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On the Complex Zeros of Solutions of Linear Differential Equations (*) (**).

STEVEN B. BANK

Summary. – *A classical result (see* R. NEVANLINNA, Acta Math., 58 (1932), p. 345) *states that for a second-order linear differential equation,* $w'' + P(z)w' + Q(z)w = 0$, *where* $P(z)$ *and* $Q(z)$ *are polynomials, there exist finitely many rays,* $\arg z = \varphi_j$, *for* $j = 1, \ldots, m$, *such that for any solution* $w = f(z) \not\equiv 0$ *and any* $\varepsilon > 0$, *all but finitely many zeros of* f *lie in the union of the sectors* $|\arg z - \varphi_j| < \varepsilon$ *for* $j = 1, \ldots, m$. *In this paper, we give a complete answer to the question of determining when the same result holds for equations of arbitrary order having polynomial coefficients. We prove that for any such equation, one of the following two properties must hold: (a) for any ray,* $\arg z = \varphi$, *and any* $\varepsilon > 0$, *there is a solution* $f \not\equiv 0$ *of the equation having infinitely many zeros in the sector* $|\arg z - \varphi| < \varepsilon$, *or (b) there exist finitely many rays,* $\arg z = \varphi_j$, *for* $j = 1, \ldots, m$, *such that for any* $\varepsilon > 0$, *all but finitely many zeros of any solution* $f \not\equiv 0$ *must lie in the union of the sectors* $|\arg z - \varphi_j| < \varepsilon$ *for* $j = 1, \ldots, m$. *In addition, our method of proof provides an effective procedure for determining which of the two possibilities holds for a given equation, and in the case when (b) holds, our method will produce the rays,* $\arg z = \varphi_j$. *We emphasize that our result applies to all equations having polynomial coefficients, without exception. In addition, we mention that if the coefficients are only assumed to be rational functions, our results will still give precise information on the possible location of the bulk of the zeros of any solution.*

1. – Introduction.

For second-order linear differential equations, $w'' + P(z)w' + Q(z)w = 0$, where $P(z)$ and $Q(z)$ are polynomials, there is a classical result due mainly to E. HILLE (see [13; p. 345] or [20; p. 282]) which states that there are finitely many rays, $\arg z = \varphi_j$, for $j = 1, \ldots, m$ (which can be explicitely calculated from the equation), with the property that for any $\varepsilon > 0$, all but finitely many zeros of any solution $f \not\equiv 0$ must lie in the union of the sectors $|\arg z - \varphi_j| < \varepsilon$ for $j = 1, \ldots, m$. This result was proved by

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Indirizzo dell'A.: Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, U.S.A.

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using a method of asymptotic integration (see [11; Chapter 7.4], [13; p. 345] or [8; pp. 6-10]) to construct a fundamental set $\{f_1, f_2\}$ of solutions in sectorial regions, with the property that f_1 and f_2 have only finitely many zeros in these regions, and their ratio f_1/f_2 tends to ∞ as $z \to \infty$ in these regions. Then clearly any nontrivial linear combination of f_1 and f_2 can have only finitely many zeros in these regions.

In attempting to investigate the corresponding situation for higher-order equations,

$$w^{(n)} + R_{n-1}(z)\,w^{(n-1)} + \ldots + R_0(z)\,w = 0\,, \tag{1.1}$$

where $n > 2$, and the coefficients $R_j(z)$ are polynomials, it was shown in [1] and [6], that the situation in the case $n > 2$ can be far different than in the case $n = 2$. For example, it was shown that when $n > 2$, the equation,

$$w^{(n)} + z^2 w'' + zw' + w = 0\,, \tag{1.2}$$

has the property (which we will call the *global oscillation property*) that for any ray, $\arg z = \varphi$, and any $\varepsilon > 0$, there is a solution $f \not\equiv 0$ having infinitely many zeros in the sector $|\arg z - \varphi| < \varepsilon$. This fact was proved using [3; Theorem I, p. 144], which shows that in sectorial regions, the equation (1.2) possesses solutions f_1 and f_2 which are asymptotically equivalent as $z \to \infty$ to z^i and z^{-i} respectively (where $i = (-1)^{1/2}$). Using Rouché's theorem, it is then a simple matter to construct linear combinations of f_1 and f_2 which have infinitely many zeros in a sector $|\arg z - \varphi| < \varepsilon$, where φ and ε are given. Of course, the same situation will occur whenever an equation (1.1) possesses a pair of solutions in sectorial regions, whose ratio is asymptotically equivalent as $z \to \infty$ to a function of the form z^{bi} where b is a nonzero real number. (Our results here will show that this is the only way the global oscillation property can occur.)

The example (1.2) shows that an investigation of the zeros when $n > 2$ naturally divides into at least three questions, namely: (i) What are the different possibilities for the location of the bulk of the zeros of solutions of (1.1)?; (ii) Is there an effective method for deciding which possibility holds for a given equation (1.1)?; (iii) In the case where the situation is analogous to the second-order case, is there an effective method for producing the rays, $\arg z = \varphi_j$, for $j = 1, \ldots, m$? In this paper, we give a rather complete answer to all three equations. The following theorem (which is stated for the more general case where the coefficients of (1.1) are rational functions), gives an answer to (i):

THEOREM 1. – Given the equation (1.1), where $n \geq 1$ and $R_0(z), \ldots, R_{n-1}(z)$ are any rational functions. Then, one of the following holds;

(*A*) For any θ in $(-\pi, \pi)$ and any $\varepsilon > 0$, there exist positive constants δ and K, with $\delta < \min\{\varepsilon, \theta + \pi, \pi - \theta\}$, and a solution $f \not\equiv 0$ of (1.1) such that f is analytic and has infinitely many zeros $z_1, z_2, \ldots$, with $\lim_{m \to \infty} |z_m| = +\infty$, on the region defined by $|\operatorname{Arg} z - \theta| < \delta$ and $|z| > K$.

(B) There exist a positive integer λ and real numbers $\sigma_1, \dots, \sigma_\lambda$ lying in $(-\pi, \pi]$ such that for any $\varepsilon > 0$ and any solution $f \not\equiv 0$ of (1.1) which is meromorphic on the plane, all but finitely many zeros of f lie in the union for $k = 1, \dots, \lambda$, of the sectors, $|\arg z - \sigma_k| < \varepsilon$.

We remark that for solutions of (1.1) which are not meromorphic on the plane, our results in Theorem 4 (see § 12) will still apply to give precise results on the possible location of the bulk of the zeros in that part of the slit plane near infinity. In regard to the questions (ii) and (iii) mentioned above, these questions are answered in the affirmative by Theorem 4. An example illustrating these methods is worked out in § 17.

The foundation of our method is an improvement of the result in [3; Theorem I] which was mentioned earlier. This result from [3] applies to all equations (1.1) where $\{R_0(z), \dots, R_{n-1}(z)\}$ is contained in a certain type of field which was introduced by W. STRODT in [18]. Roughly speaking, the $R_j(z)$ are assumed to be analytic functions in a sectorial region which possess asymptotic expansions as $z \to \infty$ in the region in terms of functions of the form cz^α were α is real and c is a nonzero complex number. (A rigorous definition of this type of field is found in § 2 (c) below.) Associated with such an equation (1.1) are two quantities which were developed in [3]. First, there is a nonnegative integer p (called the *critical degree*), and second, there is an equation, $F(\alpha) = 0$ (called the *critical equation*), where $F(\alpha)$ is a polynomial in α of degree p, having constant coefficients. It was shown in [3; Theorem I] that if $p > 0$, then corresponding to any complex root α of the critical equation, say α is of multiplicity m, there is a solution $f_j(z)$ of (1.1) in sectorial regions which is asymptotically equivalent as $z \to \infty$ to the function $z^\alpha (\operatorname{Log} z)^j$ for each $j = 0, 1, \dots, m-1$. (Thus this result provides p distinct solutions of (1.1).) Unfortunately, the method used in the proof of [3; Theorem I] obscured the explicit domains on which the solutions f_j possess their prescribed asymptotic behavior. However, by a careful reexamination of the part of the proof which uses results of STRODT [17], we can prove a version of [3; Theorem I] in which the domains of the f_j can be given explicitely. This extension of [3; Theorem I] can be found in § 4 below. In [4], a third quantity associated with an equation (1.1) was introduced. This quantity is an algebraic polynomial $G(v)$ (called the *factorization polynomial*) whose degree in v is $n-p$, and whose coefficients belong to the same field as the coefficients of (1.1). It was shown in [4] that if $G(v)$ satisfies the condition that no two of its roots are asymptotically equivalent as $z \to \infty$, then the operator on the left side of (1.1) can be exactly factored into a composition of first-order operators. Then by successive integrations, one could produce (on explicit sectorial regions) linearly independent solutions $h_1, \dots, h_{n-p}$, where each h_k is of the form $\exp \int V_k$ for some function V_k which is asymptotically equivalent as $z \to \infty$ to a root of $G(v)$. These $n-p$ solutions, together with the p solutions previously produced, form a fundamental set for (1.1). By examining the asymptotic behavior of the ratios of these solutions to see where a linear combination could possibly have zeros, a proof was given in [1] of the special case of Theorem 1 where it is assumed that no two roots of $G(v)$ are asymptotically equivalent as $z \to \infty$.

However, when $G(v)$ has asymptotically equivalent roots, this approach in [1] of exact factorization could not be made to work. In order to obtain a result which applies to all equations (1.1) without exception, another approach is needed. Such an approach is furnished by a method developed by C. POWDER in his dissertation [14]. The benefit of Powder's method (see § 7) is that it shows that even in the case where $G(v)$ has asymptotically equivalent roots, one can explicitely find functions $W(z)$, which are asymptotically equivalent to roots of $G(v)$, and which have the property that under the change of dependent variable $w = (\exp \int W) u$ (followed by division by $\exp \int W$), the equation (1.1) is transformed into an equation in u which has a strictly positive critical degree. Thus, the extension of [3; Theorem I] can be applied to give solutions (together with explicit domains) of the u-equation. This method can be shown to yield the $n-p$ additional solutions $h_1, \ldots, h_{n-p}$, needed for a fundamental set. This result is stated in Theorem 3 below (see § 9). By examining the asymptotic behavior of the ratios of these solutions, we arrive at our result on zeros given in Theorem 4 below.

We conclude with the following remark: If the classical Wiman-Valiron theory (see [19; pp. 106-109] or [9; p. 341]) is applied to an equation (1.1) when the $R_j(z)$ are polynomials, then it is well-known that a finite set of positive rational numbers is produced which represent the possible orders of growth of a transcendental solutions. Using Theorem 3 below, we can show that the largest number produced by the Wiman-Valiron theory is actually the order of growth of some solution of (1.1). (This fact is proved in § 18 below.)

2. - Concepts from the Strodt theory [16].

(*a*) [16; § 94]: *The neighborhood system* $F(a, b)$. Let $-\pi \leqslant a < b \leqslant \pi$. For each nonnegative real-valued function g on $(0, (b-a)/2)$, let $V(g)$ be the union (over all $\delta \in (0, (b-a)/2)$) of all sectors, $a + \delta < \operatorname{Arg}(z - h(\delta)) < b - \delta$, where $h(\delta) = g(\delta)\, e^{i(a+b)/2}$. The set of all $V(g)$ (for all choices of g) is denoted $F(a, b)$, and is a filter base which converges to ∞. Each $V(g)$ is a simply-connected region (see [16; § 93]), and we require the following simple fact which is proved in [6; § 2]:

LEMMA 2.1. - Let V be an element of $F(a, b)$, and let $\varepsilon > 0$ be arbitrary. Then there is a constant $R_0(\varepsilon) > 0$ such that V contains the set, $a + \varepsilon \leqslant \operatorname{Arg} z \leqslant b - \varepsilon$, $|z| \geqslant R_0(\varepsilon)$.

