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On Some Abstract Variable Domain Hyperbolic Differential Equations (*).

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Abstract. – *The Cauchy problem is studied for a class of linear abstract differential equations of hyperbolic type with variable domain. Existence and uniqueness results are proved for (suitably defined) weak solutions. Some applications to P.D.E. are also given: they concern linear hyperbolic equations either in non-cylindrical regions or with mixed variable lateral conditions.*

1. – Introduction.

This paper is devoted to the study of the following abstract Cauchy problem. Let $T > 0$ be given. Let $V \subseteq H \equiv H^* \subseteq V^*$ be the standard complex Hilbert triplet. Let moreover $\{V(t)\}_{t \in [0, T]}$ be a family of closed subspaces of V. We are also given a «sufficiently smooth» operator function «$t \to A(t)$» from $[0, T]$ into $\mathcal{L}(V, V^*)$, such that, for a.a. $t \in [0, T]$, $A(t)$ is a hermitian and V-coercive operator. Let finally $u_0 \in V(0)$, $u_1 \in H$, and $f(t)$ be given, where «$t \to f(t)$» is a «sufficiently smooth» V^*-valued function. Then, we look for a V-valued function «$t \to u(t)$», which solves, *in some suitable weak sense*, the following Cauchy problem:

$$u(t) \in V(t), \qquad \text{for a.a. } t \in]0, T[; \tag{1.1}$$

$$u''(t) + A(t)\,u(t) = f(t), \qquad \text{in }]0, T[; \tag{1.2}$$

$$u(0) = u_0\,, \qquad u'(0) = u_1\,. \tag{1.3}$$

We remark that, in the previous setting (of variational type), (1.1)-(1.2)-(1.3) provides an abstract unified formulation for various important problems concerning linear hyperbolic equations. So, by this way, we can treat e.g. (for linear hyperbolic P.D.E.) the

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Cauchy-Dirichlet problems (with e.g. homogeneous Dirichlet boundary conditions) *in non-cylindrical regions*, and the Cauchy-mixed problems *with mixed variable lateral conditions*. (See Section 5 for such particular cases). On the other hand, such problems could also be treated by using other abstract approaches (e.g. the semigroups approach). In fact, we could consider the problem (1.2)-(1.3), where, for a.a. $t \in]0, T[$, $A(t)$ is an unbounded (self-adjoint positive) operator in the Hilbert space H, *and*, moreover, *the domain* $D(A(t))$ *of* $A(t)$ (*and also* $D(A^{1/2}(t))$) *can vary with* t (while, e.g., $u_0 \in D(A(0))$, $u_1 \in H$, and a H-valued $f(t)$ are given).

Bearing in mind this last approach, we are considering, in fact, linear *abstract variable domain hyperbolic problems*. For the sake of brevity, we will *also* use this expression for the «concrete» problems considered above, and for the abstract (variational) formulation (1.1)-(1.2)-(1.3).

It is well known that, for linear abstract variable domain hyperbolic problems, various important results were proved (in various settings, and under different assumptions) by KREĬN [19], KATO [18], DA PRATO [12], CARROLL and STATE [10], GOLDSTEIN [16], MAZUMDAR [25]. Afterwards, such problems were investigated by AROSIO [1], who generalized the previous results on this subject (except for the fact that $f(t) \equiv 0$ is taken in [1]). In fact, he considered the problem (1.2)-(1.3) (with $f(t) \equiv 0$), starting from the semigroups approach. Next, by using the fractional powers of $A(t)$, he introduced a continuous one-parameter family of weak problems (related, in a natural way, to (1.2)-(1.3)), where the extremal cases correspond, in fact, to the semigroups point of view and to a variational approach. Then he proved, for such a family of problems, various results concerning existence, uniqueness, and regularity of the solutions. (Notice that the paper [1] also provides a large bibliography, and a careful comparison with previous work on this subject). The abstract theory of [1] applies well to Cauchy-mixed problems for linear hyperbolic P.D.E., with mixed variable lateral conditions (see § 2.3 in [1]). However, it seems that such a theory is *not* suitable for applications to initial-boundary value problems *in non-cylindrical regions*.

The aim of our present paper is to study the problem (1.1)-(1.2)-(1.3) (with $f(t) \neq 0$, in general) in the variational setting we introduced at the beginning of this section. Since we do *not* require, in general, that $V(t)$ is dense in H, our abstract results *also* apply to initial boundary value problems in non-cylindrical regions (as well as they apply to Cauchy-mixed problems with mixed variable lateral conditions).

Let us describe the structure of our present paper. Section 2 concerns, at first, the notation and the main assumptions. Remark that, in particular, we require that «$t \to A(t)$» is a BV-function on $]0, T[$ (as AROSIO did in [2]), and that, for a.a. $t \in]0, T[$, $A(t)$ is a hermitian and *only weakly* V-coercive operator. Then, in Section 2, we give a (natural) notion of a *weak* (variational) solution to (1.1)-(1.2)-(1.3) (see Definition 2.1). The most part of Section 2 concerns various preparatory results, which will be used in the sequel. In particular, we consider the projection operator $P(t)$ of V onto $V(t)^{\perp}$, and we approximate it through a suitable regularization by convolution. We also approximate by convolution the data $A(t)$ and $f(t)$, by proceeding as in [2].

In Section 3 we prove, through Theorem 3.1, that a weak solution to (1.1)-(1.2)-(1.3) *actually exists*, when we assume (besides the «natural» hypotheses on $A(t), f(t), u_0, u_1$) that $\{V(t)\}$ is *a non-decreasing family with* t. (Observe that a similar monotonicity condition is needed for the existence results in [1]). Our main tool, in the proof, is a suitable procedure of penalization, which is based on the (regularized) projection operators $P(t)$. (Such a procedure was previously employed in [6], in a less refined form, for some linear abstract variable domain *parabolic* problems). Moreover, by means of Theorem 3.2, we obtain that the weak solutions have a further regularity property, when we *also* assume (besides the previous hypotheses) that the spaces $V(t)$ are dense in H.

Section 4 concerns our uniqueness result, which we keep quite separate from the previous ones. In fact, we prove, through Theorem 4.1, the *uniqueness* of the weak solution to (1.1)-(1.2)-(1.3), when we assume (besides the other «natural» hypotheses) that $\{V(t)\}$ is *a non-increasing family with t*. Our proof is based on a classical argument due to LADYZENSKAJA [20] (which we employ suitably, by also using some procedures of [2]). (Observe that, in [1], the uniqueness result, which corresponds to the variational point of view in [1], requires, in fact, that $V(t) \equiv$ *constant*).

Now, considering our Theorems 3.1 and 4.1 *together*, we see that we can get the existence *and* the uniqueness of the weak solution *only* in the case where $V(t) \equiv$ *constant*. However, *in our general abstract framework*, our results seem to be *rather sharp*: indeed, we do *not* expect, *in general*, the existence or the uniqueness of the weak solution, if the family $\{V(t)\}$ does *not* fulfil the appropriate monotonicity condition. A full account of this fact will be given in subsection 5.2 (see, in particular, Remark 5.6), by means of some «concrete» examples, to which our abstract theory applies. (On the other hand, it is clear (also see subsection 5.2) that, in some special and «concrete» cases (e.g. in the case of the classical (linear) wave operator), one can obtain sometimes even sharper results, by using a more direct approach).

Section 5 of our present paper concerns the main examples of applications of our abstract results: in particular, subsection 5.1 is devoted to Cauchy-mixed problems, with mixed variable lateral conditions, for linear hyperbolic P.D.E., while subsection 5.2 concerns initial-boundary value problems, for linear hyperbolic P.D.E., in non-cylindrical regions.

(As for further regularity of our weak solutions, we expect (by considering the results in [1]) that the assumption $V(t) \equiv V$ is needed. Then, in this case, such results are well known: we refer e.g. to [3], [15], [1], [2], and to the references therein).

Let us also remark that, for linear abstract variable domain problems, the literature is much wider in the parabolic case than in the hyperbolic one: let us only mention the very recent paper by SAVARÉ [27] (and the references therein). (Observe that we will also use, in the sequel, some preparatory results contained in [27]).

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2. – Notation and main assumptions; the weak solutions. Some preliminary results.

2.1. Let T be given, with $0 < T < +\infty$. We will use the well known spaces $C^k([0, T]; X)$, $L^p(0, T; X)$, $W^{k,p}(0, T; X)$, where X is some Banach space, $1 \leq p \leq + \infty$, and k is some non-negative integer.

Let now (as e.g. in LIONS and MAGENES [24])

$$V \subseteq H \equiv H^* \subseteq V^*, \quad \text{with } V \text{ separable}, \tag{2.1}$$

be the standard *complex* Hilbert triplet (i.e. V and H are two complex Hilbert spaces, with $V \subseteq H$, and such inclusion is continuous and dense; H is identified with its antidual space H^*, so that H can be continuously and densely imbedded in V^*). $(\cdot, \cdot)$ denotes both the scalar product in H and the antiduality pairing between V^* and V; $((\cdot, \cdot))$ denotes the scalar product in V. $\|\cdot\|$, $|\cdot|$, and $\|\cdot\|_*$ are respectively the norms in V, H, and V^*. We are also given

$$\text{a family } \{V(t)\}_{t \in [0,T]} \text{ of closed subspaces of } V. \tag{2.2}$$

We will use the following notation. Let p be given, with $1 \leq p \leq +\infty$; then:

$$L^p(0, T; V(t)) \equiv \{w(t) \in L^p(0, T; V) \,|\, w(t) \in V(t), \text{ for a.a. } t \in]0, T[\}. \tag{2.3}$$

Next, let us introduce a suitable function $A(t)$ of bounded variation on $]0, T[$, with values in the space $\mathcal{L}(V, V^*)$ of linear and continuous operators from V into V^*. So, throughout the present paper, we assume that:

$$A(t) \in BV(0, T; \mathcal{L}(V, V^*)); \tag{2.4}$$

$$(A(t)\,u, v) = \overline{(A(t)\,v, u)}, \quad \forall u, v \in V, \quad \text{and for a.a. } t \in]0, T[\tag{2.5}$$

(hermitian symmetry of $A(t)$);

(2.6) $\exists c > 0$, and $\exists \lambda \geq 0$ such that:

$$(A(t)\,u, u) + \lambda |u|^2 \geq c\|u\|^2, \quad \forall u \in V, \quad \text{and for a.a. } t \in]0, T[$$

(weak V-coerciveness of $A(t)$).

We will also consider functions $f(t)$, such that

$$f(t) = f_1(t) + f_2(t), \quad \text{where } f_1(t) \in L^1(0, T; H), \quad \text{and } f_2(t) \in BV(0, T; V^*). \tag{2.7}$$

Now, let us introduce a natural notion of weak solution to the problem (1.1)-(1.2)-(1.3). Towards this aim, let us firstly recall that $W^{1,p}(0, T; X) \subset C^0([0, T]; X)$, where $1 \leq p \leq +\infty$, and X is any Banach space. Then, we define

$$W \equiv \{w(t) \,|\, w(t) \in L^1(0, T; V(t)) \cap W^{1,1}(0, T; H); \, w(T) = 0\}. \tag{2.8}$$

Next, suppose that $u(t)$ satisfies «formally» (1.1)-(1.2)-(1.3), and «multiply» (in the antiduality pairing between V^* and V) both sides of (1.2) by any $w(t) \in W$; then, integrate from 0 to T. By integrating by parts the term $\int_0^T (u''(t), w(t))\,dt$, and taking into account (1.3), we are led, in a natural way, to give the following definition of weak solution to the problem (1.1)-(1.2)-(1.3).

DEFINITION 2.1. – Let (2.1)-(2.2) and (2.4)-(2.5)-(2.6) hold. Let $u_0 \in V(0)$, $u_1 \in H$, and $f(t)$ be given, where $f(t)$ satisfies (2.7). Then, we say that $u(t)$ *is a weak solution to the problem* (1.1)-(1.2)-(1.3), *if and only if*

$$\text{(2.9)} \qquad a)\ u(t) \in L^{\infty}(0, T; V(t)) \cap W^{1,\infty}(0, T; H), \qquad b)\ u(0) = u_0,$$

and the following equality holds:

$$\text{(2.10)} \qquad \int_0^T [(A(t)\, u(t), w(t)) - (u'(t), w'(t))]\, dt = \int_0^T (f(t), w(t))\, dt + (u_1, w(0)),$$

for every $w(t) \in W$ (where W is defined in (2.8)).

