Annali di Matematica pura ed applicata
(IV), Vol. CLXXI (1996), pp. 379-394

Interfacial Phenomenon for One-Dimensional Equation of Forward-Backward Parabolic Type (*).

VLADIMIR N. GREBENEV

Summary. – *An interfacial phenomenon for a class of the solutions of a nonlinear forward-backward parabolic equation in $R \times (0, T)$ is investigated. In general, short time-period of interfaces is considered. This inner analysis allows to construct on some time interval a solution of the Cauchy problem for certain initial data.*

1. – Introduction.

We study of the emergence of interfaces associated with *a continuous weak solution* of the equation

$$u_t = uu_{xx} + u_x^2 \ (\equiv 1/2(u^2)_{xx}), \qquad (x, t) \in Q_T = R \times (0, T), \tag{1.1}$$

without any sign restrictions for the function u. First we shall focus our attention on a generation mechanism of the interfaces between regions where $u < 0$ and $u > 0$ in certain ranges (rectangles). We consider several cases in connection with different distributions of $u(x, t)$ on the sections of a rectangle boundary. On some sections the equation (1.1) is parabolic and for the other sections it is backward parabolic. It will be shown that in certain cases, a set that splits the regions of parabolicity and backward parabolicity is a curve in R^2. In another cases, these special regions may be separated by the bounded are which will be further referred as *an interfacial layer region*.

We show that *a continuous weak solution* of the Cauchy problem, in general, on a short time interval is decomposed into two generalized solutions of the Cauchy problem, for which equation (1.1) is parabolic on the first solution and backwqrd parabolic on the second one. This result provides in particular the method for constructing *a*

(*) Entrata in Redazione il 20 aprile 1994 e, in versione finale il 25 novembre 1995.
Indirizzo dell'A.: Institute of Computational Technologies, Siberian Branch of Russian Academy of Sciences, Lavrentjeva Ave. 6, Novosibirsk 630090, Russia.

continuous weak solution to the Cauchy problem for (1.1). We end the paper by two simple examples of seeking our solution with various behaviour for the interfaces.

Let us dwell briefly on the background of this paper. There are two problems which traditionally are considered for equations with changing directions of parabolicity. The first one is based on setting the «initial» data on $t = 0$ and $t = T$. It depends on parabolicity or backward parabolicity of the equation on the «initial» data. In the simple case when the regions of parabolicity and backward parabolicity are known a priori, this problem has been studied by several authors (see, for instance [1]). The other one is the Cauchy problem. It will be convenient to designate them as Problem 1 (P1) and Problem 2 (P2). In the present problems we comment the following:

a) Questions on existence, uniqueness and non-uniqueness.

b) The emergence of interfacial layers and curves.

As for Problem 1, it should be mentioned here (see [2], the abstract of [2] is in *Soviet Advances in Math.*, 1987, N. 2, p. 156) that under special assumptions about the «initial» functions, we recall that they are given on lines $t = 0$ and $t = T$ in agreement with the phase states, *a continuous weak solution* of P1 can be viewed as the sum of generalized solutions of two evolution problems for corresponding porous medium equation. Note that their interfacial curves are straight lines connecting the points $(0, 0)$ and $(0, T)$. As is well-known in this case [3], the «initial» functions must possess the suitable behavior near interface. Also, it was shown that a uniqueness of solutions occurs.

Problem 2 has not been sufficiently studied at present time, a possibility of changing the parabolicity direction implies the ill-posedness of the classical problems for a forward-backward parabolic equation (1.1). In particular, it has been shown [4, 5] that the nonstationary Neumann problem for a forward-backward heat equation has a continuum of generalized solutions. Moreover, a set of generalized solutions is precompact but not closed in the space C and is closed but not precompact in W_2^1 (see [6]). The solvability of the boundary-value problems in the class of the measure-valued solutions in the sense of TARTAR - DIPIERNA [7] was proved in [8]. Among the latest papers on this topic we could mention [9], where the non-classical boundary-value problem for a nonlinear forward-backward parabolic equation with the hysteresis effect was studied. The author shows that solutions of this problem are obtained as a limit of generalized solutions of the Neumann boundary-value problem for a regularizing equation when a small parameter in a regularizator tends to zero. classical solutions were studied by LAIR [10, 11], who proved the uniqueness for a forward-backward diffusion equation with smooth constitutive function. Some results concerning self-similar solutions to a nonlinear diffusion equation with a variable-sign diffusion coefficients can be found in [12].

Also, it is interesting to interpretate the equation (1.1) in a sense of a limit case of the Cahn-Hilliard's type equation

$$\nu_t + \varepsilon \nu_{xxxx} = [\nu^2]_{xx}.$$

The Cahn-Hilliard equation was introduced by CAHN and HILLIARD [13] to describe the motion of an antiphase boundary separating two phases of a polycrystalline material. Some of the result concerning the Cahn-Hilliard equation are reviewed in [14].

Many physics models are described by forward-backward parabolic equations. Equations with a changing direction of parabolicity arise in investigation of fluids with super anomalous viscosity, in the description of shock waves in fluids with gas bubbles, multiphase thermomechanics with interfacial structure, the ocean thermocline, etc. N. N. YANENKO seems to be the first to have introduced the equation with viscosity coefficient of variable sign into mathematical use [15] with the aim to model complex (turbulent) flows. The equation (1.1) can be considered as a model equation for several of physical, mechanical and other processes.

Thus, the main purpose of this paper is the study of geometrical properties of the solutions of forward-backward parabolic equation (1.1) i.e. we wish to fill a gap in the point *b*) and additionally to suggest some recipe for the constructing *a continuous weak solution* to the Cauchy problem for (1.1) within of approach which will be used here.

2. – Some auxiliary lemmas and notation.

In this paragraph the terminology from the Stefan problem is introduced and some properties of *a continuous weak solution* of the equation (1.1) are established.

Further we will consider *a continuous weak solution* u of the equation (1.1) in $Q_T = R \times (0, T)$ for some $T > 0$.

We understand a continuous weak solution of (1.1) *as a function* u *on* Q_T *with the following*:

$$u \in C(Q_T) \cap L^\infty(Q_T),$$

$$u_x \in L^\infty_{\mathrm{loc}}(Q_T),$$

$$\int_{x_1}^{x_2} u(x, t_2)\, \psi(x, t_2)\, dx\, dt - \int_{x_1}^{x_2} u(x, t_1)\, \psi(x, t_1)\, dx\, dt - \int_{t_1}^{t_2} \int_{x_1}^{x_2} \{u\psi_t - uu_x \psi_x\}\, dx\, dt = 0,$$

for arbitrary $t_1 < t_2$ *and* $x_1 < x_2$ *such that the rectangle* $[x_1, x_2] \times [t_1, t_2]$ *is in* Q_T *and any* $\psi \in C^{1,1}(Q_T)$ *having compact support for all* $t \in [t_1, t_2]$.