(*b*) [16; § 13]: *The relation of asymptotic equivalence.* If $f(z)$ is an analytic function on some element of $F(a, b)$, then $f(z)$ is called *admissible* in $F(a, b)$. If c is a complex number, then the statement $f \to c$ in $F(a, b)$ means (as is customary) that for any

$\varepsilon > 0$, there exists an element V of $F(a, b)$ such that $|f(z) - c| < \varepsilon$ for all $z \in V$. The statement $f \ll 1$ in $F(a, b)$, means that in addition to $f \to 0$, all the functions $\theta_j^k f \to 0$ in $F(a, b)$, where θ_j denotes the operator $\theta_j f = z(\text{Log } z) \dots (\text{Log}_{j-1} z) f'(z)$, and where (for $k \geq 0$), θ_j^k is the k-th iterate of θ_j. The statements $f_1 \ll f_2$ and $f_1 \sim f_2$ in $F(a, b)$ mean respectively $f_1/f_2 \ll 1$ and $f_1 - f_2 \ll f_2$. (As usual, z^α and $\text{Log } z$ will denote the principal branches of these functions on $|\text{Arg } z| < \pi$.) We will write $f_1 \approx f_2$ to mean $f_1 \sim cf_2$ for some nonzero constant c. (We remark that this strong relation of asymptotic equivalence is designed to ensure that if $f \ll 1$ in $F(a, b)$, then $\theta_j f \ll 1$ in $F(a, b)$ for all $j \geq 1$. (See [16; § 28]).) If $f \sim cz^{-1+d}$ in $F(a, b)$, where $c \neq 0$ and $d \geq 0$, then the indicial function of f is the function,

$$(2.1) \qquad IF(f, \phi) = \text{Cos}\,(d\phi + \arg c) \quad \text{for } a < \phi < b\,.$$

If g is any admissible function in $F(a, b)$, we will denote by $\int g$, a primitive of g in an element of $F(a, b)$. We will require the following two results, the first of which is proved in [2: § 10]:

LEMMA 2.2. – Let $f \sim cz^{-1+d}$ in $F(a, b)$, where $c \neq 0$ and $d > 0$. If (a_1, b_1) is any subinterval of (a, b) on which $IF(f, \phi) < 0$ (respectively, $IF(f, \phi) > 0$), then for all real α, $\exp \int f \ll z^\alpha$ (respectively, $\exp \int f \gg z^\alpha$) in $F(a_1, b_1)$.

LEMMA 2.3. – Let $\alpha = a + bi$ be a complex number. Then for any $\varepsilon > 0$, we have $z^{a-\varepsilon} \ll z^\alpha$ and $z^\alpha \ll z^{a+\varepsilon}$ in $F(-\pi, \pi)$.

PROOF. – It suffices to prove the second estimate, and it suffices to prove it for $a = 0$. But this follows immediately from [7; Lemma δ, p. 271] since $f = z^{bi-\varepsilon}$ satisfies $zf' - (bi - \varepsilon)f = 0$, and so $f \ll 1$ in $F(a, b)$.

(c) [18; p. 244]: *Logarithmic fields.* A function of the form cz^α, for complex $c \neq 0$ and real α, is called a logarithmic monomial of rank zero. The set of all logarithmic monomials of rank zero will be denoted Φ_0. A *logarithmic differential field of rank zero over* $F(a, b)$ is a set Γ of functions, each defined and admissible in $F(a, b)$, with the following properties: (i) Γ is a differential field (where, as usual, we identify two elements of Γ if they agree on an element of $F(a, b)$); (ii) Γ contains Φ_0; (iii) for every element f in Γ except zero, there exists M in Φ_0 such that $f \sim M$ over $F(a, b)$. (The simplest example of such a field is the set of rational combinations of the elements of Φ_0.) If $f \sim cz^\alpha$ over $F(a, b)$, we will denote α by $\delta_0(f)$. If $f \equiv 0$, we will set $\delta_0(f) = -\infty$.)

Now let $G(z, v) = \sum_{j=0}^{n} f_j(z) v^j$ be a polynomial in v of degree $n \geq 1$ whose coefficients belong to a logarithmic differentialfield of rank zero in $F(a, b)$. A logarithmic monomial $M = cz^\alpha$ of rank zero is called a *critical monomial* of G if there exists an admissible function $h \sim M$ in $F(a, b)$ for which $G(z, h(z))$ is not $\sim G(z, M(z))$ in $F(a, b)$. The multiplicity of M is the smallest positive integer j such that M is not a critical monomial of $\partial^j G/\partial v^j$. There is an algorithm (see [5; § 26]) which produces the sequence (counting multiplicity) of critical monomials of $G(z, v)$. (By [5; § 29], the sequence has $n-d$ members, where d is the smallest $k \geq 0$ for which $f_k \not\equiv 0$.) The algorithm is based on a Newton polygon method (e.g., [12; p. 105]). (One simply finds the values of α which have the following properties: When $v = cz^{\alpha}$ is inserted into the individual terms in G, at least two such terms have the same δ_0, and this value of δ_0 is at least as large as the other terms produce. The constant c is then determined by requiring that the terms with the largest δ_0 cancel.) The critical monomials of G give the first terms of the asymptotic expansions of the roots of G. This is shown by the following fact:

LEMMA 2.4. – Let $G(z, v) = \sum_{j=0}^{n} f_j v^j$ be a polynomial in v of degree $n \geq 1$, whose coefficients $f_0, \ldots, f_n$ are elements of a logarithmic differential field of rank zero over $F(a, b)$. Then

(*a*) There exists an extension logarithmic differential field of rank zero over $F(a, b)$, in which $G(z, v)$ factors completely.

(*b*) If M is a simple critical monomial of $G(z, v)$, then there exists a unique admissible function $h(z)$ in $F(a, b)$ having the following two properties: (i) $h \sim M$ in $F(a, b)$, and (ii) $G(z, h(z)) \equiv 0$. In addition, the function $h(z)$ belongs to a logarithmic differential field of rank zero over $F(a, b)$.

PROOF. – Part (*a*) follows from [18; Theorem II, p. 244] and [14; § 2.6]. Part (*b*) follows easily from [18; §§ 24, 26] and from Part (*a*).

When $f_0 \not\equiv 0$ in $G(z, v)$, the polynomial $G(z, v)$ possesses one or more special critical monomials, $M = cz^{\alpha}$, called *principal monomials* (see [16; § 67]) which arise as follows: When $v = cz^{\alpha}$ is inserted into the individual terms of $G(z, v)$, the power $\delta_0(f_0)$ is at least as large as the δ_0 produced by the other terms. The principal monomials are the critical monomials which are of minimal rate of growth in $F(-\pi, \pi)$ (see [16; § 67]). We will require the following facts which are proved in [5; §§ 3, 31 (*c*)]:

LEMMA 2.5. – Let M be a simple critical monomial of a polynomial $G(z, v)$ whose coefficients belong to a logarithmic differential field of rank zero over $F(a, b)$, and assume $G(z, M(z)) \not\equiv 0$. Let $G_1(z, w) = G(z, M(z) + w)$. Then $G_1(z, w)$ possesses a unique principal monomial M_1. In addition, M_1 is simple, and $M_1 \ll M$ in $F(-\pi, \pi)$.

3. – Preliminaries.

Given an equation (1.1) where the $R_j(z)$ are functions which belong to a logarithmic differential field of rank zero over $F(a, b)$, we first rewrite the equation in terms of the operator θ which is defined by $\theta w = zw'$. (It is easy to prove by induction thatfor each $m = 1, 2, \ldots,$

$$w^{(m)} = z^{-m}\left(\sum_{j=1}^{m} b_{jm}\,\theta^j w\right), \tag{3.1}$$

where θ^j is the j-th iterate of the operator θ, and where the b_{jm} are integers with $b_{mm} = 1$. In fact, as polynomials in x,

$$\sum_{j=1}^{n} b_{jn}\,x^j = x(x-1)\ldots(x-(n-1))\,.\,) \tag{3.2}$$

When written in terms of θ, let (1.1) have the form

$$\sum_{j=0}^{n} B_j(z)\,\theta^j w = 0\,. \tag{3.3}$$

(Of course, the $B_j(z)$ belong to the same field as the $R_j(z)$.) By dividing equation (3.3) through by z^d where d is the maximum of $\delta_0(B_j)$ for $j = 0, \ldots, n$, we may assume that for each j, we have either $B_j \ll 1$ or $B_j \approx 1$ in $F(a, b)$, and there exists an integer $p \geqslant 0$ such that $B_j \ll 1$ for $j > p$, while B_p is $\sim$ to a nonzero constant (denoted $B_p(\infty)$). The integer p is called the *critical degree* of the equation (1.1). The equation,

$$F^*(\alpha) = \sum_{j=0}^{n} B_j(\infty)\,\alpha^j = 0\,, \tag{3.4}$$

is called the *critical equation* of (1.1). Clearly, $F^*(\alpha)$ is a polynomial in α, of degree p, having constant coefficients. Let the distinct roots of $F^*(\alpha)$ be $\alpha_0, \ldots, \alpha_r$, with α_k having multiplicity m_k. (Thus, $\sum m_k = p$.) Let $M_1, \ldots, M_p$ be the p distinct functions of the form $z^{\alpha_k}(\operatorname{Log} z)^j$ for $0 \leqslant k \leqslant r$, and integers j satisfying $0 \leqslant j \leqslant m_k - 1$. We call the set $\{M_1, \ldots, M_p\}$, the *logarithmic set* for (1.1).

When (1.1) is written in the form (3.3), we form the algebraic polynomial in v,

$$H(z, v) = \sum_{j=0}^{n} z^j B_j(z)\,v^j, \tag{3.5}$$

which we will call the *full factorization polynomial* for (1.1). Clearly, the coefficients of $H(z, v)$ belong to the same logarithmic differential field as do the coefficients of (1.1). If p is the critical degree of (1.1), it is shown in [14; Lemma 6.1], that $H(z, v)$ possesses precisely $n-p$ critical monomials $N_1, \ldots, N_{n-p}$, (counting multiplicity) satisfying $\delta_0(N_j) > -1$. We will call the set $\{N_1, \ldots, N_{n-p}\}$, the *exponential set* for (1.1). If T_j is the set of zeros on (a, b) of the function $IF(N_j, \varphi)$ (see (2.1)), then the union of the sets T_j for $j = 1, \ldots, n-p$, will be called the *transition set* for (1.1) on (a, b).

4. – We now state our second main result.

THEOREM 2. – Let $n \geqslant 1$, and let $R_0, R_1, \dots, R_{n-1}$ belong to a logarithmic differential field of rank zero over $F(a, b)$. Let $\Lambda(w)$ be the n-th order linear differential operator,

$$\Lambda(w) = w^{(n)} + R_{n-1}(z)\, w^{(n-1)} + \dots + R_0(z)\, w\,. \tag{4.1}$$

Let p be the critical degree of $\Lambda(w) = 0$, and let $\{M_1, \dots, M_p\}$ be the logarithmic set for this equation. Let $r_1 < r_2 <, \dots, r_t$ be the transition set for $\Lambda(w) = 0$, and set $r_0 = a$ and $r_{t+1} = b$. (If the transition set is empty, set $t = 0$.) Then, in each of $F(r_0, r_1)$, $F(r_1, r_2), \dots, F(r_t, r_{t+1})$ separately, the following conclusion holds: For each j, with $1 \leqslant j \leqslant p$, there exists an admissible solution $\varphi_j(z)$ of $\Lambda(w) = 0$, satisfying $\varphi_j \sim M_j$.

REMARK In view of Theorem 2, we will make the following definition:

DEFINITION 4.1. – Under the hypothesis and notation of Theorem 2, if $\{\psi_1, \dots, \psi_p\}$ is a set of admissible functions in some $F(c, d)$, such that ψ_j is a solution of $\Lambda(w) = 0$ and satisfies $\psi_j \sim M_j$ in $F(c, d)$ for $j = 1, \dots, p$, then we will call $\{\psi_1, \dots, \psi_p\}$ a *complete logarithmic set of solutions* of $\Lambda(w) = 0$ in $F(c, d)$. (Thus Theorem 2 asserts the existence of complete logarithmic sets of solutions in each of $F(r_0, r_1), \dots, F(r_t, r_{t+1})$ separately.)

