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Sobolev Spaces Associated to a Polyhedron and Fourier Integral Operations in $\boldsymbol{R}^n$ (*).

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Abstract. – *The theory of the Sobolev spaces $H^m_{\mathcal{P}}(\boldsymbol{R}^n)$ ($m \in \boldsymbol{R}$, $\mathcal{P}$ polyhedron in $\boldsymbol{R}^{2n}$) of* [BG] *is revisited here in the frame of new classes of pseudodifferential operators related to the same polyhedron $\mathcal{P}$. These operators generalize to corresponding classes of Fourier integral operators, for which we present the main lines of a symbolic calculus and results of continuity on the $H^m_{\mathcal{P}}(\boldsymbol{R}^n)$ spaces.*

0. – Introduction.

Let $P = \sum a_\gamma D^\gamma$ be a general partial differential operator with constant coefficients; consider its symbol $p(\zeta) = \sum a_\gamma \zeta^\gamma$ and the related characteristic polyhedron $\mathcal{P}$, that is the convex hull of the set $\{\gamma\colon a_\gamma \neq 0\} \cup \{0\}$. The polynomial $p(\zeta)$, as well as the operator P, is said to be multi-quasi-elliptic if the polyhedron $\mathcal{P}$ is complete and $w_{\mathcal{P}}(\zeta) \leq C|p(\zeta)|$ where $w_{\mathcal{P}}(\zeta) = \sqrt{\sum_{\gamma \in \mathcal{P}} \zeta^{2\gamma}}$; cf. the next Section 1 for more detailed definitions.

Operators with constant coefficients of multi-quasi-elliptic type have been studied by FRIBERG [F], CATTABRIGA [CT1], [CT2], PINI [P] and, in the frame of Gevrey classes, by ZANGHIRATI [Z] and CORLI [C].

With respect to these works in BOGGIATTO [BG] «half» of the dual variables in the symbol $p(\zeta)$ are «turned» into x variables. More precisely, considering only even dimensions $d = 2n$, $\zeta = (x, \xi) \in \boldsymbol{R}^n_x \times \boldsymbol{R}^n_\xi$, $\gamma = (\alpha, \beta)$, to the multi-quasi-elliptic polynomial

$$p(\zeta) = \sum_{(\alpha, \beta) \in \mathcal{P}} a_{(\alpha, \beta)} x^\alpha \xi^\beta$$

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we associate

$$P = \sum_{(\alpha, \beta) \in \mathcal{P}} a_{(\alpha, \beta)} x^\alpha D_x^\beta .$$

The operators defined in this way include for instance the Schrödinger operator $\Delta_x + V(x)$ when the potential $V(x)$ is assumed to be a negative multi-quasi-elliptic polynomial in the x variables, a representative example in $\boldsymbol{R}^2$ being:

$$V(x_1, x_2) = -(x_1^6 + x_1^4 x_2^4 + x_2^6).$$

In [BG] parametrices for operators in $\boldsymbol{R}^n$ of the preceding form are constructed in the frame of the classes $H\Gamma_\varrho^{m, m_0}(\boldsymbol{R}^{2n})$ of SHUBIN [S]. Moreover, weighted Sobolev spaces, denoted by $H_{\mathcal{P}}^m(\boldsymbol{R}^n)$, are associated to the polyhedron $\mathcal{P}$; in particular $u \in H_{\mathcal{P}}(\boldsymbol{R}^n) = H_{\mathcal{P}}^1(\boldsymbol{R}^n)$ means that

$$\|u\|_{H_{\mathcal{P}}(\boldsymbol{R}^n)} = \sum_{(\alpha, \beta) \in \mathcal{P}} \|x^\alpha D^\beta u\|_{L^2(\boldsymbol{R}^n)} < \infty .$$

As shown in [BG], an operator with polynomial coefficients P, with multi-quasi-elliptic symbol $p(\zeta)$, $\zeta = (x, \xi)$, is Fredholm as a map

$$P : H_{\mathcal{P}}^m(\boldsymbol{R}^n) \to H_{\mathcal{P}}^{m-1}(\boldsymbol{R}^n),$$

in particular

$$P : H_{\mathcal{P}}(\boldsymbol{R}^n) \to L^2(\boldsymbol{R}^n).$$

The first aim of the present paper is to revisit the theory of the Sobolev spaces $H_{\mathcal{P}}^m(\boldsymbol{R}^n)$ making use of new classes of hypoelliptic pseudodifferential operators on $\boldsymbol{R}^n$, whose symbol $p(\zeta)$, $\zeta = (x, \xi)$, satisfies estimates of the type:

$$|\partial^\gamma p(\zeta)| \leq C_\gamma w_{\mathcal{P}}^{m - \varrho|\gamma|}(\zeta), \qquad \zeta \in \boldsymbol{R}^{2n}$$

for some $m \in \boldsymbol{R}$, $\varrho > 0$.

We denote these classes by $\Lambda_{\varrho, \mathcal{P}}^m(\boldsymbol{R}^{2n})$ and by $H\Lambda_{\varrho, \mathcal{P}}^m(\boldsymbol{R}^{2n})$ the corresponding hypoelliptic classes where we require $w_{\mathcal{P}}^m(\zeta) \leq C|p(\zeta)|$.

In Sections 1 and 2 we study the related pseudodifferential operators in $H\mathrm{L}_{\varrho, \mathcal{P}}^m(\boldsymbol{R}^n)$ and, in terms of them, we re-define the spaces $H_{\mathcal{P}}^m(\boldsymbol{R}^n)$, following a general idea of BEALS [B] (see also BONY-CHEMIN [BC]).

In this first part of the paper we do not go into great detail as it is just a rearrangement of the results of [BG] into a modified symbols setting.

This new setting leads, in the second part of the paper, to a natural generalization of the $H\mathrm{L}_{\varrho, \mathcal{P}}^m(\boldsymbol{R}^n)$ pseudodifferential operators to Fourier integral operators of the form

$$Au(x) = (2\pi)^{-n} \int_{\boldsymbol{R}^n} e^{i\phi(x, \xi)} a(x, \xi) \hat{u}(\xi) \, d\xi ,$$

with amplitude $a(\zeta) \in \Lambda_{\varrho, \mathcal{P}}^m(\boldsymbol{R}^{2n})$ and a real valued phase function satisfying $\partial_\zeta^\gamma \phi(\zeta) \in \Lambda_{\varrho, \mathcal{P}}^0(\boldsymbol{R}^{2n})$ for $|\gamma| = 2$.

In Sections 3 and 4 we give a formula for the composition of a Fourier integral operator with a pseudodifferential operator and results of continuity in $S(\boldsymbol{R}^n)$ and $S'(\boldsymbol{R}^n)$. These results are analogous to those of other classes of Fourier integral operators: see for instance HELFFER [H], HELFFER-ROBERT [HRB] for the case when $w_{\mathcal{P}}(\zeta) \sim |\zeta|$, PARENTI-SEGALA [PS], RODINO [R], LIESS-RODINO [LR] for non homogeneous settings in the local frame, MOHAMED [M] for more general amplitudes under restrictive assumptions on the phase function. We do not know whether our results can be extended to the general case, when $w_{\mathcal{P}}(\zeta)$ is replaced by an arbitrary weight function of BEALS [B], HÖRMANDER [HÖ], globally defined in $\boldsymbol{R}^n$. In fact, in our proof we use the peculiar property that $w_{\mathcal{P}}^2(\zeta)$ is a polynomial.

In Section 5, finally, we prove the continuity of our Fourier integral operators with respect to the $H_{\mathcal{P}}^m(\boldsymbol{R}^n)$ spaces.

In conclusion we remark that the results of Section 3 and 4 provide the necessary analytical tools for the study of the spectral properties of certain operators arising from quantum mechanics; see the above-mentioned Schrödinger operator. These applications will be detailed considered in a future paper.

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1. - Definitions.

Let $\{s^1, \ldots, s^k\}$ be k vectors in $\boldsymbol{R}^d$ $(d > 1)$.

The convex hull of the set $\{s^1, \ldots, s^k\}$ is said «convex polyhedron» generated by $\{s^1, \ldots, s^k\}$ and the k vectors are said «vertices» if they are convex-linearly indipendent.

Let P_d be the family of the convex polyhedrons $\mathcal{P}$ such that:

(1.1) — $\mathcal{P} \subset \boldsymbol{R}_+^d = \{s \in \boldsymbol{R}^d : s_j \geq 0 \text{ for } j = 1, \ldots, d\}$;

(1.2) — $\mathcal{P}$ has dimension $d = \dim \boldsymbol{R}^d$;

— if s^i $(i = 0, \ldots, N(\mathcal{P}))$ are the vertices of $\mathcal{P}$ then:

(1.3) $\qquad Q(s^i) = \{r \in \boldsymbol{R}^d : 0 \leq r \leq s^i\} \subset \mathcal{P}$

where $r \leq s^i$ means that $r_j \leq s_j^i$ $(j = 1, \ldots, d)$; we set also $s^0 = 0$.

For every $\mathcal{P} \in P_d$ there exists a non empty finite set $A(\mathcal{P}) \subset \boldsymbol{R}_+^d - \{0\}$ such that:

$$\mathcal{P} = \bigcap_{a \in A(\mathcal{P})} \{s \in \boldsymbol{R}_+^d : \langle a, s\rangle \leq 1\}$$

$\left(\text{where } \langle a, s\rangle = \sum_{j=1}^{d} a_j s_j\right)$.

Let $F_a(\mathcal{P}) = \{s \in \mathcal{P} : \langle a, s\rangle = 1\}$, $F(\mathcal{P}) = \bigcup_{a \in A(\mathcal{P})} F_a(\mathcal{P})$. A polyhedron $\mathcal{P} \in P_d$ is said to be «complete» if for every $r, s \in \mathbf{R}^d$ with $s \in \mathcal{P}$ and $0 \leq r < s$ we have $r \in \mathcal{P} - F(\mathcal{P})$. That means that the polyhedron has no faces parallel to the coordinate hyperplanes, i.e. $a_j > 0$ $(j = 1, \ldots, d)$ for every $a \in A(\mathcal{P})$,

For $p(\zeta) = \sum a_\alpha \zeta^\alpha$; $\zeta \in \mathbf{R}^d$; $a_\alpha \in \mathbf{C}$; $\alpha \in \mathbf{N}_0^d$ $(\mathbf{N}_0 = \{0, 1, 2, \ldots\})$, the convex hull of the set $\{0\} \cup \{\alpha \in \mathbf{N}_0^d \colon a_\alpha \neq 0\}$ is said «characteristic polyhedron» of $p(\zeta)$.

Since we are interested essentially in characteristic polyhedrons of polynomials, we shall consider from now on only polyhedrons with integer vertices also when we won't state it explicitly.

A polynomial $p(\zeta)$ is said «multi-quasi-elliptic» if its characteristic polyhedron belongs to P_d, is complete and if exist two constants $C, R > 0$ such that

$$\sum_{\gamma \in V(\mathcal{P})} |\zeta^\gamma| \leq C |p(\zeta)|, \qquad |\zeta| \geq R,$$

where $V(\mathcal{P})$ is the set of the vertices of $\mathcal{P}$. For references see FRIBERG [F], PINI [P], ZANGHIRATI [Z], CORLI [C], CATTABRIGA [CT1], [CT2].

