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Nonuniqueness of Solutions of a Degenerate Parabolic Equation (*)

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Summary. – *We give some results about nonuniqueness of the solutions of the Cauchy problem for a class of nonlinear degenerate parabolic equations arising in several applications in biology and physics. This phenomenon is a truly nonlinear one and occurs because of the degeneracy of the equation at the points where $u = 0$. For a given set of values of the parameter involved, we prove that there exists a one parameter family of weak solutions; moreover, restricting the parameter set, nonuniqueness appears even in the class of classical solutions.*

1. – Introduction.

In this paper we are interested in nonnegative solutions of the problem

$$\text{(1.1)}\qquad \text{(I)}\qquad \begin{cases} u_t = u\Delta u - \gamma|\nabla u|^2 & \text{in } Q = \mathbb{R}^N \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

where $\gamma \geq 0$ is a constant and where u_0 is a nonnegative continuous function on $\mathbb{R}^N$.

Equation (1.1) arises in several applications in biology and physics. References can be found in [1, 5, 17].

In general Problem I does not possess classical solutions because of the degeneracy of equation (1.1) at points where $u = 0$. Therefore we define solutions in a weaker sense.

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DEFINITION 1.1. – $u \in L^\infty(Q) \cap L^2_{\mathrm{loc}}([0,\infty)\colon H^1_{\mathrm{loc}}(\mathbb{R}^N))$ *is called a solution of Problem* I *if* $u \geq 0$ *a.e. in* Q *and*

$$\int u_0 \psi(0)\,dx + \iint_\Omega \{u\psi_t - u\nabla u\cdot\nabla\psi - (\gamma+1)|\nabla u|^2\psi\}\,dx\,dt = 0\,,$$

for every $\psi \in C^{1,1}(\overline{Q})$ *with compact support in* $\overline{Q}$.

First let u_0 be bounded in $\mathbb{R}^N$. Then we can use the classical viscosity method to construct a solution of Problem I, i.e. we add the artificial viscosity term $\varepsilon\Delta u$ to the right-hand side of equation (1.1) and take the limit $\varepsilon \searrow 0$. In [2] it was shown that this limiting procedure gives us a uniquely determined solution $u(x,t)$ of Problem I, which from now on we call the *viscosity solution* of Problem I.

In the case $\gamma = 0$ it was discovered independently by DAL PASSO and LUCKHAUS [5] and by UGHI [17] that Problem I may possess other solutions than the viscosity solution.

To explain their results we have to describe some of the properties of the viscosity solution u. In particular, because of the degeneracy of (1.1) at $u=0$, it is important to describe the sets where $u=0$ respectively $u>0$. Since u is not necessarily continuous in Q [2], we need to define what we mean by $u>0$.

We define the positivity set $P \subset \overline{Q}$ by

(1.2) $$P = \{(x,t)\in\overline{Q}\colon \operatorname{essinf}\{u(\xi,\tau)\colon (\xi,\tau)\in U\} > 0 \text{ for some neighbourhood } U \text{ of } (x,t) \text{ which is open in } \overline{Q}\}\,,$$

where u is the viscosity solution, and set

(1.3) $$P(\tau) = P \cap \{t=\tau\} \qquad \text{for } \tau \geq 0\,.$$

Then P and $P(t)$ are open in $\overline{Q}$ respectively $\mathbb{R}^N$, and it has been shown in [1] that

(1.4) $$\overline{P(t)} = \overline{P(0)} \qquad \text{for all } t\geq 0$$

and

(1.5) $$P(t_1) \subset P(t_2) \qquad \text{if } 0 \leq t_1 < t_2\,.$$

Now we are ready to describe the nonuniqueness result by DAL PASSO and LUCKHAUS. By (1.4), the «support» $\overline{P(t)}$ of the viscosity solution u is neither expanding nor shrinking as time evolves. DAL PASSO and LUCKHAUS have constructed, if $\gamma = 0$, an infinite number of solutions with shrinking support. It turns out that their construction can be carried over to the case $\gamma > 0$. Actually, in Section 5 we shall sketch the proof of a more general result which, roughly speaking, says that for any prescribed «smoothly shrinking support» Problem I possesses a solution.

Ughi's proof of nonuniqueness was entirely different. To understand her construction we need even more information about the positivity set P. Observe that P is not always entirely determined by (1.4) and (1.5). If u_0 has for example an isolated zero at $x=0$, it is not clear if and for which $t>0$, $0 \in P(t)$. Therefore we define the *waiting-time* t^* at $x=0$ by

$$t^* = \sup\{t \geq 0 \colon 0 \notin P(t)\}. \tag{1.6}$$

For a given u_0 there exists a unique «viscosity» solution u, in the sense which we mentioned before and which we shall make precise in Proposition 2.1 (i). Then t^* is the waiting time for u and therefore it is uniquely determined by u_0, $t^* = t^*(u_0)$.

In turns out [1] that depending on γ, N and the local behaviour of u_0 near $x=0$, t^* may be zero, nonzero and finite, respectively infinite.

If $\gamma=0$ and $N=1$, UGHI [17] has constructed an initial function u_0 such that $t^*=0$, i.e. the viscosity solution becomes immediately positive at $x=0$ for $t>0$. On the other hand she constructed a second solution which does satisfy the positivity properties (1.4) and (1.5), but which vanishes at $x=0$ for all $t \geq 0$.

Ughi's construction can be extended to the case $\gamma \geq 0$ and $N=1$. Actually, if $0 \leq \gamma < 1/2$, it can be modified to construct a one-parameter family of continuous solutions, where the parameter is the time at which the solution becomes positive at $x=0$.

It is a natural question to ask whether these continua of solutions also exist for different values of γ and N. In Section 3 we shall prove that for any $\gamma \geq 0$ and $N \geq 1$ there exist initial functions such that Problem I has a one-parameter family of solutions, each of which satisfying the positivity properties (1.4) and (1.5). Our main result however concerns the particular case that

$$\gamma > \frac{1}{2}N.$$

Let u_0 have an isolated zero at $x=0$ and let t^* be defined by (1.6). If $\gamma > (1/2)N$ and $t^* < \infty$, then Problem I has a one-parameter family of continuous solutions which are all positive at $x=0$ for $t>0$, except of one solution which vanishes at $x=0$ for al $t \geq 0$.

In particular, if in addition $u_0 > 0$ in $\mathbb{R}^N \setminus \{0\}$, we arrive at a rather striking nonuniqueness result: if $\gamma > (1/2)N$ there exist initial functions u_0 such that Problem I has infinitely many *classical* solutions in $C(\overline{Q}) \cap C^{2,1}(Q)$.

If $N=1$ and $\gamma<1$, the results of Section 3 can be obtained in an alternative way, based on a transformation of coordinates. Because of its constructive character, we describe this method in Section 4. In addition it tells us how the solutions of the one-parameter families behave in the critical case $\gamma = 1/2$.

In Section 6 we collect some examples of nonuniqueness in the case that u_0 is unbounded.

Finally we say a few words about the concept of viscosity solution. The definition which we use in this paper is quite natural, but from a mathematical point of view rather unsatisfactory in the sense that it is based on the very special property of equation (1.1) that it is easy to prove that the limiting procedure with the artificial viscosity yields a uniquely determined limit function u.

We point out however that we cannot avoids this problem by using for example the definition of viscosity solutions given by P. L. LIONS [11], which is an immediate generalization of the definition given by CRANDALL and LIONS [3] (see also [4]) for first-order Hamilton-Jacobi equations. The reason is the fact that all the classical solutions which we construct in this paper are obviously viscosity solutions in the sense of Lions and hence we cannot expect uniqueness in the class of viscosity solutions in the sense of Lions. Actually only the solutions with shrinking support constructed by DAL PASSO and LUCKHAUS [5] which we described in the beginning, are no viscosity solutions in the sense of Lions.

2. – Preliminaries.