5. – Preliminaries for Theorem 2.

DEFINITION [17; § 13]. – Let $\Lambda(w) = \sum_{j=0}^{n} A_j(z)\, w^{(j)}$ be an n-th order linear differential operator whose coefficients are admissible functions in $F(a, b)$. Then $\Lambda(w)$ is called *unimajoral* in $F(a, b)$ if $\Lambda(1) \sim 1$, and $\Lambda(E) \ll 1$ whenever $E \ll 1$.

LEMMA 5.1 [17; §§ 27, 44, 99]. – Let $\Lambda(w) = \sum_{j=0}^{n} A_j(z)\, w^{(j)}$ be a unimajoral operator in $F(a, b)$ whose coefficients $A_j(z)$ belong to a logarithmic differential field of rank zero over $F(a, b)$, and assume $A_n \not\equiv 0$. Then:

(*A*) There is a sequence $(W_1, \dots, W_n)$ of logarithmic monomials of rank zero, with $\delta_0(W_j) \geqslant -1$ for all j, and a sequence $(E_0, \dots, E_n)$ of functions satisfying $E_j \ll 1$ in $F(a, b)$ for each j, such that

$$\Lambda = \dot{W}_n \dots \dot{W}_1 + \sum_{j=1}^{n} E_j \dot{W}_j \dots \dot{W}_1 + E_0\,, \tag{5.1}$$

where $\dot{W}_j$ denotes the first-order operator $\dot{W}_j(w) = w - (w'/W_j)$, and where $\dot{W}_j \dots \dot{W}_1$ denotes the composition of these operators.

(*B*) A sequence $(W_1, \dots, W_n)$ satisfying Part (*A*) can be found by finding the sequence $(L_1, \dots, L_n)$ of critical monomials (counting multiplicity) of the full factorization polynomial for $\Lambda(w) = 0$, and taking $(W_1, \dots, W_n)$ to be any permutation of $(L_1, \dots, L_n)$ satisfying the condition that for each j, either $W_j \ll W_{j+1}$ or $W_j \approx W_{j+1}$ in $F(a, b)$.

(*C*) Let $(W_1, \dots, W_n)$ be as in Part (*A*), and assume that (c, d) is a subinterval of (a, b) on which none of the functions $IF(W_j, \varphi)$ have a zero for $j = 1, \dots, n$. Then, for any admissible function g in $F(c, d)$ satisfying $g \ll 1$ in $F(c, d)$, the equation $\Lambda(w) = g$ possesses a solution $w_0(z)$ satisfying $w_0 \ll 1$ in $F(c, d)$.

LEMMA 5.2. – Let $B_0, \dots, B_n$ be functions belonging to a logarithmic differential field of rank zero over $F(a, b)$, and assume that $B_n \not\equiv 0$ and that

$$\max\{\delta_0(B_j) : j = 0, \dots, n\} = 0. \tag{5.2}$$

Let $\Lambda(w)$ denote the operator $\sum_{j=0}^{n} B_j \theta^j w$ (where θ is as in (3.1)), and let $F^*(\alpha) = 0$ be the critical equation of $\Lambda(w) = 0$. Let D be the set of roots of $F^*(\alpha) = 0$. Then, for any real number α not lying in D, the operator,

$$\Gamma_\alpha(u) = z^{-\alpha} \Lambda(c_\alpha z^\alpha u), \qquad \text{where } c_\alpha = (F^*(\alpha))^{-1}, \tag{5.3}$$

is unimajoral in $F(a, b)$.

PROOF. – Under the change of dependent variable $w = c_\alpha z^\alpha u$, it is easy to verify by induction that $\theta^j w$ becomes $c_\alpha z^\alpha (\theta + \alpha)^j (u)$, where $(\theta + \alpha)^j$ is the j-th iterate of the operator $\theta + \alpha$. Thus,

$$\Gamma_\alpha(u) = c_\alpha \sum_{j=0}^{n} B_j (\theta + \alpha)^j (u). \tag{5.4}$$

Since $(\theta + \alpha)^j(1) = \alpha^j$, it follows from (3.4) that $\Gamma_\alpha(1) \sim 1$. In view of (5.2), it also follows that $\Gamma_\alpha(E) \ll 1$ whenever $E \ll 1$ proving the lemma.

6. – Proof of Theorem 2.

We assume the hypothesis and notation of Theorem 2. As in § 3, we write the equation $\Lambda(w) = 0$ in the form (3.3), and we may assume as in § 3 that $\Lambda(w) = \sum_{j=0}^{n} B_j \theta^j w$ where (5.2) holds. Since the B_j belong to a logarithmic differential field, $\mathcal{K}$ of rank zero over $F(a, b)$, we can write $B_j = a_j + h_j$, where a_j is a complex number, and h_j is an element of $\mathcal{K}$ such that $\delta_0(h_j) < 0$. Thus, we may write $\Lambda = \Phi + \Psi$, where,

$$\Phi(w) = \sum_{j=0}^{n} a_j \theta^j w \quad \text{and} \quad \Psi(w) = \sum_{j=0}^{n} h_j \theta^j w. \tag{6.1}$$

Now let M be any element of the logarithmic set for $\Lambda(w) = 0$. We will produce in each of $F(r_0, r_1), \ldots, F(r_t, r_{t+1})$ separately, an admissible solution φ of $\Lambda(w) = 0$, satisfying $\varphi \sim M$. This will prove the theorem.

If $\Lambda(M) \equiv 0$, we can take $\varphi = M$, so we can assume that $\Lambda(M) \not\equiv 0$. We now assert that

$$\Phi(M) \equiv 0 . \tag{6.2}$$

To prove (6.2), we consider the linear differential equation $L(v) = 0$, where $L = \sum_{j=0}^{n} a_j d^j / d\zeta^j$. Clearly L has constant coefficients, and the characteristic equation of L is $\sum_{j=0}^{n} a_j \alpha^j = 0$, which is just the critical equation, $F^*(\alpha) = 0$ of Λ. Now $M = z^\alpha (\text{Log } z)^q$, where α is a root of multiplicity m of $F^*(\alpha)$, and where the integer q satisfies $0 \leq q \leq m - 1$. Thus clearly, $f(\zeta) = \zeta^q e^{\alpha\zeta}$ is an entire solution of $L(f) = 0$. However, $M(z) = f(\text{Log } z)$, and it is easy to verify that it follows that for each $j = 1, 2, \ldots$, we have $\theta^j M(z) = f^{(j)}(\text{Log } z)$. Since $L(f(\zeta)) \equiv 0$, we clearly obtain (6.2).

Since $M = z^\alpha (\text{Log } z)^q$, a simple computation shows that for each j, the function $\theta^j M$ is of the form $z^\alpha Q_j(\text{Log } z)$, where $Q_j(u)$ is a polynomial in u, of degree at most q, having constant coefficients. In view of (6.2), we thus obtain,

$$\Lambda(M) = \Psi(M) = z^\alpha \sum_{j=0}^{n} h_j Q_j(\text{Log } z) . \tag{6.3}$$

Let $\varepsilon_1 > 0$ be such that for all j, we have $\delta_0(h_j) < -\varepsilon_1$. Collecting powers of $\text{Log } z$ in (6.3), we may write,

$$\Lambda(M) = z^\alpha \sum_{j=0}^{q} f_j(z)(\text{Log } z)^j , \tag{6.4}$$

where the f_j are linear combinations (with constant coefficients) of the h_j. Thus $\delta_0(f_j) < -\varepsilon_1$ for all j. By assumption, $\Lambda(M) \not\equiv 0$, and hence some $f_j \not\equiv 0$. Set

$$\varepsilon = -\max\{\delta_0(f_j) \colon j = 0, 1, \ldots, q\} . \tag{6.5}$$

Thus $\varepsilon > \varepsilon_1$, and since the f_j belong to $\mathcal{K}$, we can write $f_j = b_j z^{-\varepsilon} + E_j$, where b_j is a constant, and where E_j is an element of $\mathcal{K}$ such that $\delta_0(E_j) < -\varepsilon$. Thus by (6.4), we have

$$\Lambda(M) = z^{\alpha-\varepsilon} \left(\sum_{j=0}^{q} b_j (\text{Log } z)^j + z^\varepsilon E \right) , \tag{6.6}$$

where E is an admissible function in $F(a, b)$ which satisfies $E \ll z^{-\varepsilon}$. Hence from (6.6), we obtain in $F(a, b)$,

$$\Lambda(M) = z^{\alpha-\varepsilon/2} H , \quad \text{where } H \ll z^{-\varepsilon/3} . \tag{6.7}$$

We now consider the nonhomogeneous equation,

$$\Lambda(v) = -z^{\alpha-\varepsilon/2}H, \tag{6.8}$$

and we choose a real number σ which is not a root of the critical equation $F^*(\beta)=0$ for $\Lambda(w)=0$. By Lemma 5.2, if $d_\sigma = (F^*(\sigma))^{-1}$, then the operator,

$$\Gamma_\sigma(w) = z^{-\sigma}\Lambda(d_\sigma z^\sigma w), \tag{6.9}$$

is unimajoral in $F(a, b)$. If $\Gamma_\sigma(w) = \sum_{j=0}^{n} U_j\theta^j w$, then by Lemma 5.1, there is a sequence $(W_1, \ldots, W_n)$ of logarithmic monomials of rank zero, with $\delta_0(W_j) \geqslant -1$, such that $\Gamma_\sigma(w)$ satisfies (5.1) for some $E_j \ll 1$ in $F(a, b)$. Such a sequence can be taken to be the sequence of critical monomials of the full factorization polynomial $T(u)$ for $\Gamma_\sigma(w)$, ordered by $W_j \ll W_{j+1}$ or $W_j \approx W_{j+1}$. But $\Gamma_\sigma(w)$ has the form (5.4) (with α replaced by σ), and the full factorization polynomial $T(u)$ can easily be read off from (5.4) to be,

$$T(u) = d_\sigma \sum_{j=0}^{n} B_j(z)(zu+\sigma)^j. \tag{6.10}$$

Thus, if $G(v)$ denotes the full factorization polynomial (3.5) for the original equation, $\Lambda(w)=0$, then,

$$T(u) = d_\sigma G(u+\sigma z^{-1}). \tag{6.11}$$

Now consider any W_j satisfying $\delta_0(W_j) > -1$. Thus there exists $h \sim W_j$ for which $T(h)$ is not $\sim T(W_j)$. It follows from (6.11) that W_j is a critical monomial of $G(v)$, and hence if $\{N_1, \ldots, N_{n-p}\}$ denotes the exponential set for $\Lambda(w)=0$, then we have shown,

$$W_j \in \{N_1, \ldots, N_{n-p}\} \quad \text{if } \delta_0(W_j) > -1. \tag{6.12}$$

Let the set of W_j with $\delta_0(W_j) = -1$ be denoted $\{c_1 z^{-1}, \ldots, c_s z^{-1}\}$. We now choose a real number λ which is not a root of $F^*(\beta)=0$, and which satisfies both of the following conditions:

$$\operatorname{Re}(\alpha) - (13\varepsilon/18) < \lambda < \operatorname{Re}(\alpha) - (10\varepsilon/18), \tag{6.13}$$

$$\lambda \notin \{\operatorname{Re}(c_1)+\sigma, \ldots, \operatorname{Re}(c_s)+\sigma\}. \tag{6.14}$$