According to [BG], in the preceding estimate we may replace $\sum_{\gamma \in V(\mathcal{P})} |\zeta^\gamma|$ with the «weight function»:

$$w_{\mathcal{P}}(\zeta) = \sqrt{\sum_{\gamma \in \mathcal{P}} \zeta^{2\gamma}}\,.$$

In [BG] we showed that $w_{\mathcal{P}}(\zeta)$ belongs to one of the Shubin classes (see [S]) and we used the corresponding pseudodifferential operators to define the Sobolev spaces $H^m_{\mathcal{P}}(\mathbf{R}^n)$.

Here we prefer to define symbol classes that are directly «shaped» on the weight function $w_{\mathcal{P}}(\zeta)$ and are therefore more suitable to develop the theory of pseudodifferential and Fourier integral operators we are interested in. They can be regarded as a particular case of the general classes of BEALS [B], HÖRMANDER [HÖ].

DEFINITION 1.1. – For $m \in \mathbf{R}$, $\varrho \in]0, 1]$ we denote with $\Lambda^m_{\varrho, \mathcal{P}}(\mathbf{R}^d)$ the set of the functions $a(\zeta) \in C^\infty(\mathbf{R}^d)$ that satisfy the following estimates:

$$|\partial^\alpha_\zeta a(\zeta)| \leq C_\alpha w_{\mathcal{P}}^{m - \varrho|\alpha|}(\zeta), \qquad \zeta \in \mathbf{R}^d$$

for any multiindex α and for suitable constants $C_\alpha > 0$.

We have $w_{\mathcal{P}} \in \Lambda^1_{\varrho_0, \mathcal{P}}(\mathbf{R}^d)$ for a certain $\varrho_0 > 0$, whose largest value can be computed in terms of $\mathcal{P}$; see [BG]. In Definition 1.1 we shall understand $0 < \varrho \leq \varrho_0 < 1$.

We will often omit the dependence from $\mathbf{R}^d$ simply writing $\Lambda^m_{\varrho, \mathcal{P}}$ instead of $\Lambda^m_{\varrho, \mathcal{P}}(\mathbf{R}^d)$.

In the case of even dimension $d = 2n$, $n \in \mathbf{N}$, $\zeta = (x, \xi)$, $x \in \mathbf{R}^n$, $\xi \in \mathbf{R}^n$, we associate to each symbol $a \in \Lambda^m_{\varrho, \mathcal{P}}(\mathbf{R}^{2n})$ the pseudodifferential operator:

$$Au(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\xi} a(x, \xi) \hat{u}(\xi)\, d\xi, \qquad u \in S(\mathbf{R}^n). \tag{1.4}$$

We say that $a(x, \xi)$ is the symbol of operator A (writing $a = \sigma(A)$, $A = Op(a)$) and we indicate by $\mathrm{L}^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$ the class of operators with symbol in $\Lambda^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^{2n})$. We shall be particularly interested in hypoelliptic symbols defined as follows:

DEFINITION 1.2. – A function $a \in C^\infty(\boldsymbol{R}^d)$ is a hypoelliptic symbol in the class $H\Lambda^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^d)$ if the following conditions are satisfied:

1) there exist posive constants C_1, C_2, R such that:

$$C_1 w^m_{\mathcal{P}}(\zeta) \leqslant |a(\zeta)| \leqslant C_2 w^m_{\mathcal{P}}(\zeta) \quad \text{if } |\zeta| \geqslant R.$$

2) for each multiindex γ exist $C_\gamma > 0$, $R > 0$ such that:

$$|\partial^\alpha_\zeta a(\zeta)| \leqslant C_\alpha |a(\zeta)| w_{\mathcal{P}}^{-\varrho|\alpha|}(\zeta) \quad \text{if } |\zeta| \geqslant R.$$

That is, in view of condition 1): $a(\zeta) \in \Lambda^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^d)$.

If $d = 2n$, $n \in \boldsymbol{N}$, then $H\mathrm{L}^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$ denotes the correspondent class of hypoelliptic pseudodifferential operators.

The properties of the classes $\mathrm{L}^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$ and $H\mathrm{L}^m_{\varrho\mathcal{P}}(\boldsymbol{R}^n)$ are standard. We postpone their detailed list to Section 2.

We give now a the definition of the Sobolev spaces $H^m_{\mathcal{P}}(\boldsymbol{R}^n)$ by means of hypoelliptic operators of type $H\mathrm{L}^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$.

DEFINITION 1.3. – Let $\mathcal{P} \in P_{2n}$ be a fixed complete polyhedron and $a_m \in H\Lambda^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^{2n})$ be a hypoelliptic symbol; then we set:

$$H^m_{\mathcal{P}}(\boldsymbol{R}^n) = A_m^{-1}(L^2(\boldsymbol{R}^n)) \quad \text{where } A_m = Op(a_m).$$

The definition coincides with that in [BG], but there we regard A_m as an operator in the classes of SHUBIN [S].

We remark that $H^m_{\mathcal{P}}(\boldsymbol{R}^n)$ depends neither on ϱ nor on the particular symbol $a_m \in H\Lambda^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^{2n})$ choosen, but only on $m \in \boldsymbol{R}$ and $\mathcal{P} \in P_{2n}$.

To complete the description of $H^m_{\mathcal{P}}(\boldsymbol{R}^n)$ we recall from [BG] the main features of these spaces; we refer to this paper for more detailed statements and proofs.

PROPOSITION 1.4. – $H^m_{\mathcal{P}}(\boldsymbol{R}^n)$ has a Hilbert space structure given by the inner product:

$$(u, v)_{\mathcal{P}} = (A_m u, A_m v)_{L^2} + (Ru, Rv)_{L^2}$$

where A_m is a hypoelliptic operator defining the space $H^m_{\mathcal{P}}(\boldsymbol{R}^n)$ according to Definition 1.3 and R is a regularizing operator, $R = I - \tilde{A}_m A_m$, with $\tilde{A}_m$ left parametrix of A_m according to Definition 2.9 and Proposition 2.10 below.

We will indicate for short with $\|u\|_m$ the norm of an element u in the space $H^m_{\mathcal{P}}(\boldsymbol{R}^n)$.

PROPOSITION 1.5. – The topological dual $H^m_{\mathcal{P}}(\boldsymbol{R}^n)^*$ of $H^m_{\mathcal{P}}(\boldsymbol{R}^n)$ coincides as Hilbert space with $H_{\mathcal{P}}^{-m}(\boldsymbol{R}^n)$.

PROPOSITION 1.6. – Let $S(\boldsymbol{R}^n)$ be the Frechet space of «rapidly decreasing» functions on $\boldsymbol{R}^n$; we have the following continuous immersions:

$$id: S(\boldsymbol{R}^n) \to H^m_{\mathcal{P}}(\boldsymbol{R}^n); \qquad id: H^m_{\mathcal{P}}(\boldsymbol{R}^n) \to S'(\boldsymbol{R}^n)$$

for any $m \in \boldsymbol{R}$, and the compact immersions:

$$id: H^t_{\mathcal{P}}(\boldsymbol{R}^n) \to H^s_{\mathcal{P}}(\boldsymbol{R}^n)$$

if $t > s$.

We recall also that an equivalent definition of the spaces $H_{\mathcal{P}}(\boldsymbol{R}^n) = H^1_{\mathcal{P}}(\boldsymbol{R}^n)$ is:

$$H^1_{\mathcal{P}}(\boldsymbol{R}^n) = \{u \in S'(\boldsymbol{R}^n) : x^\alpha D^\beta u \in L^2(\boldsymbol{R}^n) \text{ for } (\alpha, \beta) \in \mathcal{P}\}$$

and the inner product:

$$\langle u, v \rangle_{\mathcal{P}} = \sum_{(\alpha, \beta) \in \mathcal{P}} (x^\alpha D^\beta u, x^\alpha D^\beta v)_{L^2}$$

gives $H^1_{\mathcal{P}}(\boldsymbol{R}^n)$ an equivalent Hilbert space structure.

We generalize now the pseudodifferential operators to the following type of Fourier integral operators.

DEFINITION 1.7. – Let $a \in \Lambda^m_{\varrho, \mathcal{P}}(\boldsymbol{R}^{2n})$ and ϕ be a smooth real valued function satisfying the following condition:

$$\partial^\gamma_\xi \phi \in \Lambda^0_{\varrho, \mathcal{P}}(\boldsymbol{R}^{2n}) \qquad \text{for every } \gamma \text{ with } |\gamma| = 2 .$$

Then we define the Fourier integral operator:

$$A_{a, \phi} u(x) = (2\pi)^{-n} \int_{\boldsymbol{R}^n} e^{i\phi(x, \xi)} a(x, \xi) \hat{u}(\xi)\, d\xi \qquad \text{for } u \in S(\boldsymbol{R}^n).$$

We shall come back to this type of Fourier integral operators in Section 3.

2. – Classes of symbols $\Lambda^m_{\varrho, \mathcal{P}}(\boldsymbol{R}^d)$ and $H\Lambda^m_{\varrho, \mathcal{P}}(\boldsymbol{R}^d)$.

We summarize now the main properties of the classes of symbols $\Lambda^m_{\varrho, \mathcal{P}}(\boldsymbol{R}^d)$ and hypoelliptic symbols $H\Lambda^m_{\varrho, \mathcal{P}}(\boldsymbol{R}^d)$.

Their properties are very similar to those of other classes of pseudodifferential operators (in particular the Shubin classes) and can be easily recaptured from the general calculus in BEALS [B], HÖRMANDER [HÖ], so we don't go into the details of the *proof*. For the quasi-elliptic case, i.e. the case when the polyhedron $\mathcal{P}$ has a single face, see for example GRUSHIN [G], HELFFER-RODINO [HRD], and BOVE-FRANCHI-OBRECHT [BFO].

PROPOSITION 2.1. – Let $m \in \boldsymbol{R}$, $\varrho \in]0, 1]$; then:

1) $\Lambda^m_{\varrho, \wp}(\boldsymbol{R}^d)$ is a vector space;

2) if $a_1 \in \Lambda^{m_1}_{\varrho_1, \wp}(\boldsymbol{R}^d)$, $a_2 \in \Lambda^{m_2}_{\varrho_2, \wp}(\boldsymbol{R}^d)$ then $a_1 a_2 \in \Lambda^{m_1+m_2}_{\varrho, \wp}(\boldsymbol{R}^d)$ with $\varrho = \min(\varrho_1, \varrho_2)$;

3) for every multiindex $\alpha \in \boldsymbol{N}^d_0$: $\partial^\alpha_\zeta a \in \Lambda^{m-\varrho|\alpha|}_{\varrho, \wp}(\boldsymbol{R}^d)$;

4) $\bigcap_{m \in \boldsymbol{R}} \Lambda^m_{\varrho, \wp}(\boldsymbol{R}^d) = S(\boldsymbol{R}^d)$.

DEFINITION 2.2. – Let $a_j \in \Lambda^{m_j}_{\varrho, \wp}(\boldsymbol{R}^d)$ and $m_j \to -\infty$ for $j \to +\infty$. We write

$$a \sim \sum_{j=1}^{\infty} a_j$$

if $a \in C^\infty(\boldsymbol{R}^d)$ and $a - \sum_{j=1}^{r-1} a_j \in \Lambda^{\bar{m}_r}_{\varrho, \wp}(\boldsymbol{R}^d)$ where $\bar{m}_r = \max_{j \geq r} m_j$. We have then $a \in \Lambda^{m_1}_{\varrho, \wp}(\boldsymbol{R}^d)$.