Throughout this section we assume that u_0 is bounded in $\mathbb{R}^N$, and we introduce the following hypothesis.

$$H_1) \qquad u_0 \in C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N), \qquad u_0 \geq 0 \text{ in } \mathbb{R}^N.$$

The following basic result says that the viscosity solution is well-defined, and can be approximated from above by classical solutions.

PROPOSITION 2.1 [2]. – *Let $\gamma \geq 0$ and let u_0 satisfy H_1).*

(i) *The viscosity procedure defined in Section 1 yields a unique pointwise-limit function $u(x, t)$ and u is a solution of Problem I. We call u the viscosity solution of Problem I.*

(ii) *Let $\{u_{0n}\}_{n=1,2,\dots}$ be a monotone decreasing sequence of initial functions which are continuous, uniformly bounded in $\mathbb{R}^N$ and strictly positive such that*

$$u_{0n} \searrow u_0 \ \textit{pointwise in } \mathbb{R}^N \ \textit{as } n \to \infty .$$

Then the corresponding classical and strictly positive solutions $u_n \in C(\overline{Q}) \cap C^{2,1}(Q)$ satisfy

$$u_n \searrow u \ \textit{pointwise in } \overline{Q} \ \textit{as } n \to \infty ,$$

where u is the viscosity solution of Problem I.

In the following proposition we collect some regularity results for the viscosity solution.

PROPOSITION 2.2 [2]. – *Let u_0 satisfy H_1) and let u be the viscosity solution of Problem I. Then*

(i) $u_t \geq -(1/t)u$ *and* $\Delta u \geq -1/t$ *in* $\mathcal{D}'(Q)$;

(ii) *if* $\gamma > (1/2)N$, *then* $\Delta u \leq N/(2\gamma - N)t$,

$$|\nabla u|^2 \leq 2u/(2\gamma - N)t \quad \text{and} \quad u_t \leq Nu/(2\gamma - N)t \text{ in } \mathcal{D}'(Q);$$

(iii) *u is continuous at points $(x, 0)$ for all $x \in \mathbb{R}^N$*;

(iv) $u \in C(P) \cap C^{2,1}(P \cap Q)$ *and* $u > 0$ *in P, where P is defined by* (1.2);

(v) *if either* $N = 1$ *or* $\gamma > (1/2)N$, $u \in C(\overline{Q})$.

3. – The main result.

In this section we prove our main result about the nonuniqueness of solutions of Problem I.

THEOREM 3.1. – *Let u_0 satisfy H_1), let $\gamma \geq 0$ and $N \geq 1$, and let $\gamma > N/2$ if $N \geq 2$.*

(i) *There exists a continuous solution $u^*(x, t)$ of Problem I which satisfies for all $x \in \mathbb{R}^N$ and $0 \leq t_1 \leq t_2$*

$$u^*(x, t_1) = 0 \Leftrightarrow u^*(x, t_2) = 0 .$$

(ii) *Let $u_0(0) = 0$ and $t^* < \infty$, where t^* is defined by* (1.6). *Let $u(x, t)$ be the viscosity solution of Problem I. Then there exists a one-parameter family (which is continuous in the topology of $C_{\mathrm{loc}}(\overline{Q})$) of continuous solutions $u^\alpha(x, t)$, $0 \leq \alpha \leq 1$, of Problem I such that*

$$u^0 \equiv u^* \quad \text{and} \quad u^1 \equiv u \text{ in } Q,$$

and, for all $0 \leq \alpha \leq \beta \leq 1$,

$$u^\alpha \leq u^\beta \quad \text{and} \quad u^\alpha \not\equiv u^\beta \text{ in } Q.$$

In addition, for any $0 \leq \alpha \leq 1$, u^α satisfies the positivity properties (1.4) *and* (1.5), *u^α can be approximated by strictly positive and classical solutions of Problem I, and u^α satisfies the regularity properties given in Proposition* 2.2 (i)-(ii). *Finally, if $\gamma > (1/2)N$, then*

$$u^\alpha(0, t) > 0 \quad \text{for all } t > 0 \text{ and } \alpha \in (0, 1]. \tag{3.1}$$

The last property has a striking consequence. If $u_0 > 0$ in $\mathbb{R}^N \setminus \{0\}$ and $u^\alpha(0, t) > 0$ for $t > 0$, then u is bounded away from zero in compact subsets of Q, and hence, by standard theory of uniformly parabolic equations [12], $u^\alpha \in C^{2,1}(Q)$, i.e. we have obtained the following result.

COROLLARY 3.2 (*Nonuniqueness of classical solutions*). – *If $\gamma > (1/2)N$, there exist initial functions u_0 satisfying H_1), for which Problem I has infinitely many classical solutions which belong to $C^{2,1}(Q) \cap C(\overline{Q})$.*

REMARK 3.3. – In [1] the reader can find conditions on u_0 which guarantee that $t^* < \infty$. In particular, for any $\gamma \geq 0$ and $N \geq 1$ there exist initial functions u_0 such that $u_0(0) = 0$ and $t^* < \infty$. If $\gamma > N/2$ and u_0 has an isolated zero at $x = 0$, a necessary and sufficient condition for $t^* < \infty$ is the local integrability of $u_0^{-\gamma}$ near $x = 0$. Moreover, both from the results of [1] and Theorem 3.1 (ii) it follows that if $\gamma > N/2$ and $t^* < \infty$, then $t^* = 0$.

PROOF OF THEOREM 3.1. – (i) Let

$$\Omega = \{x \in \mathbb{R}^N : u_0(x) = 0\} \tag{3.2}$$

and define a nested sequence of open neighbourhoods Ω_n of Ω by

$$\Omega_n = \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Omega) < 1/n\}, \qquad n = 1, 2, \ldots . \tag{3.3}$$

Then there exists a sequence of continuous initial data $u_{0n}(x)$ such that for any $n = 1, 2, \ldots$

$$u_{0n} \leq u_{0n+1} \leq u_0 \quad \text{in } \mathbb{R}^N,$$

$u_{0n} \to u_0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}^N$, and

$$u_{0n} > 0 \quad \text{in } \mathbb{R}^N \setminus \Omega_n, \qquad u_{0n} = 0 \quad \text{in } \Omega_n .$$

Let u_n be the viscosity solution of Problem I with initial function u_{0n}. By Proposition 2.2 (v) $u_n \in C(\overline{Q})$, and, by (1.4) and (1.5),

$$u_n > 0 \quad \text{in } \{\mathbb{R}^N \setminus \Omega_n\} \times \mathbb{R}^+$$

and

$$u_n \equiv 0 \quad \text{in } \Omega \times \mathbb{R}^+ .$$

By the Comparison Principle for viscosity solutions, $u \geq u_{n+1} \geq u_n$ in Q; and hence we can define

$$u^*(x, t) = \lim_{n \to \infty} u_n(x, t) \leq u(x, t) . \tag{3.4}$$

It follows from the construction that

$$u^* > 0 \quad \text{in } \{\mathbb{R}^N \setminus \Omega\} \times \mathbb{R}^+$$

and

$$u^* \equiv 0 \quad \text{in } \Omega \times \mathbb{R}^+ . \tag{3.5}$$

It remains to show that u^* is a continuous solution of Problem I. Since either

$N=1$ or $\gamma>(1/2)N$, the functions u_n are locally equicontinuous in Q [2, Proposition 2.4] (this is an almost immediate consequence of the estimates given in Proposition 2.2 (i)-(ii)). By [2, Lemma 6.1] this equicontinuity holds also near $t=0$, i.e. the u_n are locally equicontinuous in $\overline{Q}$, and thus $u^*\in C(\overline{Q})$.