It will be convenient to use the following notations:

$$\mathfrak{N} = \{(x, t) \in Q_T :\ u(x, t) = 0\}$$

stands for the interfacial layer region,

$$\mathfrak{M}_+ = \{(x, t) \in Q_T \colon u(x, t) > 0\}$$

designates the positive phase,

$$\mathfrak{M}_- = \{(x, t) \in Q_T \colon u(x, t) < 0\}$$

is the negative phase.

The interfacial layer region that evolves as a curve we will be referred to as *interface.*

The classical regularity results (see, e.g. [16]), using a (forward) parabolicity of (1.1) on $\mathfrak{M}_-$ in the direction of a decreasing time, readily shows that *a continuous weak solution* u is a classical one in $Q_T \backslash \mathfrak{N}$.

Now we present some auxiliary lemmas which rely on *the implicit function theorem* and *a maximum principle argument.* These lemmas are extracted from the papers [17-19]. Lemma 1 is important for the study the behavior of the set $\{(x, t) \in \mathfrak{M}_+ : u(x, t) = \varepsilon\}$ where ε is a noncritical value of u, i.e. $(u_x, u_t) \neq 0$ on the set $u^{-1}(\varepsilon) \cap Q_T$

LEMMA 1. – Let $A \subset Q_T$ be an open set such that $\tilde{A} = \operatorname{int}(\partial A \cap \{t = \sigma, \sigma_1 \leqslant \sigma < T, \sigma_1 \geqslant 0\})$ is a nonempty open set in R. Also suppose that u is positive on $A \cup \tilde{A}$. Let $(x_\varepsilon, t_\varepsilon) \in A$, $u(x_\varepsilon, t_\varepsilon) = \varepsilon$ where ε is a noncritical value of u on the set $u^{-1}(\varepsilon) \cap A$. Suppose $\varepsilon < \min u(x, t)$ on $\partial A \cap \{\sigma_1 \leqslant t \leqslant t_\varepsilon\}$. Then there exists a smooth level curve $x = \beta_\varepsilon(t)$, $\sigma_1 < t \leqslant t_\varepsilon$, lying in A, along which $u = \varepsilon$ and u_x never vanishes, except perhaps at $t = t_\varepsilon$.

The proof of Lemma 1 consists in [17] (see Lemma 5.3, [17]).

REMARK 1. – Except for at most a countable number of values ε, $\lim\limits_{t \to \sigma_1} \beta_\varepsilon(t)$ exist.

REMARK 2. – The properties of equipotential curves for a case of the smooth functions defined on manifolds with boundary are well studied. It is known as *Cronrod's theorem.* Some new facts about the behaviour of a level lines for generalized solutions of the potous medium equation were obtained by B. KNERR (see [17]).

The following lemmas are based on *the maximum principle.*

LEMMA 2. – Let B be a subdomain of Q_T bounded by 4 curves: an interval γ_0 lying on $t = \sigma_2$, an interval γ_1 lying on $t = \sigma_3$ $(\sigma_2 < \sigma_3)$, a continuously differentiable closed curve γ_2 connecting the right end-points of γ_0, γ_1 and lying (except for its end-points) in the rectangle S_{σ_2, σ_3} $(S_{\sigma_2, \sigma_3} = \{(x, t) \in Q_T \colon \sigma_2 < t < \sigma_3)$, and a continuously differentiable closed curve γ_3 connecting the left end-points of γ_0, γ_1 and lines γ_2, γ_3 are never parallel to the x-axis. If $u(x, t) > 0$ on $\gamma_0 \cup \gamma_2 \cup \gamma_3$ then $u(x, t) > 0$ in $\bar{B}$.

LEMMA 3. – Let $P \in Q_T$ be a rectangule given by $\alpha < x < \beta$, $\tau_1 < t < \tau_2$ such that $u(x, t) \geq 0$ in $\bar{P}$ and $u(x, \tau_1) > 0$ for $\alpha < x < \beta$ then $u(x, t) > 0$ in P.

The proof of Lemma 3 is based on *the maximum principle argument* of the porous medium equation and given in [18].

LEMMA 4. – Let conditions of Lemma 2 be satisfied. And only for γ_2, γ_3 we have a continuity. Suppose that $u(x, t) \geq 0$ on $\gamma_1 \cup \gamma_2 \cup \gamma_3$ then $u(x, t) \geq 0$ in $\bar{B}$.

The above lemma implies that no *a backward parabolic region* appears in $\bar{B}$ if $u(x, t) \geq 0$ on $\gamma_1 \cup \gamma_2 \cup \gamma_3$. Lemma 4 is proved by employing *the strong maximum principle* as in Lemma 2 from [19].

LEMMA 4′. – If in Lemma 4 the assumption that $u(x, t) \geq 0$ on $\gamma_1 \cup \gamma_2 \cup \gamma_3$ is replaced by the assumption $u(x, t) \leq 0$ on $\gamma_0 \cup \gamma_2 \cup \gamma_3$, then $u(x, t) \leq 0$ in $\bar{B}$.

3. – The emergence of interfacial curves.

We start with an investigation of interfacial phenomenon for *a continuous weak solution* u of equation (1.1) from the prescirbed cases under the assumption that this solution exists.

Consider a rectangle $P \subset Q_T$ ($P = (x_1, x_1 + 2\delta) \times (t_1, t_2)$ for some $\delta > 0$) and introduce the following cases with the special distribution of the phases of u on boundary of P:

Case (a). $u(x_1, t) > 0$, $u(y_1, t) < 0$ for $t \in [t_1, t_2]$, $y_1 = x_1 + 2\delta$; $u(x, t_1) > 0$ if $x \in [x_1, x_1 + \delta)$, $u(x, t_1) < 0$ if $x \in (x_1 + \delta, y_1]$.

Case (a_1). Between phases where $u(x, t_1) > 0$, and $u(x, t_1) < 0$ there exists *an interfacial layer region* $\Omega(t_1)$ *such that* $\Omega = [x_1 + \delta, y_1 - \delta/2]$;

Case (b). $u(x_1, t) > 0$, $u(y_1, t) > 0$ *for* $t \in [t_1, t_2]$; $u(x, t_1) > 0$ *if* $x \in [x_1, x_1 + \delta/2)$, $u(x, t_1) < 0$ *if* $x \in (x_1 + \delta/2, y_1 - \delta/2)$ *and* $u(x, t_1) > 0$ *if* $x \in (y_1 - \delta/2, y_1]$.