PROPOSITION 2.3. – If $a_j \in \Lambda^{m_j}_{\varrho, \wp}(\boldsymbol{R}^d)$ with $m_j \to -\infty$ for $j \to +\infty$ then there exists $a \in C^\infty(\boldsymbol{R}^d)$ such that:

$$a \sim \sum_{j=1}^{\infty} a_j .$$

Furthermore if b is another function such that $b \sim \sum_{j=1}^{\infty} b_j$ holds then $a - b \in S(\boldsymbol{R}^d)$.

If $d = 2n$, according to (1.4), a pseudodifferential operator $A = Op(a) \in \mathrm{L}^m_{\varrho, \wp}(\boldsymbol{R}^n)$ with a symbol $a \in \Lambda^m_{\varrho, \wp}(\boldsymbol{R}^{2n})$ is defined. The following properties hold for the operator A.

PROPOSITION 2.4. – Let $a \in \Lambda^m_{\varrho, \wp}(\boldsymbol{R}^{2n})$, then A defines a continuous map:

$$A : S(\boldsymbol{R}^n) \to S(\boldsymbol{R}^n), \tag{2.1}$$

that extends to a continuous map:

$$A : S'(\boldsymbol{R}^n) \to S'(\boldsymbol{R}^n). \tag{2.2}$$

Furthermore:

if $a \in \Lambda^0_{\varrho, \wp}(\boldsymbol{R}^{2n})$ then A extends to a continuous map:

$$A : L^2(\boldsymbol{R}^n) \to L^2(\boldsymbol{R}^n); \tag{2.3}$$

if $a \in \Lambda^{-m}_{\varrho, \wp}(\boldsymbol{R}^{2n})$, $m > 0$, then A extends to a compact operator:

$$A : L^2(\boldsymbol{R}^n) \to L^2(\boldsymbol{R}^n). \tag{2.4}$$

DEFINITION 2.5. – Let $A \in \mathrm{L}^m_{\varrho, \wp}(\boldsymbol{R}^n)$, we indicate with ${}^tA$ the operator defined by the condition $\langle Au, v\rangle = \langle u, {}^tAv\rangle$, $u, v \in S(\boldsymbol{R}^n)$, where $\langle u, v\rangle = \int_{\boldsymbol{R}^n} u(x) v(x)\, dx$, and with

A^* the operator defined by the condition $(au, v) = (u, A^* v)$, $u, v \in S(\boldsymbol{R}^n)$, where $(u, v) = \int_{\boldsymbol{R}^n} u(x)\bar{v}(x)\,dx$.

PROPOSITION 2.6. – If $A \in \mathrm{L}^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$ then ${}^tA \in \mathrm{L}^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$ and $A^* \in \mathrm{L}^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$, furthermore the following asymptotic expansions hold:

$$\sigma({}^tA) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha a(x, -\xi),$$

$$\sigma(A^*) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha D_x^\alpha \bar{a}(x, -\xi).$$

PROPOSITION 2.7. – Let $A_1 \in \mathrm{L}^{m_1}_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$ and $A_2 \in \mathrm{L}^{m_2}_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$; then $A_1A_2 \in \mathrm{L}^{m_1+m_2}_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$ and

$$\sigma(A_1A_2) \sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha a_1(x, \xi) D_x^\alpha a_2(x, \xi)$$

where $a_1(x, \xi) = \sigma(A_1)$, $a_2(x, \xi) = \sigma(A_2)$.

We state now the main properties of hypoelliptic symbols and operators introduced in Definition 1.2.

PROPOSITION 2.8.

1) If $a \in HA^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^d)$ then $a^{-1} \in HA^{-m}_{\varrho,\mathcal{P}}(\boldsymbol{R}^d)$ and $(\partial^\alpha a)/a \in HA^{-\varrho|\alpha|}_{\varrho,\mathcal{P}}(\boldsymbol{R}^d)$ for all α (eventually after a modification of $a(\zeta)$ on a compact set).

2) If $a_1 \in HA^{m_1}_{\varrho,\mathcal{P}}(\boldsymbol{R}^d)$ and $a_2 \in HA^{m_2}_{\varrho,\mathcal{P}}(\boldsymbol{R}^d)$ then $a_1 a_2 \in HA^{m_1+m_2}_{\varrho,\mathcal{P}}(\boldsymbol{R}^d)$.

3) If $A_1 \in H\mathrm{L}^{m_1}_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$ and $A_2 \in H\mathrm{L}^{m_2}_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$ then $A_1A_2 \in H\mathrm{L}^{m_1+m_2}_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$.

4) If $A \in H\mathrm{L}^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$ then ${}^tA \in H\mathrm{L}^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$ and $A^* \in H\mathrm{L}^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$.

DEFINITION 2.9. – An operator $R \in \bigcap_{m \in \boldsymbol{R}} \mathrm{L}^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$ is said «regularizing».

Regularizing operators define continuous maps $R\colon S'(\boldsymbol{R}^n) \to S(\boldsymbol{R}^n)$.

PROPOSITION 2.10 (Existence of the parametrix). – Let $A \in H\mathrm{L}^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$; then there exist an operator $B \in H\mathrm{L}^{-m}_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$ and two regularizing operators R_1 and R_2 such that:

$$AB = I + R_1, \qquad BA = I + R_2.$$

B is said «parametrix» of A. If B' is another parametrix of the same operator A then $B - B'$ is a regularizing operator.

PROPOSITION 2.1 (Regularity of hypoelliptic operators). – Let $A \in H\mathrm{L}^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$. If $Au \in S(\boldsymbol{R}^n)$ for some $u \in S'(\boldsymbol{R}^n)$ then necessarily $u \in S(\boldsymbol{R}^n)$.

An asymptotic expansion formally identical to that given, for example, in KUMANO-GO [K] (cap. 2, par. 5) could be given for the parametrix of an operator $A \in H\mathrm{L}^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$; for brevity we won't rewrite here this formula.

The proof of the following proposition will be detailed, since the argument will be useful for the proof of Theorem 5.5 in the sequel.

PROPOSITION 2.12. – If $A \in \mathrm{L}^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$ then it defines a continuous operator:

$$A : H^s_{\mathcal{P}}(\boldsymbol{R}^n) \to H^{s-m}_{\mathcal{P}}(\boldsymbol{R}^n).$$

PROOF. – Let $A_s \in H\mathrm{L}^s_{\varrho,\mathcal{P}}$ and $A_{s-m} \in H\mathrm{L}^{s-m}_{\varrho,\mathcal{P}}$ the operators defining the spaces $H^s_{\mathcal{P}}(\boldsymbol{R}^n)$ and $H^{s-m}_{\mathcal{P}}(\boldsymbol{R}^n)$ respectively, and $\tilde{A}_s$, $\tilde{A}_{s-m}$ be two parametrices (of A_s and A_{s-m} respectively).

Then $\tilde{A}_{s-m} \in H\mathrm{L}^{-s}_{\varrho,\mathcal{P}}$ and $\tilde{A}_{s-m} \in H\mathrm{L}^{-(s-m)}_{\varrho,\mathcal{P}}$, so the operator $A_{s-m}A\tilde{A}_s$ belongs to the class $\mathrm{L}^0_{\mathcal{P}}(\boldsymbol{R}^n)$ and defines a continuous map of $L^2(\boldsymbol{R}^n)$.

The conclusion follows from the definition of the norms of $H^s_{\mathcal{P}}(\boldsymbol{R}^n)$ and $H^{s-m}_{\mathcal{P}}(\boldsymbol{R}^n)$.

PROPOSITION 2.13. – Let $A \in H\mathrm{L}^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$; then it defines a Fredholm operator:

$$A : H^s_{\mathcal{P}}(\boldsymbol{R}^n) \to H^{s-m}_{\mathcal{P}}(\boldsymbol{R}^n).$$

The proof is the same as in [BG], where also we proved a formula for the calculation of the index.

3. – Fourier integral operators.

We consider now Fourier integral operators according to Definition 1.7.

We show first of all that, under additional assumptions on the phase function, our operators define continuous maps from $S(\boldsymbol{R}^n)$ to $S(\boldsymbol{R}^n)$.

We set $f \sim g$ if $f(\zeta)$ and $g(\zeta)$ are two functions satisfying the condition:

$$C_1 f(\zeta) \leq g(\zeta) \leq C_2 f(\zeta) \qquad (\text{for every } \zeta \in \boldsymbol{R}^{2n})$$

where C_1, C_2 are suitable positive constants.

We begin with two technical lemmas.

LEMMA 3.1. – Let $\phi(x, \xi)$ satisfy the assumptions in Definition 1.7. Let $p(x, \xi)$ be given in $\Lambda^m_{\varrho,\mathcal{P}}$.

If $w_{\mathcal{P}}(x, \nabla_x \phi(x, \xi)) \sim w_{\mathcal{P}}(x, \xi)$ then $p(x, \nabla_x \phi(x, \xi)) \in \Lambda^m_{\varrho,\mathcal{P}}$.

If $w_{\mathcal{P}}(\nabla_\xi \phi(x, \xi), \xi) \sim w_{\mathcal{P}}(x, \xi)$ then $p(\nabla_\xi \phi(x, \xi), \xi) \in \Lambda^m_{\varrho,\mathcal{P}}$.

PROOF. – We show that $p(x, \nabla_x \phi(x, \xi)) \in \Lambda^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^{2n})$.

The following estimate holds for the function $p(x, \nabla_x \phi(x, \xi))$:

$$|p(x, \nabla_x \phi(x, \xi))| \leq C' w^t_{\mathcal{P}}(x, \nabla_x \phi(x, \xi)) \leq C w^t_{\mathcal{P}}(x, \xi)$$

where C', $C > 0$ are suitable constants. In order to estimate the derivatives of the function $p(x, \nabla_x \phi(x, \xi))$, preliminarly we show by induction that:

$$\partial^\gamma_z [p(x, \nabla_x \phi(x, \xi))] = \sum_{0 < \beta \leq \gamma} (\partial^\beta_z p)(x, \nabla_x \phi(x, \xi)) \lambda_\beta(x, \xi) \tag{3.8}$$

for some $\lambda_\beta \in \Lambda^{\varrho(|\beta| - |\gamma|)}_{\varrho, \mathcal{P}}(\boldsymbol{R}^{2n})$.

If $|\gamma| = 1$ then (3.8) holds. Assume now that it holds for every γ with $|\gamma| = m_0$ ($m_0 \in \boldsymbol{N}$) and let $\partial^{\tilde\gamma}_z = \partial_{x_i} \partial^\gamma_z$ (the same calculation, even somewhat semplified, holds if $\partial^{\tilde\gamma}_z = \partial_{\xi_i} \partial^\gamma_z$), then we have:

$$\partial^{\tilde\gamma}_z [p(x, \nabla_x \phi(x, \xi))] = \partial_{x_i} \partial^\gamma_z [p(x, \nabla_x \phi(x, \xi))] =$$

$$= \sum_{0 < \beta \leq \gamma} \Big[\partial_{x_i} \big[(\partial^\beta_z p)(x, \nabla_x \phi(x, \xi)) \big] \lambda_\beta(x, \xi) + (\partial^\beta_z p)(x, \nabla_x \phi(x, \xi)) \, \partial_{x_i} \lambda_\beta(x, \xi) \Big] =$$

$$= \sum_{0 < \beta \leq \gamma} \Big[(\partial_{x_i} \partial^\beta_z p)(x, \nabla_x \phi(x, \xi)) \lambda_\beta(x, \xi) +$$

$$+ \sum_{k=1}^n (\partial_{\eta_k} \partial^\beta_z p)(x, \eta) \Big|_{\eta = \nabla_x \phi(x, \xi)} \cdot (\partial_{x_i} \partial_{x_k} \phi)(x, \xi) \lambda_\beta(x, \xi) +$$

$$+ (\partial^\beta_z p)(x, \nabla_x \phi(x, \xi)) \, \partial_{x_i} \lambda_\beta(x, \xi) \Big].$$

But $\partial_{x_i} \partial_{x_k} \phi \in \Lambda^0_{\varrho, \mathcal{P}}$ so $(\partial_{x_i} \partial_{x_k} \phi) \times \lambda_\beta \in \Lambda^{\varrho(|\beta| - |\gamma|)}_{\varrho, \mathcal{P}}$ and $\partial_{x_i} \lambda_\beta \in \Lambda^{\varrho(|\beta| - |\gamma|) - \varrho}_{\varrho, \mathcal{P}} = \Lambda^{\varrho(|\beta| - |\tilde\gamma|)}_{\varrho, \mathcal{P}}$.

