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Banach-Saks Operators on Spaces of Continuous Functions (*).

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Summary. – *Let K be a compact Hausdorff space and let E be a Banach space. We denote by $C(K, E)$ the Banach space of all E-valued continuous functions defined on K, endowed with the supremum norm. We study in this paper Banach-Saks operators defined on $C(K, E)$ spaces. We characterize these operators for a large class of compacts K (the scattered ones), or for a large class of Banach spaces E (the superreflexive ones). We also show by some examples that our theorems can not be extended directly.*

1. – Introduction.

In 1953, GROTHENDIECK [11] began the study of linear operators defined on $C(K)$ spaces. Later, in 1962, PELCZYNSKI [13] initiated the characterization of classes of linear operators defined on $C(K, E)$ spaces, E being a Banach space.

Since then, weakly compact, DUNFORD-PETTIS (or completely continuous) and unconditionally converging operators have been intensely studied. Less work has been done on the study of Banach-Saks operators defined on $C(K)$ spaces (see, for instance [5], [8] and [10]) and even less if you fix your interest in these operators defined on $C(K, E)$ spaces.

Throughout this paper E and F are Banach spaces, K is a compact Hausdorff space and Σ is the σ-field of the Borel subsets of K. We recall that any operator $T: C(K, E) \to F$ may be represented as an integral with respect to a finitely additive set function $m: \Sigma \to L(E, F'')$ having finite semivariations on $K(|m|(K) < +\infty)$ such that $\|T\| = |m|(K)$ (see for example [9], pag. 182) m is called the representing measure of T.

A compact space K is called scattered if every subset A of K has a point relatively isolated in A. The class of scattered compact spaces includes all countable compact spaces, all compact ordinals (with the order topology) and all the one-point compacti-

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fications of sets with the discrete topology. We send the reader to [14] for any detail referred to scattered compacts.

We recall the definition of the Banach space $L^p(\mu, X)$, the space of all (equivalence classes of) X-valued Bochner integrable functions, with norm

$$\|f\|_p = \left(\int \|f\|^p d\mu\right)^{1/p}$$

if $1 \leq p < \infty$. $L^\infty(\mu, X)$ is the space of all (equivalence classes of) X-valued Bochner integrable functions, with norm

$$\|f\|_\infty = \text{essential supremum } \{\|f(\omega)\| \colon \omega \in K\}.$$

Information about these spaces is very well explained in [9]. We also use in Theorem 4 some results about Orlicz spaces of vector valued functions. I recommend [12] for details concerning these spaces, but, perhaps, the best thing the reader can do is to take for granted some results of that kind.

We say that a Banach space is superreflexive if it is isomorphic to a uniformly convex Banach space. The class of superreflexive Banach spaces include all the finite-dimensional Banach spaces, and it is contained in the class of the Banach spaces with the Banach-Saks property. This last class is included in the class of the reflexive Banach spaces.

Finally we recall that a Banach space E has the Banach-Saks property if I_E, the identity map, is a Banach-Saks operator. And $T\colon X \to Y$ is an operator of this kind if T satisfies one of the two equivalent conditions:

a) for every bounded sequence (x_n) of X you can choose a subsequence (x'_n) of (x_n) such that the sequence $(T(x'_1 + \ldots + x'_n)/n)$ converges.

b) For every bounded sequence (x_n) of X you can choose a subsequence (x'_n) of (x_n) such that for every subsequence (x''_n) of (x'_n) the sequence $(T(x''_1 + \ldots + x''_n)/n)$ converges.

It is also known that T'' is a Banach-Saks operator if T is, and that every Banach-Saks operator is weakly compact.

We send the reader to [15] and [2] for questions about superreflexivity and the Banach-Saks property, respectively [15], is also a good reference for a theorem about Schauder basis on reflexive spaces which we use in the proof of Proposition 7.

2. – Banach-Saks operators on $C(K, E)$, K scattered.

We begin this section remembering basic facts about $C(K, E)$ spaces. We have said that any operator

$$T\colon C(K, E) \to F,$$

can be represented by

$$T(f) = \int_K f\,dm\,,$$

being $m\colon \Sigma \to L(E, F'')$ a finitely additive set function. This means that, for any finite set $B = \{t_1, \dots, t_n\}$, we have

$$T(f) = \int_{K-B} f\,dm + \sum_{i \leq n} m(\{t_i\})\,f(t_i)\,.$$

We also have that, for any Borel set A and $x \in E$, the function $x \cdot \chi_A \in C(K, E'')$ and

$$T''(x \cdot \chi_A) = \int_A x\,dm = m(A)(x)\,.$$

That's to say, the equality $T(f) = \int f\,dm$ is also valid for some $f \in C(K, E'') \setminus C(K, E)$.