Case (c). $u(x_1, t) > 0$, $u(y_1, t) < 0$ *for* $t \in [t_1, t_2]$; $u(x, t_1) > 0$ *if* $x \in [x_1, x_1 + \delta)$, $u(x, t_2) < 0$ *if* $x \in (x_1 + \delta, y_1]$ *and* $u(x_1 + \delta, t_1) = u(x_1 + \delta, t_2) = 0$.

Case (c_1). *In this subcase there exists an interfacial layer region* $\Omega(t_2)$ on the line $t = t_2$ such that $\Omega(t_2) = [x_1 + \delta, y_1 - \delta/2]$.

Let us study the emergence of *an interfacial layer region* for each case. The emergence of the interfaces $\xi_+^{(a)}$, $\xi_-^{(a)}$ for Case (a) will be our initial aim. From conditions on $u(x_1, t)$, $u(y_1, t)$ follows that there exist functions $h^+(t)$, $h^-(t)$, such that $u(x, t) > 0$ if $x_1 \leq x < h^+(t)$, and $u(x, t) < 0$ if $h^-(t) < x \leq y_1$. The continuity of u implies that $h^+(t)$ is a lower semicontinuous function and $h^-(t)$ is an upper semicontinuous function. First we prove that they are continuous functions.

Let $\mathfrak{D} = \{(x, t) \in P\colon -1 < x < h^+(t)\}$. Since u continuous in Q_T, $\mathfrak{D}$ is an open subset of R^2. Taking into account the idea which was used in [17, 18, 20] we apply *the level-set approach* for an approximation of h^+, h^- by the level curves $\xi_+^{(n)}$, $\xi_-^{(n)}$ of u. Thus, we consider u on the open set $\mathfrak{D}$. *Sard's theorem* and Lemmas 1, 2 guarantee the existence of smooth level curves $\xi_+^{(j)}$ lying in $\mathfrak{D}$ and connecting points $(x_{\varepsilon_j}, t) \in \mathfrak{D}$ to some point (y_{ε_j}, t_1) of interval $(x_1, x_1 + \delta)$ along which $u(\xi_+^{(j)}(t), t) = \varepsilon_j$, $j = 1, 2, 3, \dots, \dots$. Here $\{\varepsilon_j\}$ are noncritical values of u with the following properties:

$$\lim_{j \to \infty} \varepsilon_j = 0, \qquad \varepsilon_{j+1} < \varepsilon_j, \qquad \varepsilon_1 < \min u(x_1, t),\ t \in [t_1, t_2].$$

The sequence $\{\xi_+^{(j)}\}$ is a monotonously increasing. Now consider a piecewise smooth loop L_{ε_j} bounded above by $t = \tau$, below by $t = t_1$ and laterally by $\xi_+^{(j)}, \xi_+^{(j+s)}$. It is easy to show that

$$\int_{L_j} u\,dx + 1/2\partial/\partial x(u)^2\,dt = 0. \tag{3.1}$$

Immediately from (3.1) follows

$$\int_{\xi_+^{(j)}(\tau)}^{\xi_+^{(j+s)}(\tau)} u(x, \tau)\,dx - \int_{\xi_+^{(j)}(t_1)}^{\xi_+^{(j+s)}(t_1)} u(x, t_1)\,dx + \varepsilon_j \int_{\xi_+^{(j)}} dx + u_x\,dt - \varepsilon_{j+s} \int_{\xi_+^{(j+s)}} dx + u_x\,dt.$$

or

$$\begin{aligned} \xi_+^{(j)}(\tau) - \xi_+^{(j)}(t_1) + \int_{t_1}^{\tau} u_x(\xi_+^{(j)}(t), t)\,dt + {}& \\ +\varepsilon_j^{-1} \left\{ \int_{\xi_+^{(j)}(\tau)}^{\xi_+^{(j+s)}(\tau)} u(x, \tau)\,dx - \int_{\xi_+^{(j)}(t_1)}^{\xi_+^{(j+s)}(t_1)} u(x, t_1)\,dx \right\} - {}& \\ -\varepsilon_{j+s} \times \varepsilon_j^{-1} \left\{ \int_{t_1}^{\tau} u_x(\xi_+^{(j+s)}(t), t)\,dt + \xi_+^{(j+s)}(\tau) - \xi_+^{(j)}(t_1) \right\} = 0&. \end{aligned} \tag{3.2}$$

Now we pass to the limit assuming $s \to \infty$ in (3.2) and obtain

$$\xi_+^{(j)}(\tau) - \xi_+^{(j)}(t_1) + \int_{t_1}^{\tau} u_x(\xi_+^{(j)}(t), t)\,dt + \varepsilon_j^{-1} \int_{\xi_+^{(j)}(\tau)}^{h_{(\tau)}^+} u(x, \tau)\,dx = \varepsilon_j^{-1} \int_{\xi_+^{(j)}(t_1)}^{t_1} u(x, t_1)\,dx. \tag{3.3}$$

We will prove that the derivatives $\xi_+^{(j)\prime}$ are uniformly bounded relatively to j. From

(3.3) it follows that

$$\xi_+^{(j)\prime}(\tau) + u_x(\xi_+^{(j)}(\tau), \tau)\,dt + \left\{\varepsilon_j^{-1} \int_{\xi_+^{(j)}(\tau)}^{h^+(\tau)} u(x, \tau)\,dx\right\}' = 0$$

where the last function is bounded. Indeed, take a set

$$\mathfrak{J} = \{(x, t) \in \mathfrak{D}\colon t_1 < \tau \leqslant t \leqslant \tau + \Delta\tau < t_2,\ \xi_+^{(j)}(t) \leqslant x \leqslant \xi_+^{(j+s)}(t)\}.$$

Then

$$\int_{\xi_+^{(j)}(\tau+\Delta\tau)}^{\xi_+^{(j+s)}(\tau+\Delta\tau)} u(x, \tau + \Delta\tau)\,dx - \int_{\xi_+^{(j)}(\tau)}^{\xi_+^{(j+s)}(\tau)} u(x, \tau)\,dx = \iint_{\mathfrak{J}} u_t\,dx\,dt =$$

$$= \int_{\tau}^{\tau+\Delta\tau} uu_x(\xi_+^{(j+s)}(t), t)\,dt - \int_{\tau}^{\tau+\Delta\tau} uu_x(\xi_+^{(j)}(t), t)\,dt,$$

passing to the limit when $s \to \infty$, and using the relation

$$\lim_{s\to\infty} uu_x(\xi_+^{(j+s)}(t), t) = 0. \tag{3.4}$$

We get

$$\left|\int_{\xi_+^{(j)}(\tau+\Delta\tau)}^{h^+(\tau+\Delta\tau)} u(x, \tau + \Delta\tau)\,dx - \int_{\xi_+^{(j)}(\tau)}^{h^+(\tau)} u(x, \tau)\,dx\right| \leqslant \Delta\tau\varepsilon_j K_P$$

where K_P is some positive number which in general depends on P. From here the above result follows.