Then for suitable $\tilde\lambda_{\tilde\beta} \in \Lambda^{\varrho(|\tilde\beta| - |\tilde\gamma|)}_{\varrho, \mathcal{P}}$:

$$(\partial^{\tilde\gamma}_z p)(x, \nabla_x \phi(x, \xi)) = \sum_{0 < \tilde\beta \leq \tilde\gamma} (\partial^{\tilde\beta}_z p)(x, \nabla_x \phi(x, \xi)) \tilde\lambda_{\tilde\beta}(x, \xi),$$

i.e. (3.8) holds for $\tilde\gamma \in \boldsymbol{N}$ with $|\tilde\gamma| = m_0 + 1$.

We can now estimate the derivatives of $p(x, \nabla_x \phi(x, \xi))$ as follows:

$$\big|\partial^\gamma_z [p(x, \nabla_x \phi(x, \xi))]\big| \leq \sum_{0 < \beta \leq \gamma} |(\partial^\beta_z p)(x, \nabla_x \phi(x, \xi))| \, |\lambda_\beta(x, \xi)| \leq$$

$$\leq C \sum_{0 < \beta \leq \gamma} w^{t - \varrho|\beta|}_{\mathcal{P}}(x, \nabla_x \phi(x, \xi)) \, w^{\varrho(|\beta| - |\gamma|)}_{\mathcal{P}}(x, \xi) \leq C' w^{t - \varrho|\gamma|}_{\mathcal{P}}(x, \xi),$$

This proves that $p(x, \nabla_x \phi(x, \xi)) \in \Lambda^m_{\varrho, \mathcal{P}}(\boldsymbol{R}^{2n})$.

LEMMA 3.2. – Let $\phi(x, \xi)$ satisfy the assumptions in Definition 1.7. Let $(x, \xi) \in \boldsymbol{N}^{2n}_0$ be given, and assume that x^α, ξ^β can be regarded as symbols in $\Lambda^M_{\varrho, \mathcal{P}}$ for some $M \in \boldsymbol{R}$.

If $w_{\mathscr{P}}(x, \nabla_x \phi(x, \xi)) \sim w_{\mathscr{P}}(x, \xi)$ then

$$x^\alpha D_x^\beta e^{i\phi(x, \xi)} = e^{i\phi(x, \xi)} b(x, \xi) \tag{3.1}$$

where $b(x, \xi)$ is in $\Lambda^m_{\varrho, \mathscr{P}}$ with principal symbol $x^\alpha (\nabla_x \phi(x, \xi))^\beta$.

PROOF. – Let us write $\zeta = (x, \xi)$. We begin by proving that:

$$\partial_\zeta^\beta e^{i\phi(\zeta)} = e^{i\phi(\zeta)} \sum_{\alpha^{(1)} \dots \alpha^{(m)}} c_{\alpha^{(1)}, \dots \alpha^{(m)}} \partial_\zeta^{\alpha^{(1)}} \phi(\zeta), \dots, \partial_\zeta^{\alpha^{(m)}} \phi(\zeta) \tag{3.2}$$

where $c_{\alpha^{(1)}, \dots, \alpha^{(m)}}$ are suitable coefficients and the sum ranges over all the m-tuples of multiindices $(\alpha^{(1)}, \dots, \alpha^{(m)})$ such that $1 \leq m \leq |\beta|$ and $\alpha^{(1)} + \dots + \alpha^{(m)} = \beta$, $\alpha^{(h)} \neq 0$ for $h = 1, \dots, m$.

The identity (3.2) holds for $|\beta| = 1$.

Suppose now it holds for a fixed $\delta \in N_0^{2n}$.

If $j = 1, \dots, 2n$, then:

$$\partial_{\zeta_j} \partial_\zeta^\delta e^{i\phi(\zeta)} = e^{i\phi(\zeta)} \Bigg\{ i\partial_{\zeta_j} \phi(\zeta) \sum_{\alpha^{(1)}, \dots, \alpha^{(m)}} c_{\alpha^{(1)}, \dots, \alpha^{(m)}} \partial_\zeta^{\alpha^{(1)}} \phi(\zeta), \dots, \partial_\zeta^{\alpha^{(m)}} \phi(\zeta) +$$

$$+ \sum_{\alpha^{(1)}, \dots, \alpha^{(m)}} \Bigg[c_{\alpha^{(1)}, \dots, \alpha^{(m)}} \sum_{h = 1, \dots, m} \partial_\zeta^{\alpha^{(1)}} \phi(\zeta) \dots \partial_{\zeta_j} \partial_\zeta^{\alpha^{(h)}} \phi(\zeta) \dots \partial_\zeta^{\alpha^{(m)}} \phi(\zeta) \Bigg] \Bigg\}$$

This expression is of the same form as (3.2) with β such that $\partial_\zeta^\beta = \partial_{\zeta_j} \partial_\zeta^\delta$; so (3.2) is proved.

In (3.2) we further recognize that all the terms, but the term $i^{|\alpha|} (\nabla_x \phi(x, \xi))^\beta$, contain at least one second order derivate of $\phi(x, \xi)$. The conclusion follows then from the properties of Definition 1.7 and from Lemma 3.1, by observing that if $\gamma < \beta$ we have $x^\alpha \xi^\beta \in \Lambda^{M - \varepsilon}_{\varrho, \mathscr{P}}$ for some $\varepsilon > 0$, and therefore also $x^\alpha (\nabla_x \phi(x, \xi))^\gamma \in \Lambda^{M - \varepsilon}_{\varrho, \mathscr{P}}$.

PROPOSITION 3.3. – Let $a(\zeta)$ and $\phi(\zeta)$, $(\zeta = (x, \xi) \in R_x^n \times R_\xi^n)$, satisfy the condition of Definition 1.7.

If $w_{\mathscr{P}}(\nabla_\xi \phi(x, \xi), \xi) \sim w_{\mathscr{P}}(x, \xi)$, then the Fourier operator

$$A_{a, \phi} u(x) = (2\pi)^{-n} \int_{R^n} e^{i\phi(x, \xi)} a(x, \xi) \hat{u}(\xi) \, d\xi$$

defines a continuous map:

$$A_{a, \phi} : S(R^n) \to S(R^n).$$

PROOF. – (I) We begin by showing that $A_{a, \phi} u(x)$ is a bounded function whenever $u \in S(R^n)$. We associate here to the function $w_{\mathscr{P}}^2(x, \xi) \in \Lambda^2_{\varrho, \mathscr{P}}(R^{2n})$ the following operator:

$$w_{\mathscr{P}}^2(D_\xi, \xi) = \sum_{(\alpha, \beta) \in \mathscr{P}} \xi^{2\beta} D_\xi^{2\alpha}.$$

Then we have, exchanging the role of the x and the ξ variables in Lemma 3.2:

$$w_{\mathcal{P}}^2(D_\xi, \xi)[e^{i\phi(x,\xi)}] = e^{i\phi(x,\xi)} \sum_{(\alpha,\beta)\in\mathcal{P}} \xi^{2\beta}\left(\sum_{\substack{\alpha^{(1)}+\ldots+\alpha^{(m)}=2\alpha\\ 1\le m\le|2\alpha|}} c_{\alpha^{(1)}\ldots\alpha^{(m)}}\partial_\xi^{\alpha^{(1)}}\phi\ldots\partial_\xi^{\alpha^{(m)}}\phi\right) =$$

$$= e^{i\phi(x,\xi)} \sum_{(\alpha,\beta)\in\mathcal{P}} (\xi^{2\beta}(\nabla_\xi\phi(x,\xi))^{2\alpha} + R_\alpha(x,\xi)) = e^{i\phi(x,\xi)}(w_{\mathcal{P}}^2(\nabla_\xi\phi(x,\xi),\xi) + R(x,\xi)),$$

where $R_\alpha(x,\xi)$ and $R(x,\xi)$ are suitable functions and the latter belongs to the class $\Lambda_{\varrho,\mathcal{P}}^{2-\varepsilon}(\boldsymbol{R}^{2n})$ $(\varepsilon>0)$. Note that $w_{\mathcal{P}}^2(\nabla_x\phi(x,\xi),\xi)\in\Lambda_{\varrho,\mathcal{P}}^2(\boldsymbol{R}^{2n})$, in view of Lemma 3.1 and our assumption $w_{\mathcal{P}}(\nabla_x\phi(x,\xi),\xi)\sim w_{\mathcal{P}}(x,\xi)$.

In follows that $w_{\mathcal{P}}^2(\nabla_\xi\phi(x,\xi),\xi)+R(x,\xi)\neq 0$ for large (x,ξ) so that we can write:

$$e^{i\phi(x,\xi)} = \frac{\sum_{(\alpha,\beta)\in\mathcal{P}} \xi^{2\beta} D_\xi^{2\alpha}[e^{i\phi(x,\xi)}]}{w_{\mathcal{P}}^2(\nabla_\xi\phi(x,\xi),\xi)+R(x,\xi)} = \mathcal{M}_{\mathcal{P},\phi}(e^{i\phi(x,\xi)})$$

where we have set:

$$\mathcal{M}_{\mathcal{P},\phi} = \sum_{(\alpha,\beta)\in\mathcal{P}}\left(\frac{\xi^{2\beta}}{w_{\mathcal{P}}^2(\nabla_\xi\phi(x,\xi),\xi)+R(x,\xi)}\right)D_\xi^{2\alpha}.$$

We can crite:

$$A_{a,\phi}u(x) = \frac{1}{(2\pi)^n}\int_{\boldsymbol{R}^n} e^{i\phi(x,\xi)}a(x,\xi)\hat{u}(\xi)\,d\xi =$$

$$= \frac{1}{(2\pi)^n}\int_{\boldsymbol{R}^n}\mathcal{M}_{\mathcal{P},\phi}[e^{i\phi(x,\xi)}]a(x,\xi)\hat{u}(\xi)\,d\xi = \frac{1}{(2\pi)^n}\int_{\boldsymbol{R}^n} e^{i\phi(x,\xi)}\mathcal{M}_{\mathcal{P},\phi}^t[a(x,\xi)\hat{u}(\xi)]\,d\xi =$$

$$= \frac{1}{(2\pi)^n}\int_{\boldsymbol{R}_n} e^{i\phi(x,\xi)}\left[\sum_{(\alpha,\beta)\in\mathcal{P}} D_\xi^{2\alpha}\left(\frac{a(x,\xi)\hat{u}(\xi)\xi^{2\beta}}{w_{\mathcal{P}}^2(\nabla_\xi\phi(x,\xi),\xi)+R(x,\xi)}\right)\right]d\xi =$$

$$= \frac{1}{(2\pi)^n}\int_{\boldsymbol{R}^n}\left[\sum_{(\alpha,\beta)\in\mathcal{P}}\sum_{\gamma\le2\alpha}\binom{2\alpha}{\gamma}D_\xi^\gamma\left(\frac{a(x,\xi)}{w_{\mathcal{P}}^2(\nabla_\xi\phi,\xi)+R}\right)D_\xi^{2\alpha-\gamma}(\xi^{2\beta}\hat{u}(\xi))\right]d\xi.$$

So if we set:

$$a_{\gamma,\alpha}(x,\xi) = \binom{2\alpha}{\gamma}D_\xi^\gamma\left(\frac{a(x,\xi)}{w_{\mathcal{P}}^2(\nabla_\xi\phi(x,\xi),\xi)+R(x,\xi)}\right),$$

$$\hat{u}_{\gamma,\beta}(x,\xi) = D_\xi^{2\alpha-\gamma}(\xi^{2\beta}\hat{u}(\xi)),$$

then we have:

$$A_{a,\phi}u(x) = \sum_{(\alpha,\beta)\in\mathcal{P}} \sum_{\gamma\le 2\alpha} \int_{\mathbf{R}^n} e^{i\phi(x,\xi)} a_{\gamma,\alpha}(x,\xi)\, \widehat{u}_{\gamma,\beta}(\xi)\, d\xi ,$$

with $a_{\gamma,\alpha} \in \Lambda^{m-2}_{\varrho,\mathcal{P}}(\mathbf{R}^{2n})$ and $\widehat{u}_{\gamma,\beta} \in S(\mathbf{R}^n_\xi)$.