If $T\colon C(K, E) \to F$ is a weakly compact operator, then m has three very good properties:

i) In fact, $m(\Sigma) \subset L(E, F)$, so we can define $m\colon \Sigma \to L(E, F)$.

ii) For every $A \in \Sigma$, $m(A)\colon E \to F$ is a weakly compact operator.

iii) m has semivariation continuous at ϕ. That's to say, for any sequence of Borel sets $(A_n) \downarrow \phi$ $\left(\text{and this means that } (A_n) \text{ satisfies } A_n \supset A_{n+1} \text{ and } \bigcap_n A_n = \phi\right)$, we have that

$$\lim_n |m|(A_n) = 0\,.$$

Let's recall that, for any Borel set A, we define the semivariation of m in A by

$$|m|(A) = \operatorname{Sup}\left\{\left\|\sum_{i \leq n} m(A_i)(x_i)\right\|\colon \{A_1, \dots, A_n\} \text{ partition of } A \text{ in } \Sigma;\ x_1, \dots, x_n \in B(E)\right\}.$$

Very easily, we have our first result.

THEOREM 1. – Let $T\colon C(K, E) \to F$ be a Banach-Saks operator. Then the representing measure m satisfies:

i) $m(\Sigma) \subset L(E, F)$ and m has semivariation continuous at ϕ.

ii) $m(A)\colon E \to F$ is a Banach-Saks operator, for every Borel set A.

PROOF. – First of all, we observe that T is a weakly compact operator, for T is a Banach-Saks one. So, ii) is immediately obtained. Also, we have said that T'' is a Banach-Saks operator due to the fact that T is. Then, if you notice that

$$m(A) = T'' \circ j_A \quad \text{where, for every Borel set } A,$$

$$j_A : E \to C(K, E)'', \qquad j_A(x) = x \cdot \chi_A,$$

it is clear that j_A is a linear and continuous operator, and so $m(A)$ is a Banach-Saks operator.

Our target is: when do i) and ii) assure us that T is a Banach-Saks operator? Of course, if F has the Banach-Saks property, every operator T which arrives to F is of the Banach-Saks type, so i) and ii) assure us that T is a Banach-Saks operator. We are going to see in this section that if K is a scattered compact i) and ii) imply that T is a Banach-Saks operator. In the next section we will examine conditions on E.

We need first the following result.

LEMMA 2. – Let T: $C(K, E) \to F$ be a continuous linear operator. Then, T is a Banach-Saks operator if and only if for every metrizable quotient $\hat{K}$ of K, the induced operator $\hat{T}$: $C(\hat{K}, E) \to F$ is Banach-Saks.

PROOF. – We recall that $\hat{K}$ is a quotient of K if there exists p: $K \to \hat{K}$ continuous and onto, such that $\hat{K}$ has the finest topology which makes p continuous. Given p, we define p': $C(\hat{K}, E) \to C(K, E)$ by

$$p'(f) = f \circ p.$$

In this way, we also define $\hat{T}$: $C(\hat{K}, E) \to F$ by $\hat{T} = T \circ p'$.

It is obvious that if T is a Banach-Saks operator, then so is $T \circ p' = \hat{T}$. Let's see now that if $\hat{T}$ is a Banach-Saks operator for every $\hat{K}$, so is T.

Let (ϕ_n) be a bounded sequence of $C(K, E)$. Similarly as in [1], pag. 236, we build a metrizable quotient space $\hat{K}$ of K, and a sequence $(\hat{\phi}_n) \subset C(\hat{K}, E)$ such that, if p: $K \to \hat{K}$ is the canonical mapping:

$$\hat{\phi}_n(p(t)) = \phi_n(t), \qquad \text{for every } t \in K \text{ and } n \in \boldsymbol{N}.$$

So, $(\hat{\phi}_n)$ is a bounded sequence of $C(K, E)$. Due to the fact that $\hat{T}$ is a Banach-Saks operator, we can choose a subsequence $(\hat{\phi}_{n(k)})$ of $(\hat{\phi}_n)$ such that the sequence

$$(\hat{T}(\hat{\phi}_{n(1)} + \ldots + \hat{\phi}_{n(k)})/k\colon k \in \boldsymbol{N})$$

converges. But $\hat{T}(\hat{\phi}_{n(i)}) = T \circ p'(\hat{\phi}_{n(i)}) = T(\hat{\phi}_{n(i)} \circ p) = T(\phi_{n(i)})$. So the sequence

$$(T(\phi_{n(1)} + \ldots + \phi_{n(k)})/k\colon k \in \boldsymbol{N})$$

converges. That's to say, T is a Banach-Saks operator.