Thus the Ascoli-Artsel theorem guarantees convergence of the family $\{\xi_+^{(j)}\}$ in the uniform norm to a certain continuous function $\xi_+^{(a)}$ which coincides with h^+. It is easy to verify that $h^+(t)$ is in fact a continuous function up to $t = t_2$. The proof of the continuity of h^- is similar to the proof of this property for h^+.

The function $\partial/\partial x(u)^2$ is continuous on $\bar{\mathfrak{D}}$. The continuity of $\partial/\partial x(u)^2$ up to h^+ is a consequence of the fact that $\partial/\partial x(u(\cdot, t))^2$ are equicontinuous in the interval $(-1, h^+)$ relatively to x which follows from (3.4) and $\partial/\partial x(u(h^+(t), t))^2 = 0$.

Denote by $\mathfrak{T}$ the set

$$\mathfrak{T} = \{(x, t) \in P\colon \xi_+^{(a)} \leqslant x \leqslant \xi_-^{(a)}\}.$$

Note that in a general case $\bar{\mathfrak{T}}$ is a triangle with curvilinear boundaries. Here $\xi_-^{(a)}$ is «continuous realization» of h^- by level lines of u. We show that $\mathfrak{T}$ is *an interfacial layer region* of u. Immediately from Lemma 4 it follows $u \leqslant 0$ on $\mathfrak{T}$. To prove that $\mathfrak{T}$ is *an interfacial layer region* we define u^- on $\{(x, t) \in P\colon \xi_+^{(a)}(t) < x < y_1,\ t_1 < t < t_2\}$

by a formula

$$u^-(x,t) = u(x,t) \quad \text{if } \xi_+^{(a)}(t) \leq x \leq y_1, \qquad t_1 \leq t \leq t_2.$$

Consider the transformation $\tilde{u}^-$ of the function u^-

$$\tilde{u}^-(x,t) = -u^-(x, t_2 + t_1 - t).$$

Then $\tilde{u}^-(x,t)$ is a solution of the problem:

$$\tilde{u}_t^- = (\tilde{u}^{-2})_{xx} \quad \text{on} \quad \{(x,t) \in P \colon \tilde{\xi}_+^{(a)}(t) < x < y_1,\, t_1 < t < t_2\}, \tag{3.5}$$

$$\tilde{u}^-(\tilde{\xi}_+^{(a)}(t), t) = 0, \qquad \tilde{u}^-(y_1, t) = -u^-(y_1, t_2 + t_1 - t),\, t_1 < t < t_2,$$

$$\tilde{u}^-(x, t_1) = -u^-(x, t_2), \tag{3.7}$$

where $\tilde{\xi}_+^{(a)}(t) = \xi_+^{(a)}(t_2 + t_1 - t)$. The proof of this statement is trivial. We understand a solution of (3.5)-(3.7) in the generalized sense as in [21] (the integral identity is understood as the fulfillment of it for every interior rectangle). We introduce the following notation. Let γ be an interval lying on $t = \tau$, $t_1 < \tau < t_2$ between the points $\tilde{\xi}_+^{(a)}(\tau)$, $\tilde{\xi}_-^{(a)}(\tau)$ ($\tilde{\xi}_-^{(a)}(\tau) = \xi_-^{(a)}(t_2 + t_1 - t)$) and $\tilde{u}^- > 0$ on γ. Then by Lemma 3 for nonnegative solutions of (3.5)-(3.7) we have $\tilde{u}^- > 0$ in $\gamma \times [\tau, t_2]$, that contradicts the equality $\tilde{u}^-(\xi_-^{(a)}(t), t) = 0$. Therefore the proof of the result on a coincidence of $\mathfrak{N}$ and $\mathfrak{T}$ is complete. Note that here we, generally speaking, can not cause $\mathfrak{T}$ to have a nonempty interior. In Case (a_1), in contrast to the previous situation we may ensure that indeed $\mathring{\mathfrak{T}} \neq \emptyset$ and the set $\Omega(t_1)$ expands with time. Thus we proved the following theorem.

THEOREM 1. – Consider the rectangle $P \subset Q_T$ ($P = (x_1, x_1 + 2\delta) \times (t_1, t_2)$ for some $\delta > 0$) and let u be *a continuous weak solution* of (1.1). Assume that hypotheses of Case (a) are satisfied. Then there are continuous functions $\xi_+^{(a)}(t)$, $\xi_-^{(a)}(t)$ defined on $[t_1, t_2]$ such that $x_1 < \xi_+^{(a)}(t) \leq \xi_-^{(a)}(t) < x_1 + 2\delta$ (see Fig. 1) and

$$\mathfrak{M}_+ = \{(x,t) \in \bar{P} \colon x_1 \leq x < \xi_+^{(a)}(t)\},$$

$$\mathfrak{M}_- = \{(x,t) \in \bar{P} \colon \xi_-^{(a)}(t) < x \leq x_1 + 2\delta\}.$$

Under hypothesis of Case (a_1), *an interfacial layer region* $\mathfrak{N}$ emerges; i.e. $\mathring{\mathfrak{N}} \neq \emptyset$ and (see Fig. 2)

$$\mathfrak{N} = \{(x,t) \in \bar{P} \colon x_1 < \xi_+^{(a_1)}(t) \leq x \leq \xi_-^{(a_1)}(t) < x_1 + 2\delta\}.$$

REMARK 3. – In fact the properties of interface $\xi_+^{(a)}$ ($\xi_-^{(a)}$) are the same that occur when we consider an interface for the porous medium equation.

Now consider Case (b). It is of interest to consider the situation about disappear-

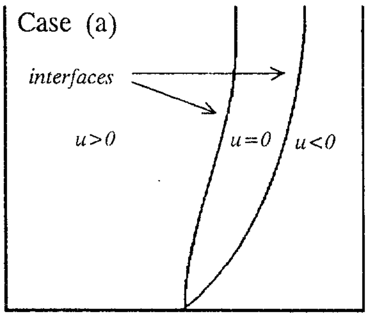

Figure 1

ance of *the phase* $\mathfrak{M}_-$ in P. By Lemma 4, we can easily settle this matter. We shall prove that *the backward parabolic region* i.e. the set $\mathfrak{M}_- \neq \emptyset$.

Assume the contrary, i.e. that $\mathfrak{M}_- \cap P \cap \{t = \sigma\} = \emptyset$ for some $t_1 < \sigma \leq t_2$. From Lemma 4 we have that $u \geq 0$ in $P_\sigma = P \cap \{t \leq \sigma\}$. The existence of such a rectangle contradicts the conditions of Case (b).