Let $u_{0n,\varepsilon}$ be a sequence of positive functions, monotone decreasing with respect to ε, such that $u_{0n,\varepsilon}\searrow u_{0n}$ as $\varepsilon\searrow 0$. Let $u_{n\varepsilon}$ be the corresponding solutions of Problem I. Then, by Proposition 2.1 (ii) $u_{n\varepsilon}\to u_n$ as $\varepsilon\to 0$ uniformly on compact subsets of $\overline{Q}$. Using a standard diagonal procedure we can construct a sequence $u_{n\varepsilon_n}$ which we denote by u_ε^*, such that

$$u_\varepsilon^*\to u^* \text{ uniformly on compact subsets of } \overline{Q} \text{ as } \varepsilon\searrow 0. \tag{3.6}$$

Observe that u_ε^* is not monotone in ε, but we may assume that

$$u_{0\varepsilon}^*\equiv u_\varepsilon^*(\cdot,0)\leq u_{0\varepsilon} \quad \text{in } \mathbb{R}^N, \tag{3.7}$$

where $u_{0\varepsilon}\searrow u_0$ as $\varepsilon\searrow 0$ and hence, by Proposition 2.1 (ii), the corresponding solutions $u_\varepsilon(x,t)$ satisfy

$$u_\varepsilon\searrow u \quad \text{as} \quad \varepsilon\searrow 0 \text{ in } \overline{Q}. \tag{3.8}$$

where u is the viscosity solution of Problem I.

Applying [2, Remark 6.7] to the sequences $u_\varepsilon^*\leq u_\varepsilon$, it follows that u^* is a solution of Problem I.

(ii) We choose $T>t^*$. Since $u\in C(\overline{Q})$, this implies that

$$A\equiv u(0,T)>0. \tag{3.9}$$

Let $u_{0\varepsilon}^*$ and $u_{0\varepsilon}$ be defined as in (3.7), and consider the family of initial functions

$$u_{0\varepsilon}^\delta=\delta u_{0\varepsilon}+(1-\delta)\,u_{0\varepsilon}^*, \quad 0\leq\delta\leq 1, \tag{3.10}$$

with corresponding solutions $u_\varepsilon^\delta(x,t)$. By (3.5), (3.6) and (3.8),

$$u_\varepsilon^0(0,T)\to u^*(0,T)=0 \quad \text{as } \varepsilon\searrow 0$$

and

$$u_\varepsilon^1(0,T)\to u(0,T)=A \quad \text{as } \varepsilon\searrow 0.$$

We fix $a\in(0,A)$. Since u_ε^δ are positive smooth solutions which depend continuously on δ, there exists for any $\varepsilon>0$ small enough a $\delta_\varepsilon\in(0,1)$ such that

$$u_\varepsilon^{\delta_\varepsilon}(0,T)=a.$$

We define

$$u_\varepsilon(x,t;a)=u_\varepsilon^{\delta_\varepsilon}(x,t) \quad \text{in } Q.$$

Then, as in the proof of (i), it follows that

$$u_\varepsilon(x, t; a) \to u(x, t; a) \quad \text{as } \varepsilon \to 0,$$

where $u(x, t; a)$ is a continuous solution of Problem I. From the construction it follows that

$$u(0, T; a) = a.$$

We claim that the family $\{u^\alpha\}_{0 \leqslant \alpha \leqslant 1}$ defined by

$$u^\alpha(x, t) = u(x, t; \alpha A) \quad \text{for } (x, t) \in \overline{Q},$$

satisfies all the properties of Theorem 3.1 (ii).

The continuous dependence on α in $C_{\text{loc}}(\mathbb{R}^N \times [0, \infty))$ follows again from the equicontinuity properties. The remaining properties follow at once from the construction, except of property (3.1).

To prove (3.1), let $\gamma > (1/2)N$ and $0 < \alpha \leqslant 1$. Then, by Proposition 2.2 (ii), $u_\varepsilon(x, t; \alpha A)$ satisfies the estimate

$$u_t \leqslant Nu/(2\gamma - N)t \quad \text{in } Q,$$

for all $\varepsilon > 0$. Integration over (t, T) at $x = 0$ yields

$$0 < \alpha A = u_\varepsilon(0, T; \alpha A) \leqslant (T/t)^{N/(2\gamma - N)} u_\varepsilon(0, t; \alpha A) \quad \text{for } 0 < t \leqslant T$$

and hence, taking the limit $\varepsilon \searrow 0$,

$$u^\alpha(0, t) > 0 \quad \text{for } 0 < t \leqslant T.$$

Finally, by (1.5), this implies that $u^\alpha(0, t) > 0$ for all $t > 0$.

This completes the proof of Theorem 3.1.

REMARK 3.4. – Let $\gamma > 1$ and let u be a smooth and strictly positive function on Q satisfying equation (1.1) in Q. If we define

$$v = u^{-\gamma} \quad \text{in } Q \text{ and } m = 1 - \frac{1}{\gamma} \in (0, 1),$$

then v satisfies the so-called Porous Medium equation

$$v_t = \operatorname{div}(v^{m-1} \nabla v) \quad \text{in } Q.$$

Observe that

$$\gamma > \frac{N}{2} \Leftrightarrow \frac{N-2}{N} < m < 1 \quad \text{if } N \geqslant 2$$

and, by Remark 3.3,

$$\gamma > \frac{N}{2} \quad \text{and} \quad t^* < \infty \Leftrightarrow u_0^{-\gamma} \text{ is locally integrable near } x = 0.$$

HERRERO and PIERRE [8] have shown that, if $(N-2)/N < m < 1$, the Cauchy problem

$$\text{(PM)} \qquad \begin{cases} v_t = \operatorname{div}(v^{m-1}\nabla v) & \text{in } Q, \\ v(x,0) = v_0(x) & x \in \mathbb{R}^N, \end{cases}$$

has a unique solution satisfying $v_t \in L^1_{\text{loc}}(Q)$ if

$$0 \leqslant v_0 \in L^1_{\text{loc}}(\mathbb{R}^N).$$

Hence one may wonder if this uniqueness result for Problem PM is a contradiction with our nonuniqueness result for Problem I (substituting $v_0 = u_0^{-\gamma}$).

To understand this better we argue the other way around and indicate how Problem PM could be used to prove Theorem 3.1 (ii) (except the results about u^*). Instead of $v_0 = u_0^{-\gamma}$, we substitute into Problem PM

$$v_0(x) = u_0^{-\gamma}(x) + \beta\delta(x), \qquad \beta \geqslant 0,$$

i.e., v_0 is a finite measure. PIERRE [14] has shown that Problem PM has a solution v_β, satisfying

$$v \in C^\infty(Q); \qquad v > 0 \quad \text{in } Q.$$

It can be shown that $u_\beta = v_\beta^{-1/\gamma}$ is a solution of Problem I with initial function u_0. Since v_β is totally ordered, i.e. pointwise strictly increasing in β, we have found an ordered continuum $\{u_\beta\}_{\beta \geqslant 0}$ of solutions of Problem I.

The condition that $\gamma > (1/2)N$ if $N \geqslant 2$ is not necessary for the existence of continua of solutions.

THEOREM 3.5. – *Let $\gamma \geqslant 0$ and $N \geqslant 1$. Then there exist initial functions u_0 satisfying H_1, such that Problem I possesses a one-parameter family of solutions which satisfy the positivity properties* (1.4) *and* (1.5).

If $N = 1$, the result follows at once from Theorem 3.1 and Remark 3.3.

If $N > 1$, we can reduce Theorem 3.5 to the one-dimensional case by choosing an initial function u_0 which only depends on one variable. Below however we shall show that we can also choose u_0 for example radially symmetric.