We can demonstrate now the main result of this section.

THEOREM 3. – Let $T: C(K, E) \to F$ be a continuous linear operator, and K a scattered compact. Then, T is a Banach-Saks operator if and only if the representing measure m satisfies.

i) $m(\Sigma) \subset L(E, F)$ and m has semivariation continuous at ϕ.

ii) $m(A): E \to F$ is a Banach-Saks operator, for every Borel set A of K.

PROOF. – We have seen half of this theorem in Theorem 1. So, we only have to prove that i) and ii) imply that T is a Banach-Saks operator. By Lemma 2, it is enough to prove that $\hat{T}$ is a Banach-Saks operator, for every metrizable quotient $\hat{K}$ of K.

Let $\hat{m}$ be the representing measure of $\hat{T}$. It is quite clear that $\hat{m}(A) = m(p^{-1}(A))$, being A any Borel set of K. So $\hat{m}$ satisfies:

i) If $\hat{\Sigma}$ is the σ-algebra of the Borel sets of $\hat{K}$, then $\hat{m}(\hat{\Sigma}) \subset L(E, F)$ and $\hat{m}$ has semivariation continuous at ϕ.

ii) $\hat{m}(A): E \to F$ is a Banach-Saks operator, for every Borel set A of $\hat{K}$.

Now, by 8.6 of [14], if K is a scattered compact and $\hat{K}$ a metrizable quotient of K, then $\hat{K}$ is a countable compact. If $\hat{K}$ is finite, we have

$$\hat{T}(f) = \hat{m}(\{t_1\})(f(t_1)) + \ldots + \hat{m}(\{t_n\})(f(t_n)),$$

being $\hat{K} = \{t_1, \ldots, t_n\}$, and as $\hat{m}(\{t_i\})$ is a Banach-Saks operator, so is $\hat{T}$.

So let's suppose $\hat{K}$ infinite. Let $\hat{K} = \{t_i : i \in N\}$. We now take any bounded (for example, by 1) sequence (f_n) of $C(\hat{K}, E)$. As $\hat{m}(\{t_1\})$ is a Banach-Saks operator, and $(f_n(t_1): n \in N)$ is a bounded (in fact, by 1) sequence of E, we can choose $x_1 \in F$ and a subsequence (f'_n) of (f_n) such that, for every subsequence $(f'_{n(k)})$ of (f'_n) we have

$$(\hat{m}(\{t_1\})(f'_{n(1)}(t_1) + \ldots + f'_{n(k)}(t_1))/k : k \in N) \to x_1$$

in the F-norm.

We call the sequence $(f'_n) = (f^{(1)}_n)$. In this way we can build sequences $(f^{(s)}_n : n \in N)$ satisfying:

a) $(f^{(s)})$ is a subsequence of $(f^{(s-1)})$.

b) For every subsequence $(f^{(s)}_{n(k)} : k \in N)$ of $(f^{(s)}_n : n \in N)$ we have

$$(\hat{m}(\{t_s\})(f^{(s)}_{n(1)}(t_s) + \ldots + f^{(s)}_{n(k)}(t_s))/k : k \in N) \to x_s .$$

Noticing that, if $1 \leq i \leq s$, $(f^{(s)}_{n(1)})$ is a subsequence of $(f^{(i)}_n : n \in N)$ it is obvious that

$$(\hat{m}(\{t_i\})(f^{(s)}_{n(1)}(t_i) + \ldots + f^{(s)}_{n(k)}(t_i))/k : k \in N) \to x_s .$$