Let us investigate the distribution of *the phases* $\mathfrak{M}_+ \cap P$, $\mathfrak{M}_- \cap P$ on $t = \sigma$. It is clear to see that on $t = \sigma$, the set where $u(x, \sigma) > 0$ consists of only two disjoint intervals: $\mathfrak{I}_1$, initiating on $x = x_1$ and $\mathfrak{I}_2$, ending at $x = x_1 + 2\delta$; and the set $\mathfrak{M}_- \cap P \cap \{t = \sigma\}$ is connected for all $\sigma \in [t_1, t_2]$. The proof of this statement can be obtained by the same manner as Lemma 2 of [19] and Theorem 1.

THEOREM 2. – Let u be *a continuous weak solution* of (1.1). Assume that hypotheses of Case (b) are satisfied. Then the set $\mathfrak{M}_- \cap P \cap \{t = \sigma\}$ is not empty and consists of the connected component $\mathfrak{I}_0$, $t_1 \leq \sigma \leq t_2$, $\mathfrak{M}_+ \cap P \cap \{t = \sigma\} = \mathfrak{I}_1 \cup \mathfrak{I}_2$, where $\mathfrak{I}_0$, $\mathfrak{I}_1$, $\mathfrak{I}_2$ are some intervals from R^1 and what split them is interfacial layer region. Moreover, $\mathfrak{M}_+ \cap P = \{(x, t) \in P \colon x_1 \leq x < \xi_+^{(b)}(t)\} \cup \{(x, t) \in P \colon \zeta_+^{(b)} < x \leq x_1 + 2\delta\}$,

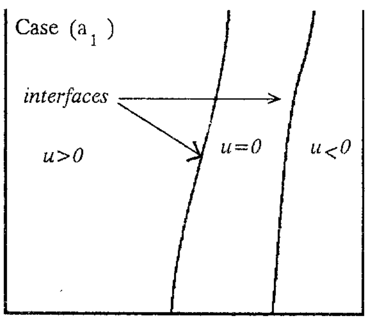

Figure 2

$\mathfrak{M}_- \cap P = \{(x, t) \in P: \xi_-^{(b)}(t) < x < \zeta_-^{(b)}(t)\}$ where $\xi_+^{(b)}$, $\xi_-^{(b)}$, $\zeta_+^{(b)}$, $\zeta_-^{(b)}$ are the interfacial curves such that $\xi_+^{(b)} \leqslant \xi_-^{(b)} < \zeta_-^{(b)} \leqslant \xi_+^{(b)}$.

Now, we study the interfacial phenomenon in Case (*c*). Using *the level-set approach* as above, we will see that $\mathfrak{M}_+ = \{(x, t) \in P: x_1 \leqslant x < x_1 + \delta\}$, $\mathfrak{M}_- = \{(x, t) \in P: x_1 + \delta < x \leqslant x_1 + 2\delta\}$. It follows from a fact that $\mathfrak{N}$ is a line connecting the points $(x_1 + \delta, t_1)$ and $(x_1 + \delta, t_2)$. We give briefly outline of proof of this statement. The details may be easily recovered.

Consider the rectangle P. By contradiction, assume that $\mathfrak{N} \cap \{(x, t) \in P: x < x_1 + \delta\} \neq \emptyset$. Fix some $\bar{x}$, from the interval $(x_1, x_1 + 2\delta]$ lying on the line $\{t = t_1\}$ such that, the value $t_{\bar{x}} = \tau(\bar{x}) = \inf\{t \in [t_1, t_2]: u(\bar{x}, t) = 0, (x, t) \in P\}$ is defined. Let us denote by

$$\mathfrak{X} = \left\{x \in P \cap \{t = t_1\} \text{ and } \{x_1 < x \leqslant x_1 + \delta\}: \exists t_x \in [t_1, t_2], t_x = \tau(x)\right\}.$$

We may assume that $\mathfrak{X}$ is maximal $\mathfrak{X}$-set. In view of Lemma 21 it is easy to verify that the function $\tau = \tau(x)$ is decreasing on $\mathfrak{X}$. Let $\bar{x}, \bar{y} \in \mathfrak{X}$. Let

$$\mathfrak{I}_{\bar{x}} = \{(x, t_{\bar{x}}) \in P: t_{\bar{x}} = \tau(x), x_1 < x < x\},$$

$$\mathfrak{I}_{\bar{y}} = \{(x, t_{\bar{y}}) \in P: t_{\bar{y}} = \tau(\bar{y}), x_1 < x < \bar{y}, \bar{y} > \bar{x}\}.$$

Rescaling as above in Case (*a*) we can construct the level curves $\xi_+^{(i)}(t)$, connecting points $(x_{\varepsilon_i}, i_{\bar{x}}) \in \mathfrak{I}_{\bar{x}}$ to $(y_{\varepsilon_i}, i_{\bar{y}}) \in \mathfrak{I}_{\bar{y}}$ along which $u(\xi_+^{(i)}(t), t) = \varepsilon_i$, $i = 1, 2, 3, \ldots, \ldots$ for an approximation of $\tau(x)$ on $\mathfrak{X} \cap [\bar{x}, \bar{y}]$. Here $\{\varepsilon_i\}$ are noncritical values of u with the following properties:

$$\lim_{i \to \infty} \varepsilon_i = 0, \qquad \varepsilon_{i+1} < \varepsilon_i, \qquad \varepsilon_1 < \min u(x_1, t), \qquad t \in [t_{\bar{y}}, t_{\bar{x}}].$$

By Lemmas 3, 4, $u(x, t) > 0$ on $\{(x, t) \in P;\ x_1 \leqslant x < \xi_+^{(i)}(t), \tau(\bar{y}) \leqslant t \leqslant \tau(\bar{x})\}$. Now let $i \to \infty$, We will find

$$\xi_+^{(c)}(t) = \lim_{i \to \infty} \xi_+^{(i)}(t) \quad \text{for} \quad t \in [t_{\bar{y}}, t_{\bar{x}}],$$

such that $\xi_+^{(c)}(t_{\bar{x}}) = \bar{x}$, $\xi_+^{(c)}(t_{\bar{y}}) = \bar{y}$. then, by the same argument as in Case (*a*) we claim: first, the front curve $\xi_+^{(c)}$ is a continuous curve; second, the function

$$u^{+(c)}(x, t) = \begin{cases} u(x, t) & \text{for } x_1 \leqslant x < \xi_+^{(c)}(t), \\ 0 & \text{for } \xi_+^{(c)}(t) < x < \infty, \qquad t \in [t_{\bar{y}}, t_{\bar{x}}], \end{cases}$$

is a nonnegative generalized solution (as in [21] of the Cauchy problem in $R \times (t_{\bar{y}}, t_{\bar{x}})$ for an equation (1.1). We recall that the function $\partial/\partial x(u^{+(c)})^2$ is continuous through the curve $\xi_+^{(c)}$. This last curve is (as we can see by standard interfacial result for nonnegative solutions of the porous medium equation) nondecreasing on $[t_{\bar{y}}, t_{\bar{x}}]$. In view of this, subset $\mathfrak{N}$-type in $\{(x, t) \in P: x_1 \leqslant x < x_1 + \delta\}$ does not exist.