We introduce some notation. Let $r_0 > 0$. We shall assume that u_0 satisfies the hypothesis

$H(r_0)$. *u_0 is radially symmetric, $u_0(0) > 0$, and $u_0(x) = 0$ if $|x| = r_0$.*

If u_0 satisfies $H(r_0)$, the viscosity solution u is radially symmetric: $u = u(r, t)$, $r = |x|$. We set $\tilde{Q} = \mathbb{R}^+ \times \mathbb{R}^+$ and define the positivity sets

$$(3.11) \qquad \tilde{P} = \{(r, t) \in \tilde{Q}: \operatorname{essinf}\{u(\rho, \tau) > 0, (\rho, \tau) \in U\} > 0 \text{ for some neighbourhood } U \text{ of } (r, t) \text{ which is open in } \tilde{Q}\}$$

and

$$(3.12) \qquad \tilde{P}(\tau) = \tilde{P} \cap \{t = \tau\}.$$

Finally we define the waiting-time $t^*(r_0)$ at $|x| = r_0$ by

$$(3.13) \qquad t^*(r_0) = \sup\{t \geq 0: r_0 \notin \tilde{P}(t)\}.$$

Then Theorem 3.5, restricted to radially symmetric initial functions u_0, is an immediate consequence of the following result.

LEMMA 3.6. – *Let $\gamma \geq 0$ and $N \geq 2$, and let u_0 satisfy H_1) and $H(r_0)$ for some $r_0 > 0$.*

(i) *There exists a continuous radially symmetric solution $u^*(x, t)$ satisfying the properties of Theorem* 3.1 (i).

(ii) *Let $t^*(r_0) < \infty$, where $t^*(r_0)$ is defined by* (3.13). *Then there exists a one-parameter family (which is continuous in $C_{\mathrm{loc}}(Q)$) of continuous radially symmetric solutions u^α, $0 \leq \alpha \leq 1$, of Problem I such that*

$$u^0 \equiv u^* \quad \text{and} \quad u^1 \equiv u \quad \text{in } Q,$$

and for all $0 \leq \alpha < \beta \leq 1$

$$u^\alpha \leq u^\beta \quad \text{and} \quad u^\alpha \not\equiv u^\beta \quad \text{in } Q.$$

In addition u^α satisfies the positivity properties (1.4) *and* (1.5), *u^α can be approximated by strictly positive and classical solutions of Problem I, and u^α satisfies the regularity properties given in Proposition* 2.2 (i)-(ii). *Finally, if $\gamma > (1/2)N$, then for each $\alpha \in (0, 1]$, $u^\alpha(x, t) > 0$ if $|x| = r_0$ and $t > 0$.*

(iii) *There exist initial functions satisfying H_1) and $H(r_0)$ such that $t^*(r_0) < < \infty$.*

PROOF. – The proof of (i) and (ii) is almost identical to the proof of Theorem 3.1. The only nontrivial property to prove is the equicontinuity on compact subsets of $\overline{Q}$ of a sequence of uniformly bounded, classical and radially symmetric solutions u_n, satisfying $u_n(0, 0) \geq \varepsilon > 0$ for some $\varepsilon > 0$.

To prove this equicontinuity, we distinguish three regions in $\overline{Q}$.

Near $t = 0$ the equicontinuity (locally with respect to x) follows, as before, from [2, Lemma 6.1].

Near $x=0$ we use that $u_n(0,0)$ is bounded away from zero. Hence, for a bounded time interval the solutions u_n are, near $x=0$, bounded away from zero. Thus equation (1.1) is uniformly parabolic near $x=0$ (locally with respect to t) and the equicontinuity follows.

Finally, away from $x=0$ and $t=0$, it follows easily from the lower bound of Δu_n, given by Proposition 2.2 (i), and the radial symmetry, that $|\nabla u_n|$ is locally uniformly bounded. Hence, by a result by GILDING [7], u_n is locally uniformly Hölder continuous in t (with Hölder exponent 1/2), and the equicontinuity follows.

We do not prove (iii) here. We only remark that a straightforward modification of the proofs in [1] yields that a bounded radially symmetric initial function u_0 satisfies all the required conditions if $u_0(0)>0$ and if it behaves like

$$\mathcal{C}\big||x|-r_0\big|^{\mu}$$

near the sphere $|x|=r_0$ for some positive constants $\mathcal{C}$ and μ, provided that $\mu\gamma<1$. We leave the details to the reader.

REMARK 3.7. – If we replace the sphere $|x|=r_0$ in Lemma 3.6 by a smooth, closed and bounded $(N-1)$-surface Γ homeomorphic to a sphere, we loose the radial symmetry which we needed to prove the equicontinuity property. Therefore we are not able to prove the existence of the continuum u^α. However, the existence of a solution $u^*(x,t)$ which satisfies

$$u^*=0 \quad \text{on} \quad \Gamma\times\mathbb{R}^+, \tag{3.14}$$

can still be proved, using the idea's of Ughi's original nonuniqueness proof [17].

Let Γ satisfy the inner and outer sphere condition, let $u_0=0$ on Γ, and let Ω denote the open interior of Γ. We define $u_{01}, u_{02}\in C(\mathbb{R}^N)$ by

$$u_0=u_{01}+u_{02}; \qquad u_{01}\equiv 0 \quad \text{in } \mathbb{R}^N\setminus\Omega; \qquad u_{02}\equiv 0 \quad \text{in } \Omega.$$

Let u_1 and u_2 be the viscosity solutions of Problem I with u_0 replaced by u_{01} respectively u_{02}. Then, by (1.4),

$$u_1\equiv 0 \quad \text{in } \{\mathbb{R}^N\setminus\Omega\}\times\mathbb{R}^+; \qquad u_2\equiv 0 \quad \text{in } \overline{\Omega}\times\mathbb{R}^+.$$

Using the estimates in the existence proof in [2], it follows easily that u^*, defined by

$$u^*\equiv u_1+u_2 \quad \text{in } \overline{Q},$$

is a solution of Problem I with initial datum u_0. Clearly u^* satisfies (3.14). In particular, if the viscosity solution u does not satisfy (3.14), there exists at least two solutions of Problem I.

4. – $N=1$: a coordinate transformation.

In this section we give an alternative proof of some of the results in Section 3 if

$$N=1 \quad \text{and} \quad 0 \leqslant \gamma < 1 .$$

The proof has a constructive nature. In addition it will show us that the continuum u^α still satisfies property (3.1) if $\gamma = 1/2$, but it does not if $0 \leqslant \gamma < 1/2$.

The proof is based on a coordinate transformation $(x, t) \to (y, \tau)$. It turns out to be easier to discuss first the inverse transformation $(y, \tau) \to (x, t)$.

We consider the nonlinear diffusion problem

$$\text{(4.1)} \quad \text{(II)} \qquad \begin{cases} v_\tau = \varphi(v)_{yy} & y \in \mathbb{R},\ t>0, \\ v(y, 0) = v_0(y) & y \in \mathbb{R}, \end{cases}$$

where v_0 is a bounded, continuous and nonnegative function, and $\varphi \in C([0, \infty)) \cap \cap C^\infty(\mathbb{R}^+)$ satisfies

$$\varphi(0) = 0 ; \qquad \varphi' > 0 \quad \text{in } \mathbb{R}^+ .$$

It is well-known [13] that Problem II has a unique (weak) solution, i.e. a nonnegative function $v \in C(\mathbb{R} \times [0, \infty])$ such that

(i) $\varphi(v)_y \in L^2(\mathbb{R} \times [0, T])$ for all $T > 0$;

(ii) for all $\psi \in C^{1,1}(\mathbb{R} \times [0, \infty))$ with compact support

$$\int_{\mathbb{R}} v_0(y)\psi(y, 0)\, dy + \int_{\mathbb{R}\times\mathbb{R}^+} \{v\psi_\tau - \varphi(v)_y \psi_y\}\, dy\, d\tau = 0 .$$

For the moment we assume that $v(y, \tau)$ is a classical solution of Problem II. Let

$$\text{(4.2)} \qquad L = \int_{-\infty}^{\infty} v_0(y)\, dy < \infty .$$

We define the transformation $(y, \tau) \to (x, t)$ by

$$\text{(4.3)} \qquad x = \int_{-\infty}^{y} v(s, t)\, ds + C \quad \text{and} \quad t = \tau ,$$

for some $C \in \mathbb{R}$. Then (cf. [15])

$$x_y = v \quad \text{and} \quad x_\tau = \varphi(v)_y = v\varphi(v)_x ,$$

from which we derive that

$$v_\tau = v_t + v_x x_\tau = v_t + v\varphi(v)_x v_x ,$$

and, on the other hand,

$$v_\tau = \varphi(v)_{yy} = (v\varphi(v)_x)_x v = v^2 \varphi(v)_{xx} + v\varphi(v)_x v_x .$$

Hence v satisfies, as a function of x and t, the equation,

$$v_t = v^2 \varphi(v)_{xx}, \qquad C < x < C + L, \ t > 0, \tag{4.4}$$

where L and C are defined by (4.2) and (4.3).