Let us observe (see Remark 5.6 in subsection 5.2 below) that, under the only previous assumptions, we have, in general, neither the existence nor the uniqueness of the weak solution to (1.1)-(1.2)-(1.3). We shall prove, in Section 3, an existence result, by assuming (besides the previous hypotheses) that $\{V(t)\}$ is a *non-decreasing* family with t, i.e. that

$$\text{(2.11)} \qquad V(t_1) \subseteq V(t_2), \qquad \forall t_1, t_2 \text{ such that } 0 \leqslant t_1 \leqslant t_2 \leqslant T.$$

The following subsections are devoted to some preparatory results for the proofs in Sections 3 and 4.

2.2. We assume, throughout this subsection, that (2.1), (2.2), and (2.11) hold. We extend the definition of $\{V(t)\}$ to all of $\mathbb{R}$, by setting

$$\text{(2.12)} \qquad V(t) \equiv V(0), \qquad \forall t < 0; \qquad V(t) \equiv V(T), \qquad \forall t > T.$$

Then, $\{V(t)\}_{t \in \mathbb{R}}$, is also a non-decreasing family with t. Next, we define, for every $t \in \mathbb{R}$:

$$\text{(2.13)} \qquad \begin{cases} \pi(t) \equiv \text{projection operator from } V \text{ onto } V(t); \\ P(t) \equiv I - \pi(t) \ \ (I = \text{identity operator}). \end{cases}$$

(Hence, $P(t)$ is the projection operator from V onto $V(t)^{\perp}$, where $V(t)^{\perp}$ denotes the orthogonal subspace of $V(t)$ in V. Of course, thanks to the previous assumptions, $\{V(t)^{\perp}\}_{t \in \mathbb{R}}$, is a *non-increasing* family with t). Now, consider, for every $u, v \in V$, the function «$\mathbb{R} \ni t \to ((P(t)u, v))$». Thanks to a general result due to SAVARÈ [27] (see Prop. 2.1 in [27]), we have that

(2.14) there exists a (at most) countable set $S_p \subset \mathbb{R}$, such that the function «$\mathbb{R} \ni t \to ((P(t)\, u, v))$» is continuous in $\mathbb{R} \setminus S_p$, $\quad \forall u, v \in V$.

Moreover, since $\{V(t)^{\perp}\}_{t \in \mathbb{R}}$ is a non-increasing family, it is clear that

(2.15) $\forall v \in V$, the function «$\mathbb{R} \ni t \to \|P(t)\, v\|^2 = ((P(t)\, v, v))$» is non-increasing, and hence differentiable for a.a. $t \in \mathbb{R}$, with a non-positive derivative.

We remark that the operator function «$\mathbb{R} \ni t \to P(t)$» (with values in $\mathcal{L}(V, V)$) has only, in general, little regularity (even in some «concrete» cases, where the family $\{V(t)\}$ «seems to depend smoothly on t»; see [7]). For our purposes, we need to approximate such a function by more regular maps: we do it by refining an argument used in [6]. So, let $\{\varphi_k(t)\}_{k \geq 1}$ a sequence of «smoothing kernels», i.e. such that ($\forall k \geq 1$)

$$\begin{cases} \varphi_k(t) \in C_0^\infty(\mathbb{R}); \quad \varphi_k(t) \geq 0, \quad \forall t \in \mathbb{R}; \\ \operatorname{supp}(\varphi_k(t)) \subset \left[-\frac{1}{k}, 0\right]; \quad \int_{-\infty}^{+\infty} \varphi_k(t)\, dt = 1. \end{cases} \tag{2.16}$$

Then, we define ($\forall k \geq 1$)

$$P_k(t) \equiv P(t) * \varphi_k(t) \equiv \int_{-\infty}^{+\infty} P(t-\tau)\, \varphi_k(\tau)\, d\tau, \qquad \forall t \in \mathbb{R}. \tag{2.17}$$

(The Bochner-type integral in (2.17) exists, thanks to the previous assumptions on the family $\{V(t)\}$. On the other hand, observe that the definition (2.17) makes sense, in general, by requiring only that the $\mathcal{L}(V, V)$-function «$t \to P(t)$» is strongly measurable). It is clear that ($\forall k \geq 1$)

$$«t \to P_k(t)» \in C^\infty(\mathbb{R}; \mathcal{L}(V, V)). \tag{2.18}$$

Now, we collect some properties of $P_k(t)$ in the following lemma.

LEMMA 2.1. – Let (2.1), (2.2), (2.11), and (2.12) hold. Let $P_k(t)$ be defined through (2.17) and (2.16). Then, for every $k \geq 1$, for any $t \in \mathbb{R}$, and any $u, v \in V$, it results that:

$$((P_k(t)\, u, v)) = ((u, P_k(t)\, v)); \tag{2.19}$$

$$\|P_k(t) v\|^2 \leq ((P_k(t)\, v, v)); \tag{2.20}$$

$$\|P_k(t)\, v\| \leq \|P(t)\, v\| \quad \text{(and, of course, } \|P(t)\, v\| \leq \|v\|\text{)}; \tag{2.21}$$

$$\frac{d}{dt}((P_k(t)\, v, v)) \equiv ((P_k'(t)\, v, v)) \leq 0. \tag{2.22}$$

PROOF. – (2.19) follows easily from the definition (2.17), and the fact that $P(t)$ is a projection operator. (2.20) results from the following calculation, thanks also to the Jensen inequality (see, e.g., RUDIN [26]):

$$\|P_k(t)\, v\|^2 = \left\| \int_{-\infty}^{+\infty} P(t-\tau)\, v \varphi_k(\tau)\, d\tau \right\|^2 \leq \left(\int_{-\infty}^{+\infty} \|P(t-\tau)\, v\|\, \varphi_k(\tau)\, d\tau \right)^2 \leq$$

$$\leq \int_{-\infty}^{+\infty} \|P(t-\tau)\, v\|^2 \varphi_k(\tau)\, d\tau = \int_{-\infty}^{+\infty} ((P(t-\tau)\, v, v))\, \varphi_k(\tau)\, d\tau = ((P_k(t)\, v, v)).$$

(2.21) follows readily from (2.17), by using the fact that $\operatorname{supp}(\varphi_k(t)) \subset [-1/k, 0]$ (see (2.16)), and that the function «$t \to \|P(t) v\|$» is non-increasing (see (2.15)). Finally, (2.22) results obviously from (2.17), (2.18), and (2.15). (Observe that, in this proof, (2.19) and (2.20) were deduced without using the monotonicity of the family $\{V(t)\}$).

The following remark will also be important in the sequel.

REMARK 2.1. – Under the assumptions in Lemma 2.1, consider any $t \in \mathbb{R}$, and any $w \in V(t)$. Then, it results that

$$((P_k(t)\, v, w)) = 0\,, \qquad \forall v \in V\,, \qquad \text{and } \forall k \geq 1\,. \tag{2.23}$$

In fact, $P_k(t)\, v$ belongs to $V(t)^\perp$: this follows from (2.17), since the family $\{V(t)^\perp\}$ is non-increasing, and $\mathrm{supp}(\varphi_k(t)) \subset [-1/k, 0]$ (see (2.16)).

In the sequel, we will also use the following lemma.

LEMMA 2.2. – Under the assumptions in Lemma 2.1, take any $u(t) \in L^2(0, T; V)$. Then, it results that

$$P_k(t)\, u(t) \to P(t)\, u(t) \ \text{ strongly in } L^2(0, T; V), \text{ as } k \to +\infty\,. \tag{2.24}$$

PROOF. – We have, from (2.14), that there exists a (at most) countable set $\tilde{S}_p \subset]0, T[$, such that

$$\forall \bar{t} \in]0, T[\setminus \tilde{S}_p, \quad \lim_{t \to \bar{t}} ((P(t)\, u, v)) = ((P(\bar{t})\, u, v))\,, \qquad \forall u,\ v \in V\,. \tag{2.25}$$

On the other hand, from (2.16) and (2.17), we get that ($\forall k \geq 1$, $\forall t \in \mathbb{R}$, $\forall u,\ v \in V$)

$$|((P_k(t)\, u, v)) - ((P(t)\, u, v))| = \left| \left(\left(\int_{-\infty}^{+\infty} [P(t-\tau)\, u - P(t) u]\, \varphi_k(\tau)\, d\tau, v \right) \right) \right| \leq \tag{2.26}$$

$$\leq \int_{-1/k}^{0} |((P(t-\tau)\, u - P(t)\, u, v))|\, \varphi_k(\tau)\, d\tau\,.$$

Hence, from (2.25) and (2.26), we deduce that

(2.27) $\forall \bar{t} \in]0, T[\setminus \tilde{S}_p$, $\forall u,\ v \in V$, $((P_k(\bar{t})\, u, v)) \to ((P(\bar{t})\, u, v))$, as $k \to +\infty$ (i.e. that, for a.a. $t \in]0, T[$, and $\forall u \in V$, $P_k(t)\, u \rightharpoonup P(t)\, u$ weakly in V, as $k \to +\infty$).

Take now any $u(t), v(t) \in L^2(0, T; V)$. Thanks to (2.27), (2.21), and to the Lebesgue dominated convergence theorem, we get that

$$\int_0^T ((P_k(t)\, u(t), v(t)))\, dt \to \int_0^T ((P(t)\, u(t), v(t)))\, dt\,, \qquad \text{as } k \to +\infty\,,$$

and hence that

$$\text{(2.28)} \qquad \begin{cases} \forall u(t) \in L^2(0, T; V), \\ P_k(t)\, u \rightharpoonup P(t)\, u \text{ weakly in } L^2(0, T; V), \qquad \text{as } k \to +\infty . \end{cases}$$

On the other hand, from (2.21), we also have that

$$\text{(2.29)} \qquad \forall u(t) \in L^2(0, T; V), \qquad \limsup_{k \to +\infty} \left(\int_0^T \|P_k(t)\, u(t)\|^2\, dt \right)^{1/2} \leqslant \left(\int_0^T \|P(t)\, u(t)\|^2\, dt \right)^{1/2} .$$

Hence, since $L^2(0, T; V)$ is a Hilbert space, (2.24) follows from (2.28) and (2.29).

REMARK 2.2. – In the paper [6] (devoted to a class of abstract variable domain differential equations of *parabolic* type), the author assumed, besides the monotonicity hypothesis (2.11), that, $\forall v \in V$, the V-valued function «$t \to P(t)v$» had to be strongly continuous in $[0, T]$. Now, we remark that the results in [6] still hold, if we *drop* there this last assumption (as we can see by reviewing the proofs in [6], in the light of Lemmas 2.1 and 2.2 above). Anyway, the recent results by SAVARÉ [27] generalized and improved considerably those in [6], even in the case where this assumption is removed.

2.3. For our proofs in the following sections, we also need to approximate the given functions «$t \to A(t)$» (see (2.4), (2.5), (2.6)), and «$t \to f_1(t)$», «$t \to f_2(t)$» (see (2.7)) by more regular maps. We do it in the present subsection, by also using the convolution method, and proceeding as AROSIO did in [2]. Let us introduce the following notation:

(2.30) if X is a Banach space, $a, b \in [-\infty, +\infty]$, and $g(t) \in BV(a, b; X)$, $\mathcal{V}(a, b; g(t); X)$ denotes the *total variation* of $g(t)$ in $]a, b[$.