Iterating this procedure we obtain analogous expressions where $a_{\gamma,\alpha} \in \Lambda^{m-k}_{\varrho,\mathcal{P}}(\mathbf{R}^{2n})$ with arbitrarily great k.

This shows that $A_{a,\phi}u(x)$ is a bounded function on $\mathbf{R}^n$.

(II) Using part (I) we show now that $x^\alpha(\partial_x^\beta A_{a,\phi}u)(x)$ with $u \in S(\mathbf{R}^n)$ is a bounded function for every couple of multiindices $(\alpha,\beta) \in \mathbf{N}_0^{2\alpha}$. This implies $A_{a,\phi}u \in S(\mathbf{R}^n)$.

We can formally write:

$$x^\alpha(\partial_x^\beta A_{a,\phi}u)(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} x^\alpha \partial_x^\beta [e^{i\phi(x,\xi)} a(x,\xi)]\, \widehat{u}(\xi)\, d\xi$$

so it follows from the Leibnitz rule and Lemma 3.2:

$$x^\alpha(\partial_x^\beta A_{a,\phi}u)(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\phi(x,\xi)} b_{\alpha,\beta}(x,\xi)\, \widehat{u}(\xi)\, d\xi$$

where $b_{\alpha,\beta} \in \Lambda^{m+M}_{\varrho,\mathcal{P}}(\mathbf{R}^{2n})$ for some $M \in \mathbf{R}$ depending on α and β.

We conclude from part (I) with $a(x,\xi)$ replaced by $b_{\alpha,\beta}(x,\xi)$, that $x^\alpha(\partial_x^\beta A_{a,\phi}u)(x)$ is a bounded function.

(III) Finally we show that $A_{a,\phi}: S(\mathbf{R}^n) \to S(\mathbf{R}^n)$ is continuous. We make use of the closed graph theorem.

Let $u_k \to u$ in $S(\mathbf{R}^n)$ and $A_{a,\phi}u_k \to v$ in $S(\mathbf{R}^n)$. Then $f_k(x,\xi) = e^{i\phi(x,\xi)} a(x,\xi)\, \widehat{u}_k(\xi)$ converges pointwise to $f(x,\xi) = e^{i\phi(x,\xi)} a(x,\xi)\, \widehat{u}(\xi)$.

As $\widehat{u}_k \to \widehat{u}$ in $S(\mathbf{R}^n)$, for fixed $\varepsilon > 0$ and $s \in \mathbf{N}$, we have:

$$\big|\, |\widehat{u}_k(\xi)| - |\widehat{u}(\xi)| \,\big| \le |\widehat{u}_k(\xi) - \widehat{u}(\xi)| \le \frac{\varepsilon}{\langle\xi\rangle^s}, \qquad \left(\langle\xi\rangle = \sqrt{1+|\xi|^2}\right),$$

for great k.

It follows

$$|\widehat{u}_k(\xi)| \le \frac{\varepsilon}{\langle\xi\rangle^s} + |\widehat{u}(\xi)| \quad \text{and} \quad |f_k(x,\xi)| \le |a(x,\xi)| \left(\frac{\varepsilon}{\langle\xi\rangle^s} + |\widehat{u}(\xi)| \right).$$

The righthand side of the above estimate, for fixed $x \in \mathbf{R}^n$, is a function belonging to $L^1(\mathbf{R}^n_\xi)$ provided that s is taken greater then $m+n$.

The Lebesgue bounded convergence theorem shows then that $A_{a,\phi}u_k$ converges pointwise to $A_{a,\phi}u$.

For uniqueness of the limit in $S(\mathbf{R}^n)$ we conclude that $v = A_{a,\phi}u$, so the continuity of $A_{a,\phi}$ follows from the closed graph theorem.

In order to consider Fourier integral operators acting on the $H^s_{\mathcal{P}}(\boldsymbol{R}^n)$ spaces we extend now these operators to the space $S'(\boldsymbol{R}^n)$ topological dual of $S(\boldsymbol{R}^n)$.

THEOREM 3.4. – Let $a(\zeta)$ and $\phi(\zeta)$, $(\zeta = (x, \xi) \in \boldsymbol{R}^n_x \times \boldsymbol{R}^n_\xi)$, satisfy the condition of Definition 1.7 and suppose that:

$$w_{\mathcal{P}}(\nabla_\xi \phi(x, \xi), \xi) \sim w_{\mathcal{P}}(x, \xi) \tag{3.3}$$

$$w_{\mathcal{P}}(x, \nabla_x \phi(x, \xi)) \sim w_{\mathcal{P}}(x, \xi). \tag{3.4}$$

Then the Fourier integral operator

$$(A_{a,\phi} u)(x) = (2\pi)^{-n} \int_{\boldsymbol{R}^n} e^{i\phi(x,\xi)} a(x, \xi) \hat{u}(\xi) \, d\xi \qquad (u \in S(\boldsymbol{R}^n))$$

has a continuous extension:

$$A_{a,\phi} : S'(\boldsymbol{R}^n) \to S'(\boldsymbol{R}^n).$$

PROOF. – It is sufficient to show that the transposed operator ${}^tA_{a,\phi}$ maps continuously $S(\boldsymbol{R}^n)$ into itself.

For $u, \in S(\boldsymbol{R}^n)$ we have:

$$(A_{a,\phi} u, v) = (2\pi)^{-n} \int_{\boldsymbol{R}^{3n}} e^{i\phi(x,\xi)} a(x, \xi) e^{-i\xi y} u(y) v(x) \, dx \, d\xi \, dy =$$

$$= (2\pi)^{-n} \int_{\boldsymbol{R}^n_y} u(y) \, dy \int_{\boldsymbol{R}^n_\xi} e^{-i\xi y} \, d\xi \int_{\boldsymbol{R}^n_x} e^{i\phi(x,\xi)} a(x, \xi) \hat{w}(x) \, dx$$

where we have set $\hat{w}(x) = v(x)$.

If we denote by $\mathcal{F}$ the Fourier transform, we have:

$$(A_{a,\phi} u, v) = (u, \mathcal{F} \circ G \circ \mathcal{F}^{-1}(v))$$

where:

$$(Gw)(\xi) = (2\pi)^{-n} \int_{\boldsymbol{R}^n} e^{i\phi(x,\xi)} a(x, \xi) \hat{w}(x) \, dx .$$

If we set ${}^t\phi(x, \xi) = \phi(\xi, x)$ and ${}^t a(x, \xi) = a(\xi, x)$, we remark that the hypotheses assumed for ϕ and a remain valid for the functions ${}^t\phi$ and ${}^ta$.

Then G is a Fourier integral operator of phase ${}^t\phi$ and symbol ${}^ta$, that is $G = A_{{}^ta, {}^t\phi}$, and for the transposed operator we have the expression:

$${}^tA = \mathcal{F} \circ A_{{}^ta, {}^t\phi} \circ \mathcal{F}^{-1}$$

which shows the continuity of ${}^tA$.

We have the following result about the composition of a Fourier integral operator and a pseudodifferential operator,

THEOREM 3.5. – Let A be a Fourier integral operator with amplitude $a(\zeta) \in \Lambda^m_{\varrho,\mathcal{P}}(\boldsymbol{R}^{2n})$ $m \in \boldsymbol{R}$, $\varrho \in]0, 1]$), $\zeta = (x, \xi) \in \boldsymbol{R}^{2n}$ and phase $\phi(\zeta)$. We assume that:

(3.5) — ϕ is a real valued function such that: $\partial^\gamma_\zeta \phi(x, \xi) \in \Lambda^0_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$,

for $|\gamma| = 2$,

(3.6) — $w_{\mathcal{P}}(\nabla_\xi \phi(x, \xi), \xi) \sim w_{\mathcal{P}}(x, \xi)$,

(3.7) — $w_{\mathcal{P}}(x, \nabla_x \phi(x, \xi)) \sim w_{\mathcal{P}}(x, \xi)$.

Let P be a pseudodifferential operator with symbol $p(\zeta) \in \Lambda^t_{\varrho,\mathcal{P}}(\boldsymbol{R}^n)$. Then $P \circ A$ is a Fourier integral operator of phase $\phi(\zeta)$ and amplitude $h(\zeta)$ defined, modulo regularizing terms, by the following asymptotic expansion:

$$h(x, \xi) \sim \sum_{\alpha \in \boldsymbol{N}^n_0} c_\alpha(x, \xi)$$

where

$$c_\alpha(x, \xi) = \left(\frac{1}{\alpha!}\right) \partial^\alpha_\xi p(x, \nabla_x \phi(x, \xi)) D^\alpha_z [e^{i\psi(x, z, \xi)} a(z, \xi)]_{z = x}$$

with $\psi(x, z, \xi) = \phi(z, \xi) - \phi(x, \xi) - \langle \nabla_x \phi(x, \xi), z - x \rangle$.

We prove this theorem in the next section. Here are Lemmas that we will need; some of them will used also in Section 5.

LEMMA 3.6. – Under the same hypotheses of Theorem 3.5 we have:

$$c_\alpha \in \Lambda^{m+t-\varrho|\alpha|}_{\varrho,\mathcal{P}}(\boldsymbol{R}^{2n}).$$

PROOF. – We prove that for every multiindex $\beta \in \boldsymbol{N}^n_0$:

$$[\partial^\beta_z e^{i\psi(x, z, \xi)}]_{z = x} \in \Lambda^0_{\varrho,\mathcal{P}}(\boldsymbol{R}^{2n}). \tag{3.9}$$

This follows easily from the following facts:

— $[e^{i\psi(x, z, \xi)}]_{z = x} \equiv 1$;

— $[\partial_{z_j} e^{i\psi(x, z, \xi)}]_{z = x} = i[e^{i\psi(x, z, \xi)}]_{z = x} \Big[(\partial_{z_j} \phi(z, \xi) - \partial_{x_j} \phi(x, \xi))\Big]_{z = x} = 0$

for $j = 1, \ldots, n$;

— $[\partial^\beta_z e^{i\psi(x, z, \xi)}]_{z = x}$ with $|\beta| \geq 2$ is a sum of products of derivatives of $\phi(x, \xi)$ of order greater than 2, so it is a function belonging to $\Lambda^0_{\varrho,\mathcal{P}}$.