Let's define the diagonal sequence

$$(g_n : n \in N) = (f_n^{(n)} : n \in N).$$

We want to prove that $(\hat{T}(g_1 + \dots + g_n)/n) \to \sum_{1 \leqslant i} x_i$. First of all, let's see that the series $\sum_{1 \leqslant i} x_i$ converges. Let $\delta > 0$, and define $B_k = \{t_i : i \geqslant k\}$. As $(B_k) \downarrow \phi$, there exists $p \in N$ such that

$$|\hat{m}|(B_n) < \delta \quad \text{if } n > p.$$

Let $k' > k > p$. For any $n \in N$, we have

$$\left\| \sum_{k \leqslant i \leqslant k'} \hat{m}(\{t_i\}) \left(\sum_{j \leqslant n} g_j(t_i)/n \right) \right\| \leqslant$$

$$\leqslant \sup \left\{ \left\| \sum_{j \leqslant n} g_j(t_i)/n \right\| : k \leqslant i \leqslant k' \right\} \cdot |\hat{m}|(\{t_k, \dots, t_{k'}\}) \leqslant |\hat{m}|(\{t_k, \dots\}) = |\hat{m}|(B_k) < \delta,$$

so we can deduce that

$$(*) \qquad \left\| \sum_{k \leqslant i \leqslant k'} x_i \right\| = \lim_n \left\| \sum_{k \leqslant i \leqslant k'} \hat{m}(\{t_i\}) \left(\sum_{j \leqslant n} g_j(t_i)/n \right) \right\| \leqslant \delta$$

and we have seen that the series $\sum_{1 \leqslant i} x_i$ converges.

Now, we can prove that

$$(\hat{T}(g_1 + \dots + g_n)/n) \to \sum_{1 \leqslant i} x_i$$

and, by this, $\hat{T}$ is a Banach-Saks operator. With this intention let $\varepsilon > 0$. As $(B_k) \downarrow \phi$, there exists $p \in N$ such that

$$(**) \qquad |\hat{m}|(B_n) < \varepsilon/4 \quad \text{if } n > p.$$

We fix p. Then, for every $n > p$, p fixed, we obtain

$$\left\| \hat{T}(g_1 + \dots + g_n)/n - \sum_{1 \leqslant i} x_i \right\| = \left\| \sum_{j \leqslant n} \left(\sum_{i \leqslant p} \hat{m}(\{t_i\})(g_j(t_i)) + \int_{B_{p+1}} g_j \, d\hat{m} \right) \Big/ n - \sum_{1 \leqslant i} x_i \right\| \leqslant .$$

The equality before is trivial because of

$$\hat{T}(g_j) = \sum_{i \leqslant p} \hat{m}(\{t_i\})(g_j(t_i)) + \int_{B_{p+1}} g_j \, d\hat{m}$$

and now, the inequality follows:

$$(1)\qquad \leqslant \left\|\sum_{j\leqslant n}\sum_{i\leqslant p}\widehat{m}(\{t_i\})(g_j(t_i))/n-\sum_{i\leqslant p}x_i\right\|+\sum_{j\leqslant n}\left\|\int_{B_{p+1}}g_j\,d\widehat{m}\right\|\Big/n+\left\|\sum_{p+1\leqslant i}x_i\right\|\leqslant .$$

First of all, we have by (**) that

$$\left\|\int_{B_{p+1}}g_j\,d\widehat{m}\right\|\leqslant\|g_j\|_\infty\,|\widehat{m}|(B_{p+1})<\varepsilon/4 .$$

In second place, by (*) and (**) we deduce

$$\left\|\sum_{p+1\leqslant i}x_i\right\|=\lim_k\left\|\sum_{p+1\leqslant i\leqslant k}x_i\right\|=\lim_k\left[\lim_n\left\|\sum_{p+1\leqslant i\leqslant k}\widehat{m}(\{t_i\})\Big(\sum_{j\leqslant n}g_j(t_i)/n\Big)\right\|\right]\leqslant$$

$$\leqslant\lim_k\left[\lim_n|\widehat{m}|(\{t_{p+1},\dots,t_k\})\left\|\sum_{j\leqslant n}g_j/n\right\|_\infty\right]\leqslant|\widehat{m}|(B_{p+1})<\varepsilon/4 .$$

So, the inequality (1) follows:

$$(2)\qquad\leqslant\left\|\sum_{i\leqslant p}\Big(\sum_{j\leqslant n}\widehat{m}(\{t_i\})(g_j(t_i))/n-x_i\Big)\right\|+\varepsilon/2\leqslant$$

$$\leqslant\sum_{i\leqslant p}\left\|\sum_{j\leqslant n}\widehat{m}(\{t_i\})(g_j(t_i))/n-x_i\right\|+\varepsilon/2\leqslant$$