Consider now the BV-functions «$t \to A(t)$» and «$t \to f_2(t)$». The left (resp. right) limits $A(t^-)$, $f_2(t^-)$ (resp. $A(t^+)$, $f_2(t^+)$) exist for every $t \in]0, T]$ (resp. $t \in [0, T[$). Then, denote by $\tilde{A}(t)$ (resp. $\tilde{f}_2(t)$) the extension of $A(t)$ (resp. $f_2(t)$) to all of $\mathbb{R}$, such that $\tilde{A}(t) = A(0^+)$ (resp. $\tilde{f}_2(t) = f_2(0^+)$) for every $t < 0$, and such that $\tilde{A}(t) = A(T^-)$ (resp. $\tilde{f}_2(t) = f_2(T^-)$) for every $t > T$. On the other hand, denote by $\tilde{f}_1(t)$ the extension of $f_1(t)$ to all of $\mathbb{R}$, such that $\tilde{f}_1(t) = 0$ for every $t < 0$ and every $t > T$. Consider now a sequence $\{\chi_k(t)\}_{k \geqslant 1}$ of «smoothing kernels», such that ($\forall k \geqslant 1$)

$$\text{(2.31)} \qquad \begin{cases} \chi_k(t) \in C_0^\infty(\mathbb{R}); \qquad \chi_k(t) \geqslant 0, \qquad \forall t \in \mathbb{R}; \\ \operatorname{supp}(\chi_k(t)) \subset \left[0, \dfrac{1}{k}\right]; \qquad \displaystyle\int_{-\infty}^{+\infty} \chi_k(t)\, dt = 1 . \end{cases}$$

Then, we define ($\forall k \geqslant 1$, $\forall t \in \mathbb{R}$):

$$\text{(2.32)} \qquad A_k(t) \equiv \tilde{A}(t) * \chi_k(t); \qquad f_{1k}(t) \equiv \tilde{f}_1(t) * \chi_k(t); \qquad f_{2k}(t) \equiv \tilde{f}_2(t) * \chi_k(t),$$

where the convolution product $*$ is meant as in (2.17). (The definitions in (2.32) make sense, since the convolution products are, in fact, Bochner-type integrals, which exist, thanks to the properties of «$t \to \tilde{A}(t)$», «$t \to \tilde{f}_1(t)$», and «$t \to \tilde{f}_2(t)$»). It is clear that ($\forall k \geq 1$):

$$\begin{cases} «t \to A_k(t)» \in C^\infty(\mathbb{R}; \mathcal{L}(V, V^*)); \quad «t \to f_{1k}(t)» \in C^\infty(\mathbb{R}; H); \\ «t \to f_{2k}(t)» \in C^\infty(\mathbb{R}; V^*). \end{cases} \tag{2.33}$$

We list now various properties of such functions: some of them are obvious; the others are proved in AROSIO [2]. Firstly, consider the functions «$t \to A_k(t)$». Thanks to (2.4), (2.5), (2.6), (2.31), (2.32), it results that ($\forall k \geq 1$):

$$(A_k(t)\, u, v) = \overline{(A_k(t)\, v, u)}, \qquad \forall u, v \in V, \qquad \forall t \in \mathbb{R}; \tag{2.34}$$

$$(A_k(t)\, u, u) + \lambda |u|^2 \geq c\|u\|^2, \qquad \forall u \in V, \qquad \forall t \in \mathbb{R}; \tag{2.35}$$

$$A_k(0) = A(0^+); \quad A_k'(0) = 0. \tag{2.36}$$

Moreover, by denoting by $|||\cdot|||$ the usual operator norm in $\mathcal{L}(V, V^*)$, and using the notation (2.30), we also have that:

$$\lim_{k \to +\infty} |||A_k(t) - A(t)||| = 0, \qquad \text{for a.a. } t \in]0, T[; \tag{2.37}$$

$$|||A_k(t)||| \leq \operatorname*{ess\,sup}_{\tau \in]0, T[} |||A(\tau)||| \equiv M, \qquad \forall t \in \mathbb{R}, \qquad \forall k \geq 1, \tag{2.38}$$

and hence, in particular,

$$A_k(t) \to A(t) \text{ strongly in } L^1(0, T; \mathcal{L}(V, V^*)), \text{ as } k \to +\infty; \tag{2.39}$$

$$\int_0^t |||A_k'(\tau)|||\, d\tau = \mathcal{V}(0, t; A_k(\tau); \mathcal{L}(V, V^*)) \leq \mathcal{V}(0, t; A(\tau); \mathcal{L}(V, V^*)), \qquad \forall t \in [0, T], \qquad \forall k \geq 1. \tag{2.40}$$

Next, let us consider the functions «$t \to f_{1k}(t)$», and «$t \to f_{2k}(t)$». Thanks to (2.7), (2.31) and (2.32), it results that:

$$f_{1k}(0) = 0, \quad f_{2k}(0) = f_2(0^+), \qquad \forall k \geq 1; \tag{2.41}$$

$$\begin{cases} a) \displaystyle\int_0^t |f_{1k}(\tau)|\, d\tau \leq \int_0^t |f_1(\tau)|\, d\tau, \qquad \forall t \in [0, T], \qquad \forall k \geq 1; \\ b) \; f_{1k}(t) \to f_1(t) \text{ strongly in } L^1(0, T; H), \text{ as } k \to +\infty; \end{cases} \tag{2.42}$$

$$\lim_{k \to +\infty} \|f_{2k}(t) - f_2(t)\|_* = 0, \qquad \text{for a.a. } t \in]0, T[; \tag{2.43}$$

$$\|f_{2k}(t)\|_* \leq \operatorname*{ess\,sup}_{\tau \in]0, T[} \|f_2(\tau)\|_* \equiv N, \qquad \forall t \in \mathbb{R}, \qquad \forall k \geq 1, \tag{2.44}$$

and hence, in particular,

(2.45) $$f_{2k}(t) \to f_2(t) \text{ strongly in } L^1(0, T; V^*), \text{ as } k \to +\infty ;$$

(2.46) $$\int_0^t \|f'_{2k}(\tau)\|_* \, d\tau = \mathcal{V}(0, t; f_{2k}(\tau); V^*) \leq \mathcal{V}(0, t; f_2(\tau); V^*), \qquad \forall t \in [0, T], \quad \forall k \geq 1 .$$

2.4. This last subsection is devoted, for the most part, to another approximation lemma, which we will use in Section 3, for the proof of our existence result. Firstly, let us introduce the following notation. Since (2.1) holds,

(2.47) we denote by J the canonical antiduality operator from V into V^*, which is defined by $(Ju, v) = ((u, v))$, $\forall u, v \in V$.

It is obvious that J is an isometric isomorphism of V onto V^*.

Now, we can state the following approximation lemma.

LEMMA 2.3. – Let (2.1), (2.2), (2.11), and (2.12) hold, and consider the definitions (2.13), (2.16), and (2.17). Let $A(t)$ (resp. $f_2(t)$) satisfy (2.4), (2.5), and (2.6) (resp. (2.7)), and consider $A(0^+)$, and $f_2(0^+)$ (as we did in subsection 2.3). Take any $u_0 \in V(0)$. Then, there exists a sequence $\{u_{0k}\}_{k \geq 1}$ such that

(2.48) $$\begin{cases} a) \ u_{0k} \in V, \quad \forall k \geq 1 ; \\ b) \ u_{0k} \to u_0 \text{ strongly in } V, \text{ as } k \to +\infty ; \\ c) \ [f_2(0^+) - A(0^+)\, u_{0k} - kJP_k(0)\, u_{0k}] \in H, \quad \forall k \geq 1 ; \\ d) \ k\|P_k(0) u_{0k}\| \leq c^{-1}, \quad \forall k \geq 1 \text{ (where } c \text{ is given in (2.6))} . \end{cases}$$

PROOF. – Firstly, for the sake of brevity, we define ($\forall k \geq 1$):

(2.49) $$f \equiv f_2(0^+); \qquad A \equiv A(0^+); \qquad A_{k,\lambda} \equiv A + \lambda I + kJP_k(0),$$

where λ is taken (and henceforth is *fixed*) as in (2.6). Thanks to (2.4), (2.6), (2.20), and (2.47), it is clear that ($\forall k \geq 1$):

(2.50) $$A_{k,\lambda} \in \mathcal{L}(V, V^*); \qquad (A_{k,\lambda} v, v) \geq c\|v\|^2, \qquad \forall v \in V \text{ (where } c \text{ is as in (2.6))} .$$

Hence, $\forall k \geq 1$, $A_{k,\lambda}$ is an *isomorphism of* V *onto* V^*. Moreover, we observe that, if $A_{k,\lambda}^{-1}$ denotes the inverse operator of $A_{k,\lambda}$, (2.50) also gives that ($\forall k \geq 1$):

(2.51) $$\|A_{k,\lambda}^{-1} w\| \leq c^{-1} \|w\|_*, \qquad \forall w \in V^* .$$

Next, by considering (2.49) and $u_0 \in V(0)$, we define:

(2.52) $$g \equiv f - Au_0 - \lambda u_0 \text{ (where } \lambda \text{ is as above)}.$$

Now, since $g \in V^*$, and since H is dense in V^*, we have that:

(2.53) there exists a sequence $\{g_k\}_{k \geq 1}$, such that: $g_k \in H$, $\forall k \geq 1$; $g_k \to g$ strongly in V^*, as $k \to +\infty$, and, in particular, $\|g_k - g\|_* \leq k^{-1}$, $\forall k \geq 1$.

Next, let us fix *any* $k \geq 1$, and consider the vector $f - g_k$ ($\in V^*$). Thanks to the properties of the operator $A_{k,\lambda}$, we get that:

(2.54) there exists a unique $u_{0k} \in V$, which satisfies $A_{k,\lambda} u_{0k} = f - g_k$.

Hence, we obtain, in fact, a sequence $\{u_{0k}\}_{k \geq 1}$ of elements of V. Let us show that such a sequence fulfils (2.48) *b*), *c*), and *d*). Firstly, from (2.54) and (2.49), it results that:

$$f - Au_{0k} - kJP_k(0)\, u_{0k} = g_k + \lambda u_{0k}\,, \qquad \forall k \geq 1\,, \tag{2.55}$$

and hence (thanks also to (2.53)) (2.48) *c*) is proved. Next, since $u_0 \in V(0)$, and since $P_k(0)\, v \in V(0)^{\perp}$, $\forall v \in V$, $\forall k \geq 1$ (see Remark 2.1), we also have that:

$$P_k(0) u_0 = 0\,, \qquad \forall k \geq 1\,. \tag{2.56}$$

Now, by using (2.51), (2.54), (2.52), (2.56), and (2.53), we obtain that:

$$\|u_{0k} - u_0\| = \|A_{k,\lambda}^{-1}(A_{k,\lambda}(u_{0k} - u_0))\| \leq c^{-1}\|A_{k,\lambda} u_{0k} - A_{k,\lambda} u_0\|_* = \tag{2.57}$$

$$= c^{-1}\|f - g_k - [A + \lambda I + kJP_k(0)]\, u_0\|_* = c^{-1}\|g - g_k\|_* \leq c^{-1} k^{-1}\,, \qquad \forall k \geq 1\,.$$

Hence, (2.48) *b*) is also proved (with, moreover, an estimate for $\|u_{0k} - u_0\|$; such an estimate is, in particular, a consequence of the estimate in (2.53)). Finally, thanks to (2.56), (2.21), and (2.57), we get that:

$$k\|P_k(0)\, u_{0k}\| = k\|P_k(0)(u_{0k} - u_0)\| \leq k\|u_{0k} - u_0\| \leq c^{-1}\,, \qquad \forall k \geq 1\,, \tag{2.58}$$

and (2.48) *d*) is also proved.

We shall also use, in the sequel, a suitable generalized Gronwall lemma. Now, we recall it (see e.g. BAIOCCHI [4], also for more general results in this direction).

LEMMA 2.4. – Let $g(t)$, $\alpha(t)$, $\beta(t)$ be given with: $0 \leq g(t) \in C^0([0, T])$; $0 \leq \alpha(t) \in L^1(0, T)$; $0 \leq \beta(t) \in L^1(0, T)$. Let g_0 be a non-negative constant. Assume that, for every $t \in [0, T]$,

$$g^2(t) \leq g_0^2 + \int_0^t \alpha(s)\, g(s)\, ds + \int_0^t \beta(s)\, g^2(s)\, ds \tag{2.59}$$

holds. Then, it results that

$$g(t) \leqslant 2\left[g_0 + \int_0^T \alpha(s)\,ds\right]\cdot \exp\left(2\int_0^T \beta(s)\,ds\right), \qquad \forall t \in [0, T]. \tag{2.60}$$

3. – Existence of weak solutions.

We prove, in this section, that a weak solution to (1.1)-(1.2)-(1.3) *actually exists*, when we assume that (besides the other «natural» hypotheses) *the monotonicity condition* (2.11) *holds*. (Such a solution is *not* unique, in general, as we shall see in Remark 5.6 of subsection 5.2 below).