We again invoke Lemma 5.2, and form the operator $\Gamma_\lambda(w)$ given by (6.9) with λ in place of σ. As before, $\Gamma_\lambda(w)$ is unimajoral in $F(a, b)$, and a sequence $(W_{11}, W_{22}, \ldots, W_{nn})$ satisfying (5.1) for $\Gamma_\lambda(w)$ can be found by taking the sequence of critical monomials (in an ascending order) of the full factorization polynomial $T_1(u)$ for $\Gamma_\lambda(w)$. But as before, $T_1(u)$ has the form (6.10) with σ replaced by λ, and it follows that

$$T_1(u) = (d_\lambda/d_\sigma)\, T(u+(\lambda-\sigma)z^{-1}), \tag{6.15}$$

and

$$T_1(u) = d_\lambda G(u + \lambda z^{-1}). \tag{6.16}$$

As in the case for σ, it follows that (6.12) holds also for λ, so that

$$W_{jj} \in \{N_1, \ldots, N_{n-p}\} \quad \text{if } \delta_0(W_{jj}) > -1. \tag{6.17}$$

Now consider any W_{jj} with $\delta_0(W_{jj}) = -1$, say $W_{jj} = rz^{-1}$. Thus there exists $h \sim W_{jj}$ such that $T_1(h)$ is not $\sim T_1(W_{jj})$. Set $c = r + \lambda - \sigma$, so from (6.15) we obtain

$$T(h + (\lambda - \sigma) z^{-1}) \quad \text{is not} \quad \sim T(cz^{-1}). \tag{6.18}$$

We assert that $c \neq 0$. We assume the contrary, and set $g = h + (\lambda - \sigma) z^{-1}$. Then $g \ll z^{-1}$, and (6.18) asserts that $T(g)$ is not $\sim T(0)$. Thus from (6.11), it follows that $G(g + \sigma z^{-1})$ is not $\sim G(\sigma z^{-1})$, which we will show is impossible. To see this, we note that $G(v)$ is given by the right side of (3.5), and hence it follows that $G(\sigma z^{-1}) \sim F^*(\sigma)$ since $F^*(\sigma) \neq 0$ by our choice of σ. Since $g \ll z^{-1}$, it again follows from (3.5) that $G(g + \sigma z^{-1}) \sim F^*(\sigma)$ which proves our assertion. This contradiction proves that $c \neq 0$.

Since $c \neq 0$ in (6.18), we see that cz^{-1} is a critical monomial of $T(u)$, and thus c must be one of the numbers $c_1, \ldots, c_s$, say c_k. Hence, $r = c_k + \sigma - \lambda$, so from (6.14) we see that $\mathrm{Re}(r) \neq 0$. Hence if $\delta_0(W_{jj}) = -1$, clearly $IF(W_{jj}, \varphi)$ is a nonzero constant function. Thus from (6.17) (and the definition of transition set), it follows that with $r_0, r_1, \ldots, r_{t+1}$ as defined in the statement, none of the functions $IF(W_{jj}, \varphi)$ (for $j = 1, \ldots, n$) have a zero on any of the intervals $(r_0, r_1), (r_1, r_2), \ldots, (r_t, r_{t+1})$. Hence from Part (C) of Lemma 5.1, for each $j = 0, 1, \ldots, t$, and any admissible function $g_j \ll 1$ in $F(r_j, r_{j+1})$, the equation

$$\Gamma_\lambda(w) = g_j, \tag{6.19}$$

possesses a solution $w_j \ll 1$ in $F(r_j, r_{j+1})$.

We return now to equation (6.8). Setting $v = d_\lambda z^\lambda w$, this equation takes the form (6.19) where $g_j = -z^{\alpha - \lambda - \varepsilon/2} H$. In view of (6.7), we have $g_j \ll z^{\alpha - \lambda - (5\varepsilon/6)}$ in $F(a, b)$. From our choice of λ in (6.13), clearly $\lambda + (5\varepsilon/6) - \mathrm{Re}(\alpha) > \varepsilon/9$, and from Lemma 2.3, we know $z^\alpha \ll z^{\mathrm{Re}(\alpha) + (\varepsilon/9)}$ in $F(-\pi, \pi)$. It easily follows that $g_j \ll 1$ in $F(a, b)$. Thus,for each $j = 0, 1, \ldots, t$, the equation (6.19) possesses a solution $w_j \ll 1$ in $F(r_j, r_{j+1})$. Hence $v_j = d_\lambda z^\lambda w_j$ is a solution of (6.8) in $F(r_j, r_{j+1})$. But from our choice of λ in (6.13) and Lemma 2.3, we have $v_j \ll z^\alpha$ in $F(r_j, r_{j+1})$, and hence $v_j \ll M$ in $F(r_j, r_{j+1})$. Thus from (6.7) and (6.8), we have $\Lambda(M + v_j) \equiv 0$, and so $M + v_j$ is an admissible solution of $\Lambda(w) = 0$ in $F(r_j, r_{j+1})$ satisfying $M + v_j \sim M$. This concludes the proof of Theorem 2.

7. – Concepts and notation from [14].

Let,

$$\Omega(w) = \sum_{j=0}^{n} B_j(z)\,\theta^j w\,, \tag{7.1}$$

be an n-th order linear differential operator whose coefficients $B_0, \ldots, B_n$ belong to a logarithmic differential field $\mathcal{K}$ of rank zero over $F(a, b)$ and assume $B_n \not\equiv 0$. (As in § 3, $\theta w = zw'$.) Let W belong to an extension logarithmic differential field $\mathcal{K}_1$ of rank zero over $F(a, b)$, and assume $W \gg z^{-1}$ in $F(a, b)$. Set $h = \exp\int W$, and let $\Lambda(v)$ be the operator defined by $\Lambda(v) = \Omega(hv)/h$. Then $\Lambda(v)$ has coefficients belonging to $\mathcal{K}_1$, and we denote,

$$\Lambda(v) = \sum_{j=0}^{n} B_j[W]\,\theta^j v\,. \tag{7.2}$$

Let $H(u)$ and $K(u)$ denote respectively, the full factorization polynomials for $\Omega(w)$ and $\Lambda(v)$, so that,

$$H(u) = \sum_{j=0}^{n} z^j B_j u^j \quad \text{and} \quad K(u) = \sum_{j=0}^{n} z^j B_j[W]\,u^j. \tag{7.3}$$

In [14; § 10], the following concept is introduced: W is said to have transform type (m, q) with respect to H (briefly, $\operatorname{trt}(W, H) = (m, q)$) if Λ has critical degree m, and if q is the minimum multiplicity of all critical monomials M of $K(u)$ which satisfy $z^{-1} \ll M \ll W$ in $F(a, b)$. (If there are no such M, then we set $q = 0$.) The following results are proved in [14; § 10]:

LEMMA 7.1. – With the above notation, assume $W \sim N$ in $F(a, b)$ where N is a critical monomial of $H(u)$ of multiplicity d, satisfying $N \gg z^{-1}$, and assume that $\operatorname{trt}(W, H) = (m, q)$. Then:

(*a*) $K(u)$ has precisely $d - m$ critical monomials L satisfying $z^{-1} \ll L \ll W$, counting multiplicity.

(*b*) We have $m + q \leq d$.

(*c*) If $q = 0$, then $m = d$.

(*d*) If $(m, q) = (0, d)$, and we set,

$$G(u) = \sum_{k=d-1}^{n} \binom{k}{d-1} B_k(z)\, z^{k-(d-1)} (W + u)^{k-(d-1)}, \tag{7.4}$$

then $G(u)$ possesses a unique principal monomial V. In addition, V has the following properties: (i) V is a simple critical monomial of G; (ii) $V \ll W$; (iii) There is a unique function g satisfying $g \sim V$ in $F(a, b)$ and $G(g) \equiv 0$; (iv) If $U = W + g$, then $U \sim W$ in $F(a, b)$, and $\operatorname{trt}(U, H) = (m_1, q_1)$ where $q_1 < d$.

REMARK. – Conclusions (*a*)-(*c*) are proved in [14; Lemma 10.3]. The conclusion (*d*) follows from [14; Lemmas 10.5, 8.5] and from Lemma 2.4 above.

In view of Parts (*b*) and (*d*) of Lemma 7.1, we introduce the following notation:

DEFINITION 7.2. – With the above notation, let N be a critical monomial of $H(u)$ of multiplicity d, satisfying $N \gg z^{-1}$, and let $\operatorname{trt}(N, H) = (m, q)$. By Part (*b*), we have $q \leqslant d$. If $q < d$, set $N^* = N$. If $q = d$, then by Part (*b*), we have $m = 0$. We set $N^* = U$, where U is the function in Part (*d*) which is constructed by taking W equal to N. Hence, in all cases, we have,

$$N^* \sim N \quad \text{and} \quad \operatorname{trt}(N^*, H) = (m_1, q_1) \text{ where } q_1 < d. \tag{7.5}$$

REMARK. – The *-operation to form N^* depends upon the polynomial $H(u)$, and we will indicate this, where necessary, by saying that it is relative to $H(u)$.

The significance of N^* is indicated by the following result:

LEMMA 7.3. – Let $\Omega(w)$ and $H(u)$ be as in (7.1) and (7.3). Let N be a critical monomial of $H(u)$ of multiplicity d, satisfying $N \gg z^{-1}$. Let $\Lambda_1(v) = \sum_{j=0}^{n} B_j[N^*]\theta^j v$, and let $K_1(u)$ be the full factorization polynomial for Λ_1. Then, any critical monomial L of $K_1(u)$, which satisfies $z^{-1} \ll L \ll N^*$, has multiplicity strictly less than d.

PROOF. – By (7.5), if $(m_1, q_1) = \operatorname{trt}(N^*, H)$, then $q_1 < d$. By Lemma 7.1, Part (*a*), the polynomial $K_1(u)$ has precisely $d - m_1$ critical monomials L, satisfying $z^{-1} \ll L \ll$ $\ll N^*$, counting multiplicity. If one of these monomials has multiplicity $r \geqslant d$, then clearly $m_1 = 0$ and $r = d$. Thus $K_1(u)$ would have one distinct critical monomial L satisfying $z^{-1} \ll L \ll N^*$, and L would be of multiplicity d. Then, by definition of q_1, we would have $q_1 = d$, contradicting (7.5). This proves Lemma 7.3.

DEFINITION 7.4. – Let $\Omega(w)$ and $H(u)$ be as in (7.1) and (7.3), and let $N \gg z^{-1}$ be a critical monomial of $H(u)$ of multiplicity d. A finite sequence $(V_0, V_1, \ldots, V_r)$, where r is a nonnegative integer, and where the V_j are elements of an extension logarithmic differential field of $\mathcal{K}$, of rank zero over $F(a, b)$, will be called an *N-sequence for* Ω if and only if the following conditions are satisfied: (i) $V_0 = N^*$; (ii) If $r \geqslant 1$, then there is a critical monomial M_1 of

$$K_1(u) = \sum_{j=0}^{n} z^j B_j[V_0]\, u^j, \tag{7.6}$$

satisfying $z^{-1} \ll M_1 \ll V_0$, such that $V_1 = M_1^*$ (where the *-operation is relative to

K_1), and in general, for $1 \leqslant k \leqslant r$, there is a critical monomial M_k of

$$K_k(u) = \sum_{j=0}^{n} z^j B_j[V_0 + V_1 + \ldots + V_{k-1}]\, u^j \tag{7.7}$$

satisfying,

$$z^{-1} \ll M_k \ll V_{k-1} \quad \text{and} \quad V_k = M_k^*, \tag{7.8}$$

where the *-operation in (7.8) is relative to K_k. The set of all N-sequences for Ω will be denoted $\mathcal{D}(N, \Omega)$. If $V^\# = (V_0, \ldots, V_r)$ is an N-sequence for Ω, let $\Lambda_0 = \Omega$, $K_0 = H$, and for $1 \leqslant k \leqslant r+1$, set

$$\Lambda_k(v) = \sum_{j=0}^{n} B_j[V_0 + V_1 + \ldots + V_{k-1}]\, \theta^k v. \tag{7.9}$$

Let $K_k(u)$ be given by (7.7) for $1 \leqslant k \leqslant r+1$. We call the sequences $(\Lambda_0, \Lambda_1, \ldots, \Lambda_{r+1})$ and $(K_0, K_1, \ldots, K_{r+1})$ respectively, the *operator sequence* for $V^\#$, and the *polynomial sequence* for $V^\#$. The equation, $\Lambda_{r+1}(v) = 0$, will be called the *terminal* equation for $V^\#$, and its critical degree will be called the *terminal index* for $V^\#$, and will be denoted $t(V^\#)$. We will say that $V^\#$ is *active* if $t(V^\#) > 0$, and we denote the set of all active N-sequences for Ω by $\mathcal{D}_1(N, \Omega)$.