In particular, it follows that $\xi_+^{(c)}$ is nondecreasing function. In order to complete

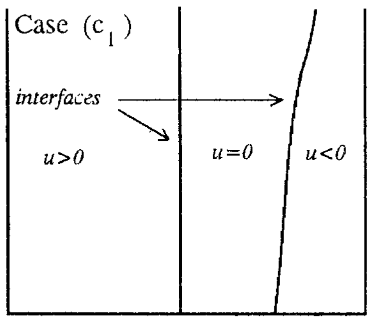

Figure 3

the Case (c) it remains to show that

(3.8) $$\mathfrak{N} \cap \{(x, t) \in P \colon x > x_1 + \delta\} = \emptyset .$$

The proof of (3.8) is the same as above.

The previous consideration obviously may be extended to Case (c_1). By the same manner, we will obtain that the set $\mathfrak{N} \cap \{(x, t) \in P \colon x < x_1 + \delta\}$ is empty. From here it follows that $u(x, t_2) > 0$ if $x_1 \leqslant x < x_1 + \delta$. This means that we can proceed as in Case (a_1) if the direction of the time t is reversed. In other words, if we consider particles which move along level curves $\xi_-^{(s)}$ of u then $(\xi_-^{(s)}(t_2), t_2)$ is the final position of the particles which occupied an initial positions $(\xi_-^{(s)}(t), t) \in P$, $s = 1, 2, 3, \ldots$, whereas the particles corresponding to this description for the level curves $\xi_+^{(s)}(t)$ have the final position at the time $t = t_1$. Thus we have proved (we leave the details to the reader) for Case (c_1) that (see Fig. 3)

$$\mathfrak{M}_+ \cap P = \{(x, t) \in P \colon x_1 \leqslant x < \xi_+^{(c_1)}(t),\ t_1 \leqslant t \leqslant t_2\},$$

$$\mathfrak{M}_- \cap P = \{(x, t) \in P \colon \xi_-^{(c_1)}(t) < x \leqslant y_1,\ t_1 \leqslant t \leqslant t_2\},$$

$$\mathfrak{N} \cap P = \{(x, t) \in P \colon \xi_+^{(c_1)}(t) \leqslant x \leqslant \xi_-^{(c_1)}(t),\ t_1 \leqslant t \leqslant t_2\}.$$

Here $\xi_+^{(c_1)}$, $\xi_-^{(c_1)}$ represent the interface curves which divide P into $\mathfrak{M}_+$, $\mathfrak{M}_-$, $\mathfrak{N}$ and interfacial layer region $\mathfrak{N}$ for this case has a positive measure (the phase fron $\xi_+^{(c_1)}$ becomes straight segment connecting $(x_1 + \delta, t_1)$ to $(x_1 + \delta, t_2)$.

4. – Solutions of the one-dimensional problems.

In the spirit of the preceding chapter let us look for solutions of Problem 1 and Problem 2.

Problem 1

$$\begin{cases} u_t = uu_{xx} + u_x^2, & (x, t) \in Q_T, \\ u(x, 0) = u_1(x), \quad -\infty < x \leqslant 0, & u(x, T) = u_2(x), \quad 0 \leqslant x < \infty, \end{cases}$$

where as in [1, Ch. 3] we demand that $u_1(x)$, $u_2(x)$ satisfy the following assumptions:

$$\text{(C}_1\text{)} \qquad \begin{cases} u_1(x) \geqslant 0 & \text{if } -\infty < x \leqslant 0, \\ u_2(x) \leqslant 0 & \text{if } 0 \leqslant x < \infty, \\ u_1(0) = u_2(0) = 0. \end{cases}$$

Problem 2

$$\begin{cases} u_t = uu_{xx} + u_x^2, & (x, t) \in Q_T, \\ u(x, 0) = u_0(x), & -\infty < x < \infty, \end{cases}$$

where $u_0(x)$ is a given function, satisfying the assumptions:

$$\text{(C}_2\text{)} \qquad \begin{cases} u_0(x) \geqslant 0 & \text{if } -\infty < x \leqslant 0, \\ u_0(x) \leqslant 0 & \text{if } 0 \leqslant x < \infty, \\ u_0(0) = 0. \end{cases}$$

We make the hypothesis:

(H) u_0, u_1, u_2 are assumed continuous bounded functions.

We define what we mean by solutions of these problems.

A continuous weak solution of (1.1) *is a continuous weak solution of Problem* 1 (*Problem* 2), *if* $u(x, t)$ *is continuous in* $\bar{Q}_T$ *and takes the value* $u_1(x)$, $u_2(x)$ *at the time* $t = 0$ *and* $t = T$ (*accordingly for Problem* 2, $u(x, 0) = u_0(x)$).

Problem 1 has been studied in [2] where we showed that *a continuous weak solution* of Problem 1 can be decomposed into two generalized solutions of the corresponding evolution problems. As for an interfacial phenomenon then in particular, if $u_1(x) > 0$ for $-\infty < x < 0$, $u_2(x) < 0$ for $0 < x < \infty$ and $u_1(0) = u_2(0) = 0$, their interfacial curves become the straight segment connectin $(0, 0)$ to $(0, T)$. For this reason the functions u_1, u_2 must possess the suitable behavior near zero i.e.

$$\text{(4.1)} \qquad \begin{cases} A^-(u_1, [x, 0)) = \limsup\limits_{x \to -0} |x|^{-3} \int\limits_x^0 u_1(x)\, dx < \infty, \\[2ex] A^+(|u_2|, (0, x]) = \limsup\limits_{x \to +0} |x|^{-3} \int\limits_0^x |u_2(x)|\, dx < \infty. \end{cases}$$

In the question about solvability (in the class of *the continuous weak solutions*) of Problem 1, the above consideration leads us to assumptions about local effects (4.1). The precise estimates for T may be represented. For the details we refer the reader to [2].