Now we choose the function φ such that the equations (4.4) and (1.1) can be identified. Observe that (4.4) and (1.1) can be rewritten, at least formally, as

$$-(v^{-1})_t = \varphi(v)_{xx},$$

respectively, if $0 < \gamma < 1$,

$$-(u^{-\gamma})_t = \frac{\gamma}{1-\gamma}(u^{1-\gamma})_{xx}.$$

Hence, defining

$$\begin{cases} u(x,t) = \begin{cases} v^{1/\gamma}(y,\tau) & \text{if } C < x < C+L, \\ 0 & \text{otherwise}, \end{cases} \\ u_0(x) = \begin{cases} v_0^{1/\gamma}(y) & \text{if } C < x < C+L, \\ 0 & \text{otherwise}, \end{cases} \end{cases} \tag{4.5}$$

it follows that u is formally a solution of Problem I if we set

$$\varphi(s) = \frac{\gamma}{1-\gamma} s^{(1-\gamma)/\gamma}, \qquad s \geq 0. \tag{4.6}$$

If $\gamma = 0$ we find in the same way that the correspondence between u and v is given by

$$\varphi(s) = \exp[-1/s] \quad \text{for } s > 0,$$

and

$$u(x,t) = \begin{cases} \exp[-1/v(y,\tau)] & \text{if } C < x < C+L, \\ 0 & \text{otherwise}. \end{cases} \tag{4.7}$$

We now arrive at the key observation which will enable us to prove nonuniqueness. Let $\chi \in C(\mathbb{R})$ satisfy for some $a < 0 < b$

$$\begin{cases} \chi > 0 & \text{in } (a,0) \cup (0,b), \\ \chi = 0 & \text{otherwise}. \end{cases} \tag{4.8}$$

We define for any constant $h \geq 0$ the function v_0^h by

$$v_0^h(y) = \begin{cases} \chi(y) & \text{if } y \leq 0, \\ 0 & \text{if } 0 < y < h, \\ \chi(y-h) & \text{if } y \geq h. \end{cases}$$

We substitute $v_0 = v_0^h$ into Problem II with φ given by (4.6) (respectively (4.7) if $\gamma = 0$), we denote its solution by $v^h(y,t)$, and apply the transformation (4.3) in which we choose

$$C = L_1 = -\int_{-\infty}^{0} \chi(y)\, dy .$$

Finally we define $u^h(x,t)$ and $u_0^h(x)$ by (4.5) (respectively (4.7)).

The main point is now that

$$u_0(x) \equiv u_0^h(x) \text{ does not depend on } h \geq 0 .$$

Indeed, this follows at once from the construction of $v_0^h(y)$ and the nature of the transformation (4.3). Observe that u_0 satisfies the condition

$H_2)$ $\quad$ *$u_0 > 0$ in $(L_1, 0) \cup (0, L_2)$ for some $L_1 < 0 < L_2$, $u_0 = 0$ otherwise, and*

$$u_0^{-\gamma} \in L^1(L_1, L_2) \quad \textit{if } 0 < \gamma < 1 ,$$

respectively

$$|\log u_0| \in L^1(L_1, L_2) \quad \textit{if } \gamma > 0 .$$

Here we have set $L_2 = \int_0^\infty \chi(y)\, dy = L + L_1$.

The nonuniqueness of solutions of Problem I for this initial function u_0 follows from the following result.

THEOREM 4.1. – *Let, for given $\chi \in C(\mathbb{R})$ which satisfies (4.8) for some $a < 0 < b$, $u^h(x,t)$ be constructed as above.*

(i) *For any $h \geq 0$, u^h is a solution of Problem I.*

(ii) *If $1/2 \leq \gamma < 1$, then $u^h > 0$ in $(L_1, L_2) \times \mathbb{R}^+$ and $u^{h_1} \not\equiv u^{h_2}$ in $\mathbb{R} \times [0, T]$ for all $T > 0$ if $h_1 \neq h_2$.*

(iii) *If $0 \leq \gamma < 1/2$, then $u^h > 0$ in $(L_1, 0) \cup (0, L_2) \times \mathbb{R}^+$, and there exists a $T_h \geq$ ≥ 0 such that*

$$u^h(0,t) = 0 \quad \textit{if } 0 \leq t \leq T_h \quad \textit{and} \quad u^h(0,t) > 0 \quad \textit{if } t > T_h .$$

In addition T_h is strictly increasing with respect to h, and

$$u^{h_1} \equiv u^{h_2} \quad \text{in } \mathbb{R} \times [0, T] \text{ if } T \leqslant T^* \equiv \min\{T_{h_1}, T_{h_2}\},$$

$$u^{h_1} \not\equiv u^{h_2} \quad \text{in } \mathbb{R} \times [0, T] \text{ if } T > T^* \text{ and } h_1 \neq h_2.$$

PROOF. – First let $\gamma = 1/2$. Then, by (4.6), $\varphi(s) = s$, and (4.1) is nothing else than the heat equation.

Hence $v^h \in C^{2,1}(\mathbb{R} \times \mathbb{R}^+)$ and satisfies

$$v^h > 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^+, \tag{4.9}$$

and the formal argument above to show that u^h is a solution of Problem I is actually rigorous. Clearly $u^h > 0$ in $(L_1, L_2) \times \mathbb{R}^+$. It follows easily from the construction of u^h that for all $t > 0$

$$u^{h_1}(\cdot, t) \not\equiv u^{h_2}(\cdot, t) \quad \text{in } \mathbb{R} \text{ if } h_1 \neq h_2.$$

We leave the proof of the reader.

If $1/2 < \gamma < 1$, the proof is similar. By (4.6), $\varphi'(0) = \infty$, and it follows from [16] that v^h satisfies (4.9) and hence, by standard theory for uniformly parabolic equations, $v^h \in C^{2,1}(\mathbb{R} \times \mathbb{R}^+)$.

Finally let $0 \leqslant \gamma < 1/2$. Then $\varphi'(0) = 0$, (4.9) fails if $h > 0$, and we have to be more careful. By standard results about degenerate parabolic problems [10] there exist a number $T_h \geqslant 0$ and functions $\xi_1, \xi_4 \in C([0, \infty))$ and $\xi_2, \xi_3 \in C([0, T_h])$ such that

T_h is strictly increasing with respect to h,

$$\xi_2(T_h) = \xi_3(T_h), \qquad \xi_1(0) = a, \qquad \xi_2(0) = 0,$$

$$\xi_3(0) = h \quad \text{and} \quad \xi_4(0) = h + b,$$

$$v^h(y, \tau) \begin{cases} > 0 & \text{if } \begin{cases} y \in (\xi_1(\tau), \xi_2(\tau)) \cup (\xi_3(\tau), \xi_4(\tau)) & \text{if } \tau \leqslant T_h, \\ y \in (\xi_1(\tau), \xi_4(\tau)) & \text{if } \tau > T_h, \end{cases} \\ = 0 & \text{otherwise}, \end{cases}$$

$$\int_{-\infty}^{\infty} v^h(y, \tau)\, dy = L_1 + L_2 \quad \text{for } \tau \geqslant 0,$$

$$\int_{-\infty}^{\xi_2(\tau)} v^h(y, T)\, dy = L_1, \qquad \int_{\xi_3(\tau)}^{\infty} v^h(y, \tau)\, dy = L_2 \quad \text{if } \tau \leqslant T_h.$$