REMARKS. – (1) We remark that $\Lambda_{k+1}(v)$ in (7.9) can be formed directly from $\Lambda_k(v)$ by the obvious formula,

$$\Lambda_{k+1}(v) = \Lambda_k\left(\left(\exp\int V_k\right) v\right) \Big/ \left(\exp\int V_k\right). \tag{7.10}$$

(2) It is easy to see that if N is a critical monomial of $H(u)$ of multiplicity d, then for any N-sequence $(V_0, \ldots, V_r)$ we have $r \leqslant d-1$. This follows since $V_j = M_j^*$, and if s_j denotes the multiplicity of the critical monomial M_j of $K_j(u)$, then by Lemma 7.3, we have $d > s_1 > s_2 > \ldots > s_r$. Since $s_r \geqslant 1$, we obtain $d \geqslant r+1$.

8. – A crucial lemma.

LEMMA 8.1. – Let $\Omega(w) = \sum_{j=0}^{n} B_j \theta^j w$ where the B_j belong to a logarithmic differential field of rank zero over $F(a, b)$, and assume $B_n \not\equiv 0$. Let $N \gg z^{-1}$ be a critical monomial of multiplicity d of the full factorization polynomial for Ω. Then,

$$\sum \{t(V^\#) \colon V^\# \in \mathcal{D}(N, \Omega)\} = d, \tag{8.1}$$

and

$$\sum \{t(V^\#) \colon V^\# \in \mathcal{D}_1(N, \Omega)\} = d. \tag{8.2}$$

PROOF. – Since $t(V^{\#}) = 0$ if $V^{\#} \notin \mathcal{O}_1(N, \Omega)$, it clearly suffices to prove (8.1). The proof will be by induction on d. However, we will give a direct proof of (8.1) for both $d = 1$ and $d = 2$ in order to better illustrate the concept of N-sequence for the reader.

Case I: $d = 1$. By (7.5), $\operatorname{trt}(N^*, K_0) = (m_1, 0)$ where $K_0(u)$ is the full factorization polynomial for Ω. By Lemma 7.1(*c*), we have $m_1 = 1$. Hence, if we set $V_0 = N^*$, then (V_0) is an N-sequence with terminal index 1. However, since $\operatorname{trt}(N^*, K_0) = (1, 0)$, the polynomial $K_1(u)$ in (7.6) has no critical monomial M_1 satisfying $z^{-1} \ll M_1 \ll N^*$. Thus (V_0) is the only N-sequence for Ω, and so (8.1) holds.

Case II: $d = 2$. By (7.5) and Lemma 7.1(*b*), if $\operatorname{trt}(N^*, K_0) = (\sigma_0, \lambda_0)$, then we have,

$$\sigma_0 + \lambda_0 \leq 2 \quad \text{and} \quad \lambda_0 \leq 1\,. \tag{8.3}$$

We distinguish the two cases, $\lambda_0 = 0$ and $\lambda_0 = 1$. Assume first that $\lambda_0 = 0$. Then by Lemma 7.1(*c*), we have $\sigma_0 = 2$. Thus the N-sequence (N^*) has terminal index 2, and as in Case I, this is the only N-sequence. Thus (8.1) holds in this case.

Now assume $\lambda_0 = 1$. Thus by (8.3), either $\sigma_0 = 0$ or $\sigma_0 = 1$. We distinguish these two cases.

Subcase A: $\sigma_0 = 0$ and $\lambda_0 = 1$. Since $\sigma_0 = 0$, the N-sequence (N^*) has terminal index zero. By Lemma 7.1(*a*), the polynomial $K_1(u)$ in (7.6) (where $V_0 = N^*$) has two critical monomials R_1 and R_2 satisfying $z^{-1} \ll R_j \ll V_0$, and both have multiplicity 1 since $\lambda_0 = 1$. Thus by (7.5) and Lemma 7.1(*c*), we have $\operatorname{trt}(R_j^*, K_1) = (1, 0)$ for $j = 1, 2$. It easily follows that (V_0, R_1^*) and (V_0, R_2^*) are both N-sequences for Ω having terminal indices equal to 1, and as in Case I, there are no other N-sequences. Thus (8.1) holds.

Subcase B: $\sigma_0 = 1$ and $\lambda_0 = 1$. Thus (V_0) is an N-sequence with terminal index 1. By Lemma 7.1(*a*), the polynomial $K_1(u)$ possesses one critical monomial M_1 satisfying $z^{-1} \ll M_1 \ll V_0$, and M_1 is simple. By (7.5) and Lemma 7.1(*c*), we have $\operatorname{trt}(M_1^*, K_1) = (1, 0)$, which shows that (V_0, M_1^*) is an N-sequence having terminal index 1, and there are no other N-sequences. Hence (8.1) holds.

Case III: $d \geq 3$. We proceed by induction, assuming that (8.1) holds whenever a critical monomial of a full factorization polynomial has multiplicity $d_1 < d$. Now let $N \gg z^{-1}$ be a critical monomial of $K_0(u)$ having multiplicity d. By (7.5), we have $\operatorname{trt}(N^*, K_0) = (m, q)$ where $q < d$. If $m = d$, then $q = 0$ by Lemma 7.1, so there is only one N-sequence, namely (N^*), and its terminal index is d. Thus (8.1) holds if $m = d$. Thus we may assume $m < d$. Letting $V_0 = N^*$, we have from Lemma 7.1(*a*) that the polynomial $K_1(u)$ in (7.6) possesses distinct critical monomials $L_1, \ldots, L_\lambda$ such that $z^{-1} \ll L_j \ll V_0$ for each j, and if m_j denotes the multiplicity of L_j, then

$$m_1 + m_2 + \ldots + m_\lambda = d - m\,. \tag{8.4}$$

In addition, by Lemma 7.3 we have $m_j < d$ for each $j = 1, \ldots, \lambda$. Thus if Λ_1 is given by (7.9) for $k = 1$, we have by the induction hypothesis that

$$\sum \{t(W^{\#}): W^{\#} \in \mathcal{D}(L_j, \Lambda_1)\} = m_j, \tag{8.5}$$

for each $j = 1, \ldots, \lambda$. Howeer, it is obvious that if $W^{\#} = (W_0, \ldots, W_r)$ belongs to $\mathcal{D}(L_j, \Lambda_1)$, then $V^{\#} = (N^*, W_0, \ldots, W_r)$ belongs to $\mathcal{D}(N, \Omega)$, and $t(W^{\#}) = t(V^{\#})$. Conversely, it is clear that if $V^{\#} = (V_0, \ldots, V_r)$ belongs to $\mathcal{D}(N, \Omega)$, and if $r \geq 1$, then V_1 must be L_j^* for some j, and hence $(V_1, \ldots, V_r)$ belongs to $\mathcal{D}(L_j, \Lambda_1)$ and has the same terminal index as $V^{\#}$. Hence, it follows that if the sum on the left side of (8.1) is written as,

$$t((V_0)) + \sum_{j=1}^{\lambda} \left(\sum \{t(V^{\#}): V^{\#} \in \mathcal{D}(N, \Omega);\ V_1 = L_j^*\}\right), \tag{8.6}$$

then by (8.5), this sum is,

$$t((V_0)) + \sum_{j=1}^{\lambda} m_j. \tag{8.7}$$

But since $\operatorname{trt}(N^*, K_0) = (m, q)$, clearly the terminal index of the N-sequence (V_0) is m. Thus by (8.5) and (8.7), the left side of (8.1) equals d. This proves the lemma by induction.

9. - Theorem 3.

Given the equation (1.1) where $n \geq 1$ and where the functions $R_0(z), \ldots, R_{n-1}(z)$ belong to a logarithmic differential field of rank zero over $F(a, b)$. When written in terms of the operator θ (where $\theta w = zw'$) let (1.1) have the form $\Omega(w) = 0$, where $\Omega(w) = \sum_{j=0}^{n} B_j(z)\,\theta^j w$. Let p be the critical degree of (1.1), and let $N_1, \ldots, N_s$ be the distinct elements (if any) of the exponential set (see § 3) for (1.1). Let E_1 denote the transition set for (1.1) on (a, b). For each $k = 1, \ldots, s$, and each active N_k-sequence, $V^{\#}$, for Ω, let $E(V^{\#})$ denote the transition set for the terminal equation for $V^{\#}$ (see § 7) on (a, b). Let $E = \{r_1, \ldots, r_q\}$, where $r_1 < r_2 < \ldots < r_q$, denote the union of E_1 and all the sets $E(V^{\#})$ as $V^{\#}$ ranges over the sets $\mathcal{D}_1(N_k, \Omega)$ for $k = 1, \ldots, s$. Let (c, d) denote any of the intervals $(a, r_1), (r_1, r_2), \ldots, (r_{q-1}, r_q), (r_q, b)$. Then each of the following conclusions holds:

(*A*) The equation (1.1) possesses a complete logarithmic set of solutions $\{\varphi_1, \ldots, \varphi_p\}$ in $F(c, d)$.

(*B*) If $k \in \{1, ..., s\}$, and $V^{\#} = (V_0, \ldots, V_r)$ is an element of $\mathcal{D}_1(N_k, \Omega)$, then the equation (1.1) possesses $t(V^{\#})$ admissible solutions, $h_1, \ldots, h_{t(V^{\#})}$, in $F(c, d)$ of the form

$$h_j(z) = \psi_j(z)\left(\exp\int(V_0 + \ldots + V_r)\right), \tag{9.1}$$

where $\{\psi_1, \ldots, \psi_{t(V^{\#})}\}$ is a complete logarithmic set of solutions of the terminal equation for $V^{\#}$ in $F(c, d)$.

(*C*) The total number of solutions represented in Parts (*A*) and (*B*) is precisely n, and these n solutions form a fundamental set of solutions for (1.1) in some element of $F(c, d)$.

10. – Preliminaries for Theorem 3.

As in [16, § 14(*a*)], we will say that an admissible function f in $F(a, b)$ is *trivial in* $F(a, b)$ if the relation $f \ll z^{\beta}$ holds for each real number β. We will require the following simple facts:

LEMMA 10.1. – Let n be a positive integer. Let $\{f_1, \ldots, f_n\}$ be a set of admissible functions in some $F(a, b)$, with the following property: For any two distinct elements k and j in $\{1, ..., n\}$, either $f_k \ll f_j$ or $f_j \ll f_k$ in $F(a, b)$. Then, there is an element m in $\{1, ..., n\}$ such that $f_j \ll f_m$ for each j in $\{1, ..., n\}$, distinct from m.

PROOF. – The proof is by induction on n, being trivial for $n = 1$. Assuming the statement for n, assume $\{f_1, \ldots, f_{n+1}\}$ satisfies the hypothesis of the lemma. Then there exists m in $\{1, ..., n\}$ such that $f_j \ll f_m$ for $1 \leq j \leq n$, $j \neq m$. If $f_m \ll f_{n+1}$, then f_{n+1} is the desired element. If $f_{n+1} \ll f_m$, then f_m is the desired element, and the proof is complete.

LEMMA 10.2. – Let n be a positive integer, and let $\{f_1, \ldots, f_n\}$ be a set of admissible functions in some $F(a, b)$, with the following property: For any two distinct elements k and j in $\{1, ..., n\}$, either f_k/f_j or f_j/f_k is trivial in $F(a, b)$. Then, there is an element m in $\{1, ..., n\}$, such that f_j/f_m is trivial in $F(a, b)$ for each element j in $\{1, ..., n\}$, distinct from m.

PROOF. – The proof is similar to Lemma 10.1.

