\|_{q/q-2}, R.$$

The other steps of the proof is similar to the steps of Lemma 1 in [10]. Thus we have shown the conclusion of Lemma 5.1.

REMARK. – If the assumption $|D^2 f| \leq L$ is not true, we may make the same transformation as in Section 4. Then the conclusion of Lemma 5.1 is also true.

We suppose that

$$G^{\alpha\beta}(0) = \delta_{\alpha\beta} \tag{5.9}$$

and moreover we assume that

$$\int_0^1 \frac{\omega(t^2)}{t} \, dt < +\infty. \tag{5.10}$$

We have

LEMMA 5.2. – Let u be a minimum of the obstacle problem (1.6) for the functional (1.1) with an obstacle (1.9), and assume that f is a function independent of x, and let

(1.2), (1.3) and (1.5) hold. Then for every ρ, R with $0<\rho<R$ we have

$$\int_{B_R} |u(Rx)-u(\rho x)|^2\, d\boldsymbol{H}_{n-1}(x) \leq c \log\frac{R}{\rho}[\Phi(R)-\Phi(\rho)] \tag{5.11}$$

where

$$\Phi(t) = t^{2-n}\exp\left(c\int_0^1 \frac{\omega(s^2)}{s}\,ds\right)\int_{B_t} A(x,u)\,Du\,Du\,dx\,.$$

The proof of Lemma 5.2 is similar to the proof of Lemma 2 in [10], we only notice that

$$x_t = t\left(\frac{x}{|x|}\right), \qquad u_t = (x_t) \in \mu\,.$$

we have

THEOREM 5.3. – Let u be a bounded minimizer of the obstacle problem (1.6) with the obstacle (1.9) and let (1.2), (1.3) and (1.5) hold, and suppose that the conclusion of Lemma 5.2 hold. Then the dimension of the singular set Σ of u can not exceed $n-3$. If $n=3$, u may have at most isolated singular points.

Finally we want to mention some open problems:

PROBLEM 1. – If we do not assume that the assumption (1.11) holds, we do not know whether the conclusion of Theorem 3.1 or Theorem 4.1 is true or not.

PROBLEM 2. – If we consider the obstacle of the form

$$\mu = \{u \in H^{1,2}(\Omega, \boldsymbol{R}^N)|\, u^i(x) \geq \psi^i(x),\ i=1,\dots,N \text{ a.e. on } \Omega\}\,,$$

we don't know whether the conclusion of Theorem 3.1 or Theorem 4.1 is true or not. (This problem was mentioned by GIAQUINTA in [5]).

REFERENCES

[1] F. DUZAAR - M. FUCHS, *Variational problems with non-convex obstacles and integrals constraint*, Math. Z., **191** (1986), pp. 585-591.

[2] F. DUZAAR - M. FUCHS, *Optimal regularity theorem for variational problems with obstacles*, Manuscripta Math., **56** (1986), pp. 209-234.

[3] M. FUCHS, *Variational inequalities for vector-values functions with non-convex obstacles*, Analysis, 5 (1985), pp. 223-238.

[4] M. FUCHS, *Some remarks on the boundary regularity for minima of variational problems with obstacles*, Manuscripta Math., **54** (1985), pp. 107-119.

[5] M. GIAQUINTA, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Annals of Math. Studies, no. 105, Princeton Univ. Press (1983).

[6] M. GIAQUINTA, *The problem of the regularity of minimizers, Proceedings of the International Congress of Mathematicians*, Berkeley, California U.S.A. (1986), pp. 1072-1083.

[7] M. GIAQUINTA, *Regularity results for weak solutions to variational equations and inequalities for nonlinear elliptic systems*, No. 54 SFB 123, Heidelberg (1980).

[8] M. GIAQUINTA, *Remarks on the regularity of weak solutions to some variational inequalities*, Math. Z., **177** (1981), pp. 15-31.

[9] M. GIAQUINTA - E. GIUSTI, *On the regularity of the minima of variational integrals*, Acta Math., **148** (1982), pp. 31-46.

[10] M. GIAQUINTA - E. GIUSTI, *The singular set of the minima of certain quadratic functional*, Ann. Sc. Norm. Pisa, **11** (1984), pp. 45-55.

[11] S. HILDEBRANDT - K.-O. WIDMAN, *Variational inequalities for vector valued function*, J. für Reine u. Angew. Math., **309** (1979), pp. 181-220.

[12] M. C. HONG, *Existence and partial regularity in the calculus of variations*, Ann. Mat. Pura Appl., **149** (1987), pp. 311-328.

[13] M. C. HONG, *Regularity results for the minimizers of certain quadratic functionals with obstacles*, to appear.

[14] D. KINDERLEHRER - G. STAMPACCHIA, *An Introduction to Variational Inequalities and their Applications*, Academic Press, New York (1980).

[15] J. F. RODRIGUES, *Obstacle Problems in Mathematical Physics*, North-Holland Math. Studies, **134**, Notas de Mathematica, **114** (1987).

[16] F. TOMI, *Variations problem vom Dirichlet-type mit einner Ungleichung als Nebendedingwng*, Math. Z., **128** (1972), pp. 43-74.

[17] M. WIEGNER, *On minima of variational problems with some non-convex constrains*, Manuscripta Math., **57** (1987), pp. 149-168.