Remembering that $\Big(\sum_{j\leqslant n}\widehat{m}(\{t_i\})(g_j(t_i))/n\colon n\in\boldsymbol{N}\Big)\to x_i$ for $i=1,\dots,p$ there exists N such that, if $n>N$ we have

$$\left\|\sum_{j\leqslant n}\widehat{m}(\{t_i\})(g_j(t_i))/n-x_i\right\|<\varepsilon/2p,\qquad\text{for } i=1,\dots,p$$

and so, the inequality (2) follows (for $n>\sup(N,P)$):

$$\leqslant\sum_{i\leqslant p}\varepsilon/2p+\varepsilon/2=\varepsilon .$$

We have shown that $\widehat{T}$ is a Banach-Saks operator, and the theorem is finished.

3. – Banach-Saks operators on $C(K,E)$, E superreflexive.

We have said in Section 1 that a Banach-Saks operator is always weakly compact. Sometime the reverse is true, as in this theorem which has the flavour of DIESTEL-SEIFERT [8].

THEOREM 4. – Let $T\colon C(K,E)\to F$ be a weakly compact operator, and E a superreflexive Banach space. Then, T is a Banach-Saks operator.

PROOF. – Let's remember that $C(K, E)'$ can be represented by

$$\text{rcabv}(\Sigma, E') = \{\nu: \Sigma \to E'/\nu \text{ is a Borel regular measure of bounded variation}\}$$

and because of this, for any $y' \in F'$, we can define $m_{y'}$ as the regular Borel measure of bounded variation which satisfies the following equalities:

$$\langle T'(y'), f \rangle = \langle y', T(f) \rangle = \left\langle y', \int f\, dm \right\rangle = \int f\, dm_{y'}$$

for any $f \in C(K, E)$. See [6] for details.

Due to the fact that T is weakly compact, so is T', and by definition the set

$$T'(B(F')) = \{m_{y'}: y' \in B(F')\}$$

is a relatively weakly compact set of $\text{rcabv}(\Sigma, E')$. By [6], there exists $\mu \in \text{rcabv}(\Sigma)$ satisfying

$$T'(B(F')) \subset L^1(\mu, E')$$

and, by this inclusion, for any $y' \in B(F')$, there is a $g_{y'} \in L^1(\mu, E')$ such that

$$m_{y'}(A) = \int_A g_{y'}\, d\mu, \qquad \text{for any } A \in \Sigma.$$

Let's call $H = \{g_{y'}: y' \in B(F')\}$. As H is a relatively weakly compact set of $L^1(\mu, F')$, it is uniformly integrable (see [9]). This means that

$$\lim_n \sup \left\{ \int_{\{\omega: \|f(\omega)\| > n\}} \|f\|\, d\mu: f \in H \right\} = 0$$

and this is equivalent to the fact that $|H|$ is uniformly integrable, being

$$|H| = \{f: K \to \mathbf{R}/\text{there is a } y' \in B(F') \text{ such that } f(\omega) = \|g_{y'}(\omega)\|, \text{ for every } \omega \in K\}.$$

A classical result (see [7] p. 24) says that $|H| \subset L^1(\mu)$ is a uniformly integrable set if and only if there exists a function G: $[0, \infty) \to [0, \infty)$ satisfying:

a) G is a convex, increasing function.

b) $G(0) = 0$ and $G(t)/t \to \infty$ as $t \to \infty$.

c) $M' = \sup\{\int G(|f|)\, d\mu: f \in |H|\} < \infty$.

The canonical injection from the vectorial Orlicz space $L^G(\mu, E')$ to $L^1(\mu, E')$ is always continuous, and the condition *c)* means that the operator

$$T_0': F' \to L^G(\mu, E'), \qquad \text{defined by } T_0'(y') = g_{y'}$$

is continuous. So, remembering that $T'(F') \subset L^1(\mu, E')$, T' admits the following factorization:

$$\begin{array}{ccc} F' & \xrightarrow{T'} & L^1(\mu, E') \\ & {\scriptstyle T_0'}\searrow \quad \nearrow{\scriptstyle j} & \\ & L^G(\mu, E') & \end{array}$$

being j the canonical injection.

