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Quasi-Minimal Enumeration Degrees and Minimal Turing Degrees (*).

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Abstract. – *We show that there exists a set A such that A has quasi-minimal enumeration degree, and there are uncountably many sets B such that A is enumeration reducible to B and B has minimal Turing degree. Answering a related question raised by Solon, we also show that there exists a nontotal enumeration degree which is not e-hyperimmune.*

1. – Introduction.

We adopt the formalization of enumeration reducibility given by Friedberg and Rogers in [3], and our exposition follows [11]. In the Friedberg-Rogers formalization, an *enumeration operator* $\Phi_z\colon 2^\omega \to 2^\omega$ is derived from a recursively enumerable set W_z by the equation

$$\Phi_z(B) = \{x : (\exists u)[\langle x, u\rangle \in W_z \text{ and } D_u \subseteq B]\},$$

where D_u is the finite set with canonical index u. Henceforth, we may write $\langle x, D\rangle$ instead of $\langle x, u\rangle$, when D is equal to D_u. Given an effective listing $\{W_z : z \in \omega\}$ of the recursively enumerable sets, we get a corresponding indexing $\{\Phi_z : z \in \omega\}$ of the e-operators.

A set A is *enumeration reducible* (*e-reducible*) to another set B (notation: $A \leqslant_e B$), if $(\exists z)[A = \Phi_z(B)]$. Let $\equiv_e$ denote the equivalence relation generated by $\leqslant_e$, and let $[A]_e$ be the *enumeration degree* (*e-degree*) of A. The degree structure $\langle \mathcal{D}_e, \leqslant_e\rangle$ is defined by setting $\mathcal{D}_e = \{[A]_e : A \subseteq \omega\}$, and setting $[A]_e \leqslant_e [B]_e$ if and only if $A \leqslant_e B$. The

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structure $\mathcal{D}_e$ is an upper semilattice with least element $\mathbf{0}_e = [A]_e$, where A is any recursively enumerable set. The operation of least upper bound is given by $[A]_e \vee [B]_e = [A \oplus B]_e$, where $A \oplus B = \{2x: x \in A\} \cup \{2x+1: x \in B\}$. By identifying partial functions with their graphs, where $\mathrm{graph}(\varphi) = \{\langle x, y\rangle: \varphi(x) = y\}$, we shall write $\varphi \leqslant_e \psi$ to mean $\mathrm{graph}(\varphi) \leqslant_e \mathrm{graph}(\varphi)$. This reducibility coincides with the reducibility between partial functions introduced by Kleene in [5]. Similarly, given any set A and any partial function ϕ, $A \leqslant_e \phi$ means $A \leqslant_e \mathrm{graph}(\phi)$.

Let $\langle \mathcal{D}_T, \vee, \mathbf{0}_T\rangle$ denote the upper semilattice of Turing degrees (T-degrees), with partial ordering relation $\leqslant_T$. It is possible to view $\mathcal{D}_T$ as a substructure of $\mathcal{D}_e$ in a sense that is made precise by the following theorem.

THEOREM 1.1 (Embedding Theorem [11]). – *Define $i([A]_T) = [c_A]_e$ (where A is any set and c_A denotes the characteristic function of A). Then i is a well-defined embedding from $\mathcal{D}_T$ into $\mathcal{D}_e$ preserving* 0 *and* $\vee$.

PROOF. – For all sets A, B, we have

$$A \leqslant_T B \Leftrightarrow c_A \leqslant_T c_B \Leftrightarrow c_A \leqslant_e c_B\,,$$

where c_A, c_B are the characteristic functions of A, B respectively. The previous implications are justified by the following observations: for all sets A and total functions f, g, we have that $A \leqslant_T f \Rightarrow A \leqslant_e f$; $f \leqslant_e A \Rightarrow f \leqslant_T A$; hence $f \leqslant_T g \Leftrightarrow f \leqslant_e g$.

It is easy now to see that i, as defined above, is an embedding preserving 0 and $\vee$. ■

DEFINITION 1.2. – An e-degree $\boldsymbol{a}$ is *total* if $\boldsymbol{a} \in \mathrm{range}(i)$.

It is easy to see that $\boldsymbol{a}$ is total if and only if $(\exists \text{ total} f)[f \in \boldsymbol{a}]$. We also say that a set A is *single-valued* if A is the graph of a partial function, and A is *total* if it is the graph of a total function. Thus $\boldsymbol{a}$ is total if and only if $\boldsymbol{a}$ contains a total set A.

The existence of non-total e-degrees is an easy consequence of the existence of quasi-minimal e-degrees, first shown by Medvedev ([9]). We recall the definition of a quasi-minimal e-degree.

DEFINITION 1.3 ([9]). – An e-degree $\boldsymbol{a}$ is called *quasi-minimal* if

- $\boldsymbol{a} \neq \mathbf{0}_e$,
- $(\forall \text{ total } \boldsymbol{b})[\boldsymbol{b} \leqslant \boldsymbol{a} \Rightarrow \boldsymbol{b} = \mathbf{0}_e]$.

It is easy to see that a non-zero e-degree $\boldsymbol{a}$ is quasi-minimal if and only if $(\forall A \in \boldsymbol{a})(\forall \text{ total } f)[f \leqslant_e A \Rightarrow f \text{ recursive}]$.

A classical result of the theory of degrees of unsolvability is Spector's Theorem ([14]) asserting the existence of minimal T-degrees, i.e. atoms of the structure $\mathcal{D}_T$.