By using results of chap. 3 to Problem 2 it is not difficult to prove that the interfacial phenomenon also continues to occur. Let us consider the following distribution of the values $u_0(x)$ (analogously to Cases (a), (a_1)):

$$(\mathrm{C}_1') \quad \begin{array}{ll} u_0(x) > 0 & \text{if } -\infty < x < 0, \; u_0(0) = 0, \\ u_0(x) < 0 & \text{if } 0 < x < \infty, \end{array}$$

$$(\mathrm{C}_2'') \quad \begin{array}{ll} u_0(x) > 0 & \text{if } -\infty < x < 0, \; u_0(0) = 0, \\ u_0(x) < 0 & \text{if } \nu < x < \infty, \\ u_0(0) \equiv 0 & \text{if } 0 < x \leqslant \nu. \end{array}$$

Obviously, the structure of the phases $\mathfrak{M}_+$, $\mathfrak{M}_-$, $\mathfrak{N}$ is the same kind for a short time as described above in the Cases (a), (a_1). Indeed, we can find a rectangle E located in a narrow strip Q_{T^*} near the line $t = 0$ with similar conditions on u as in the Cases (a), (a_1). Note that the spontaneous appearance of the interfaces in $Q_{T^*} \setminus E$ is eliminated in view of Lemma 3 and the discussion accompanying the function $\tau(x)$ in Chap. 3. Therefore there exists interfaces ξ_+, ξ_- such that they are nondecreasing continuous functions on $[0, T^*]$ dividing Q_{T^*} into $\mathfrak{M}_\pm$, $\mathfrak{N}$. Denote by

$$u^+(x, t) = \begin{cases} u(x, t) & \text{for } (\mathrm{x}, \mathrm{t}) \in Q_T^* \cap \{\mathrm{x} < \xi^+(\mathrm{t})\}, \\ 0 & \text{for } (x, t) \in Q_T^* \cap \{x \geqslant \xi^+(t)\}, \end{cases}$$

where $u(x, t)$ is *a continuous weak solution* of P2 and

$$u^-(x, t) = \begin{cases} u(x, t) & \text{for } (\mathrm{x}, \mathrm{t}) \in Q_T^* \cap \{\mathrm{x} > \xi^-(\mathrm{t})\}, \\ 0 & \text{for } (x, t) \in Q_T^* \cap \{x \leqslant \xi^-(t)\}. \end{cases}$$

It has already been noted in Chap. 3 that the function $\partial/\partial x(u^2)$ is continuous across *the interfacial curves*. From the above remark it follows that u^+ is a generalized solution of the Cauchy problem on which the equation is parabolic (accordingly, u^- is a generalized solution of the Cauchy problem on which equation (1.1) is backward parabolic). So we have

THEOREM 3. – Let u be *a continuous weak solution* of P2 under assumptions (C_1') (C_2''). Then near $t = 0$ for u it holds the following form

$$u = u^+ + u^-,$$

with the functions specified above.

Now it would be interesting to discuss the question about an existence of *a continuous weak solution* of P2. To find this solution, our strategy will be concentrated on the solutions having the form of $u = u^+ + u^-$. We illustrate this design by two example.

I) *Interface dynamics with stationary interfaces.*

The existence of a generalized solution u^+ with initial data

$$u^+(x, 0) \equiv u_0^+(x) = \begin{cases} u_0(x), & \text{for } x < 0, \\ 0, & \text{for } x \geqslant 0 \end{cases}$$

under the hypothesis (H) is well-known. In case of the assumptions from (C_1') on $u_0(x)$ for $x \leqslant 0$, this solution has interface beginning at $x = 0$ which is stationary for a positive time $t(u_0^+)$, so-called waiting time, if and only if, (see [3]) $A^-(u_{0+}, [x, 0)) < \infty$. In particular, $0 < \{A^-(u_0^+, [x, 0))\}^{-1} \leqslant \mu t(u_0^+) \leqslant \mu\{A^-(u_0^+, [x, 0))\}^{-1}$ with some constant $\mu > 1$ independent of u_0^+. As for the existence of function u^- we take the initial data u_0^- from $\{\gamma_\tau(w_\theta)$: $a \leqslant w_\theta \leqslant 0, w_\theta(x) < 0$ for $x > 0$, $|a| < \infty, w_\theta(x) \equiv 0$ for $x \leqslant 0$, $w_\theta \in C(R)\}|_{t=0}$; here $\gamma_\tau(w_\theta)$ is a semiorbit of a generalized solution $w = w(t, w_\theta)$ of the Cauchy problem for (1.1) (in the direction of decreasing time) emanating from w_θ at the time $t = \theta$. For each $\tau < \theta$ the $\gamma_\tau(w_\theta)$ is defined by $\gamma_\tau(w_\theta) = \{w(t, w_\theta)$: $t \leqslant \tau\}$. In view of the same reason as above, we consider the transformation $\tilde{w}$ of w. Under the condition $A^+(\tilde{w}(x, 0), (0, x]) < \infty$ there exists a positive waiting time $t(\tilde{w}(x, 0))$ such that $0 < \{A^+(\tilde{w}(x, t), (0, x])\}^{-1} \leqslant \mu t(u_0^+) \leqslant \mu\{A^+(\tilde{w}(x, t), (0, x])\}^{-1}$. So if $\theta \leqslant \min(t(\tilde{w}(x, 0)), t(u_0^+))$ then

$$u(x, t) = \begin{cases} u^+(x, t) & \text{for } x \leqslant 0, \ 0 \leqslant t \leqslant \theta, \\ u^-(x, t) & \text{for } x \geqslant 0, \end{cases}$$

(where $u^- \in \{\gamma_\tau(w_\theta)\}$) is *a continuous weak solution* of P2 with stationary interface (the integral identity for $u$can be readily verified). To obtain more detailed information abvout u we consider (as above) the function

$$\tilde{u}(x, t) = \begin{cases} -u^-(x, \theta - t) & \text{for } (x, t) \in Q_\theta, \ x > 0, \\ 0 & \text{for } (x, t) \in Q_\theta, \ x \leqslant 0. \end{cases}$$

According to the well-known results for the porous medium equation (see, for instance, [22]), $\{\gamma_\tau(\tilde{u}_0)$: $0 \leqslant \tilde{u}_0 \leqslant \beta, \beta < \infty, \tilde{u}_0 \in C(R)\}$ is a precompact subset of $C(R)$. This shows that $u_0(x)$ for $x \geqslant 0$ cannot be an arbitrary continuous function on R^+. Moreover, for $\tilde{u}(x, t)$ near the line $\{x = 0\}$ (see Lemma 2.1, [23]) it holds $A^+(\tilde{u}(x, t), (0, x]) \leqslant K_{\tilde{u}_0}(1 - t/t_{A^*})^{-1}$, $0 \leqslant t \leqslant \theta$, where t_{A^*} is quantity depending on $A^+(\tilde{u}(x, t), (0, x])$ such that $t_{A^*} \geqslant \theta$ with some finite constant $K_{\tilde{u}_0}$ depending only on $\tilde{u}_0$ or, roughly speaking, $\tilde{u}(x, t)$ behaves like $ax^2(1 - t/t_{A^*})^{-1} + o(x^2)$ as $x \to +0$. The

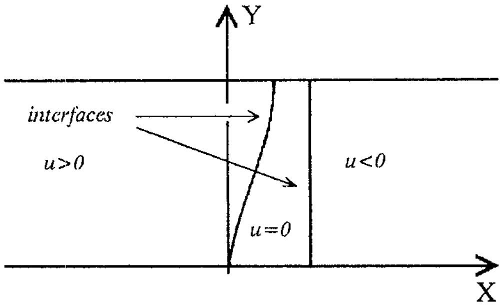

Figure 4

same representation is to be true for u^+ near $\{x = 0\}$. In view of that, it follows that the behavior of $u_0(x)$ near $\{x = 0\}$ must be specified.