If we take adjoint operators in this diagram, and if we consider T'' restricted to $C(K, E)$, (that's so to say, T'' will be, in this case, $J_F \circ T$, being $J_F\colon F \to F''$ the natural injection) the diagram above remains in this way:

$$\begin{array}{ccc} C(K,E) & \xrightarrow{J_F \circ T} & F'' \\ & {\scriptstyle j_0'}\searrow \quad \nearrow{\scriptstyle (T_0')'} & \\ & L^G(\mu, E') & \end{array}$$

where $j_0' = j'|_{C(K,E)}$. Let G' be the conjugate Young function of G. Then it is a trivial fact that $M_{G'}(\mu, E)$ is a closed subspace of $L^G(\mu, E')'$, where

$M_{G'}(\mu, E) = \{f\colon K \to E, f$ a μ-measurable function such that

i) $\int G'(\|f\|)\, d\mu < \infty$ and

ii) $\exists (f_n) \subset L^\infty(\mu, E)$ satisfying $\int G'(\|f_n - f\|)\, d\mu \to 0$ as $n \to \infty\}$.

With this definition it is clear that $L^\infty(\mu, E') \subset M_{G'}(\mu, E)$, and so $C(K, E) \subset$ $\subset M_{G'}(\mu, E)$, and this canonical injection is continuous. By this, the previous diagram can be simplified and left as

$$\begin{array}{ccc} C(K,E) & \xrightarrow{J_F \circ T} & F'' \\ & {\scriptstyle j_1'}\searrow \quad \nearrow{\scriptstyle T_0''} & \\ & M_{G'}(\mu, E) & \end{array}$$

taking j_1' as j_0', with its image included in $M_{G'}(\mu, E)$, and

$$T_0'' = (T_0')'|_{M_{G'}(\mu,E)}.$$

If we prove that j_1' is a Banach-Saks operator, then so will $T_0'' \circ j_1' = J_F \circ T$, and hence T, and the proof will be finished.

Consequently, let's see that j_1' is as we want. The only fact that we need to know of $M_{G'}(\mu, E)$ is:

(*) $(g_n) \to g$ in $M_{G'}(\mu, E)$ if and only if $\int G'(\|g_m - g\|)\, d\mu \to 0$ as $n \to \infty$.

Let (f_n) be a bounded (by 1, for example) sequence of $C(K, E)$. Then (f_n) is bounded in $L^2(\mu, E)$, and $L^2(\mu, E)$ is superreflexive (due to the fact that E is, see [10]), so $L^2(\mu, E)$ has the Banach-Saks property. Because of this, (f_n) has a subsequence (f'_n) such that the sequence:

$$(**) \qquad (g_n) = ((f'_1 + \ldots + f'_n)/n) \to f \text{ in the } L^2(\mu, E)\text{-norm}.$$

Now, as $\|g_n(t)\| \leq 1$ for all $t \in K$, (**) implies that $\|f(t)\| \leq 1$ almost everywhere (in the sense of μ), by the vector mean value theorem.

Finally, as $(\|g_n - f\|: n \in N)$ is a bounded sequence of $L^\infty(\mu)$, the following conditions are equivalent:

a) $\int \|g_n - f\|^2 d\mu \to 0$ as $n \to \infty$,

b) $\int G'(\|g_n - f\|) d\mu \to 0$ as $n \to \infty$.

By (*) *b)* is equivalent to $(g_n) = ((f'_1 + \ldots + f'_n) \to f$ in $M_{G'}(\mu, E)$ and the proof is finished.

NOTE 5. – If we revisit the proof of this theorem, we notice that only in one point we have used the fact that E is a superreflexive Banach space: to assure that $L^2(\mu, E)$ has the Banach-Saks property. So, the theorem remains true if we change the hypothesis on E by «let E be a Banach space such that $L^2([0,1], E)$ has the Banach-Sacks property», which implies $L^2(\mu, E)$ has the Banach-Saks property for any measure μ.

The following result is a improvement of Theorem 1 on the same lines as Theorem 3.

THEOREM 6. – Let E be a superreflexive Banach space (or, by the previuos note, a Banach space E such that $L^2([0,1], E)$ has the Banach-Saks property). Let $T: C(K, E) \to F$ be a linear operator whose representing measure m satisfies:

i) $m(\Sigma) \subset L(E, F)$ and m has semivariation continuous at ϕ.

ii) $m(A): E \to F$ is a Banach-Saks operator, for every Borel set A of K.

Then, T is a Banach-Saks operator.

PROOF. – Due to ii), $m(A)$ is a weakly compact operator, and this condition, united to i), implies that T is a weakly compact operator (see [6]). By Theorem 4, T is a Banach-Saks operator, and the proof is finished.