THEOREM 3.1. – Let (2.1), (2.2), (2.11), and (2.4) (2.5), (2.6) hold. Take any $f(t)$ as in (2.7), any $u_0 \in V(0)$, and any $u_1 \in H$. Then, there exists a (not necessarily unique) weak solution $u(t)$ to the problem (1.1)-(1.2)-(1.3) (i.e. a function $u(t)$ satisfying (2.9) and (2.10)).

PROOF. – Our proof consists of the following steps a), b), c). Firstly, we use a procedure of penalization (with a suitable regularization of the data), in order to approximate (1.1)-(1.2)-(1.3) through a sequence of «regular» problems. Each of them has a unique solution, which is «smooth enough». Let $\{u_k(t)\}_{k\geqslant 1}$ denote the sequence of such solutions. In the step b), we prove some estimates for the functions $u_k(t)$ (independently of k) in some suitable norms. In the step c), we pass to the limit: thanks to the estimates in b), we can extract from $\{u_k(t)\}_{k\geqslant 1}$ a subsequence, which converges *to a* function $u(t)$ in some suitable weak topologies. Then, we prove that such $u(t)$ satisfies (2.9) and (2.10).

a) Firstly, by proceeding as in subsection 2.2, consider the definition (2.17) of $P_k(t)$. Next, by proceeding as in subsection 2.3, consider the definitions (2.32) of $A_k(t)$, $f_{1k}(t)$, and $f_{2k}(t)$. Now, since $u_0 \in V(0)$, Lemma 2.3 applies: so, let $\{u_{0k}\}_{k\geqslant 1}$ be a sequence satisfying (2.48) (where J is defined in (2.47)). Moreover, since V is dense in H, take

(3.1) a sequence $\{u_{1k}\}_{k\geqslant 1}$, such that $u_{1k} \in V$, $\forall k \geqslant 1$, and $u_{1k} \to u_1$ strongly in H, as $k \to +\infty$.

Then, take *any* integer $k \geqslant 1$, and consider the following problem:

$$u_k''(t) + A_k(t)\,u_k(t) + kJP_k(t)\,u_k(t) = f_{1k}(t) + f_{2k}(t), \qquad 0 < t < T; \tag{3.2}$$

$$u_k(0) = u_{0k}, \qquad u_k'(0) = u_{1k}. \tag{3.3}$$

Now, we have to recall that: $A_k(t)$ satisfies (2.33), (2.34), (2.35), (2.36); J is defined in (2.47); $P_k(t)$ satisfies (2.18), (2.19), (2.20); $f_{1k}(t)$ and $f_{2k}(t)$ satisfy (2.33), (2.41) (so that, in particular, $f_{1k}(0) + f_{2k}(0) = f_2(0^+)$, $\forall k \geqslant 1$); $u_{1k} \in V$ (see (3.1)); u_{0k} satisfies (2.48) (in particular, (2.48) a) and c)). Then, thanks to such properties, we can use a particular case of a general result by GILARDI [15] (see Teor. 4.4 of [15] with $h = 1$) to obtain that

(3.4) there exists a unique $u_k(t) \in C^1([0, T]; V) \cap C^2([0, T]; H)$, solution to (3.2)-(3.3).

b) Consider *any* integer $k \geq 1$, and take $u_k(t)$ as in (3.4). Since $u_k(t)$ is «smooth enough», we can «multiply» (in the antiduality pairing between V^* and V) both sides of (3.2) by $u_k'(t)$. By taking the real parts, and using (2.33), (2.34), (2.47), (2.18), (2.19), we get

$$\begin{aligned}(3.5)\quad & \frac{d}{dt}|u_k'(t)|^2 + \frac{d}{dt}(A_k(t)\,u_k(t), u_k(t)) - (A_k'(t)\,u_k(t), u_k(t)) + \\ & + k\frac{d}{dt}((P_k(t)\,u_k(t), u_k(t))) - k((P_k'(t)\,u_k(t), u_k(t))) = \\ & = 2\,\mathrm{Re}\,(f_{1k}(t), u_k'(t)) + 2\,\mathrm{Re}\left[\frac{d}{dt}(f_{2k}(t), u_k(t)) - (f_{2k}'(t), u_k(t))\right], \qquad \forall t \in]0, T[\,.\end{aligned}$$

Now, we take into account (2.22), and we integrate (3.5) from 0 to t ($0 \leq t \leq T$). By also using (2.36), (2.41), and (3.3), we obtain

$$\begin{aligned}(3.6)\quad & |u_k'(t)|^2 + (A_k(t)\,u_k(t), u_k(t)) + k((P_k(t)\,u_k(t), u_k(t))) \leq \\ & \leq |u_{1k}|^2 + (A(0^+)\,u_{0k}, u_{0k}) + \int_0^t (A_k'(s)\,u_k(s), u_k(s))\,ds + \\ & + k((P_k(0)\,u_{0k}, u_{0k})) + 2\int_0^t |(f_{1k}(s), u_k'(s))|\,ds + 2\,|(f_{2k}(t), u_k(t))| + \\ & + 2\,|(f_2(0^+), u_{0k})| + 2\int_0^t |(f_{2k}'(s), u_k(s))|\,ds\,, \qquad \forall t \in [0, T]\,.\end{aligned}$$

Now, it is obvious (from (2.48) *a*) and *b*), and from (3.1)) that there exist two positive numbers $M_1 = M_1(u_0)$ and $M_2 = M_2(u_1)$, such that

$$(3.7)\qquad \|u_{0k}\| \leq M_1 \quad \text{and} \quad |u_{1k}| \leq M_2\,, \qquad \forall k \geq 1\,.$$

Then, starting from (3.6), and using (2.35), (2.20), (3.7), (2.38), (2.48) *d*), (2.44), and some standard inequalities, we get

$$\begin{aligned}(3.8)\quad & |u_k'(t)|^2 + c\|u_k(t)\|^2 + k\|P_k(t)u_k(t)\|^2 \leq \lambda\,|u_k(t)|^2 + M_2^2 + MM_1^2 + \\ & + \int_0^t |||A_k'(s)|||\cdot\|u_k(s)\|^2\,ds + M_1 c^{-1} + 2\int_0^t |f_{1k}(s)|\cdot|u_k'(s)|\,ds + \\ & + \frac{2}{c}N^2 + \frac{c}{2}\|u_k(t)\|^2 + 2NM_1 + 2\int_0^t \|f_{2k}'(s)\|_*\cdot\|u_k(s)\|\,ds\,, \qquad \forall t \in [0, T]\,.\end{aligned}$$

On the other hand, an easy calculation shows that

$$(3.9)\qquad |u_k(t)|^2 \leqslant 2\,|u_{0k}|^2 + 2t\int_0^t |u_k'(s)|^2\,ds\,, \qquad \forall t\in[0,T].$$

Hence (denoting by $\tilde c$ a positive number such that $|v| \leqslant \tilde c\,\|v\|$, $\forall v\in V$; see (2.1)), we obtain (from (3.8) and (3.9))

$$(3.10)\qquad \min(1;\,c/2)\cdot[\,|u_k'(t)|^2 + \|u_k(t)\|^2] \leqslant$$

$$\leqslant 2\lambda\tilde c^2 M_1^2 + M_2^2 + MM_1^2 + M_1 c^{-1} + \frac{2}{c}N^2 + 2NM_1 +$$

$$+\int_0^t [2\lambda T + |||A_k'(s)|||\,]\cdot[\,|u_k'(s)|^2 + \|u_k(s)\|^2]\,ds +$$

$$+2\int_0^t [\,|f_{1k}(s)| + \|f_{2k}'(s)\|_*]\cdot[\,|u_k'(s)|^2 + \|u_k(s)\|^2]^{1/2}\,ds\,, \qquad \forall t\in[0,T].$$

Then, we use here Lemma 2.4, and we get

$$(3.11)\qquad [\,|u_k'(t)|^2 + \|u_k(t)\|^2]^{1/2} \leqslant$$

$$\leqslant 2\max(1;\,2c^{-1})\left[\left(2\lambda\tilde c^2 M_1^2 + M_2^2 + MM_1^2 + M_1 c^{-1} + \frac{2}{c}N^2 + 2NM_1\right)^{1/2} + \right.$$

$$\left. +2\int_0^T (\,|f_{1k}(s)| + \|f_{2k}'(s)\|)_*\,ds\right]\cdot$$

$$\cdot\exp\left(2\max(1;\,2c^{-1})\int_0^T [2\lambda T + |||A_k'(s)|||\,]\,ds\right), \qquad \forall t\in[0,T].$$

Hence, we take into account (2.40), (2.42) a), and (2.46). So, from (3.11), we obtain that

(3.12) there exists a positive number C, depending on T, V, H, $A(t)$, $f_1(t)$, $f_2(t)$, u_0, u_1, *but independent of* k *and of* t, such that $|u_k'(t)| + \|u_k(t)\| \leqslant C$, $\forall t\in[0,T]$, $\forall k\geqslant 1$.

Now, we come back to (3.8), and we take into account (3.12), (2.1), (2.40), (2.42) a), and (2.46). Then, we also get that

(3.13) there exists a positive number D, *independent of* k *and of* t, such that $k^{1/2}\|P_k(t)\,u_k(t)\| \leqslant D$, $\forall t\in[0,T]$, $\forall k\geqslant 1$.

Hence, we have obtained that

$$\{u_k(t)\}_{k\geq 1} \text{ is bounded in } L^\infty(0, T; V); \tag{3.14}$$

$$\{u_k'(t)\}_{k\geq 1} \text{ is bounded in } L^\infty(0, T; H); \tag{3.15}$$

$$\{k^{1/2} P_k(t) u_k(t)\}_{k\geq 1} \text{ is bounded in } L^\infty(0, T; V). \tag{3.16}$$

c) Firstly, it is obvious (from (3.16)) that

$$P_k(t) u_k(t) \to 0 \text{ strongly in } L^\infty(0, T; V), \text{ as } k \to +\infty . \tag{3.17}$$

Moreover, thanks to (3.14) and to (3.15), we can extract from $\{u_k(t)\}_{k\geq 1}$ *a subsequence, still denoted by* $\{u_k(t)\}_{k\geq 1}$, such that, as $k \to +\infty$,

$$u_k(t) \rightharpoonup u(t) \text{ weakly star in } L^\infty(0, T; V), \text{ and also weakly in } L^2(0, T; V); \tag{3.18}$$

$$u_k'(t) \rightharpoonup u'(t) \text{ weakly star in } L^\infty(0, T; H), \text{ and also weakly in } L^2(0, T; H). \tag{3.19}$$

We will show that $u(t)$ satisfies, in fact, (2.9) and (2.10). Firstly, it is obvious that $u(t) \in L^\infty(0, T; V) \cap W^{1,\infty}(0, T; H)$. To verify that (2.9) *a*) holds, we need only to prove that

$$u(t) \in V(t) \quad \text{for a.a. } t \in]0, T[. \tag{3.20}$$

Towards this aim, consider the above subsequence $\{u_k(t)\}_{k\geq 1}$, and take *any* $v(t) \in L^2(0, T; V)$. By using (2.19), (3.18), and Lemma 2.2, we get that

$$\int_0^T ((P(t)\, u(t) - P_k(t)\, u_k(t), v(t)))\, dt = \int_0^T ((P(t)\, u(t) - P_k(t)\, u(t), v(t)))\, dt + \tag{3.21}$$

$$+ \int_0^T ((u(t) - u_k(t), P_k(t)\, v(t)))\, dt \to 0 , \ \textit{as} \ \mathrm{k} \to +\infty ,$$

i.e. that

$$P_k(t) u_k(t) \rightharpoonup P(t)\, u(t) \text{ weakly in } L^2(0, T; V), \text{ as } k \to +\infty . \tag{3.22}$$

Hence, from (3.17) and (3.22), we deduce that $P(t) u(t) = 0$ for a.a. $t \in]0, T[$. Then, thanks also to the definition (2.13), (3.20) is proved. (Actually (see Remark 3.2 below), we shall obtain something better than (3.20), i.e. that $u(t) \in V(t)$, *for every* $t \in [0, T]$).

Next, we verify that (2.9) *b*) also holds. Firstly, we have, obviously, that $u(t) \in C^0([0, T]; H)$. Moreover, we can readily deduce, from (3.18) and (3.19), that, in particular,

$$u_k(0) \rightharpoonup u(0) \text{ weakly in } H , \text{ as } k \to +\infty . \tag{3.23}$$

On the other hand, thanks to (3.3) and to (2.48) *b*), we also have that

$$u_k(0) \to u_0 \text{ strongly in } V , \text{ as } k \to +\infty . \tag{3.24}$$

Hence, (2.9) *b*) is proved. (Also see Remark 3.1 below, for some complementary observations).