These properties translate immediately in the properties mentioned in (iii) for $u^h(x,t)$. The fact that $u^{h_1} \equiv u^{h_2}$ in $\mathbb{R} \times [0,T]$ follows from the equalities

$$v^{h_1}(y,\tau) = v^{h_2}(y,\tau) \quad \text{if } \tau \leq T^*, \quad \int_{-\infty}^{y} v^{h_i}(s,\tau)\,ds \leq L_1 \quad (i = 1,2),$$

$$v^{h_1}(y-h_1,\tau) = v^{h_2}(y-h_2,\tau) \quad \text{if } \tau \leq T^*, \quad \int_{-\infty}^{y-h_i} v^{h_i}(s,\tau)\,ds > L_1 \quad (i = 1,2),$$

Finally (i) follows easily from the fact that v^h is smooth whenever it is positive and continuous, which implies that u^h satisfies (1.1) classically at points where $u^h > 0$; moreover, since $(u^h)_x$ is locally bounded in Q and $v_h \to 0$ as $|y| \to \infty$, u^h satisfies Definition 1.1. This completes the proof of Theorem 4.1.

REMARK 4.2. – Starting point of Theorem 4.1, was the function $\chi(y)$, arriving at some function $u_0(x)$ which satisfies H_2). On the other hand, using the inverse transformation $(x,t) \to (y,\tau)$, it is not difficult to construct for any given initial function u_0 satisfying the hypotheses H_1)-H_2), a function χ to which Theorem 4.1 applies. In particular the continuum of solutions of Problem I exists for any u_0 satisfying H_1)-H_2). We leave the details to the reader.

REMARK 4.3. – The essential parts of condition H_2) are the facts that $u_0(0) = 0$ and $u_0^{-\gamma}$ (respectively $\log u_0$) are integrable in a neighbourhood of $x = 0$. In [1] it has been shown that of $N = 1$, this integrability condition is equivalent to the condition $t^* < \infty$ which was required in Theorem 3.1 (ii).

REMARK 4.4. – The transformation can also be used to construct solutions with shrinking support, which we shall discuss in Section 5. For example, let

$$\gamma = \frac{1}{2}.$$

Then, as we have seen in the proof of Theorem 4.2, $v(y,\tau)$ satisfies the heat equation. However, instead of solving the Cauchy problem for $v(y,\tau)$ for some initial function $v_0(y)$, we consider the free boundary problem

$$\text{(III)} \qquad \begin{cases} v_\tau = v_{yy} & \text{for } \xi^-(\tau) < y < \xi^+(\tau), \ 0 < \tau \leq T, \\ v(\xi^\pm(\tau),\tau) = 0 & \text{for } 0 < \tau \leq T, \\ v_y(\xi^\pm(\tau),\tau) = \mp f^\pm(\tau) & \text{for } 0 < \tau \leq T, \\ v(y,0) = v_0(y) > 0 & \text{for } a < y < b, \\ \xi^+(0) = b, \quad \xi^-(0) = a, \quad \xi^- < \xi^+ & \text{on } [0,T], \end{cases}$$

where $a<b$, $T>0$, $f^{\pm}\in C([0,T])$, $f^{\pm}>0$ on $[0,T]$, $v_0\in C([a,b])$, $v_0(a)=v_0(b)=0$ and $v_0>0$ on (a,b).

It follows from results in [9], that Problem III has a unique classical solution (v,ξ^+,ξ^-) (with $\xi^{\pm}\in C([0,T])$), provided that

$$\int_0^T \{f^+(\tau)+f^-(\tau)\}\,d\tau<\int_a^b v_0(y)\,dy\,. \tag{4.10}$$

Since any solution of Problem III satisfies, a priori,

$$v(y,\tau)>0 \quad \text{if } \xi^-(\tau)<y<\xi^+(\tau)\,, \qquad 0<\tau\leqslant T$$

and

$$\int_{\xi_1(T)}^{\xi_2(T)} v(y,T)\,dy=\int_a^b v_0(y)\,dy-\int_0^T\{f^+(\tau)+f^-(\tau)\}\,d\tau\,,$$

it follows at once that condition (4.10) is also necessary.

Next we define the transformation $(y,\tau)\to(x,t)$ by

$$x=\int_{-\infty}^{y} w(s,t)\,ds+\int_0^{\tau} f^+(s)\,ds\,, \qquad t=\tau$$

and we define $u\in C(\mathbb{R}\times[0,T])$ by

$$u(x,t)=\begin{cases} v^2(y,\tau) & \text{if } S^-(t)<x<S^+(t)\,, \qquad 0\leqslant t\leqslant T\,,\\ 0 & \text{otherwise}\,,\end{cases}$$

where $S^{\pm}\in C^1([0,T])$ are defined by

$$S^-(t)=\int_0^t f^-(s)\,ds\,,$$

and

$$S^+(t)=\int_a^b v_0(y)\,dy-\int_0^t f^+(s)\,ds=\int_{\xi^-(t)}^{\xi^+(t)} v(y,t)\,dy+\int_0^t f^-(s)\,ds\,.$$

Observe that

$$(S^+)'=-f^+<0 \quad \text{and} \quad (S^-)'=f^->0 \quad \text{on } [0,T]\,,$$

and that hence $\operatorname{supp} u(\cdot,t)$ is strictly shrinking.

Finally, the fact that u is a solution of Problem I with initial function $u(x,0)$ follows easily from the facts that u satisfies classically the equation

$$u_t = uu_{xx} - \frac{1}{2}u_x^2 \quad \text{if } S^-(t) < x < S^+(t), \quad 0 < t \leq T,$$

and that $u_x(S^\pm(t), t) = 2v_y(\xi^\pm(\tau), \tau)$ is uniformly bounded on $[0, T]$.

Choosing different functions $f^\pm(t)$, we can obtain in this way solutions of Problem I with different strictly shrinking support.

5. – Solutions with shrinking support.

In this section we briefly sketch how solutions of Problem I can be constructed which do not satisfy the positivity property (1.4). If $\gamma = 0$, some of these solutions were found by DAL PASSO and LYCKHAUS [5], who used a different method.

For the sake of simplicity we assume in this section that u_0 satisfies the following condition.

H_3) $\quad u_0(x) > 0 \quad$ *if* $|x| < 1$, $\quad$ *and* $\quad u_0(x) = 0 \quad$ *if* $|x| \geq 1$.

Let $T > 0$ and $C_T = B_1(0) \times (0, T)$. Let $\mathcal{S}_T$ be the set of all $\Omega_T \subset C_T$ such that:

(i) Ω_T is a proper open subset of C_T.

(ii) $\overline{\Omega}_T \cap \{t = 0\} = B_1(0)$.

(iii) $\overline{\Omega}_T \cap \{t = t_1\} \subset \overline{\Omega}_T \cap \{t = t_2\}$ if $0 \leq t_2 < t_1 \leq T$.

(iv) There exists a surjective coordinate transformation $\Phi\colon \overline{\Omega}_T \to \overline{C}_T$ given by

$$\begin{cases} y_i = \varphi_i(x_1, \ldots, x_N, t) & i = 1, \ldots, N, \\ \tau = t, \end{cases}$$

such that $\varphi_i \in C^2(\overline{Q})$ and the $(N \times N)$-matrix

$$\frac{\partial \Phi}{\partial x} = \left(\frac{\partial \varphi_i}{\partial x_j}\right)$$

is non-singular on $\overline{\Omega}_T$.