DEFINITION 10.3. – Assume the hypothesis and notation of Theorem 3. For each $q = 1, ..., s$, and any $V^{\#} = (V_0, \ldots, V_r)$ belonging to $\mathcal{D}_1(N_q, \Omega)$, where $r \geq 1$, let $K_k(u)$ be given by (7.7) for $1 \leq k \leq r$. For each $k = 1, ..., r$, let $\mathcal{F}_k(V^{\#})$ denote the set of critical monomials M of $K_k(u)$ satisfying $z^{-1} \ll M \ll V_{k-1}$. We define $\mathcal{E}_k(V^{\#})$ to the union of the sets of zeros on (a, b) of all the functions, $IF(M, \varphi)$ for $M \in \mathcal{F}_k(V^{\#})$, and $IF(M -$

$-M_1, \varphi)$ for distinct elements M and M_1 in $\mathcal{F}_k(V^{\#})$. Finally, we let A denote the union of the sets of zeros on (a, b) of all the functions $IF(N_j - N_k, \varphi)$ where $1 \leq k < j \leq s$. With the set E as defined in the statement of Theorem 3, we now define the *oscillation set* on (a, b) of equation (1.1) to be the union of the sets E, A, and all the sets $\mathcal{E}_k(V^{\#})$ as $V^{\#} = (V_0, \ldots, V_r)$ ranges over $\Phi_1(N_q, \Omega)$ for all $q = 1, \ldots, s$, and k ranges over $\{1, \ldots, r\}$. (The oscillation set is clearly finite.)

11. – Proof of Theorem 3.

Part (A) follows from Theorem 2, noting that the transition set for (1.1) is included in the set E. For Part (B), if $V^{\#} = (V_0, \ldots, V_r)$ belongs to $\Phi_1(N_k, \Omega)$, then the terminal equation for $V^{\#}$ is $\Lambda_{r+1}(v) = 0$, where Λ_{r+1} is given by (7.9) for $k = r+1$. Since the critical degree of this equation is $t(V^{\#})$, and since the transition set for this equation is included in the set E, it follows from Theorem 2 that a complete logarithmic set of solutions $\{\psi_1, \ldots, \psi_{t(V^{\#})}\}$ for the terminal equation exists in $F(c, d)$. In view of (7.9), the operator $\Lambda_{r+1}(v)$ is obtained from $\Omega(w)$ by the change of variable $w = (\exp \int (V_0 + + \ldots + V_r))\, v$ (followed by division by the exponential integral), and so clearly (1.1) possesses the $t(V^{\#})$ solutions given by (9.1) proving Part (B).

Since the sum of the multiplicities of the critical monomials $N_1, \ldots, N_s$ of the full factorization polynomial for (1.1) is $n - p$ (see § 3), it now follows from Lemma 8.1 that we have produced n solutions of (1.1) in Parts (A) and (B).

It remains to prove that these n solutions form a fundamental set in any element T of $F(c, d)$ on which they all exist and are analytic. To this end, we assume the contrary. A dependence relation on T can be written in the form,

$$U_0(z) + \sum_{m=1}^{d} U_m(z) \exp\left(\int \lambda(V_m^{\#})\right) \equiv 0, \tag{11.1}$$

where $V_1^{\#}, \ldots, V_d^{\#}$ represent the distinct elements in the union of the sets $\Phi_1(N_q, \Omega)$ for $q = 1, \ldots, s$, and where $U_0(z), \ldots, U_d(z)$ are linear combinations of complete logarithmic sets of solutions, and finally, where $\lambda(V^{\#})$ denotes $V_0 + \ldots + V_r$ if $V^{\#} = (V_0, \ldots, V_r)$. Since the oscillation set (see § 10) for (1.1) on (a, b) is finite, we can find a subinterval (c_1, d_1) of (c, d) containing no points of the oscillation set. We assert that the set of functions,

$$\left\{1, \exp \int \lambda(V_1^{\#}), \ldots, \exp \int \lambda(V_d^{\#})\right\}, \tag{11.2}$$

satisfies the hypothesis of Lemma 10.2 on $F(c_1, d_1)$. First, the ratio of $\exp \int \lambda(V_m^{\#})$ to the function 1 is either trivial (as defined in § 10) in $F(c_1, d_1)$ or its reciprocal is trivial in $F(c_1, d_1)$. This follows from Lemma 2.2, since $IF(\lambda(V_m^{\#}), \varphi)$ is either strictly positive or strictly negative on (c_1, d_1) due to the fact that (c_1, d_1) contains no points of the transition set for (1.1). We now examine the ratio,

$$\exp\int \lambda(V_m^{\#})/\exp\int \lambda(V_q^{\#}) = \exp\int(\lambda(V_m^{\#}) - \lambda(V_q^{\#})), \tag{11.3}$$

for $m \neq q$. Let $V_m^{\#} = (V_0, \ldots, V_r)$ and $V_q^{\#} = (W_0, \ldots, W_\beta)$, where we may assume $r \geqslant \beta$ without loss of generality. We distinguish two cases. Assume first that $V_j \neq W_j$ for some $j \leqslant \beta$ (and we may assume j is minimal having this property). Then, clearly,

$$\lambda(V_m^{\#}) - \lambda(V_q^{\#}) \sim V_j - W_j \quad \text{in } F(c_1, d_1). \tag{11.4}$$

If $j = 0$, then $V_j - W_j$ is $\sim$ to the difference of two distinct elements of the exponential set for (1.1). Thus the indicial function for the left side of (11.4) is nonwhere zero on (c_1, d_1) since the set A introduced in Definition 10.3 is included in the oscillation set. The same conclusion holds if $j > 0$, since the set $\mathcal{E}_j(V_m^{\#})$ defined in § 10 is included in the oscillation set (and clearly, in the notation of § 10, $V_j - W_j$ is $\sim$ to the difference of two distinct elements of the set $\mathcal{T}_j(V_m^{\#})$). Hence in this case, it follows from Lemma 2.2 that the ratios (11.3) is either trivial or its reciprocal is trivial in $F(c_1, d_1)$. In the second case, $V_j = W_j$ for all $j \leqslant \beta$, and hence we must have $\beta < r$. Thus,

$$\lambda(V_m^{\#}) - \lambda(V_q^{\#}) \sim V_{\beta+1}, \tag{11.5}$$

and since $V_{\beta+1}$ is $\sim$ to an element of $\mathcal{T}_{\beta+1}(V_m^{\#})$, it follows as above that the indicial function for the left side of (11.5) is nowhere zero on (c_1, d_1). Thus, again by Lemma 2.2, the ratio (11.3) is either trivial, or its reciprocal is trivial, in $F(c_1, d_1)$. Thus the set (11.2) satisfies the hypothesis of Lemma 10.2, and so there is an element L in the set in (11.2) with the property that when any other element is divided by L, the quotient is trivial in $F(c_1, d_1)$. We now divide the relation (11.1) by L, and the resulting relation shows that the particular $U_j(z)$ corresponding to L is trivial in $F(c_1, d_1)$. Since this $U_j(z)$ is a linear combination of a complete logarithmic set of solutions, it follows from [3; Lemma 10] that all the constants in the linear combination $U_j(z)$ are zero. Thus the term in (11.1) corresponding to $U_j(z)$ has been eliminated. We now apply Lemma 10.2 to the set (11.2) with L removed, and divide the relation (11.1) by the element produced. Again [3; Lemma 10], forces all the constants in the corresponding U_k to be zero. Repetition of this argument eventually shows that all constants in the dependence relation (11.1) are all zero contradicting our assumption. Thus we have a fundamental set. This proves Theorem 3.

12. – Theorem 4.

Given the equation (1.1), where $n \geqslant 1$, and where the functions $R_0(z), \ldots, R_{n-1}(z)$ belong to a logarithmic differential field of rank zero over $F(a, b)$. Let $N_1, \ldots, N_S$ be the distinct elements (if any) of the exponential set for (1.1), and let (1.1) have the form $\Omega(w)=0$, where $\Omega(w)=\sum_{j=0}^{n} B_j\theta^j w$, when (1.1) is written in terms of $\theta w = zw'$. Then:

(*A*) Assume that (1.1) satisfies at least one of the following two conditions: (i) The critical equation for (1.1) possesses two distinct roots having the same real part; (ii) For some k, $1\leqslant k\leqslant s$, there is an element $V^{\#}$ in $\mathcal{O}_1(N_k, \Omega)$ such that the terminal equation for $V^{\#}$ has the property that its critical equation possesses two distinct roots having the same real part. Then (1.1) has the following property: For any ϕ in (a, b) and any $\varepsilon>0$, there exist positive constants δ and K, and a solution $f\not\equiv 0$ of (1.1) such that,

$$\delta<\min\{\varphi-a,\, b-\varphi,\, \varepsilon\}, \tag{12.1}$$

and such that f is analytic and has infinitely many zeros $z_1, z_2, \ldots$, with $\lim_{m\to\infty}|z_m| = +\infty$, on the region defined by,

$$|\operatorname{Arg} z-\varphi|<\delta \quad \text{and} \quad |z|>K. \tag{12.2}$$

(*B*) Assume that (1.1) satisfies neither of the conditions (i) and (ii) in Part (*A*). Let the oscillation set for (1.1) on (a, b) consist of the points $r_1<r_2<\ldots<r_q$, and set $r_0=a$ and $r_{q+1}=b$. Let $f\not\equiv 0$ be any admissible solution of (1.1) in $F(a, b)$. Then, for each j, $0\leqslant j\leqslant q$, there is an element of $F(r_j, r_{j+1})$ on which f has no zeros. (If the oscillation set is empty, the result holds when q is taken to be zero.)

13. – Preliminaries for Theorem 4.

We will require the following result from [1] (see also [6; § 8]):

LEMMA 13.1. – Given any equation (1.1), where $n\geqslant 2$, and where $R_0, \ldots, R_{n-1}$ belong to a logarithmic differential field of rank zero over $F(a, b)$. Assume that the critical equation for (1.1) possesses two distinct roots having the same real part. Then for any φ in (a, b) and any $\varepsilon>0$, there exist positive constants δ and K, and a solution $f\not\equiv 0$ of (1.1) such that,

$$\delta<\min\{\varphi-a,\, b-\varphi,\, \varepsilon\}, \tag{13.1}$$

and such that f is analytic and has infinitely many zeros $z_1, z_2, \ldots$, with $\lim_{m\to\infty}|z_m| = +\infty$, on the region defined by,

$$|\operatorname{Arg} z-\varphi|<\delta \quad \text{and} \quad |z|>K. \tag{13.2}$$

REMARK. – The proof given in [1] was for the case when the $R_j(z)$ are rational functions, but the exact same proof holds for the general case.

14. – Proof of Theorem 4.

Part (*A*). If (1.1) satisfies condition (i), then the conclusion follows immediately from Lemma 13.1. If (1.1) satisfies condition (ii), say for $V^{\#} = (V_0, \ldots, V_r)$, then the conclusion of Lemma 13.1 holds for the terminal equation, $\Lambda_{r+1}(v) = 0$, for $V^{\#}$. Since any solution v of this equation gives rise to the solution of (1.1) defined by,

$$w = \left(\exp\left(\int (V_0 + \ldots + V_r)\right)\right) v, \tag{14.1}$$

and since any zero of v is a zero of w, clearly the conclusion of Part (*A*) holds in this case too.