Finally, we show that $u(t)$ satisfies (2.10). Towards this aim, take *any* $w(t) \in W$ (where W is defined in (2.8)), and «multiply» (in the antiduality pairing between V^* and V) both sides of (3.2) by $w(t)$. Thanks to Remark 2.1, we get

$$(3.25)\qquad (u_k''(t), w(t)) + (A_k(t)\, u_k(t), w(t)) = (f_{1k}(t) + f_{2k}(t), w(t))\,, \qquad \text{for a.a } t \in]0, T[\,.$$

Next, integrate (3.25) from 0 to T. Then, by making an integration by parts, and using (2.8) and (3.3), we obtain

$$(3.26)\qquad \int_0^T (A_k(t)\, u_k(t), w(t))\, dt - \int_0^T (u_k'(t), w'(t))\, dt = $$
$$= (u_{1k}, w(0)) + \int_0^T (f_{1k}(t), w(t)) dt + \int_0^T (f_{2k}(t), w(t))\, dt\,.$$

Take now, in (3.26), *any element* $u_k(t)$ *of the above subsequence* $\{u_k(t)\}_{k \geq 1}$, *satisfying* (3.18) *and* (3.19). Thanks to (2.8), (3.1), (3.19), (2.42) *b*), it is clear that, as $k \to +\infty$,

$$(3.27)\qquad \begin{cases} (u_{1k}, w(0)) \to (u_1, w(0))\,; \\ \displaystyle\int_0^T (u_k'(t), w'(t))\, dt \to \int_0^T (u'(t), w'(t))\, dt\,; \\ \displaystyle\int_0^T (f_{1k}(t), w(t))\, dt \to \int_0^T (f_1(t), w(t))\, dt\,. \end{cases}$$

On the other hand, we claim that, as $k \to +\infty$,

$$(3.28)\qquad \begin{cases} a) \displaystyle\int_0^T (A_k(t)\, u_k(t), w(t))\, dt \to \int_0^T (A(t)\, u(t), w(t))\, dt\,; \\ b) \displaystyle\int_0^T (f_{2k}(t), w(t))\, dt \to \int_0^T (f_2(t), w(t))\, dt\,. \end{cases}$$

(Bearing in mind (2.8), it is clear that *only* $w(t) \in L^1(0, T; V)$ can be used in (3.28), as *only* $w(t) \in W^{1,1}(0, T; H)$ was used in (3.27)). To verify (3.28) *a*), we use firstly (2.34), and we take into account (3.18). Then (3.28) *a*) is proved, if we are able to show that

$$(3.29)\qquad A_k(t)\, w(t) \to A(t)\, w(t) \ \text{ strongly in } L^1(0, T; V^*),\ \text{ as } k \to +\infty\,.$$

Now, (3.29) is, in fact, true, thanks to (2.37), (2.38), and to the Lebesgue dominated convergence theorem. Similarly, (3.28) *b*) can also be proved, by using (2.43), (2.44), and the Lebesgue theorem again.

Hence, from (3.26), (3.27), and (3.28), we get that $u(t)$ satisfies (2.10). So, Theorem 3.1 is completely proved.

For the sequel, we also need the following notation:

(3.30) if X is any Banach space, then X_w denotes the space X endowed with its weak topology.

REMARK 3.1. – Bearing in mind (2.9), we observe that, in particular, any such $u(t)$ satisfies $u(t) \in L^\infty(0, T; V) \cap C^0([0, T]; H)$. Hence (thanks also to (2.1)), we can use Lemma 8.1 of chap. 3 of LIONS and MAGENES [24] to obtain that

(3.31) any weak solution $u(t)$ to (1.1)-(1.2)-(1.3) (defined through Def. 2.1 above) satisfies, moreover, $u(t) \in C^0([0, T]; V_w)$.

Then, note that the initial condition (2.9) *b*) can also be meant in the sense of $C^0([0, T]; V_w)$ (and not only in the sense of $C^0([0, T]; H)$). On the other hand, in the context of Definition 2.1, the initial condition $u'(0) = u_1$ of (1.3) can only be meant, *in general*, in the sense that (2.10) holds true. However, we will show (by means of Theorem 3.2 below) that, *under a suitable additional assumption* (i.e. (3.33) below), any weak solution $u(t)$, obtained through Theorem 3.1, satisfies moreover $u'(t) \in C^0([0, T]; H_w)$ (and in this sense $u'(0) = u_1$ can also be meant).

REMARK 3.2. – The weak solutions $u(t)$ (defined through Def. 2.1) satisfy, in particular, (3.20). Now, we can prove that, moreover,

(3.32) *when* (2.11) *also holds* (e.g. in the case of Theorem 3.1 above), *it results that* $u(t) \in V(t)$, *for every* $t \in [0, T]$.

Towards this aim, let us firstly observe that $u(0) = u_0 \in V(0)$. Next, define $E = \{t \in [0, T] \mid u(t) \in V(t)\}$. Since (3.20) holds, we have that E is dense in $[0, T]$. Now, take any $\bar{t} \in]0, T]$, such that $\bar{t} \notin E$. Then, there exists a *non-decreasing* sequence $\{t_n\}_{n \geq 1}$, with $t_n \in E$, $\forall n \geq 1$, and such that $t_n \to \bar{t}$, as $n \to +\infty$. On the other hand, thanks to (3.31), it results that $u(t_n) \rightharpoonup u(\bar{t})$ weakly in V, as $n \to +\infty$. Now, we have that $u(t_n) \in V(t_n)$, $\forall n \geq 1$, and hence, thanks to (2.11), that $u(t_n) \in V(\bar{t})$, $\forall n \geq 1$. Then, since $V(\bar{t})$ is weakly closed, we obtain that $u(\bar{t}) \in V(\bar{t})$, and the proof of (3.32) is complete. (We refer to SAVARÉ [27] for the proof of several interesting properties of the families $\{V(t)\}$, also under more general assumptions).

We want to stress the fact that, in Theorem 3.1, the *only* assumptions on the family $\{V(t)\}$ are (2.2) and (2.11). Hence, the family $\{V(t)\}$ can actually have, in particular, «jump discontinuities».

We want also to stress the fact that, in Theorem 3.1, we do *not* make *any density assumption of the spaces* $V(t)$ *in the space* H (so that, in particular, Theorem 3.1 *also* applies well to initial-boundary value problems for linear hyperbolic P.D.E. *in non-cylindrical regions*; see subsection 5.2 below). On the other hand, we can prove that, when we make such a density assumption too, then any weak solution, obtained through Theorem 3.1, has an additional regularity property.

THEOREM 3.2. – Let all of the assumptions of Theorem 3.1 hold. Suppose, moreover, that

$$V(0) \text{ is dense in } H \tag{3.33}$$

(and hence, thanks to (2.11), *any* $V(t)$ $(0 \leq t \leq T)$ is dense in H). Then, every weak solution $u(t)$ to (1.1)-(1.2)-(1.3), obtained through Theorem 3.1, satisfies moreover $u'(t) \in C^0([0, T]; H_w)$ (and in this sense $u'(0) = u_1$ can also be meant).

PROOF. – We adapt here, to our framework, a technique which was used by AROSIO [1], [2] (see, in particular, p. 159 in [2] and p. 191 in [1]).

We start as we did in the part a) of the proof of Theorem 3.1. So, let us consider the sequences $\{P_k(t)\}$, $\{A_k(t)\}$, $\{f_{1k}(t)\}$, $\{f_{2k}(t)\}$, $\{u_{0k}\}$, $\{u_{1k}\}$. Firstly, let us remark (as in [2]) that

(3.34) for every $k \geq 1$, the modulus $\sigma_k(\delta)$ of uniform continuity in $[0, T]$ of

$F_k(t) \equiv \int_0^t |f_{1k}(s)|\, ds$ does not exceed the modulus $\sigma(\delta)$ of uniform continuity of

$F(t) \equiv \int_0^t |f_1(s)|\, ds$ in $[0, T]$.

Next, let us proceed as in the proof of Theorem 3.1. So, consider the problems (3.2)-(3.3), the corresponding results (3.4), and the sequence $\{u_k(t)\}_{k \geq 1}$ of the corresponding solutions $u_k(t) \in C^1([0, T]; V) \cap C^2([0, T]; H)$ (satisfying, in particular, (3.12)). Let $\{u_k(t)\}_{k \geq 1}$ also denote a subsequence satisfying (3.18) and (3.19) (where, as we know, $u(t)$ fulfils (2.9) and (2.10)). We will show that

$$\forall h \in H, \text{ the sequence } \{(u_k'(t), h)\}_{k \geq 1} \text{ is equicontinuous on } [0, T]. \tag{3.35}$$

Now, since also (3.12) holds, we can use the Ascoli-Arzelà theorem, to deduce that there exists a subsequence, still denoted by $\{(u_k'(t), h)\}_{k \geq 1}$, which converges (to $(u'(t), h)$, of course) in $C^0([0, T])$, as $k \to +\infty$. Hence, we have that $u'(t) \in C^0([0, T]; H_w)$, and (thanks also to (3.3) and to (3.1)) it is clear that in this sense $u'(0) = u_1$ can also be meant.

Then, the present theorem is proved, if we verify that (3.35) holds. Towards this aim, take *any* $v \in V(0)$, and «multiply» by v (in the antiduality pairing between V^* and V) both sides of (3.2). Then, since $P_k(t)\, u_k(t) \in V(t)^\perp \subseteq V(0)^\perp$ (see Remark 2.1), we get

$$\frac{d}{dt}(u_k'(t), v) + (A_k(t)\, u_k(t), v) = (f_{1k}(t) + f_{2k}(t), v)\,, \qquad \forall t \in]0, T[\,. \tag{3.36}$$

Next, we integrate (3.36) from t_1 to t_2 $(0 \leq t_1 \leq t_2 \leq T)$, and we take into account (2.38), (2.44), (3.12), and (3.34). Hence, we obtain

$$(3.37) \qquad |(u_k'(t_2) - u_k'(t_1), v)| \leqslant \left| \int_{t_1}^{t_2} (A_k(t)\, u_k(t), v)\, dt \right| + \left| \int_{t_1}^{t_2} (f_{1k}(t) + f_{2k}(t), v)\, dt \right| \leqslant$$

$$\leqslant \|v\| \cdot \left[\int_{t_1}^{t_2} |||(A_k(t)\; ||| \cdot \|u_k(t)\|\, dt + \int_{t_1}^{t_2} \|f_{2k}(t)\|_*\, dt \right] +$$

$$+ |v| \int_{t_1}^{t_2} |f_{1k}(t)|\, dt \leqslant \|v\| \cdot (MC + N)(t_2 - t_1) + |v|\, \sigma(t_2 - t_1), \qquad \forall k \geqslant 1 .$$

Take now *any* $h \in H$. Then, by using (3.12) and (3.37), we get, *for every* $v \in V(0)$, *and for any* t_1 *and* t_2 *with* $0 \leqslant t_1 \leqslant t_2 \leqslant T$,

$$(3.38) \qquad |(u_k'(t_2) - u_k'(t_1), h)| \leqslant |(u_k'(t_2) - u_k'(t_1), h - v)| + |(u_k'(t_2) - u_k'(t_1), v)| \leqslant$$

$$\leqslant 2C|h - v| + \|v\|(MC + N)(t_2 - t_1) + |v|\, \sigma(t_2 - t_1), \qquad \forall k \geqslant 1.$$

Hence, thanks to (3.38) and to assumption (3.33), it is clear that (3.35) holds true.

Finally, by reviewing the proof of Theorem 3.2, we see that (by the same method) we can obtain, in fact, the following *more general* result.

PROPOSITION 3.1. – Let all of the assumptions of Theorem 3.1 hold. Suppose, moreover, that there exists $\bar{t} \in [0, T[$ such that

$$(3.39) \qquad V(\bar{t}) \text{ is dense in } H$$

(and hence, thanks to (2.11), *any* $V(t)$, with $\bar{t} \leqslant t \leqslant T$, is also dense in H). Then, every weak solution $u(t)$ to (1.1)-(1.2)-(1.3), obtained through Theorem 3.1, satisfies moreover $u'(t) \in C^0([\bar{t}, T]; H_w)$.