4. – Some examples and questions.

In this last section, we would like to revisit our results, to see whether or not they can be improved. We begin with Theorem 3. The next proposition claims that Theorem 3 characterizes the scattered compacts.

PROPOSITION 7. – Let K be a non scattered compact. Then, there are Banach spaces E, F and a non Banach-Saks operator $T: C(K, E) \to F$ such that its representing measure m satisfies the conditions i) and ii) of Theorem 3.

PROOF. – First of all, we study the case $K = [0, 1]$. In [2], chapter V, the authors show a Banach space E with the Banach-Saks property, and a sequence (f_n), $f_n: [0, 1] \to E$, which satisfies:

a) $\|f_n(t)\| = 1$, for every $t \in K$.

b) $f_n((j/2^n, (j+1)/2^n)) = e_{2^n+j}$, being (e_i) a Schauder basis of E. This implies $(f_n(t)) \xrightarrow{\omega} 0$ for any $t \in [0, 1]$.

c) For any subsequence (f'_n) of (f_n), we have

$$\lim_n \|(f'_1 + \dots + f'_n)/n\|_2 = 1$$

being $\|\cdot\|_2$ the norm of $L^2([0, 1], E)$.

We can modify slightly each f_n to make it continuous. If λ is the Lebesgue measure of $[0, 1]$, we can take open sets G_n such that

i) $G_n \supset \{j/2^n : j = 0, \dots, 2^n\}$:

ii) $G_n \supset G_{n+1}$ for every $n \in N$, and $\lambda(G_n) < 1/n$.

By i), it is obvious that

$$\bar{f}_n: [0, 1] \setminus G_n \to X_n, \qquad \bar{f}_n(t) = f_n(t)$$

(where $X_n = \text{span}\{e_{2^n+j} : j = 0, \dots, 2^n - 1\}$) is a continuous function.

Then, by Tietze extension theorem, there is a continuous function $g_n: [0, 1] \to X_n$ such that

iii) $g_n(t) = \bar{f}_n(t) = f_n(t)$ for all $t \in [0, 1] \setminus G_n$, and

a') $\|g_n\|_\infty \leq \sup\{\|\bar{f}_n(t)\| : t \in [0, 1] \setminus G_n\} \leq 1$.

Then, as $X_n \subset E$, if we consider $g_n: [0, 1] \to E$, we have:

b') $(g_n(t)) \xrightarrow{\omega} 0$, for all $t \in [0, 1]$.

This is trivial because E is a reflexive Banach space, so given any Schauder basis (for instance, (e_i)), *b')* is equivalent (for all $t \in [0, 1]$) to:

$$\|g_n(t)\| \leq M(t) \quad \text{and} \quad (e'_i, g_n(t)) \to 0 \text{ as } n \to \infty, \ \forall i.$$

And these two conditions are obviously satisfied.

Finally, (g_n) satisfies:

c') For any subsequence (g'_n) of (g_n), we have

$$\lim_n \|(g'_1 + \dots + g'_n)/n\|_2 = 1.$$

The proof of c') is left as an exercise. The reader must only use a), c), ii), iii) and a').

Now, if the reader assumes a'), b') and c') as a starting point, it is clear that the natural injection

$$j\colon C([0,1],E) \to L^2([0,1],E)$$

has the following properties:

α) As E is a reflexive Banach space, so is $L^2([0,1],E)$. Because of this, j is a weakly compact operator, and so the representing measure m of j satisfies the condition i) of Theorem 3.

β) Because of E has the Banach-Saks property, $m(A)$ satisfies the condition ii) of Theorem 3.

γ) j is not a Banach-Saks operator. Given the bounded sequence (g_n) of $C([0,1],E)$, you cannot choose a subsequence (g'_n) whose arithmetic means converge in $L^2([0,1],E)$. b') says that the sequence of the arithmetic means, if it converges, it must be to 0, and c') says that, if that sequence converges, it must be to a function of norm 1.

Finally, let K be any non scattered compact. A very important result (see 8.5.4. of [14]) says that there is a function

$$\phi\colon K \to [0,1]$$

continuous and onto. So, for any continuous function $f\colon [0,1] \to E$, we can define the function $\phi^*(f)\colon K \to E$ by $\phi^*(f) = f \circ \phi$. Then, taking E and j exactly as in the paragraph before, we have the following diagram:

$$\begin{array}{ccc} C(K,E) & \dashrightarrow & L^2([0,1],E) \\ {\scriptstyle \phi^*}\nwarrow & & \nearrow{\scriptstyle j} \\ & C([0,1],E) & \end{array}$$

Then, a very important result in [3] claims that the weakly compact operator j can be extended to a weakly compact operator $\bar{j}\colon C(K,E) \to L^2([0,1],E)$ such that diagram is commutative.