Finally from (3.9) we have:

$$[\partial_z^\beta e^{i\psi(x,z,\xi)} \partial_z^{\alpha-\beta} a(z,\xi)]_{z=x} \in \Lambda_{\varrho,\mathcal{P}}^{m-\varrho|\alpha-\beta|},$$

so that $D_z^\alpha [e^{i\psi(x,z,\xi)} a(z,\xi)]_{z=x} \in \Lambda_{\varrho,\mathcal{P}}^m$ by the Leibniz rule.

This together with the fact that, in view of Lemma 3.1, we have $p(x, \nabla_x \phi(x,\xi)) \in \Lambda_{\varrho,\mathcal{P}}^t(\boldsymbol{R}^{2n})$, shows that $c_\alpha \in \Lambda_{\varrho,\mathcal{P}}^{m+t-\varrho|\alpha|}$.

In the proof of Theorem 3.5 we will make use of following fact that we also formulate here as a lemma.

LEMMA 3.7. – Let $C > 0$ be a suitable small constant. If $|\tau| \leq C w_{\mathcal{P}}^\varrho(\zeta + \tau)$ we have:

$$w_{\mathcal{P}}(\zeta + \tau) \sim w_{\mathcal{P}}(\zeta).$$

PROOF. – If we consider the function $w_{\mathcal{P}}^\varrho(\zeta) \in \Lambda_{\varrho,\mathcal{P}}^\varrho$, thanks to the estimates defining the classes $\Lambda_{\varrho,\mathcal{P}}^m$ we have:

$$|\nabla_\zeta w_{\mathcal{P}}^\varrho(\zeta)| \leq \text{const},$$

so from the following Proposition 3.8 we can assert that:

$$w_{\mathcal{P}}^\varrho(\zeta + \tau) \sim w_{\mathcal{P}}^\varrho(\zeta)$$

as long as we take $|\tau| \leq C w_{\mathcal{P}}^\varrho(\zeta + \tau)$ with C sufficiently small (that is we satisfy (3.10) below with $f = w_{\mathcal{P}}^\varrho$ as a particular case).

The conclusion follows taking the $1/\varrho$ power of $w_{\mathcal{P}}^\varrho(\zeta)$.

PROPOSITION 3.8. – Let f be a complex valued function on $\boldsymbol{R}^d$ whose derivatives of order one are bounded, say $\max\limits_{u \in \boldsymbol{R}^d} |\nabla_u f(u)| \leq K$. If for $h \in \boldsymbol{R}^d$:

$$|h| \leq C \inf_{u \in \boldsymbol{R}^d} (|f(u)| + |f(u+h)|) \tag{3.10}$$

with $C < 1/K$, then:

$$|f(u+h)| \sim |f(u)|.$$

PROOF. – Observe that:

$$\big| |f(u+h)| - |f(u)| \big| \leq |f(u+h) - f(u)| \leq$$
$$\leq \max_\xi |(\nabla f)(\xi)|\,|h| \leq KC(|f(u)| + |f(u+h)|),$$

provided that (3.10) holds, so that

$$|f(u+h)| \leq \left(\frac{1+KC}{1-KC}\right) |f(u)|. \tag{3.11}$$

In the same way we prove:

$$|f(u)| \leqslant \left(\frac{1+KC}{1-KC}\right)|f(u+h)| . \tag{3.12}$$

From (3.11) and (3.12) we have

$$|f(u+h)| \sim |f(u)| .$$

Lemma 3.7 will be used also in the proof of Theorem 5.4. It states essentially that $w_{\varrho}^{g}(\zeta)$ is a weight function in the sense of BEALS [B].

4. – Proof of the composition theorem.

The proof of the composition Theorem 3.5 we give here is rather long and intricate but we follow a standard technique used by many authors in analogous theorems for other classes of Fourier integral operators. For reference see KUMANO-GO [K], PARENTI-SEGALA [PS], RODINO [R], HELFFER-ROBERT [HRB], MOHAMED [M]. The difficulty consists only in the adaptation of this technique to our case.

From the functional composition of P with A we have:

$$PAu(x) = \int_{R^n} e^{i\phi(x,\xi)}\hat{u}(\xi)\,d\xi \int_{R^{2n}} e^{-i\phi(x,\xi)}e^{ix\eta}p(x,\eta)\,e^{iz\eta}e^{i\phi(z,\xi)}a(z,\xi)\,dz\,d\eta .$$

Then we can write formally $PA = A_{h,\phi}$ with

$$h(x,\xi) = e^{-i\phi(x,\xi)} \int_{R^{2n}} e^{ix\eta}p(x,\eta)\,e^{-iz\eta}e^{i\phi(z,\xi)}a(z,\xi)\,dz\,d\eta = e^{-i\phi(x,\xi)}P(e^{-i\phi(\cdot,\xi)}a(\cdot,\xi)) .$$

Let $\varepsilon > 0$ be fixed and $\varphi \in C_0^\infty(\boldsymbol{R}^n)$ be a real function satisfying the following conditions:

— $0 \leqslant \varphi \leqslant 1$;

— $\varphi(t) = 0 \quad$ for $|t| \geqslant \dfrac{\varepsilon}{2}$;

— $\varphi(t) = 1 \quad$ for $|t| \leqslant \dfrac{\varepsilon}{4}$.

If we set $\chi(x,z) = \varphi(|x-z|^2/\langle(x,z)\rangle^{2\varrho})$, then $\chi(x,z)$ is a bounded function together with its derivatives of every order.

We set now:

$$h_1(x,\xi) = e^{-i\phi(x,\xi)} \int_{R^{2n}} e^{i\langle x-z,\eta\rangle}p(x,\eta)\,e^{i\phi(z,\xi)}\chi(x,z)\,a(z,\xi)\,dz\,d\eta =$$

$$= e^{-i\phi(x,\xi)}P(e^{i\phi(\cdot,\xi)}\chi(\cdot,\xi)\,a(\cdot,\xi)) ;$$

$$h_2(x, \xi) = e^{-i\phi(x, \xi)} \int_{R^{2n}} e^{i\langle x - z, \eta\rangle} p(x, \eta)\, e^{i\phi(z, \xi)} [1 - \chi(x, z)]\, a(z, \xi)\, dz\, d\eta =$$

$$= e^{-i\phi(x, \xi)} P(e^{i\phi(\cdot, \xi)} [1 - \chi(\cdot, \xi)]\, a(\cdot, \xi)) .$$

Then $h(x, \xi) = h_1(x, \xi) + h_2(x, \xi)$ and consequently $PA = A_{h_1, \phi} + A_{h_2, \phi}$.

(I) We will show that $h_2 \in \Lambda_{\varrho, \mathcal{P}}^{-\infty}$ and this will allow us to prove the theorem with $h(x, \xi)$ replaced by $h_1(x, \xi)$.

Consider the operator

$$\mathcal{M} = -i \sum_{j=1}^{n} \left(\frac{x_j - z_j}{|x - z|^2} \right) \partial_{\eta_j} .$$

A direct calculation shows that $\mathcal{M}[e^{i\langle x - z, \eta\rangle}] = e^{i\langle x - z, \eta\rangle}$.

Since $1 - \chi(x, z) = 0$ in a neighbourhood of the hyperplane $x = z$, we can use the operator $\mathcal{M}$ to write $h_2(x, \xi)$ as follows, with an arbitrary integer $r > 0$:

$$|h_2(x, \xi)| = \left| \int_{R^{2n}} e^{i\langle x - z, \eta\rangle} e^{i\phi(z, \xi)} (1 - \chi(x, z))\, a(z, \xi)\, {}^t\mathcal{M}^r [p](x, z, \eta)\, dz\, d\eta \right| .$$

We define now:

$$\mathcal{L}_{\mathcal{P}, \phi} = \sum_{(\alpha, \beta) \in \mathcal{P}} \left(\frac{z^{2\alpha}}{w_{\mathcal{P}}^2(z, \nabla_z \phi(z, \xi)) + R(z, \xi)} \right) D_z^{2\beta}$$

with $R(x, \xi)$ as in part I of the proof of Proposition 3.3, the role of the variables z and ξ being now exchanged.

We observe that $\mathcal{L}_{\mathcal{P}, \phi} [e^{i\phi(z, \xi)}] = e^{i\phi(z, \xi)}$, so:

$$(4.1) \quad |h_2(x, \xi)| = \left| \int_{R^{2n}} e^{ix\eta} e^{i\phi(z, \xi)} {}^t\mathcal{L}_{\mathcal{P}, \phi}^s \left[e^{-iz\eta} (1 - \chi(x, z))\, a(z, \xi)\, {}^t\mathcal{M}^r[p](x, z, \eta) \right] dz\, d\eta \right| \leq$$

$$\leq \int_{R^{2n}} \left| {}^t\mathcal{L}_{\mathcal{P}, \phi}^s \left[e^{-iz\eta} (1 - \chi(x, z))\, a(z, \xi)\, {}^t\mathcal{M}^r [p](x, z, \eta) \right] \right| dz\, d\eta$$

for every $s \in N$.

It is possible to give an explicit expression for ${}^t\mathcal{M}^r [p](x, z, \eta)$ as sum of terms of type $\partial_\eta^\theta p(x, \eta) \times \prod_{j=1}^{r} (x_{k_j} - z_{k_j}) / |x - z|^2$, where $|\theta| = r$.

In (4.1) we apply then ${}^t\mathcal{L}_{\mathcal{P}, \phi}^s$ to the product $a(z, \xi)\, b(x, z, \eta)$ where we have set:

$$b(x, z, \eta) = \partial_\eta^\theta p(x, \eta)\, e^{-iz\eta} (1 - \chi(x, z)) \prod_{j=1}^{r} \frac{x_{k_j} - z_{k_j}}{|x - z|^2} .$$

We show that for every multiindex α we have for all $k > 0$:

$$|\partial_z^\alpha b(x, z, \eta)| \leq c_{\alpha, k} \langle x \rangle^{-k} \langle z \rangle^{-k} \langle \eta \rangle^{-k} \tag{4.2}$$

for some constants $c_{\alpha, k}$.

In fact it is easy proved by induction that for every δ:

$$\left| \partial_z^\delta \left[(1 - \chi(x, z)) \prod_{j=1}^{r} \frac{x_{k_j} - z_{k_j}}{|x - z|^2} \right] \right| \leq \tilde{c}_{r, \delta} \langle (x, z) \rangle^{-r\varrho}$$

for suitable positive constants $\tilde{c}_{r, \delta}$ and for every $(x, z) \in \boldsymbol{R}^{2n}$.

Then using the preceding estimate and the fact that $p \in \Lambda_{\varrho, \mathscr{P}}^t$, we have from the Leibnitz rule:

$$|\partial_z^\alpha b(x, z, \eta)| \leq C \sum_{\beta + \gamma + \delta = \alpha} |\partial_\eta^\theta p(x, \eta)| \, |\partial_z^\gamma e^{-iz\eta}| \left| \partial_z^\delta \left[(1 - \chi(x, z)) \prod_{j=1}^{r} \frac{x_{k_j} - z_{k_j}}{|x - z|^2} \right] \right| \leq$$

$$\leq c_{r, \alpha} w_{\mathscr{P}}^{t - \varrho r}(x, \eta) \langle \eta \rangle^{|\gamma|} \langle (x, z) \rangle^{-r\varrho}$$

where C, $c_{r, \alpha}$ are positive constants.