4. – Uniqueness of the weak solution.

We prove, in this section, *the uniqueness* of the weak solution to (1.1)-(1.2)-(1.3), when we assume that (besides the other «natural» hypotheses) $\{V(t)\}$ *is a non-increasing family* (see (4.1) below). However (see Remark 5.6 in subsection 5.2 below), it can happen that, *under only such assumptions*, a weak solution to (1.1)-(1.2)-(1.3) does *not* exist.

Bearing in mind Definition 2.1 (and since our problem is a *linear* problem), we have that our uniqueness result is given by the following theorem.

THEOREM 4.1. – Let (2.1), (2.2), (2.4), (2.5), and (2.6) hold. Assume, moreover, that

$$(4.1) \qquad V(t_1) \supseteq V(t_2), \qquad \forall t_1,\ t_2 \text{ such that } 0 \leqslant t_1 \leqslant t_2 \leqslant T .$$

Let $u(t)$ satisfy (2.9) and (2.10), with $u_0 = 0$, $u_1 = 0$, and $f(t) = 0$, for a.a. $t \in]0, T[$. Then, it results that $u(t) = 0$, $\forall t \in [0, T]$.

PROOF. – We adapt, to our framework, a procedure which was used by AROSIO [2] (see p. 156 in [2]). (We have here some slight supplementary difficulties, since, differently from [2], our operator $A(t)$ is only *weakly* V-coercive). Such a procedure relies on a classical argument, which is due to LADYZENSKAJA [20] (also see, e.g., LIONS [21], LIONS and MAGENES [24]).

So, let us fix *any* $s \in]0, T]$, and define

$$v(t) = -\int_t^s u(\tau)\, d\tau\,, \qquad \forall t \in [0, s]; \qquad v(t) = 0, \ \ \forall t \in [s, T]. \tag{4.2}$$

Since $u(t)$ satisfies (2.9) (with $u_0 = 0$), and since (4.1) holds, we have that $v(t)$ is an «admissible test-function», i.e. that $v(t) \in W$ (where W is defined in (2.8)). In fact, we have something better, i.e. that $v(t) \in C^0([0, T]; V)$, $v(t) \in V(t)$, $\forall t \in [0, T]$, $v'(t) \in L^\infty(0, T; V(t))$, along with $v(T) = 0$.

Now, take $w(t) = v(t)$ in (2.10) (where $u_1 = 0$ and $f(t) \equiv 0$). So, by considering the real parts, we get (thanks also to $u(0) = 0$)

$$\operatorname{Re}\int_0^s (A(t)\, v'(t), v(t))\, dt = \operatorname{Re}\int_0^s (u'(t), u(t))\, dt = \frac{1}{2}\, |u(s)|^2 . \tag{4.3}$$

Next, go back to subsection 2.3, and recall the definition (2.32) of $A_k(t)$ ($k \geqslant 1$, $t \in \mathbb{R}$), along with the properties (2.33) and (2.34). Then, from (4.3), we obtain, for every integer $k \geqslant 1$,

$$\begin{aligned} \frac{1}{2}\, |u(s)|^2 &= \operatorname{Re}\int_0^s (A_k(t)\, v'(t), v(t))\, dt + \operatorname{Re}\int_0^s ([A(t) - A_k(t)]\, v'(t), v(t))\, dt = \\ &= \frac{1}{2}\, (A_k(s)\, v(s), v(s)) - \frac{1}{2}\, (A_k(0)\, v(0), v(0)) - \\ &\quad - \frac{1}{2}\int_0^s (A_k'(t)\, v(t), v(t))\, dt + \operatorname{Re}\int_0^s ([A(t) - A_k(t)]\, v'(t), v(t))\, dt . \end{aligned} \tag{4.4}$$

Now, by using (4.2), (2.36), (2.40) (along with the notation (2.30)), we get from (4.4), for every integer $k \geqslant 1$,

$$\begin{aligned} \frac{1}{2}\, |u(s)|^2 &\leqslant -\frac{1}{2} (A(0^+)\, v(0), v(0)) + \\ &\quad + \frac{1}{2}\left(\sup_{t \in [0, s]} \|v(t)\|\right)^2 \cdot \mathcal{V}(0, s; A(t); \mathcal{L}(V, V^*)) + \\ &\quad + \left(\operatorname*{ess\,sup}_{t \in]0, s[} \|v'(t)\|\right) \cdot \left(\sup_{t \in [0, s]} \|v(t)\|\right) \cdot \int_0^s |||A(t) - A_k(t)|||\, dt . \end{aligned} \tag{4.5}$$

Next, we pass to the limit as $k \to +\infty$, and we take into account (2.39). By also using (2.6), we obtain

$$|u(s)|^2 + c\|v(0)\|^2 \leqslant \lambda |v(0)|^2 + \Big(\sup_{t \in [0,s]} \|v(t)\|\Big)^2 \cdot \mathcal{V}(0, s; A(t); \mathcal{L}(V, V^*)) . \tag{4.6}$$

(Remark that, when $A(t)$ is «more regular» than in (2.4) (e.g. when $A(t) \in W^{1,1}(0, T; \mathcal{L}(V, V^*))$), (4.6) can be deduced directly from (4.3), (2.5), (2.6), without using the approximation through the «smooth» operators $A_k(t)$. In the general case (i.e. when (2.4) holds), the above procedure comes from AROSIO [2]). Now, we define

$$z(t) = \int_0^t u(\tau)\, d\tau , \qquad \forall t \in [0, T], \tag{4.7}$$

and we observe that (thanks also to (4.2))

$$v(t) = z(t) - z(s), \qquad \forall t \in [0, s], \text{ and, in particular, } v(0) = -z(s). \tag{4.8}$$

On the other hand, starting from (4.7), an easy calculation shows that

$$|z(s)|^2 \leqslant s \int_0^s |u(t)|^2\, dt . \tag{4.9}$$

Then, we put (4.8) and (4.9) in (4.6). So, by considering that an *arbitrary* $s \in]0, T]$ was taken, and that $u(0) = z(0) = 0$, we get

$$\begin{aligned} |u(s)|^2 + c\|z(s)\|^2 \leqslant \lambda s \int_0^s |u(t)|^2\, dt &+ \\ + 4\Big(\sup_{t \in [0,s]} \|z(t)\|^2\Big) \cdot \mathcal{V}(0, s; A(t); \mathcal{L}(V, V^*)) , &\qquad \forall s \in [0, T]. \end{aligned} \tag{4.10}$$

Next, thanks to (2.4), we have that *there exists* $\delta \in]0, T]$, *such that* $\mathcal{V}(0, \delta; A(t); \mathcal{L}(V, V^*)) < c/8$ (e.g.), and hence we obtain

$$|u(s)|^2 + c\|z(s)\|^2 \leqslant \lambda s \int_0^s |u(t)|^2\, dt + \frac{c}{2}\Big(\sup_{t \in [0,s]} \|z(t)\|^2\Big), \qquad \forall s \in [0, \delta]. \tag{4.11}$$

Now, from (4.11), we can readily deduce that

$$\sup_{t \in [0,s]} \|z(t)\|^2 \leqslant 2\lambda c^{-1} s \int_0^s |u(t)|^2\, dt , \qquad \forall s \in [0, \delta], \tag{4.12}$$

and hence, going back to (4.11), we get

$$|u(s)|^2 \leq 2\lambda\delta \int_0^s |u(t)|^2 \, dt, \qquad \forall s \in [0, \delta]. \tag{4.13}$$

Then, by using Lemma 2.4 (in a very particular case), *we obtain that* $u(t) = 0$, $\forall t \in [0, \delta]$. Let us prove that, *in fact*, $u(t) = 0$, $\forall t \in [0, T]$. By assuming that $\delta < T$, and proceeding as in [2], we define $\gamma \equiv \sup\{t \in [0, T] \,|\, u(s) = 0, \forall s \in [0, t]\}$; hence $\gamma \geq \delta$. Suppose that $\gamma < T$. Then, it results that $u(\gamma) = 0$. Hence, by also using the above procedure (when γ is taken in place of 0), we can obtain that $u(t) \equiv 0$ in a right neighbourhood of γ. Then, we have a contradiction. So, the proof of Theorem 4.1 is complete.

REMARK 4.1. – We want to stress the fact that, in Theorem 4.1, the *only* assumptions on the family $\{V(t)\}$ are (2.2) and (4.1). So, in particular, $\{V(t)\}$ can have «jump discontinuities». On the other hand, we do *not* require *any* density assumption of the spaces $V(t)$ into the space H. Moreover, by reviewing the above proof of Theorem 4.1, we can observe that the monotonicity assumption (4.1) was *only* used to get that the test function $v(t)$ (defined through (4.2)) satisfies $v(t) \in V(t)$ for (a.a.) $t \in [0, T]$. Anyway (as we will see in Remark 5.6 of subsection 5.2 below), when (4.1) does *not* hold, we *cannot* expect, *in general*, the uniqueness of the weak solution $u(t)$ to (1.1)-(1.2)-(1.3).

REMARK 4.2. – Bearing in mind the results of Theorems 3.1 and 4.1, we can obtain the existence *and* the uniqueness of the weak solution $u(t)$ to (1.1)-(1.2)-(1.3), when (besides the other assumptions) we have that $V(t) = \tilde{V}$, $\forall t \in [0, T]$, where $\tilde{V}$ is a (fixed) closed subspace of V. If, in particular, $\tilde{V} = V$, we deduce from Definition 2.1 that ($u''(t) \in L^1(0, T; V^*)$ and) $u(t)$ satisfies (1.2) in the sense (e.g.) of $\mathcal{D}'(0, T; V^*)$. In this case, Theorems 3.1, 3.2, 4.1 (along with Remark 3.1) give the existence and the uniqueness of $u(t) \in C^0([0, T]; V_w) \cap C^1([0, T]; H_w)$, solution to (1.2)-(1.3). (In fact, in this case, such result is well known, and one also has that $u(t) \in C^0([0, T]; V) \cap C^1([0, T]; H)$; see AROSIO [2], Thm. 1.1).

REMARK 4.3. – Our paper concerns the *forward* Cauchy problem (1.1)-(1.2)-(1.3). We can *also* consider the corresponding *backward* Cauchy problem, i.e. we take (1.1)-(1.2) again, but we replace (1.3) with

$$u(T) = u_0, \qquad u'(T) = u_1, \tag{4.14}$$

where $u_0 \in V(T)$ and $u_1 \in H$ are given. We can define (similarly to Definition 2.1) a natural notion of weak solution to (1.1)-(1.2)-(4.14), and we can prove (similarly to Sections 3 and 4 of the present paper) the correponding results, concerning the existence or the uniqueness. The conclusions are similar to the ones in the present paper for (1.1)-(1.2)-(1.3), *except for the fact that*, considering (1.1)-(1.2)-(4.14), *we have to assume* (4.1) (resp. (2.11)) *instead of* (2.11) (resp. (4.1)) for the existence (resp. uniqueness) results.

5. – Some examples and remarks.

This section is devoted to some examples of applications of the «abstract» results in Sections 3 and 4. In particular, we deal, in subsection 5.1, with Cauchy-mixed problems, with mixed variable lateral conditions, for linear hyperbolic P.D.E.. On the other hand, subsection 5.2 concerns Cauchy-Dirichlet problems (also for linear hyperbolic P.D.E.) in non-cylindrical regions.