Part (*B*). We assume that (1.1) satisfies neither of the conditions (i) and (ii), and we consider the fundamental set for (1.1) produced in Theorem 3. We assert that this fundamental set satisfies the hypothesis of Lemma 10.1 on each $F(r_j, r_{j+1})$. Using Lemma 2.3, and the proof in Theorem 3 concerning the set (11.2), it easily follows that the ratio of two solutions of the form (9.1) which correspond to different $V^{\#}$ is either trivial or its reciprocal is trivial in $F(r_j, r_{j+1})$. Similarly, the ratio of a solution of the form (9.1) to a solution φ_j in the complete logarithmic set of solutions for (1.1) is also either trivial or its reciprocal is trivial in $F(r_j, r_{j+1})$ by the same argument. Since (1.1) does not satisfy condition (i), clearly the ratio R of any two distinct elements in the logarithmic set for (1.1) satisfies either $R \ll 1$ or $R \gg 1$ in $F(-\pi, \pi)$ by Lemma 2.3. Thus the same property holds in $F(r_j, r_{j+1})$ for the distinct elements in a complete logarithmic set of solutions for (1.1). The same argument (in the light of the assumption that (1.1) does not possess property (ii)) shows that the ratio R of two distinct solutions of the form (9.1) which correspond to the same $V^{\#}$, also must satisfy either $R \ll 1$ or $R \gg 1$ in $F(r_j, r_{j+1})$. Thus the fundamental set satisfies the hypothesis of Lemma 10.1 in $F(r_j, r_{j+1})$. Now if $f \not\equiv 0$ is an admissible solution in $F(a, b)$ of (1.1), then in some element of $F(r_j, r_{j+1})$ we have $f = \sum_{i=1}^{m} c_i g_i$, where $\{g_1, \ldots, g_m\}$ is a subset of the fundamental set in $F(r_j, r_{j+1})$ and where the c_i are nonzero constants. Applying Lemma 10.1 to the set $\{g_1, \ldots, g_m\}$, we see that there is an index k in $\{1, \ldots, m\}$ such that $f = c_k g_k (1 + E)$ where $E \ll 1$ in $F(r_j, r_{j+1})$. Since g_k is admissible in $F(r_j, r_{j+1})$, and clearly has no zeros on some element of $F(r_j, r_{j+1})$ (as easily seen from the form of the fundamental set produced in Theorem 3), clearly the same property holds for f, and this concludes the proof of Theorem 4.

15. – Proof of Theorem 1.

We are given an equation (1.1) whose coefficients are rational functions. Since the rational functions are contained in a logarithmic differential field of rank zero over $F(-\pi, \pi)$ (namely, the set of all rational combinations of the logarithmic monomials of rank zero), Theorem 4 is applicable to (1.1) with $(a, b) = (-\pi, \pi)$. If (1.1) satisfies either of the two conditions (i) and (ii) listed in Part (A) of Theorem 4, then by Part (A) of Theorem 4, the conclusion (A) in Theorem 1 holds. If (1.1) does not satisfy either (i) or (ii) of Theorem 4, let $r_1 < r_2 < \dots r_q$ be the distinct points of the oscillation set for (1.1) on $(-\pi, \pi)$, and set $r_0 = -\pi$ and $r_{q+1} = \pi$. By Part (B) of Theorem 4, for any admissible solution $f \not\equiv 0$ of (1.1) in $F(-\pi, \pi)$, and any j, $0 \leqslant j \leqslant q$, there is an element T_j of $F(r_j, r_{j+1})$ on which f has no zeros. By Lemma 2.1, if $\varepsilon > 0$ is given, there is a constant $K_j(\varepsilon) > 0$ such that f has no zeros on the set

$$r_j + \varepsilon \leqslant \operatorname{Arg} z \leqslant r_{j+1} - \varepsilon, \qquad |z| \geqslant K_j(\varepsilon), \tag{15.1}$$

for each $j = 0, 1, \dots, q$. If f is actually meromorphic on the plane, then f can have only finitely many zeros in the bounded sets $|z| \leqslant K_j(\varepsilon)$, and so the conclusion (B) in Theorem 1 holds when we take $\{\sigma_1, \dots, \sigma_\lambda\}$ to be $\{r_1, \dots, r_{q+1}\}$. This concludes the proof of Theorem 1.

16. – Remarks.

(1) The main question left unanswered by Theorems 1 and 4 is the following: In Part (B) of Theorem 1, which of the numbers $\sigma_1, \dots, \sigma_\lambda$ (if any) which we produced in Theorem 4 are actually extraneous in the sense that in some ε-sector, $|\arg z - \sigma_j| < \varepsilon$, no solution $f \not\equiv 0$ of (1.1) has infinitely many zeros? In the case of second-order equations having polynomial coefficients, the Hille method (see [20; p. 382]) produces a list of rays none of which is extraneous in the above sense. The reason for this is that in the second-order case, the asymptotic solutions which are constructed actually exist with their prescribed asymptotic behavior in sectors which surround the special rays. However, in the case of equations of higher order, our theory has no such «continuation» results as yet, and so the solutions we construct are known to exist with their prescribed asymptotic behavior only on one side of some of the special rays. Thus the question of extraneous rays remains open.

(2) As the proof of Theorem 1 shows, in order to produce the numbers $\sigma_1, \dots, \sigma_\lambda$, it is required to find the oscillation set for (1.1). This involves the repeated use of the algorithm described in § 2 for finding the critical monomials of certain algebraic polynomials, namely the full factorization polynomial for (1.1), and all the polynomials $K_k(u)$ given by (7.7) for $1 \leqslant k \leqslant r+1$. These polynomials are derived using (7.7) from the corresponding operators $\Lambda_k(v)$ given by (7.9), and these latter operators are defined recursively by (7.10). In order to find the critical monomials of $K_k(u)$, it is obviously necessary to know the asymptotic behavior of the coefficients of $K_k(u)$ (or equivalently, the coefficients of $\Lambda_k(v)$). From the recursive relation (7.10), it is clear that we will require precise information on the asymptotic expansion of the functions V_k. From (7.8), we know that $V_k = M_k^*$, where M_k is a critical monomial of $K_k(u)$. If $M_k^* = M_k$, then V_k is just a logarithmic monomial. However, if $M_k^* \neq M_k$, then by

Definition 7.2, the function $M_k^* - M_k$ is a root of an algebraic polynomial (see (7.4)), and at first glance, the determination of its asymptotic expansion would seem to be complicated. However, since $M_k^* - M_k$ is asymptotically equivalent to a simple critical monomial of the algebraic polynomial (see Lemma 7.1(*d*)), the following result shows that the asymptotic expansion for $M_k^* - M_k$ (and hence the asymptotic expansion for V_k) can be easily determined to as many terms as desired:

LEMMA 16.1. – Let $G(v)$ be a polynomial in v whose coefficients belong to a logarithmic differential field of rank zero over $F(a, b)$. Let V be a simple critical monomial of $G(v)$, and let g be an admissible function in $F(a, b)$ such that $G(g) \equiv 0$ and $g \sim V$. Let $G_1(v) = G(v+V)$, and define S_1 to be the unique principal monomial of G_1 (see Lemma 2.5) if $G(V) \not\equiv 0$, while if $G(V) \equiv 0$, set $S_1 \equiv 0$. Let $G_2(v) = G_1(v+S_1)$, and set $S_2 \equiv 0$ if either $G(V) \equiv 0$ or $G_1(S_1) \equiv 0$. Otherwise, let S_2 be the principal monomial of $G_2(v)$. Now, set $G_3(v) = G_2(v+S_2)$, and define $S_3 \equiv 0$ if either $G(V) \equiv 0$, $G_1(S_1) \equiv 0$, or $G_2(S_2) \equiv 0$. Otherwise let S_3 denote the principal monomial of $G_3(v)$. Continue this process by induction to form $G_k(v)$ and S_k for $k = 4, 5, \ldots$, and define E_k by the equation,

$$g = V + S_1 + \ldots + S_k + E_k \qquad \text{for } k \geqslant 1. \tag{16.1}$$

Then the following hold for each $k \geqslant 1$: If $S_k \equiv 0$, then $E_k \equiv 0$. If $S_k \not\equiv 0$, then,

$$E_k \ll S_k \ll S_{k-1} \ll \ldots \ll S_1 \ll V \quad \text{in } F(a, b). \tag{16.2}$$

PROOF. – Assume $S_k \equiv 0$ for some $k \geqslant 1$. If $G(V) \equiv 0$, then by the uniqueness part of Lemma 2.4(*b*), we have $g \equiv V$. In addition, by definition, all $S_j \equiv 0$ for $j \geqslant 1$, so $E_k \equiv 0$ in (16.1). If $G(V) \not\equiv 0$, then $k > 1$, and one of the functions $G_1(S_1), \ldots, G_{k-1}(S_{k-1})$ must be identically zero. Let $j \geqslant 1$ be the minimal index $q \leqslant k-1$ such that $G_q(S_q) \equiv 0$. Then clearly $g_1 = V + S_1 + \ldots + S_j$ is a solution of $G(v) = 0$, and $S_{j+1} \equiv 0, \ldots, S_k \equiv 0$, while $S_j \not\equiv 0$. By Lemma 2.5, it follows that,

$$S_j \ll S_{j-1} \ll \ldots \ll S_1 \ll V, \tag{16.3}$$

so that $g_1 \sim V$. By the uniqueness part of Lemma 2.4(*b*), we have $g \equiv g_1$, from which it follows that $E_k \equiv 0$.

Now assume $S_k \not\equiv 0$. Then none of the functions $S_1, \ldots, S_{k-1}$ can be identically zero, and from Lemma 2.5 we can conclude that,

$$S_k \ll S_{k-1} \ll \ldots S_1 \ll V, \tag{16.4}$$

and that S_k is a simple critical monomial of $G_k(v)$. By Lemma 2.4(*b*), there is a solution $g_k \sim S_k$ in $F(a, b)$ of the equation $G_k(v) = 0$. Hence, $h = V + S_1 + \ldots + S_{k-1} + g_k$ is a solution of $G(v) = 0$, and by (16.4), clearly $h \sim V$. By the uniqueness part of Lemma 2.4(*b*), we have $g \equiv h$. But then in (16.1), we have $E_k = g_k - S_k$, and so $E_k \ll S_k$ since $g_k \sim S_k$. The conclusion (16.2) now follows from (16.4).

17. – Example.

We consider the equation,

$$w''' + R_2(z)\,w'' + R_1(z)\,w' + R_0(z)\,w = 0\,, \tag{17.1}$$

where R_2, R_1 and R_0 are the polynomials defined by z^2+3, $2z^2+z+3$, and z^2+z+ $+c+1$, respectively, where c is a complex parameter. Using (3.1), we rewrite the equation in the form (3.3), and we find $B_3 \equiv z^{-3}$, $B_0 \equiv R_0$, and

$$B_2 = R_2 z^{-2} - 3z^{-3}, \quad \text{and} \quad B_1 = R_1 z^{-1} - R_2 z^{-2} + 2z^{-3}. \tag{17.2}$$

The critical degree of (17.1) is then easily seen to be zero, and we form the full factorization polynomial,

$$H(v) = \sum_{j=0}^{3} z^j B_j(z)\, v^j, \tag{17.3}$$

given by (2.5). We note that $B_0 \sim z^2$, $zB_1 \sim 2z^2$, $z^2 B_2 \sim z^2$, and $z^3 B_3 \sim 1$ in $F(-\pi, \pi)$. We easily finds that the critical monomials of $H(v)$ are $N_2 = -z^2$ (from degree 3 and degree 2), and $N_1 = -1$ (from degree 2, 1, and 0), and clearly N_1 is of multiplicity two while N_2 is simple. Thus $\{-z^2, -1\}$ is the exponential set for (17.1). The indicial function (2.1) for $-z^2$ is $-\cos(3\varphi)$ which has zeros on $(-\pi, \pi)$ at $\pm\pi/6$, $\pm\pi/2$, and $\pm 5\pi/6$. The indicial function for N_2 is $-\cos\varphi$, which has zeros on $(-\pi, \pi)$ at $\pm\pi/2$. Thus the transition set for (17.1) on $(-\pi, \pi)$ is the set,

$$E_1 = \{-5\pi/6, -\pi/2, -\pi/6, \pi/6, \pi/2, 5\pi/6\}\,. \tag{17.4}$$

We must now construct the active N_k-sequences for (17.1). We consider first $N_1 =$ $= -1$, and we compute the operator (7.2) (where $W = N_1$) This is done by making the change of variable $w = e^{-z}v$, and we find that $B_3[N_1] = z^{-3}$, $B_1[N_1] = 2z^{-3}$, and

$$B_2[N_1] = 1 - 3z^{-3}, \quad \text{and} \quad B_0[N_1] = c\,. \tag{17.5}$$

The critical degree of (7.2) is 2, and the critical equation is $\alpha^2 + c = 0$. The full factorization polynomial for (7.2) has only one critical monomial whose δ_0 exceeds -1, namely $-z^2$. Since this critical monomial is $\gg N_1$, we see that in the notation of § 7, we have $\operatorname{trt}(N_1, H) = (2, 0)$. Thus $N_1^* = N_1$, and we see that $V_1^\# = (N_1)$ is an active N_1-sequence with terminal index 2. The transition set for the terminal equation for $V_1^\#$ is the set of zeros of $IF(-z^2, \varphi)$ on $(-\pi, \pi)$ which is included in E_1 given by (17.4). Thus the set $E(V_1^\#)$ defined in Theorem 3 is contained in the set E_1. There are no other active N_1-sequences. We remark that if c is a positive real number, then (17.1) satisfies the hypotheses (ii) in Part (A) of Theorem 4, and so (17.1) will have the global oscillation property listed in Part (A) of Theorem 1. We now assume that in (17.1), the parameter c is not a positive real number.