THEOREM 5.1. – *Let $T > 0$ and u_0 satisfy H_1) and H_3).*

(i) *If $\Omega_T \in \mathcal{S}_T$, Problem I has a solution u on $[0, T]$ such that*

$$u \begin{cases} > 0 & \text{in } \Omega_T, \\ = 0 & \text{in } \{\mathbb{R}^N \times (0, T)\} \setminus \Omega_T. \end{cases}$$

(ii) *Let $\Omega_T \subset C_T$ and $x_0 \in B_1(0)$. If, for any $\varepsilon > 0$ small enough,*

$$\Omega_T \cap \{0 \leq t \leq T - \varepsilon\} \in \mathcal{S}_{T-\varepsilon},$$

and

$$\overline{\Omega}_T \cap \{t = T\} = \{x_0\}\,, \tag{5.1}$$

then Problem I has a solution u such that

$$u\begin{cases} > 0 & \text{in } \Omega_T, \\ = 0 & \text{in } Q \setminus \Omega_T. \end{cases}$$

REMARK 5.2. – Apparently in (ii) property (5.1) allows us to extend the solution u for $t > T$ by zero. In general however we cannot expect u to be continuous at (x_0, T). In particular the continuity at (x_0, T) depends heavily on the local behaviour of $\partial\Omega_T$ near (x_0, T).

Theorem 5.1 implies that we can prescribe the support of a solution of Problem I, as long as the support is non-expanding in time and sufficiently smooth.

In particular there exist for any given $T > 0$ infinitely many solutions u of Problem I with T as «extinction time» i.e. $u(\cdot, t) \equiv 0$ for $t > T$ and $u(\cdot, t) \not\equiv 0$ for $t < T$.

Below we shall only sketch the proof of Theorem 5.1.

First let $\Omega_T \in S_T$. Consider the problem

$$\text{(IV)}\qquad \begin{cases} u_t = u\Delta u - \gamma|\nabla u|^2 & \text{in } \Omega_T, \\ u(x,0) = u_0(x) & \text{for } x \in \Omega_T \cap \{t = 0\}, \\ u(x,t) = 0 & \text{for } (x,t) \in \partial\Omega_T,\ 0 < t < T. \end{cases}$$

We claim that

(5.2) *Problem IV has a solution* $\tilde{u} \in C(\overline{\Omega}_T) \cap C^{2,1}(\Omega_T)$, *and* $\tilde{u}$ *is positive in* Ω_T.

To prove (5.2) we use the transformation $y = \Phi(x, t)$, $\tau = t$ to transform Problem IV to the region C_T. Then u, as a function of y and τ, has to be a solution of the Problem

$$\text{(V)}\qquad \begin{cases} u_\tau + \nabla u \dfrac{\partial \Phi}{\partial t} = u \operatorname{div}(A\nabla u) - u\nabla u \cdot \boldsymbol{c} - \gamma\left|\dfrac{\partial \Phi}{\partial x}\nabla u\right|^2 & \text{in } C_T, \\ u = 0 \quad \text{on } \partial B_1(0) \times [0, T], \\ u = u_0 \quad \text{on } C_T \cap \{t = 0\}, \end{cases} \tag{5.3}$$

where the matrix $A(y, \tau)$ and the vector $\boldsymbol{c}(y, \tau)$ with components $c_i(y, \tau)$ are defined by

$$A(y, \tau) = \left(\frac{\partial \Phi}{\partial x}\right)^T \left(\frac{\partial \Phi}{\partial x}\right)$$

and

$$c_i(y,\tau)=\sum_{j,k,l=1}^{N}\frac{\partial\varphi_i}{\partial x_j}\left(\left(\frac{\partial\Phi}{\partial x}\right)^{-1}\right)_{kl}\frac{\partial^2\varphi_k}{\partial x_j\partial x_l},$$

where we denote the inverse and transposed matrix of a matrix B by B^{-1} respectively B^T. Observe that, for any (y,τ), $A(y,\tau)$ is of the form B^TB and hence, symmetric and nonnegative. Indeed, since $\partial\Phi/\partial x$ is nonsingular, $A(y,\tau)$ is positive, and thus the operator $\operatorname{div}(A\Delta u)$ is uniformly elliptic in C_T.

Equation (5.3) is, although more complicated, essentially of the same type as equation (1.1).

In particular, a straightforward application of the techniques in [2, section 6] can be used to obtain the existence of a solution $\tilde{u}$ of Problem V, satisfying $\tilde{u}>0$ in C_T. Hence $\tilde{u}\in C^{2,1}(C_T)$. Also the continuity down to $t=0$ follows as in [2]. Finally, the continuity near the lateral boundary $\partial B\times[0,T]$ is nearly trivial. Considering $\tilde{u}(y,\tau)$ as a function of x and t, we arrive at (5.2).

Using that $|\nabla\tilde{u}|\in L^2(\Omega_T)$, it follows that the function u, defined by

$$u(x,t)=\begin{cases}\tilde{u}(x,t) & \text{if } (x,t)\in\overline{\Omega}_T,\\ 0 & \text{otherwise},\end{cases}$$

is a solution of Problem I, and Theorem 5.1 (i) follows.

Next let Ω_T be as in Theorem 5.1 (ii). Then by Theorem 5.1 (i), there exists a function $u\in C(\mathbb{R}^N\times[0,T))$ satisfying:

(i) u is a solution of Problem I on $[0,T-\varepsilon]$ for any $\varepsilon>0$, i.e. u is a solution on $[0,T)$;

(ii) $u>0$ in Ω_T, and $u\equiv 0$ in $\{\mathbb{R}^N\times(0,T)\}\setminus\Omega_T$.

In addition $|\nabla u|\in L^2(\mathbb{R}^N\times(0,T))$. Combining this with the fact that Ω_T shrinks continuously to the point (x_0,T) as $t\nearrow T$, it can be proved that u, extended by $u=0$ for $t>T$, is a solution of Problem I and we arrive at Theorem 5.1 (ii).

6. – Unbounded solutions.

The main purpose of this section is to give some examples of nonuniqueness if u_0 is not bounded in $\mathbb{R}^N$.

First we give a preliminary result about the existence of solutions. Here we mean by a solution of Problem I a solution in the sense of Definition 1.1, but merely requiring that $u\in L^\infty_{\mathrm{loc}}(Q)$ instead of $u\in L^\infty(Q)$.

THEOREM 6.1. – *Let* $\gamma\geq 0$, $N\geq 1$, $u_0\in C(\mathbb{R}^N)$ *and* $u_0\geq 0$ *in* $\mathbb{R}^N$. *If*

(6.1) $$|x|^{-2}u_0(x)\to 0 \text{ uniformly as } |x|\to\infty,$$

then Problem I has a solution, which satisfies the properties in Proposition 2.2.

REMARK 6.2. – In Theorem 6.1 we consider the global existence of solutions. If $\gamma < N/2$, explicit examples of solutions were constructed in [1] with initial function $u_0(x) = A|x|^2 + B$, which do blow up in finite time. Therefore a condition like (6.1) seems reasonable if $\gamma < N/2$. On the other hand, if $\gamma > N/2$, it can be shown that we do not need any growth condition on u_0, i.e. condition (6.1) can be omitted. The proof, which we do not give here, relies on the construction of a priori upper bounds, which prevent the solution to blow-up.

PROOF OF THEOREM 6.1. – Let $T > 0$ be arbitrary. It is enough to prove existence on $\mathbb{R}^N \times [0, T]$.

Assume for the moment that we know that there exists a positive, classical supersolution $U(x, t)$ of Problem I on $\mathbb{R}^N \times [0, T]$. Then the construction of a decreasing sequence of positive, classical solutions u_n of (1.1) such that $u_n(x, 0) \searrow u_0(x)$ as $u \to \infty$, is straightforward. Finally it follows from Remark 6.6 in [2] that $u(x, t) = \lim_{n \to \infty} u_n(x, t)$ is a solution of Problem I, satisfying all required properties.