5.1. Let $T > 0$ be given. Let Ω be an open bounded subset of $\mathbb{R}^N$, whose boundary $\Gamma \equiv \partial\Omega$ is a $(N-1)$-dimensional manifold of class (e.g.) C^1. Let Q denote the open cylinder $Q \equiv \Omega \times]0, T[$, and $\Sigma \equiv \Gamma \times]0, T[$ the lateral boundary of Q. Let moreover $\{\Gamma_0(t)\}_{t \in [0, T]}$ be a family of C^1-submanifolds (with boundary) of Γ, and define

$$\Sigma_0 \equiv \bigcup_{0 < t < T} \Gamma_0(t) \times \{t\}; \qquad \Sigma_1 \equiv \Sigma \backslash \overline{\Sigma}_0 . \tag{5.1}$$

Let us consider (formally) the following second order linear differential operator $\mathcal{A}$, with variable coefficients:

$$\mathcal{A}u \equiv -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}(x, t)\,\frac{\partial u}{\partial x_j}\right) + c(x, t)\,u\,, \tag{5.2}$$

where $a_{ij}(x, t)$ $(i, j = 1, \ldots, N)$ and $c(x, t)$ are given (complex-valued) functions (defined in $\overline{Q}$). Let finally $\nu_{\mathcal{A}} = \nu_{\mathcal{A}}(x, t)$ be the related conormal vector to Σ. Then, we consider, *in a formal way*, the following Cauchy-mixed problem. Given $f(x, t)$ (defined in Q), and $u_0(x)$, $u_1(x)$ (both defined in Ω), to find $u(x, t)$ (defined in $\overline{Q}$), such that:

$$\begin{cases} a) \ \dfrac{\partial^2 u}{\partial t^2}(x, t) + \mathcal{A}u(x, t) = f(x, t) \quad \text{in } Q\,; \\ b) \ u(x, t) = 0 \quad \text{on } \Sigma_0\,; \\ c) \ \dfrac{\partial u}{\partial \nu_{\mathcal{A}}}(x, t) = 0 \quad \text{on } \Sigma_1\,; \\ d) \ u(x, 0) = u_0(x) \quad \text{in } \Omega\,; \\ e) \ \dfrac{\partial u}{\partial t}(x, 0) = u_1(x) \quad \text{in } \Omega\,. \end{cases} \tag{5.3}$$

By also proceeding *in a formal way*, let $u(x, t)$ satisfy (5.3). Take any «sufficiently smooth» function $w(x, t)$ in $\overline{Q}$, such that $w(x, t) = 0$ on Σ_0, and such that $w(x, T) = 0$, $\forall x \in \Omega$. Then, multiply both sides of (5.3) $a)$ by $\overline{w(x, t)}$, and integrate on $\overline{Q}$. By using the Green theorem and the integration by parts formula, we obtain that $u(x, t)$ satisfies:

$$(5.4)\quad \begin{cases} a) \displaystyle\sum_{i,j=1}^{N} \int_0^T dt \int_\Omega a_{ij}(x, t)\, \frac{\partial u}{\partial x_i}(x, t)\, \overline{\frac{\partial w}{\partial x_j}(x, t)}\, dx + \\ \qquad + \displaystyle\int_0^T dt \int_\Omega c(x, t)\, u(x, t)\, \overline{w(x, t)}\, dx - \int_0^T dt \int_\Omega \frac{\partial u}{\partial t}(x, t)\, \overline{\frac{\partial w}{\partial t}(x, t)}\, dx = \\ \qquad = \displaystyle\int_0^T dt \int_\Omega f(x, t)\, \overline{w(x, t)}\, dx + \int_\Omega u_1(x)\, \overline{w(x, 0)}\, dx\,, \\ \qquad \text{for every «sufficiently smooth» } w(x, t) \text{ in } \overline{Q}, \text{ with } w(x, t) = 0 \text{ on } \Sigma_0, \\ \qquad \text{and with } w(x, T) = 0, \quad \forall x \in \Omega\,; \\ b)\ u(x, 0) = u_0(x) \quad \text{in } \Omega\,; \\ c)\ u(x, t) = 0 \quad \text{on } \Sigma_0\,. \end{cases}$$

(5.4) gives a *formal* definition of a weak solution u to the problem (5.3). Now, we set (5.4) in the framework of Definition 2.1. Towards this aim, we take

$$(5.5)\qquad H = L^2(\Omega), \qquad V = H^1(\Omega) \text{ (and hence } V^* = (H^1(\Omega))^*)\,,$$

so that (2.1) holds. Moreover, we take, for every $t \in [0, T]$,

$$(5.6)\qquad V(t) = H^1_{\Gamma_0(t)}(\Omega) \equiv \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_0(t) \text{ (in the sense of traces)}\}\,,$$

which is, of course, a closed subspace of V, and is *dense* in H. It is clear that

(5.7) (2.11) (resp. (4.1)) holds iff $\{\Gamma_0(t)\}_{t \in [0, T]}$ is a non-increasing (resp. non-decreasing) family with t.

Now, let us consider the operator $\mathcal{A}$. We assume that:

$$(5.8)\quad \begin{cases} a)\ «t \to a_{ij}(\cdot, t)» \ (i, j = 1, \ldots, N) \text{ and } «t \to c(\cdot, t)» \\ \quad \text{belong to the space } BV(0, T; L^\infty(\Omega))\,; \\ b)\ a_{ij}(x, t) = \overline{a_{ji}(x, t)} \text{ for a.a. } (x, t) \in Q \quad (i, j = 1, \ldots, N), \\ \quad \text{and the function } c(x, t) \text{ is real-valued}\,; \\ c)\ \exists \alpha > 0\,, \quad \text{such that } \displaystyle\sum_{i,j=1}^{N} a_{ij}(x, t)\, \xi_i \bar{\xi}_j \geq \alpha \sum_{i=1}^{N} |\xi_i|^2, \\ \qquad\qquad \forall (\xi_1, \ldots, \xi_N) \in \mathbb{C}^N, \text{ and for a.a. } (x, t) \in Q\,. \end{cases}$$

Then, we define, for a.a. $t \in]0, T[$, and for any $u, v \in V = H^1(\Omega)$:

$$(5.9) \qquad a(t; u, v) \equiv \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x, t) \frac{\partial u}{\partial x_i}(x) \overline{\frac{\partial v}{\partial x_j}(x)}\, dx + \int_\Omega c(x, t)\, u(x)\, \overline{v(x)}\, dx .$$

Thanks to (5.8) a), it is clear that $\{a(t; \cdot, \cdot) \mid \text{for a.a. } t \in]0, T[\}$ is a family of sesquilinear and continuous forms on $V \times V = H^1(\Omega) \times H^1(\Omega)$. Then, we can define, for a.a. $t \in]0, T[$, and for any $u, v \in V$:

$$(5.10) \qquad (A(t)\, u, v) \equiv a(t; u, v),$$

where $(\cdot, \cdot)$ denotes the antiduality pairing between V^* and V. We have thus defined a family $\{A(t) \mid \text{for a.a. } t \in]0, T[\}$ of linear and continuous operators from V into V^* (which are related, in a natural way, to $\mathcal{a}$ and to V). Then, it is clear that, for such a family $\{A(t)\}$, (2.4), (2.5), and (2.6) hold (thanks, respectively, to (5.8) a), b), and c)).

Take now:

$$(5.11) \qquad \begin{cases} a)\ f(x, t) = f_1(x, t) + f_2(x, t), \text{ where } «t \to f_1(\cdot, t)» \in L^1(0, T; L^2(\Omega)), \\ \quad \text{and } «t \to f_2(\cdot, t)» \in BV(0, T; (H^1(\Omega))^*); \\ b)\ u_0(x) \in V(0) = H^1_{\Gamma_0(0)}(\Omega); \qquad u_1(x) \in H = L^2(\Omega). \end{cases}$$

We now remember that we started from the «concrete» problem (5.3), and that we introduced its «formal» weak formulation (5.4). On the other hand, by means of (5.5), (5.6), and of (5.8)-...-(5.11), we can use Definition 2.1 to give *a precise weak formulation of* (5.3) (together with a precise notion of weak solution). For the sake of brevity, we do not rewrite here Definition 2.1 in the present case (i.e. by specifying V and H as in (5.5), $V(t)$ as in (5.6), etc.). Let us only remark that: (5.3) d) is given by (2.9) b) (also see (5.4) b)); (5.3) b) is contained in (2.9) a) (also see (5.4) c)); (5.3) a), c), e) are contained in (2.10) (also see (5.4) a)).

Now, by considering (5.7), we can use Theorems 3.1, 3.2 (along with Remarks 3.1 and 3.2), and Theorem 4.1, to obtain the following results.

PROPOSITION 5.1. – Let the above assumptions on Ω, Γ, $\{\Gamma_0(t)\}_{t \in [0, T]}$ hold. Take moreover: H and V as in (5.5), $V(t)$ as in (5.6), $A(t)$ as in (5.10) (where $a(t; u, v)$ is defined in (5.9), and (5.8) holds). Finally, take any f, u_0, u_1 as in (5.11). Then, the following conclusions hold.

a) Assume, moreover, that $\{\Gamma_0(t)\}_{t \in [0, T]}$ is *a non-increasing family with* t (and hence (2.11) holds). Then, there exists a (not necessarily unique) weak solution u to the problem (5.3) (in the sense of Definition 2.1). Such u satisfies $u(t) \in C^0([0, T]; V_w) \cap C^1([0, T]; H_w)$ (and in this sense (5.3) d) and e) can also be meant). Moreover, it results that $u(t) \in V(t)$ for *every* $t \in [0, T]$.

b) On the other hand, assume that $\{\Gamma_0(t)\}_{t \in [0, T]}$ is *a non-decreasing family with* t (and hence (4.1) holds). Then, if there exists a weak solution u to (5.3) (in the sense of Definition 2.1), it results that such u is unique.

REMARK 5.1. – Firstly, it is clear that, in Proposition 5.1 (a) and b)), the family $\{\Gamma_0(t)\}_{t\in[0,T]}$ (of C^1-submanifolds of Γ) is allowed to have «jump discontinuities» with respect to t.

Now, bearing in mind Definition 2.1 (in the present case of (5.3)), we recall that, in particular, (5.3) a) and c) are contained in (2.10): in such a general context, they cannot be better specified. On the other hand, it is clear that, by taking more regular data (e.g. a more regular f, etc.), we could deduce from (2.10) a more precise information on (5.3) a) and c) (e.g. the fact that (5.3) a) is satisfied in the sense of $\mathcal{D}'(Q)$, etc.). However, for the sake of brevity, we do not insist here on this point.

REMARK 5.2. – We observe that the problem (5.3) was also treated by AROSIO [1] (see §2.3 in [1]), *in the special case where* $\mathcal{A} = -\Delta_x$ *and* $f \equiv 0$. Indeed, in this case, he proved (as an application of his abstract theory [1]) a result like the one in the part a) of Proposition 5.1, i.e. that a (not necessarily unique) weak solution actually exists, when $\{\Gamma_0(t)\}_{t\in[0,T]}$ is a non-increasing family with t.

We also remark that the problem (5.3) was formerly studied by BARDOS and COOPER [5] (see Section 4 in [5]), *in the case where* $\mathcal{A}u = -\Delta_x u + u + |u|^{\varrho} u$, *with* $\varrho \geq 0$ (and where $f \in L^2(Q)$, $u_0 \in V(0) \cap L^{2+\varrho}(\Omega)$, $u_1 \in L^2(\Omega)$ are taken). They studied *directly* such a «concrete» (nonlinear) problem, and they were able to prove the existence *and* the uniqueness of the weak solution to (5.3), without assuming any monotonicity property of $\{\Gamma_0(t)\}_{t\in[0,T]}$, but under more general hypotheses on the set Σ_0. On the other hand, by means of Proposition 5.1, we have results for the problem (5.3), with a general (linear) operator $\mathcal{A}$ (although we have to require a monotonicity property of $\{\Gamma_0(t)\}_{t\in[0,T]}$).

REMARK 5.3. – Even if we concentrated on the example (5.3) (where $\mathcal{A}$ is given by (5.2)), our abstract results also apply to Cauchy-mixed problems (with mixed variable lateral conditions) for *higher order* linear differential operators of hyperbolic type (or hyperbolic in the sense of Petrowski, as e.g. $\partial^2/\partial t^2 + \Delta_x^2$). However, for the sake of brevity, we do not insist here on this point.