As r is arbitrary this prooves (4.2).

We consider now the explicit expression:

$${}^t\mathscr{L}_{\mathscr{P}, \phi}[a(x, \xi) b(x, z, \eta)] = \sum_{(\alpha, \beta) \in \mathscr{P}} D_z^{2\beta} \left[\left(\frac{z^{2\alpha}}{w_{\mathscr{P}}^2(z, \nabla_z \phi(z, \xi)) + R(z, \xi)} \right) a(z, \xi) b(x, z, \eta) \right].$$

Estimates analogous to (4.2) hold for $z^{2\alpha} b(x, z, \eta)$ and the hypothesis (3.7) implies that $(a(z, \xi))/(w_{\mathscr{P}}^2(z, \nabla_z \phi(z, \xi)) + R(z, \xi)) \in \Lambda_{\varrho, \mathscr{P}}^{m-2}$, so applying s times the operator ${}^t\mathscr{L}_{\mathscr{P}, \phi}$ we get:

$${}^t\mathscr{L}_{\mathscr{P}, \phi}^s [a(x, \xi) b(x, z, \eta)] = \sum_p [a_p(x, \xi) b_p(x, z, \eta)]$$

where $a_p(x, \xi) \in \Lambda_{\varrho, \mathscr{P}}^{m-2s}$ and $b_p(x, z, \eta)$ satisfy (4.2), involving z-derivatives of $b(x, z, \eta)$ of order not greater then Ms, M a suitable integer. From (4.2) and from the estimates defining the classes $\Lambda_{\varrho, \mathscr{P}}^{m-2s}$ we obtain that for every k exists an s such that for a suitable $C > 0$:

$$|{}^t\mathscr{L}_{\mathscr{P}, \phi}^s [a(x, \xi) b(x, z, \eta)]| \leq C \langle z \rangle^{-k} \langle \eta \rangle^{-k} \langle x \rangle^{-k} \langle \xi \rangle^{-k}.$$

This shows at the same time that the integrals defining $h_2(x, \xi)$ make sense and, since for the derivatives of $h_2(x, \xi)$ one can procede in a completely analogous, way, that $h_2(x, \xi) \in \Lambda_{\varrho, \mathscr{P}}^{-\infty}$.

(II) It remains now to proove that $h_1(x, \xi) \in \Lambda_{\varrho, \mathscr{P}}^{m+t}$ and that the required asimptotic expansion holds.

We set $\theta = \eta - \nabla_x \phi(x, \xi)$, and we get

$$h_1(x, \xi) = (2\pi)^{-n} \int_{R^n} e^{ix\theta} p(x, \nabla_x \phi(x, \xi) + \theta)\, d\eta \int_{R^n} e^{-iz\theta} e^{i\psi(x, z, \xi)} \chi(x, z)\, a(z, \xi)\, dz .$$

From the Taylor formula (see for instance PARENTI-SEGALA [PS]) we have:

$$h_1(x, \xi) = \sum_{|\alpha| \leq k} \left(\frac{1}{\alpha!}\right) (\partial_\eta^\alpha p)(x, \nabla_x \phi(x, \xi))\, D_z^\alpha [e^{i\psi(x, z, \xi)} \chi(x, z)\, a(z, \xi)]_{x=z} +$$

$$+ \sum_{|\alpha| \leq k} \left(\frac{1}{\alpha!}\right) R_\alpha(x, \xi),$$

with:

$$R_\alpha(x, \xi) = (2\pi)^{-n} \int_{R^n} e^{ix\theta} r_\alpha(x, \xi, \theta)\, \mathcal{F}_{z \to \theta} \{D_z^\alpha [e^{i\psi(x, z, \xi)} \chi(x, z)\, a(z, \xi)]\}\, d\theta ,$$

and

$$r_\alpha(x, \xi, \theta) = \int_0^1 (1 - t)^{k-1} (\partial_\eta^\alpha p)(x, \nabla_x \phi(x, \xi) + t\theta)\, dt .$$

But from Lemma 3.6:

$$\left(\frac{1}{\alpha!}\right) (\partial_\eta^\alpha p)(x, \nabla_x \phi(x, \xi))\, D_z^\alpha [e^{i\psi(x, z, \xi)} \chi(x, z)\, a(z, \xi)]_{x=z} \in \Lambda_{\varrho, \mathcal{P}}^{m+t-\varrho|\alpha|},$$

then we need only to show that for every $\mu \in N$ there is a constant $C > 0$ such that if $|\alpha|$ is sufficiently large we have:

$$|R_\alpha(x, \xi)| \leq C w_{\mathcal{P}}^{-\mu}(x, \xi). \tag{4.3}$$

Analogous estimates will be easily deduced for the derivatives of $R_\alpha(x, \xi)$.
Using Lemma 3.1 and Lemma 3.7 to estimate r_α we see that:

$$|r_\alpha(x, \xi, \theta)| \leq \tilde{C} w_{\mathcal{P}}^{t-\varrho|\alpha|}(x, \xi) \quad \text{if } |\theta| \leq C w_{\mathcal{P}}^{\varrho}(x, \xi) \tag{4.4}$$

for some suitable small constant $C > 0$ (see Lemma 3.7) and suitable great $\tilde{C} > 0$. This estimate is easily generalized to $\partial_\theta^\beta r_\alpha(x, \xi, \theta)$.
We take now $\chi^*(\eta) \in C_0^\infty(\boldsymbol{R}^n)$ such that:

$$\chi^*(\eta) = 1 \quad \text{for } |\eta| \leq \frac{C}{2},$$

$$\chi^*(\eta) = 0 \quad \text{for } |\eta| \geq C.$$

We consider first the integral:

$$I = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix\theta} \chi^* \left(\frac{\theta}{w_{\mathcal{P}}^{\varrho}(x, \xi)} \right) r_\alpha(x, \xi, \theta) \mathcal{F}_{z \to \theta} \{ D_z^\alpha [e^{i\psi(x, z, \xi)} \chi(x, z) a(z, \xi)] \} \, d\theta =$$

$$= \int_{\mathbf{R}^n} f_\alpha(x, \xi, x - u) D_u^\alpha [e^{i\psi(x, u, \xi)} \chi(x, u) a(u, \xi)] \, du$$

with

$$f_\alpha(x, \xi, v) = (2\pi)^{-n} \mathcal{F}_{\theta \to v} \left[r_\alpha(x, \xi, \theta) \chi^* \left(\frac{\theta}{w_{\mathcal{P}}^{\varrho}(x, \xi)} \right) \right].$$

Since on the support of $\chi^*(\theta/(w_{\mathcal{P}}^{\varrho}(x, \xi)))$ is $|\theta| \leq C w_{\mathcal{P}}^{\varrho}(x, \xi)$, then from (4.4) we have for every multiindex β:

$$|v^\beta f_\alpha(x, \xi, v)| \leq \text{const}\, (w_{\mathcal{P}}(x, \xi))^{t + \varrho n - \varrho|\alpha| - \varrho|\beta|}$$

(here and later on we indicate with *const* suitable positive constants).

This shows that for every positive constant L:

$$|f_\alpha(x, \xi, v)| \leq \text{const}\, (1 + |v| w_{\mathcal{P}}^{\varrho}(x, \xi))^{-2L} w_{\mathcal{P}}^{t + n - \varrho|\alpha|}(x, \xi).$$

If we set $L = L_1 + L_2$ we get for the integral I the estimate:

$$|I| \leq \text{cost}\, (w_{\mathcal{P}}(x, \xi))^{t + n - \varrho|\alpha|} \sup_u \{ |D_u^\alpha [e^{i\psi(x, u, \xi)} \chi(x, u) a(u, \xi)]| \times$$
$$\times (1 + |x - u| w_{\mathcal{P}}^{\varrho}(x, \xi))^{-2L_1} \} \int_{\mathbf{R}^n} (1 + |v| w_{\mathcal{P}}^{\varrho}(x, \xi))^{-2L_2} dv. \tag{4.5}$$

We assume L_2 great enough to let the integral converge and we give now estimates of the term with sup_u.

From the hypothesis (3.5) on ϕ we have that:

$$|\partial_{u_j} \phi(u, \xi) - \partial_{x_j} \phi(x, \xi)| \leq \text{const}\, (1 + |x - u| w_{\mathcal{P}}(x, \xi)).$$

It follows that:

$$|D_u^\alpha [e^{i\psi(x, u, \xi)} \chi(x, u) a(u, \xi)]| \leq \text{const}\, w_{\mathcal{P}}^{m + n + 1}(x, \xi)(1 + |x - u| w_{\mathcal{P}}(x, \xi))^{\varrho_\alpha} \tag{4.6}$$

wuit suitable $\varrho_\alpha \in \mathbf{R}^n$.

Finally from (4.5) and (4.6) we have:

$$|I| \leq \text{const}\, (w_{\mathcal{P}}(x, \xi))^{t + m + 2n + 1 - \varrho|\alpha|}.$$

As α can go to $-\infty$ we have obtained the conclusion for the part $|\theta| \leq \leq (C/2) w_{\mathcal{P}}^{\varrho}(x, \xi)$ of the domain of the integral defining $R_\alpha(x, \xi)$.

We consider now the case $|\theta| \geq (C/2) w_{\mathcal{P}}^{\varrho}(x, \xi)$.

We set $\omega(x, z, \xi, \theta) = z\theta - \psi(x, z, \xi)$, and we have:

$$\mathcal{F}_{z \to \theta}\{D_z^\alpha[e^{i\psi(x, z, \xi)}\chi(x, z)\, a(z, \xi)]\} =$$

$$= \sum_{\beta + \gamma + \delta = \alpha} C_{\beta\gamma\delta} \int_{R^n} e^{-iz\theta}\, \partial_z^\beta e^{i\psi(x, z, \xi)}\, \partial_z^\gamma \chi(x, z)\, \partial_z^\delta a(z, \xi)\, dz =$$

$$= \sum_{\beta + \gamma + \delta = \alpha} C_{\beta\gamma\delta} \int_{R^n} e^{-i\omega(x, z, \xi, \theta)}\, \sigma_\beta(x, z, \xi)\, \partial_z^\gamma \chi(x, z)\, \partial_z^\delta a(z, \xi)\, dz\,,$$

where σ_β are sums of products of derivatives of the function $\psi(x, z, \xi)$ that we can estimate according to (3.5).

We show now that in the region $|\theta| \geq C/2 w_{\mathscr{P}}^{\varrho}(x, \xi)$ we have:

$$|\nabla_z \omega(x, z, \xi, \theta)| \geq \text{const}\, w_{\mathscr{P}}^{\varrho}(x, \xi)\,. \tag{4.7}$$

Since, in view of (3.5), the second derivatives of the function ϕ are bounded, we have:

$$|\nabla_z \phi(z, \xi) - \nabla_x \phi(x, \xi)| \leq \text{const}\, |z - x| \tag{4.8}$$

But the presence of the cut-function $\chi(x, \xi)$ allows us to control the growth of $|z|$ by means of $|x|$, more precisely we can limit us to the domain:

$$|z - x|^2 \leq \varepsilon/2(1 + |x|^2 + |z|^2)^\varrho \tag{4.9}$$

with a suitable small $\varepsilon > 0$, and here we have: $|z| \leq a|x| + b$ where a, b are suitable positive constants.