It remains to construct $U(x, t)$. Fix $A > 0$ such that $1\text{-}2ANT > 0$, and define for $x \in \mathbb{R}^N$ and $0 \leq t \leq T$

$$U(x, t) = (1\text{-}2ANt)^{-1} (A|x|^2 + B),$$

where, in view of (6.1), $B > 0$ can be chosen so large that $U(x, 0) \geq u_0(x)$ for all $x \in \mathbb{R}^N$. Then U satisfies equation (1.1) for $\gamma = 0$, and hence U is, for all $\gamma \geq 0$, a supersolution of Problem I.

In the remainder of this section we give three examples of nonuniqueness. The first one is given in the following theorem.

THEOREM 6.3. – *Let $N = 1$ and $1/2 < \gamma < 1$. Then there exists a positive continuous function u_0 such that*

$$u_0^{-\gamma} \in L^1(\boldsymbol{R})$$

and such that exists a continuum of positive, classical solutions of Problem I.

PROOF. – By separation of variables we find for any $T^* > 0$ a solution of Problem I on $\mathbb{R}^N \times [0, T^*)$ of the form

$$u_{T^*}(x, t) = (T^* - t)^{-1} (f(x))^{1/(1-\gamma)}, \tag{6.2}$$

where $f(x)$ is the even solution of the problem

$$\begin{cases} f'' = (1-\gamma) f^{-1/(1-\gamma)} & \text{in } \boldsymbol{R}, \\ f(0) = 1, \quad f'(0) = 0. \end{cases} \tag{6.3}$$

By (6.3), f is convex and nondecreasing for $x>0$ and satisfies for $x>0$

$$f'(x)=(1-\gamma)\sqrt{\frac{2}{2\gamma-1}(1-f^{-(2\gamma-1)/(1-\gamma)})}\leq(1-\gamma)\sqrt{\frac{2}{2\gamma-1}}.$$

Since $f(x)\to\infty$ as $x\to\infty$ and $1/2<\gamma<1$, it follows that

$$\lim_{x\to\infty} f'(x)=(1-\gamma)\sqrt{\frac{2}{2\gamma-1}}.$$

Thus $f(x)$ behaves like $|x|$ as $|x|\to\infty$. In particular, defining $u_0(x)=u_{T^*}(x,0)$,

$$u_0^{-\gamma}\in L^1(\mathbb{R}^N). \tag{6.4}$$

Hence we have found a u_0 and a solution u_{T^*} of Problem I which blows up at $t=T^*$.

On the other hand, since $\gamma>N/2=1/2$, it follows from Theorem 6.1 and Remark 6.2 that Problem I also has a solution u, which exists globally in time (at this point we do not need the general result for $\gamma>N/2$ as stated in Remark 6.2; instead following the proof of Theorem 6.1, we use $u_{T^*}(x,t)$ as a supersolution of Problem I on $\mathbb{R}^N\times\times[0,T^*/2]$, arriving at a solution u on $\mathbb{R}^N\times[0,T^*/2]$; finally, since u satisfies the upperbound for u_t given by Proposition 2.2 (ii) as long as it exists, u cannot blow up in finite time).

Finally we define the continuum $\{u^\alpha\}_{\alpha\in(0,1)}$ of solutions «between u and u_{T^*}» by

$$u^\alpha=\begin{cases}u_{T^*} & \text{for } t<\alpha T^*,\\ \tilde{u}^\alpha & \text{for } t\geq\alpha T^*,\end{cases}$$

where $\tilde{u}^\alpha$ is the solution of Problem I on $\mathbb{R}^N\times[\alpha T^*,\infty)$ constructed in Theorem 6.1. This completes the proof of Theorem 6.3.

Theorem 6.3 has an interesting consequence for the porous medium equation

$$v_t=(v^{m-1}v_x)_x \quad \text{in } \boldsymbol{R}\times\boldsymbol{R}^+, \tag{6.5}$$

with initial function $v(x,0)=v_0(x)\in L^1(\boldsymbol{R})$.

Following Remark 3.4, $1/2<\gamma<1$ implies that

$$-1<m<0.$$

Translating Theorem 6.1 in terms of the porous medium equation, we arrive at the following nonuniqueness result.

COROLLARY 6.4. – *Let $N=1$ and $-1<m<0$. Then there exists a positive function $v_0\in C(\boldsymbol{R})\cap L^1(\boldsymbol{R})$ such that the porous medium equation (6.5) has infinitely many different, positive and classical solutions with initial function v_0.*

REMARK 6.5. – If $N=1$ and $-1<m<0$, the porous medium equation is studied in [6]. In particular it has been shown there that, for $v_0\in L^1(\boldsymbol{R})$, there exists precisely one solution $v(x, t)$ for which

$$\int_{R} v(x, t)\, dx = \int_{R} v_0(x)\, dx \qquad \text{for } t \geq 0 .$$

Actually, $v = u^{-\gamma}$, where u is defined in the proof of Theorem 6.3.

Also in the second example we take $N = 1$ and $1/2 < \gamma < 1$.

THEOREM 6.6. – *Let* $N = 1$, $1/2 < \gamma < 0$, *and*

$$u_0(x) = |x|^{1/(1-\gamma)} \qquad \textit{for } x \in \boldsymbol{R} .$$

Then Problem I has a continuum of solutions.

PROOF. – Observe that $u_0(x) = |x|^{1/(1-\gamma)}$ is a steady state solution of Problem I. However, since $1/(1-\gamma) > 2$, Δu_0 is unbounded and hence this solution does not satisfy Proposition 2.2 (ii).

On the other hand, by Theorem 6.1 and Remark 6.2 (which again could be avoided by constructing an explicit upperbound for this special case) there also exists a solution $U(x, t)$ which does satisfy Proposition 2.2 (ii). Finally, arguing as in the proof of Theorem 6.3, we can define the continuum $\{u_\tau\}_{\tau \geq 0}$ by

$$u_\tau(x, t) = \begin{cases} u_0(x) & \text{if } x \in \mathbb{R}^N,\ t < \tau , \\ U(x, t-\tau) & \text{if } x \in \mathbb{R}^N,\ t \geq \tau . \end{cases} \tag{6.6}$$

THEOREM 6.7. – *Let* $N \geq 3$, $\gamma > N/2$ *and*

$$u_0(x) = |x|^{(N-2)/(\gamma-1)} \qquad \textit{for } x \in \mathbb{R}^N .$$

Then Problem I has a continuum of solutions.

PROOF. – The function $|x|^{(N-2)/(\gamma-1)}$ is a steady-state solution of Problem I. Again let $U(x, t)$ denote the solution constructed in Theorem 6.1. We claim that

$$U(x, t) \not\equiv |x|^{(N-2)/(\gamma-1)} . \tag{6.7}$$

Indeed, it follows from the results of [1, cf. Figure 2], that, since

$$\frac{N-2}{\gamma-1} < \frac{N}{\gamma} < 2 ,$$

$U(0, t) > 0$ for $t > 0$, and (6.7) follows.

Finally, we define the continuum of solutions $\{u_\tau\}_{\tau \geq 0}$ by (6.6). Observe that if $\tau > 0$, $u_\tau(0, t) = 0$ if $t < \tau$ and $u_\tau(0, t) > 0$ if $t > \tau$. Hence u_τ cannot satisfy the estimate $u_t \leq Nu/(2\gamma - N)t$ of Proposition 2.2 (ii).

REMARK 6.8. – S. KAMIN pointed out to us that the solution $U(x, t)$ is a similarity solution.

Indeed it is of the form

$$t^{(N-2)/(2\gamma-N)}\, f(|x|\, t^{-(\gamma-1)/(2\gamma-N)}).$$

REMARK 6.9. – If $N=2$ and $\gamma=N/2=1$, then $u_0(x) = |x|^\alpha$ is a steady state solution of Problem I for any $\alpha>0$. Again, if $0<\alpha<2$, the existence of a continuum of solutions can be shown.

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