5.2. We now consider another application of our abstract results. Let $T > 0$ be given. Let $\{\Omega(t)\}_{t\in[0,T]}$ be a family of open bounded subsets $\Omega(t)$ of $\mathbb{R}^N$. We assume that, for every $t \in [0, T]$, the boundary $\Gamma(t) \equiv \partial\Omega(t)$ of $\Omega(t)$ is a $(N-1)$-dimensional manifold of class (e.g.) C^1. Let us define:

$$\text{(5.12)} \qquad \begin{cases} Q \equiv \bigcup\limits_{0<t<T} \Omega(t) \times \{t\}; \qquad \Sigma \equiv \bigcup\limits_{0<t<T} \Gamma(t) \times \{t\}; \\ B \equiv \mathbb{R}^N \times]0, T[. \end{cases}$$

We also suppose, for the moment, that

$$\text{(5.13)} \qquad Q \text{ is an } open \text{ subset of } \mathbb{R}^{N+1}.$$

Let us now consider (formally) the partial differential operator $\mathcal{A}$, which is given by (5.2). We assume that its coefficients $a_{ij}(x, t)$ $(i, j = 1, \ldots, N)$ and $c(x, t)$ are complex-valued functions defined in B. Then, we consider, *in a formal way*, the following Cauchy-Dirichlet problem. Given $f(x, t)$ (defined in Q), and $u_0(x)$, $u_1(x)$ (both defined in $\Omega(0)$), to find $u(x, t)$ (defined in $\overline{Q}$), such that:

$$
(5.14)\qquad \begin{cases} a) \ \dfrac{\partial^2 u}{\partial t^2}(x, t) + \mathcal{A}u(x, t) = f(x, t) \quad \text{in } Q\,; \\ b) \ u(x, t) = 0 \quad \text{on } \Sigma\,; \\ c) \ u(x, 0) = u_0(x) \quad \text{in } \Omega(0)\,; \\ d) \ \dfrac{\partial u}{\partial t}(x, 0) = u_1(x) \quad \text{in } \Omega(0)\,. \end{cases}
$$

Now, let us give a definition of a weak solution to the problem (5.14), according to our «abstract» Definition 2.1. Towards this aim, we take

$$
(5.15)\qquad H = L^2(\mathbb{R}^N), \qquad V = H^1(\mathbb{R}^N) \text{ (and hence } V^* = H^{-1}(\mathbb{R}^N))\,,
$$

so that (2.1) holds. Moreover, we take, for every $t \in [0, T]$,

$$
(5.16)\qquad V(t) = \{v \in H^1(\mathbb{R}^N) \mid \operatorname{supp}(v) \subset \overline{\Omega(t)}\}\,,
$$

which is, of course, a closed subspace of V. (Note that, if $v \in V(t)$, then its restriction to $\Omega(t)$ belongs to $H_0^1(\Omega(t))$). Remark however that $V(t)$ is *not* dense in H. It is clear that

(5.17) (2.11) (resp. (4.1)) holds here iff $\{\Omega(t)\}_{t \in [0, T]}$ is a non-decreasing (resp. non-increasing) family with t.

Now, let us consider the operator $\mathcal{A}$ (given by (5.2)), and assume that

(5.18) (5.8) holds, where we replace Ω (resp. Q) in (5.8) *a*) (resp. (5.8) *b*) and *c*)) with $\mathbb{R}^N$ (resp. B).

Next, define (for a.a. $t \in]0, T[$, and any $u, v \in V = H^1(\mathbb{R}^N)$) $a(t; u, v)$ as in (5.9) (where we replace Ω with $\mathbb{R}^N$). Clearly, $\{a(t; \cdot, \cdot) \mid \text{for a.a. } t \in]0, T[\}$ is a family of sesquilinear and continuous forms on $V \times V$. Then, starting from such forms, and proceeding as in (5.10), we can define the related family $\{A(t) \mid \text{for a.a. } t \in]0, T[\}$ of linear and continuous operators from $V = H^1(\mathbb{R}^N)$ into $V^* = H^{-1}(\mathbb{R}^N)$. (They are connected, in a natural way, with $\mathcal{A}$ and with V). Thanks to (5.18), it is clear that, for such a family $\{A(t)\}$, (2.4), (2.5), and (2.6) hold.

Take now:

$$
(5.19)\qquad \begin{cases} a) \ f(x, t) = f_1(x, t) + f_2(x, t), \text{ where } «t \to f_1(\cdot, t)» \in L^1(0, T; L^2(\mathbb{R}^N)), \\ \quad \text{and } «t \to f_2(\cdot, t)» \in BV(0, T; H^{-1}(\mathbb{R}^N))\,; \\ b) \ u_0(x) \in V(0) = \{v \in H^1(\mathbb{R}^N) \mid \operatorname{supp}(v) \subset \overline{\Omega(0)}\}\,; \qquad u_1(x) \in H = L^2(\mathbb{R}^N)\,. \end{cases}
$$

(Remark that, thanks to (2.8) and (2.10), «the values» of f (resp. of u_1) in $B \backslash \overline{Q}$ (resp. in $\mathbb{R}^N \backslash \overline{\Omega(0)}$) do not affect, in the present case, the weak solution u in Definition 2.1). Then, thanks to the previous definitions and remarks, we can use Definition 2.1 to give a precise weak formulation of the problem (5.14) (together with the corresponding notion of weak solution). For the sake of brevity, we do not rewrite here Definition 2.1 in the present case (i.e. by specifying V and H as in (5.15), $V(t)$ as in (5.16), etc.). Let us only remark that: (5.14) *c*) is given by (2.9) *b*); (5.14) *b*) is contained in (2.9) *a*); (5.14) *a*) and *d*) are both contained in (2.10). Now, by considering (5.17), we can use Theorem 3.1 (along with Remarks 3.1 and 3.2), and Theorem 4.1, to obtain the following results.

PROPOSITION 5.2. – Let the above assumptions on $\{\Omega(t)\}_{t\in[0,T]}$, and $\{\Gamma(t)\}_{t\in[0,T]}$ hold. Take moreover: H and V as in (5.15), $V(t)$ as in (5.16), $A(t)$ as in (5.10) (where $a(t; u, v)$ is defined in (5.9) (Ω replaced with $\mathbb{R}^N$) and (5.18) holds). Finally, take any f, u_0, u_1 as in (5.19). Then, the following conclusions hold.

a) Assume, moreover, that $\{\Omega(t)\}_{t\in[0,T]}$ is *a non-decreasing family with t* (and hence (2.11) holds). Then, there exists a (not necessarily unique) weak solution u to the problem (5.14) (in the sense of Definition 2.1). Such u also satisfies $u(t)\in C^0([0,T]; V_w)$ (and in this sense (5.14) *c*) can also be meant). Moreover, it results that $u(t)\in V(t)$ for *every* $t\in[0,T]$.

b) On the other hand, assume that $\{\Omega(t)\}_{t\in[0,T]}$ is *a non-increasing family with* t (and hence (4.1) holds). Then, if there exists a weak solution u to (5.14) (in the sense of Definition 2.1), it results that such u is unique.

REMARK 5.4. – Observe that, in Proposition 5.2, we did *not* use the assumption (5.13). Hence, in general, the family $\{\Omega(t)\}_{t\in[0,T]}$ is allowed to have «jump discontinuities» with respect to t. So, (5.14) *a*) and *d*) (which are here contained in (2.10)) cannot, in general, be better specified. On the other hand, by taking more regular data, we can expect to deduce from (2.10) a more precise information on (5.14) *a*) and *d*) (e.g. the fact that (5.14) *a*) is satisfied in the sense of $\mathcal{D}'(Q)$, etc.). However, we do not insist here on this point.

REMARK 5.5. – It is well known that the «concrete» problem (5.14) (in a non-cylindrical region Q) was formerly investigated by several authors, *in the special case where* $\mathcal{A}u = -\Delta_x u$ (or, more generally, where $\mathcal{A}u = -\Delta_x u + F(u)$, with suitable nonlinear functions F). We refer, in particular, to LIONS [22] (and also [23], chap. 3, §8), BARDOS and COOPER [5], INOUE [17], COOPER [11], SIKORAV [28], ZOLÉSIO [29], [30]. As for weak solutions to (5.14) (with $\mathcal{A} = -\Delta_x$), we have (from [5], [11], [17], [28]) the existence *and* the uniqueness, when (besides the other natural hypotheses) we assume that Σ is a *smooth* N-dimensional manifold in $\mathbb{R}^{N+1}$, *and* that Σ is *strictly time like*. Moreover, some *sharp* existence, uniqueness and regularity results were proved by DA PRATO and ZOLÉSIO [14] for the problem (5.14), with a general linear operator $\mathcal{A}$, in *suitably smooth* non-cylindrical regions Q of special type: they were obtained through a suitable change of variables, and by using some abstract results due to KATO [18] and to DA PRATO and IANNELLI [13]. On the other hand, our Proposition 5.2 gives results for (5.14), with a general (linear) operator $\mathcal{A}$, in a general (not necessarily smooth) non-cylindrical region Q, *but* we have to require a monotonicity property of $\{\Omega(t)\}_{t\in[0,T]}$.

Let us also mention the geometrical method by BOVE, FRANCHI, OBRECHT [8], which is useful to handle initial-boundary value problems, in non-cylindrical regions, for various types of evolution P.D.E..

REMARK 5.6. – We also consider (5.14), where $\mathcal{A} = -\Delta_x$. When Σ is *not* time like, such a problem is *not* well posed, in general (as we can deduce, e.g., from the careful study, which was performed in Section 1 of BARDOS and COOPER [5]). Now, we observe that a *similar* conclusion (and a conclusion *similar* to the one in Remark 5.5, corresponding to the case where Σ is strictly time like) can, of course, be readily obtained when we take, more generally, $\mathcal{A} = -\varepsilon\Delta_x$, where ε is an arbitrary positive constant. Let us view directly a behaviour of this type, in the following very particular case of (5.14). Fix any $l > 0$ and any $\varepsilon > 0$. Take any $a \in \mathbb{R}$, and fix any $T > 0$ (such that $l + aT > 0$, when $a < 0$). Then, our problem is to find $u(x, t)$ such that:

$$(5.20)\quad \begin{cases} a)\ \dfrac{\partial^2 u}{\partial t^2}(x, t) - \varepsilon \dfrac{\partial^2 u}{\partial x^2}(x, t) = 0\,; \\ \qquad\qquad\qquad (x, t) \in Q = \{(x, t) \in \mathbb{R}^2 \mid 0 < t < T;\ 0 < x < l + at\}\,; \\ b)\ u(0, t) = u(l + at, t) = 0\,, \quad 0 < t < T\,; \\ c)\ u(x, 0) = u_0(x)\,, \quad 0 < x < l\,; \\ d)\ \dfrac{\partial u}{\partial t}(x, 0) = u_1(x)\,, \quad 0 < x < l\,; \end{cases}$$

where $u_0(x) \in H_0^1(0, l)$, and $u_1(x) \in L^2(0, l)$ are given. Observe that, for the differential operator in (5.20) *a*), the characteristic lines are $x \pm \sqrt{\varepsilon}t = const.$ Now, considering the weak solutions to (5.20), we have the following conclusions (which can be readily deduced, by using the results of [5]). We have the *existence* of such a solution, when $a > -\sqrt{\varepsilon}$; but the existence *fails*, in general, when $a < -\sqrt{\varepsilon}$. We have that such a solution is *unique*, when $a < \sqrt{\varepsilon}$; but the uniqueness *fails*, when $a > \sqrt{\varepsilon}$. Now, if we view (5.20) in our abstract framework, we have here that, in particular, $\Omega(t) = \{x \in \mathbb{R}^1 \mid 0 < x < l + at\}$, and $V(t) = \{v \in H^1(\mathbb{R}) \mid \mathrm{supp}(v) \subset \overline{\Omega(t)}\}$. Then, the above conclusions show that the well-posedness (or not) of the problem (5.20) depends on the fact that the speed of the variation (with respect to t) of $\Omega(t)$ (and hence also of $V(t)$) is less (or greater) than the speed of the waves related to the hyperbolic differential operator in (5.20) *a*). Anyway, for every $\varepsilon > 0$, a suitable monotonicity property of $\{\Omega(t)\}$ (and hence of $\{V(t)\}$) guarantees the existence or the uniqueness of the weak solution to (5.20).

REMARK 5.7. – Even if we concentrated on the example (5.14) (where $\mathcal{A}$ is given by (5.2)), our abstract results also apply to various initial-boundary value problems, *in non-cylindrical regions*, for *higher order* linear differential operators of hyperbolic type (or hyperbolic in the sense of Petrowski, as e.g. $\partial^2/\partial t^2 + \Delta_x^2$). However, for the sake of brevity, we do not insist here on this point.

REMARK 5.8. – Let us also mention a paper by CANNARSA, DA PRATO, and ZOLESIO [9]. It concerns initial-boundary value problems, in non-cylindrical regions, for linear *damped* wave equations (hence, equations with both hyperbolic and parabolic «characters»).

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