Substituting in (4.9) we obtain:

$$|z - x| \leq (\varepsilon/2)^{1/2}(1 + |x|^2 + a^2|x|^2 + b^2 + 2ab|x|)^{\varrho/2}.$$

Then, if $|x| \geq 1$, it is:

$$|z - x| \leq C'(\varepsilon/2)^{1/2}(1 + |x|^2)^{\varrho/2} \leq C'(\varepsilon/2)^{1/2} w_{\mathscr{P}}^{\varrho}(x, \xi)\,;$$

and, if $|x| \leq 1$:

$$|z - x| \leq C'(\varepsilon/2)^{1/2} \leq C'(\varepsilon/2)^{1/2} w_{\mathscr{P}}^{\varrho}(x, \xi)$$

for some $C' > 0$. From (4.8) we obtain then:

$$|\nabla_z \phi(z, \xi) - \nabla_x \phi(x, \xi)| \leq C'(\varepsilon/2)^{1/2} w_{\mathscr{P}}^{\varrho}(x, \xi)\,. \tag{4.10}$$

As $|\theta| \geqslant (C/2) w_{\mathcal{P}}^{\varrho}(x, \xi)$ we have:

$$(4.11) \quad |\nabla_z \omega(x, z, \xi, \theta)| = |\theta - \nabla_z \phi(z, \xi) + \nabla_x \phi(x, \xi)| \geqslant$$
$$\geqslant \big| |\theta| - |\nabla_z \phi(z, \xi) + \nabla_x \phi(x, \xi)| \big| \geqslant (C/2) w_{\mathcal{P}}^{\varrho}(x, \xi) - C' \varepsilon^{1/2} w_{\mathcal{P}}^{\varrho}(x, \xi) \geqslant$$
$$\geqslant \operatorname{const} w_{\mathcal{P}}^{\varrho}(x, \xi).$$

as long as we choose ε small enough to have $C/2 > C' \varepsilon^{1/2}$.

This proves (4.7) which allows us to make use in (4.6) of the operator

$$\mathcal{L} = -i \sum_{j=1}^{n} c_j(x, z, \xi, \theta) \partial_{z_j} \quad \text{with} \quad c_j(x, z, \xi, \theta) = \frac{\partial_{z_j} \omega(x, z, \xi, \theta)}{|\nabla_z \omega(x, z, \xi, \theta)|^2},$$

integrating by parts:

$$(4.12) \quad \mathcal{F}_{z \to \theta}[D_z^{\alpha} e^{i\varphi(x, u, \xi)} \chi(x, z) a(z, \xi)] =$$
$$= \sum_{\beta + \gamma + \delta = \alpha} C_{\beta\gamma\delta} \int_{\mathbf{R}^n} e^{-i\omega(x, z, \xi, \theta)t} \mathcal{L}^s [\sigma_{\beta}(x, z, \xi) \partial_z^{\gamma} \chi(x, z) \partial_z^{\delta} a(z, \xi)] \, dz$$

where s is an arbitrary positive integer.

We set now $f(x, z, \xi) = \sigma_{\beta}(x, z, \xi) \partial_z^{\gamma} \chi(x, z) \partial_z^{\delta} a(z, \xi)$ and observe that there exists a constant $\tau \in \mathbf{R}^+$ such that for every multiindex α:

$$(4.13) \quad |\partial_z^{\alpha} f(x, z, \xi, \theta)| = \operatorname{const} \langle x \rangle^{\tau} \langle z \rangle^{\tau} \langle \xi \rangle^{\tau} \langle \theta \rangle^{\tau}.$$

Considering the operator ${}^t\mathcal{L}[\cdot] = -i \sum_{j=1}^{n} \partial_{z_j}[c_j(x, z, \xi, \theta)[\cdot]]$ we observe also that:

$$(4.14) \quad |\partial_z^{\alpha} c_j(x, z, \xi, \theta)| \leqslant \operatorname{const} \langle x \rangle^{-\nu} \langle z \rangle^{-\nu} \langle \xi \rangle^{-\nu} \langle \theta \rangle^{-\nu}.$$

for a suitable $\nu > 0$.

Therefore an iterated application of the operator ${}^t\mathcal{L}$ shows that for every $\mu > 0$ there exists an integer s such that:

$$|{}^t\mathcal{L}^s [\sigma_{\beta}(x, z, \xi) \partial_z^{\gamma} \chi(x, z) \partial_z^{\delta} a(z, \xi)]| \leqslant \operatorname{const} \langle x \rangle^{-\mu} \langle z \rangle^{-\mu} \langle \xi \rangle^{-\mu} \langle \theta \rangle^{-\mu} \leqslant$$
$$\leqslant \operatorname{const} \langle (x, \xi) \rangle^{-\mu} \langle z \rangle^{-\mu} \langle \theta \rangle^{-\mu}.$$

This guarantees the convergence of the integrals in the definition of R_{α}, and at the same time it proves (4.3).

5. – Fourier integral operators on $H_{\mathcal{P}}^m(\mathbf{R}^n)$ spaces.

So far we have defined Sobolev spaces $H_{\mathcal{P}}^m(\mathbf{R}^n) \subset S'(\mathbf{R}^n)$ ($m \in \mathbf{R}$) and Fourier integral operators $A_{a, \phi}$ defined on $S'(\mathbf{R}^n)$. Both were modelled on a fixed polyhedron $\mathcal{P} \subset \mathbf{R}^{2n}$, so it turns out quite natural to look at the behaviour of such operators with respect to the $H_{\mathcal{P}}^m(\mathbf{R}^n)$ spaces.

The result we obtain is that these operators are continuous provided that a further condition on the phase function $\phi(x, \xi)$ is assumed.

In order to prove this we will use a result of Asada-Fujiwara (for references see ASADA-FUJIWARA [AF], and HELFFER [H]). We will state it in semplified form after some preliminary definitions and then prove our theorem of continuity.

DEFINITION 5.1. – Let $a((x, \xi)) \in C^\infty(\boldsymbol{R}^{2n})$. We say that $a \in S^0_{0,0}$ if

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi)| \leq C_{\alpha\beta}, \qquad x \in \boldsymbol{R}^n, \quad \xi \in \boldsymbol{R}^n,$$

for suitable constants α, β.

DEFINITION 5.2. – We say that $\phi(x, \xi) \in C^\infty(\boldsymbol{R}^{2n})$ is an admissible phase function if:

— $\phi(x, \xi)$ is real valued;

— $\inf\limits_{(x, \xi) \in \boldsymbol{R}^{2n}} |\det[\partial_{x_j} \partial_{\xi_k} \mathcal{R}e\, \phi(x, \xi)]| \geq \delta_0$ for some positive constant δ_0;

— $\partial_x^\alpha \partial_\xi^\beta \phi \in S^0_{0,0}$ for $|\alpha + \beta| = 2$.

PROPOSITION 5.3 (ASADA-FUJIWARA [AF]). – Let $a \in S^0_{0,0}$ and let $\phi(x, \xi)$ be an admissible phase function. Then the Fourier integral operator

$$A_{a,\phi} u(x) = (2\pi)^{-n} \int\limits_{\boldsymbol{R}^n} e^{i\phi(x, \xi)} a(x, \xi) \hat{u}(\xi)\, d\xi$$

defines a continuous linear map:

$$A_{a,\phi} : L^2(\boldsymbol{R}^n) \to L^2(\boldsymbol{R}^n).$$

Coming back to our case we state first the following Lemma, whose proof is trivial:

LEMMA 5.4. – $\Lambda^0_{\varrho, \mathcal{P}} \subset S^0_{0,0}$.

We can now formulate our final result.

THEOREM 5.5. – Let $a(x, \xi)$ and $\phi(x, \xi)$ satisfy the same hypotheses as in Theorem 3.5 and suppose further that:

$$\inf_{(x, \xi) \in \boldsymbol{R}^{2n}} |\det[\partial_{x_j} \partial_{\xi_k} \phi(x, \xi)]| \geq \delta_0 \tag{5.4}$$

for some positive constant δ_0.

Then the Fourier integral operator $A_{a,\phi}$ defines a continuous linear map:

$$A_{a,\phi} : H^s_{\mathcal{P}}(\boldsymbol{R}^n) \to H^{s-m}_{\mathcal{P}}(\boldsymbol{R}^n)$$

for any $s \in \boldsymbol{R}$.

PROOF. – (I) First of all we remark that for every symbol $\sigma_0 \in \Lambda^0_{\varrho, \mathcal{P}}(\boldsymbol{R}^{2n})$ the following map is continuous:

$$A_{\sigma_0, \phi}: L^2(\boldsymbol{R}^n) \to L^2(\boldsymbol{R}^n). \tag{5.5}$$

This is an easy consequence of Asada Theorem (Proposition 5.3) as we see from Lemma 5.4 and from our assumptions on the phase function $\phi(x, \xi)$, which is admissable according to Definition 5.2.

We consider now a symbol $\sigma_s \in \Lambda^s_{\varrho, \mathcal{P}}$ and show that the composition $A_{\sigma_s, \phi} \circ A_{-s}$ is L^2-continuous (where A_{-s} is an operator defining the space $H^{-s}_{\mathcal{P}}(\boldsymbol{R}^n)$ and at the same time a parametrix of the operator A_s that defines $H^s_{\mathcal{P}}(\boldsymbol{R}^n)$).

From the proof of Theorem 3.4 we recall that

$$^tA_{\sigma_s, \phi} = \mathcal{F} \circ A_{^t\sigma_s, ^t\phi} \circ \mathcal{F}^{-1}$$

and similarly considering A_{-s} as a Fourier Integral operator:

$$^tA_{-s} = \mathcal{F} \circ \tilde{A}_{-s} \circ \mathcal{F}^{-1}$$

where $\tilde{A}_{-a}$ is the pseudodifferential operator with «transposed» symbol $^tw_{\mathcal{P}}(x, \xi) = w_{\mathcal{P}}(\xi, x)$ (incidentally we remark that $^tw_{\mathcal{P}}(x, \xi)$ is associated to the «transposed» polyhedron $^t\mathcal{P}$).

But the composition Theorem 3.5 implies that $\tilde{A}_{-s} \circ A_{^t\sigma_s, ^t\phi}$ is a Fourier integral operator with phase function $^t\phi$ and symbol in the class $\Lambda^0_{\varrho, \mathcal{P}}$, so from (5.5) we obtain that it is L^2-continuous.

Then the map $^tA^t_{-s}A_{\sigma_s, \phi} = \mathcal{F} \circ \tilde{A}_{-s} \circ A_{^t\sigma_s, ^t\phi} \circ \mathcal{F}^{-1}$ is L^2-continuous and so is its transposed $A_{\sigma_s, \phi} \circ A_{-s}$.

(II) We want to show now that there exists a positive constant C such that for any $u \in H^s_{\mathcal{P}}(\boldsymbol{R}^n)$:

$$\|A_{a, \phi} u\|_{s-m} \leq C \|u\|_s .$$

From the definition of the norms of the spaces $H^m_{\mathcal{P}}(\boldsymbol{R}^n)$ we easily see that this is equivalent to:

$$\|A_{s-m} A_{a, \phi} u\|_{L^2} \leq C \|A_s u\|_{L^2} .$$

As A_{-s} is a parametrix of A_s, we have $u = A_{-s}A_s u + Ru$ for some regularizing operator R, so (5.6) is true if $A_{s-m}A_{a, \phi}A_{-s}$ is an L^2-continuous operator.

To show this we just need to observe that $A_{s-m}A_{a, \phi}$ is, according to the composition Theorem 3.5, a Fourier integral operator with phase $\phi(x, \xi)$ and symbol in the class $\Lambda^0_{\varrho, \mathcal{P}}$, so part (I) implies the L^2-continuity of $A_{s-m}A_{a, \phi}A_{-s}$.

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