Exactly in the same way as the previous paragraph, if $\overline{m}$ is the representing measure of $\bar{j}$, $\overline{m}$ satisfies the conditions i) and ii) of the Theorem 3. And $\bar{j}$ cannot be a Banach-Saks operator because then, if $\bar{j}$ were so, so would be $\bar{j} \circ \phi^* = j$, and this is not the case. Thus, the proof is finished.

NOTE 8. – In Theorem 4 and note 5, the sufficient condition on E ($L^2([0,1],E)$ has the Banach-Saks property) is far from necessary. If you take $E = C(K')$, K' an infinite compact, then $C(K')$ is not even a reflexive space. But, if we take for granted that $C(K, C(K'))$ is isometric to $C(K \times K') = C(K \times K', \boldsymbol{R})$, then any weakly compact operator T: $C(K, C(K')) \to F$ will be a Banach-Saks operator, because $\boldsymbol{R}$ is superreflexive.

If E is reflexive, then the sufficient condition is also necessary. If we consider the natural injection

$$j\colon C([0,1],E) \to L^2([0,1],E)$$

it is evident that j is weakly compact, because $L^2([0,1],E)$ is reflexive. Now, this, and the fact that $L^2([0,1],E)$ has not the Banach-Saks property, implies that there is a bounded sequence $(f_n) \subset L^2([0,1],E)$ which satisfies:

a) $(f_n) \xrightarrow{\omega}$ in $L^2([0,1],E)$.

b) For every (f_n') subsequence of (f_n), we have

$$\lim_n \|(f_1' + \ldots + f_n')/n\|_2 = \delta > 0$$

(see [2], chapter 2.3, Theorem 8).

Now, by an important result of Dunford (see [9], chapter IV.2) *a)* implies that

$$(*) \qquad \int_A \|f_n\| \, d\lambda \to 0 \quad \text{as} \quad \lambda(A) \to 0 \text{ uniformly in } n \in \boldsymbol{N}.$$

To prove that j is not a Banach-Saks operator, we have to modify (f_n) to produce a bounded sequence (g_n) in $C([0,1],E)$ such that none of its subsequences converges (in the sense of its arithmetic means) in $L^2([0,1],E)$. Skipping the details, we take

$$\bar{f}_n = f_n \cdot \chi_{A_n}, \qquad \text{with } A_n \text{ verifying } \lambda(A_n) < \varepsilon,\ \varepsilon \text{ depending on } (*) \text{ and } \delta.$$

Later, we can approximate the bounded measurable function $\bar{f}_n$ by a continuous function g_n. That sequence (g_n) will be the one we need.

With these results, we can state:

PROBLEM 1. – Find other necessary (or sufficient) conditions on E to make Theorem 4 true.

NOTE 9. – In Theorem 6, I do not even know any Banach space which does not satisfy the sufficient condition but satisfies the theorem. In fact, $C(K')$, with K' an infinite compact, is not a good example, because in [4] the authors show that, as $C(K')$ contains a copy of c_0, there are a Banach space F and an operator

$$T\colon C(K, C(K')) \to F$$

such that the representing measure m satisfies the condition i) of Theorem 6, $m(A)$ is weakly compact for any $A \in \Sigma$, and T is not weakly compact.

Because of Theorem 4, $m(A)$: $C(K') = C(K', \boldsymbol{R}) \to F$ is a Banach-Saks operator, and so m satisfies conditions, i), ii) of Theorem 6. Finally, if T is not weakly compact, is cannot be a Banach-Saks operator.

We can state now our following

PROBLEM 2. – Find other conditions on E to make Theorem 6 true.

To finish this work, let me call your attention to what we said after Theorem 1. If F has the Banach-Saks property, any operator $T: C(K, E) \to F$ whose representing measure m satisfies i), ii) of Theorem 1 (in fact, any operator T) is of the Banach-Saks kind. The question is:

PROBLEM 3. – Is there a Banach space F, which does not possess the Banach-Saks property, and such that any Banach-Saks operator $T: C(K, E) \to F$ is characterized by conditions i), ii) of Theorem 1?

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