We now consider $N_2 = -z^2$, and we form the operator (7.2) with $W = N_2$, by making the change of variable $w = \exp(-z^3/3)\,v$ in (17.1). We find that in this case the critical degree of (7.2) is zero, and its full factorization polynomial has the critical monomials, z^2 (with multiplicity two) and -1 (with multiplicity 1). Thus $\operatorname{trt}(N_2, H)$ is $(0, 1)$, and so from Definition 7.2, we cannot take $N_2^* = N_2$. We are required to take $N_2^* = N_2 + g$, where g is the exact root of the polynomial $G(u)$ in (7.4) (where W is N_2), satisfying $g \sim S$, where S is the principal monomial of $G(u)$. Since $d = 1$ in (7.4), clearly the polynomial $G(u)$ is simply $H(N_2 + u)$, where $H(v)$ is given by (17.3). A simple computation shows that $S = -1$, and so $g = -1 + E$ where $\delta_0(E) < 0$. Thus $N_2^* = N_2 - 1 + E$. We now compute the operator (7.2) where $W = N_2^*$. However, we find that our asymptotic expansion for N_2^* is not precise enough for us to determine the asymptotic behavior of the coefficient $B_0[N_2^*]$ in (7.2). To find a more precise asymptotic expansion for N_2^*, we use Lemma 16.1. We compute the polynomial $G(S + u)$, and noting that $G(S) \not\equiv 0$, we find that the principal monomial S_1 of $G(S + u)$ is $3z^{-1}$. Thus according to Lemma 16.1, we have $g = S + S_1 + E_1$, where $E_1 \ll S_1$ in $F(-\pi, \pi)$. Thus,

$$N_2^* = N_2 + S + S_1 + E_1. \tag{17.6}$$

We now compute the operator (7.2) with $W = N_2^*$, and we find that using the representation (17.6), we can compute the asymptotic behavior of all the coefficients $B_j[N_2^*]$. We find that the operator (7.2) has critical degree equal to 1 (with critical root $\alpha = -6$), and its full factorization polynomial has the critical monomial z^2 of multiplicity two, and no other critical monomials with $\delta_0 > -1$. Thus $\operatorname{trt}(N_2^*, H) = (1, 0)$, and we can conclude that $V_2^\# = (N_2^*)$ is an active N_2-sequence with terminal index 1, and there are no other active N_2-sequences. The transition set for the terminal equation for $V_2^\#$ is again included in the set E_1 in (17.4). Thus the set E defined in Theorem 3 is just the set E_1 in (17.4). Since the active N_2-sequence and the active N_1-sequence both have only one entry, the sets $\mathcal{E}_k(V^\#)$ in Definition 10.3 are all empty. Since the set A defined in Definition 10.3 is already included in the set E_1 (since $N_2 - N_1 \sim N_2$), we see that E_1 in (17.4) is the oscillation set of (17.1) on $(-\pi, \pi)$. Hence when c is not a positive real number, it follows from the proof of Theorem 1, that the conclusion (*B*) in Theorem 1 holds when the set $\{\sigma_1, \ldots, \sigma_\lambda\}$ is taken to be the set E_1 in (17.4) with the point π adjoined.

18. – Remark.

When the classical Wiman-Valiron theory [19; pp. 106-109] is applied to an equation (1.1) where the functions $R_0(z), \ldots, R_{n-1}(z)$ are polynomials, not all identically zero, there is produced a finite nonempty set T of positive rational numbers with the property that the order of growth of any transcendental solution of (1.1) must belong to T. This is done as follows: We form the algebraic polynomial,

$$G(v) = z^{-n} v^n + \sum_{j=0}^{n-1} R_j(z) z^{-j} v^j. \tag{18.1}$$

Then, T is the set of all positive rational numbers β such that for some nonzero complex constant c, the function cz^{β} is the first term of one of the expansions at $z = \infty$ of the algebraic function defined by $G(v) = 0$. Of course, then cz^{β} will be a critical monomial of $G(v)$. With this notation, we will prove:

THEOREM 5. – Let β be the largest element of T. Then the equation (1.1) possesses a solution whose order of growth is β.

To this end, let $H(u)$ denote the full factorization polynomial for (1.1), and let $H_1(v) = H(v/z)$. Assuming the above notation, we will first prove the following:

LEMMA 18.1. – The function cz^{β} is a critical monomial of $H_1(v)$. In addition, $\beta \geqslant 1$.

PROOF. – To find $H(u)$, we rewrite (1.1) in terms of the operator θ by using (3.1) and (3.2) to obtain the form (3.3). We then form $H(u)$ using (3.5). When this is carried out, we find that,

$$H_1(v) = z^{-n} I_n(v) + \sum_{j=1}^{n-1} z^{-j} R_j(z) I_j(v) + R_0(z), \tag{18.2}$$

where the operator $I_j(v)$ is given by,

$$I_j(v) = v(v-1)\dots(v-(j-1)) \qquad \text{for } j \geqslant 1. \tag{18.3}$$

Clearly we may write $v^j - I_j(v) = J_j(v)$, where $J_j(v)$ is a polynomial in v, with constant coefficients, of degree at most $j-1$ for $j = 1, \dots, n$. Setting $R_n \equiv 1$, it is now easy to see from (18.1) and (18.2) that we can write,

$$G(v) - H_1(v) = \sum_{k=1}^{n-1} W_k(z) v^k, \tag{18.4}$$

where for each $k = 1, \dots, n-1$, we have

$$W_k(z) = \sum_{j=k+1}^{n} c_{j,k} z^{-j} R_j(z), \tag{18.5}$$

for some constants $c_{j,k}$.

Now by the Newton polygon determination (see [12; p. 105]) of the elements of T, the largest element β in T must be given by cancellation involving the term in $G(v)$ of largest degree. Hence there exists an index $q < n$ such that

$$\beta n - n = \beta q - q + \delta_0(R_q), \tag{18.6}$$

and for all $j \leqslant n$,

$$\beta j - j + \delta_0(R_j) \leqslant \beta n - n. \tag{18.7}$$

Since R_q is a polynomial, we have $\delta_0(R_q) \geqslant 0$ from which it follows from (18.6) that

$\beta \geqslant 1$. In view of (18.7), we see that for each $k = 1, \ldots, n-1$, and each $j \geqslant k+1$, we have

$$\delta_0(z^{-j} R_j) \leqslant \beta n - \beta(k+1) - n, \tag{18.8}$$

from which it follows from (18.5) that,

$$\delta_0(W_k) + \beta k \leqslant \beta n - \beta - n, \tag{18.9}$$

for each $k = 1, \ldots, n-1$. This relation shows that when $v = cz^\beta$ is inserted into the right hand side of (18.4), each term gives a value of δ_0 which is strictly less than $\beta n - n$. Of course, (18.7) shows that when $v = cz^\beta$ is inserted into $G(v)$, each term in $G(v)$ gives a value of δ_0 which is at most βn – n, and the term of degree n gives exactly $\beta n - n$. Thus from (18.4), each term in $H_1(v)$ will give a value of δ_0 which is at most $\beta n - n$, and the term of degree n in $H_1(v)$ gives precisely $\beta n - n$. It now follows immediately from (18.4) and the algorithm in [5; § 26] that since cz^β is a critical monomial of $G(v)$, it is also critical monomial of $H_1(v)$ proving Lemma 18.1.

Since $H_1(v) = H(v/z)$, it now follows from [5; § 30(b)] that $N = cz^{\beta-1}$ is a critical monomial of $H(u)$. Since $\beta - 1 \geqslant 0$, clearly N belongs to the exponential set for (1.1). By Lemma 8.1, the equation (1.1) possesses as active N-sequence, $V^{\#}$, and in view of Theorem 3, there exist real numbers $r_1 < r_2 <, \ldots, < r_q$ in $(-\pi, \pi)$ such that in each of $F(-\pi, r_1)$, $F(r_1, r_2), \ldots, F(r_q, \pi)$ separately, the equation (1.1) possesses at least one admissible solution $h(z)$ which is of the form,

$$h(z) = \psi(z)\left(\exp\int(V_0 + \ldots + V_r)\right), \tag{18.10}$$

where $V^{\#} = (V_0, \ldots, V_r)$ and where $\psi(z)$ is $\sim$ to a function of the form $z^\alpha(\text{Log } z)^j$ for some complex number α and some integer j. Since $\beta \geqslant 1$, it is clear (zsee (2.1)) that we can find a closed interval $[a_1, b_1]$ with $a_1 < b_1$, which lies entirely in one of the intervals $(-\pi, r_1)$, $(r_1, r_2), \ldots, (r_q, \pi)$, on which $IF(N, \varphi) \geqslant \varepsilon$ for some $\varepsilon > 0$. Since $V_0 + \ldots + V_r \sim N$, it follows from [7; Lemma ζ, p. 272], that the solution h on $F(a_1, b_1)$ has the form,

$$h(z) = K\psi(z)\exp(M(z) + E(z)), \tag{18.11}$$

where K is nonzero constant, $M(z) = (c/\beta) z^\beta$ and $E \ll M$ in $F(a_1, b_1)$. If W is an element of $F(a_1, b_1)$ on which ψ, M, and E are analytic, then by Lemma 2.1, the set W contains all points $z_t = te^{i\delta}$, where $\delta = (a_1 + b_1)/2$, and $t > 0$ is sufficiently large. From Lemma 2.3, it follows that if $d = \text{Re}(\alpha) - 1$, then for all sufficiently large t, we have

$$|\psi(z_t)| \geqslant t^d. \tag{18.12}$$

Since $M(z) = (c/\beta) z^\beta$, we clearly have,

$$\text{Re}(M(z_t)) = (|c|/\beta) t^\beta \text{Cos}(\beta\delta + \arg c), \tag{18.13}$$

and so by our choice of $[\alpha_1, \beta_1]$, we have

$$|\exp(M(z_t))| \geqslant \exp((|c| \in / \beta) t^{\beta}) \tag{18.14}$$

for all sufficiently large t. Finally, since $E \ll M$ in $F(a_1, b_1)$, it follows that for all sufficiently large t, we have,

$$|E(z_t)| \leqslant (\varepsilon/2)|M(z_t)| = (|c| \in / 2\beta) t^{\beta}. \tag{18.15}$$

Using the fact that $\mathrm{Re}(E) \geqslant -|E|$, it now follows from (18.11), (18.12), (18.14), and (18.15) that for all sufficiently large t we have,

$$|h(z_t)| \geqslant |K| t^d \exp((|c| \in / 2\beta) t^{\beta}). \tag{18.16}$$

Since $|z_t| = t$, and since $h(z)$ can be extended to be an entire function (since the R_j in (1.1) are polynomials), the relation (18.16) shows that the order of growth of h is at least β. Since the order of growth of h must belong to the set T, and since β is the largest element of T, it follows that the order of growth of h is precisely β which proves Theorem 5.

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