II) *Interface dynamics with an interfacial layer region.*

Assume that the initial state u_0 satisfies (C_2''). Then the appearance of the positive phase is a result of the existence of a generalized solution u^+ with initial data satisfying the hypothesis (H). Using the methods, of I), we find that the negative phase occupies the set $\{(x, t) \in Q_{\theta_1} : \nu \geq x < \infty\}$, where θ_1 is a waiting time (see Fig. 4). Therefore we can conclude that there exist phases $\mathfrak{M}_+$, $\mathfrak{M}_-$, generated by u^+ and u^-, with *an interfacial layer region* $\mathfrak{R}$ separating them such that these sets lie within some strip situated along the line $\{t = 0\}$. Note that the location of phase fronts will depend on the initial state $u_0(x)$.

In summary, we formulate the following result.

THEOREM 4. – Let u_0 be the function given on $(-\infty, 0]$ which satisfies hypothesis (H) and the relevant assumptions from (C_2''). Then there exists an extension of u_0 on $(0, \infty)$, satisfying hypothesis (H) and the rest assumptions from (C_2''), such that u_0 generates *a continuous weak solution* of the Cauchy problem for (1.1) on some interval time.

REMARK 3. – As is evident from the foregoing the function $u_0|_{R^+}$ must be from some precompact subset of the space C of continuous functions on R^+ with the compact-convergence topology.

Acknowledgements. The author is grateful to Prof. R. R. AKHMEROV who pointed out related references in Introduction and the International Science Foundation for financial support.

REFERENCES

[1] J. L. LIONS, *Quelques méthodes de resolution des problèmes aux limites non linéaires*, Paris (1968).

[2] V. N. GREBENEV, *On a solvability of non-standard problems for nonlinear parabolic equations with variable time direction*, Ph.D. Thesis, Novosibirsk State University (1987).

[3] J. L. VAZQUEZ, *The interfaces of one-dimensional flows in porous media*, Trans. Amer. Math. Soc., 285 (1984), pp. 717-737.

[4] K. HOLLIG, *Existence of infinitely many solutions for a forward-backward heat equation*, Trans. Amer. Math. Soc., 278 (1983), pp. 299-316.

[5] K. HOLLIG - J. A. NOHEL, *A diffusion equation with a nonmonotone constitutive function*, Nonlinear Partial Differential Equations, Dordrecht (1983), pp. 409-422.

[6] R. R. AKHMEROV, *On structure of a set of solutions of Dirichlet boundary value problem for stationary one-dimensional forward-backward parabolic equation*, Nonlinear Analysis, TMA, **11** (1987), pp. 1303-1316.

[7] R. J. DIPERNA, *Measure-valued solutions to conservation laws*, Arch. Ration. Mech. Anal., 88 (1985), No. 3, p. 1985.

[8] M. SLEMROD, *Measure-valued solutions to a backward-forward heat equation: a conference report*, in *Nonlinear Evolution Equations that Change Type*, The IMA Volumes in Math. and Appl., No. 27, pp. 232-242, Springer-Verlag, New York (1990).

[9] P. I. PLOTNIKOV, *Forward-backward parabolic equations and hysteresis*, Russian Math. Dokl., **330** (1993), pp. 691-694.

[10] ALAN V. LAIR, *Uniqueness for a forward backward diffusion equation*, Trans. Amer. Math. Soc., **291** (1985), pp. 311-317.

[11] ALAN V. LAIR, *Uniqueness for a forward backward diffusion equation with smooth constitutive function*, Applicable Anal., **29** (1988), pp. 177-189.

[12] R. R. AKHMEROV, *On periodic travelling waves of equations with viscosity coefficient of variable sign*, Nonlinear Analysis, TMA, **13** (19)89, pp. 803-817.

[13] J. W. CAHN, *On the spinodal decomposition*, Acta Metall., **9** (1961), pp. 795-801.

[14] C. M. ELLIOTT, *The Cahn-Hilliard model for the kinetics of phase separation*, in *Mathematical Models for Phase Change Problems*, in Int. Series of Numer. Math., No. 88, pp. 35-73, Birkhauser Verlag, Basel (1989).

[15] N. N. YANENKO - V. A. NOVIKOV, *On certain model of a fluid with viscosity of variable sign*, Chisl. Met. Mekh. Sploshn. Sredy, **11** (1973), pp. 142-147, Novosibirsk.

[16] O. A. LADYZEWNSKAJA - V. A. SOLONIKOV - N. N. URALCEVA, *Linear and Quasilinear Equations of Parabolic Type*, Trans. Math. Monogr., **23**, A.M.S., Rhode Island (1986).

[17] B. F. KNERR, *The porous medium equation in one dimension*, Trans. Amer. Math. Soc., **234** (1977), pp. 381-403.

[18] A. S. KALASHNIKOV, *On the occurence of singularities in the solutions of the equation of nonstationary filtration*, Z. Vycisl. Mat. i Mat. Fiz., **7** (1967), pp. 440-444.

[19] A. FRIEDMAN, *One dimensional Stefan problems with nonmonotone free boundary*, Trans. amer. Math. Soc., **133** (1968), pp. 89-114.

[20] M. BERTSCH - D. HILHORST, *The interface between regions where* $u > 0$ *and* $u < 0$ *in the porous medium equation*, Preprint Univer. de Paris-Sud Math., **10** (1989), pp. 1-25.

[21] A. S. KALASHNIKOV, *The effect of absorption in a medium in which thermal conductivity depends on temperature*, USSR comp. Math. and Math. Math. Phiz., **16:3** (1976), pp. 141-149.

[22] D. ARONSON - M. G. CRANDALL - L. A. PALTRIER, *Stabilization of solutions of a degenerate nonlinear diffusion problem*, Nonlinear analysis, TMA, **6** (1982), pp. 1001-1022.

[23] D. ARONSON - L. A. CAFFARELLI - J. L. VAZQUEZ, *Interfaces with a corner point in one-dimensional porous medium flow*, Commun. Pure Appl. Math., **38**, pp. 375-404.