It was an open problem for many years whether the structure $\mathcal{D}_e$ possesses minimal elements. The solution to this problem (namely $\mathcal{D}_e$ does not have minimal elements) was given by Gutteridge ([4]; see also [1]). Gutteridge's theorem can also be viewed as a measure of the extent to which the T-degrees and the e-degrees are different. It exhibits a $\forall\exists$-formula in the language of partially ordered sets which is satisfied by $\mathcal{D}_e$

and not satisfied by $\mathcal{D}_T$. The fact that $\mathcal{D}_e$ and $\mathcal{D}_T$ have the same existential theories follows from Theorem 1.1.

DEFINITION 1.4. – An e-degree $\boldsymbol{b}$ is *minimal total* if $\boldsymbol{b}$ is total and

$$(\forall \boldsymbol{a})[(\boldsymbol{a} \text{ is total and } \boldsymbol{a} < \boldsymbol{b}) \Rightarrow \boldsymbol{a} = \mathbf{0}_e].$$

In other words, $\boldsymbol{b}$ is minimal total if and only if there exists a minimal T-degree $\boldsymbol{a}$ such that $i(\boldsymbol{a}) = \boldsymbol{b}$, where i is the embedding of Theorem 1.1.

Since $\mathcal{D}_e$ does not have any minimal elements, every minimal total e-degree is preceded by quasi-minimal e-degrees:

THEOREM 1.5. – If b *is minimal total,* then

$$(\forall \boldsymbol{a})[\mathbf{0}_e < \boldsymbol{a} < \boldsymbol{b} \Rightarrow \boldsymbol{a} \textit{ quasi-minimal}].$$

PROOF. – If B has minimal T-degree and f is a total function such that $f <_e B$, then $f \leqslant_T B$; on the other hand, we can not have $B \leqslant_T f$, as this would give $B \leqslant_e f$, by totality of f. Therefore $f <_T B$, hence f is recursive. ■

Indeed, no total e-degree can be a minimal cover nor can it have a minimal cover (see [4], [1]; see also [7]). However, these properties do not characterize the total e-degrees, since, as shown by [7], they are satisfied by all e-degrees containing Σ_2 sets. On the other hand, there are Σ_2 sets whose e-degrees are not total, see e.g. [9] or [2].

We prove that there exist quasi-minimal e-degrees with as many as possible minimal total e-degrees above them.

THEOREM 1.6. – *There exist an e-degree* $\boldsymbol{a}$ *and a family of e-degrees* $\{\boldsymbol{b}_i : i \in I\}$ *of cardinality* $2^{\aleph_0}$ *such that, for all* $i \in I$,

- $\boldsymbol{b}_i$ *is minimal total;*
- $\boldsymbol{a}$ *is quasi-minimal;*
- $\boldsymbol{a} < \boldsymbol{b}_i$.

PROOF. – See next section. ■

This theorem answers a question raised by [13]. However, it leaves open the following interesting question.

PROBLEM 1. – *Prove or disprove that for every quasi-minimal e-degree* $\boldsymbol{a}$ *there is a minimal total e-degree* $\boldsymbol{b}$ *such that* $\boldsymbol{a} < \boldsymbol{b}$.

2. – Proof of Theorem 1.6.

We first review some definitions and terminology dealing with strings and trees which will be used in the proof of the theorem. For a detailed account of the use of trees in recursion theory see [8] and [10].

2.1. *String and tree notations.*

If B is any set, then the elements of the set $B^{<\omega}$, (i.e. the functions $\sigma: n \to B$, for some $n \in \omega$), are called *B-strings*. If $\sigma: n \to B$ is a B-string, then the number n is the *length* of σ (notation: $|\sigma|$). If $b \in B$ and $k \in \omega$, then the symbol $b^{[k]}$ denotes the length k string with constant value b. 2-strings will be called *binary strings*, or simply, *strings*.

Any partial order relation $\leqslant$ on B originates a corresponding partial order relation $\leqslant$ on $B^{<\omega}$, defined as follows: for all $\sigma, \tau \in B^{<\omega}$, let

$$\sigma \leqslant \tau \Leftrightarrow [\sigma \subseteq \tau \text{ or } i(\sigma, \tau)\downarrow \quad \text{and } \sigma(i(\sigma, \tau)) \leqslant \tau(i(\sigma, \tau))],$$

where,

$$i(\sigma, \tau) = \begin{cases} \text{least}\{x : x < |\sigma|, |\tau| \text{ and } \sigma(x) \neq \tau(x)\}, & \text{if such an } x \text{ exists ;} \\ \uparrow, & \text{otherwise .} \end{cases}$$

When dealing with (binary) strings, we will use $\leqslant$ to denote the lexicographic ordering on $2^{<\omega}$. If σ and τ are B-strings (for any set B), we write $\sigma | \tau$ to indicate that $\sigma \not\subseteq \tau$ and $\tau \not\subseteq \sigma$. We denote the *concatenation* of σ and τ by $\sigma * \tau$. If $n \leqslant |\sigma|$ then $\sigma\restriction_n$ denotes the restriction of σ to n; the symbol $\sigma\restriction_{\geqslant n}$ denotes the string of length equal to $|\sigma| - n$ and such that $\sigma\restriction_{\geqslant n}(x) = \sigma(n + x)$, for all x such that $n + x < |\sigma|$.

A *tree* is a function $T: 2^{<\omega} \to 2^{<\omega}$ such that, for all strings ζ and η,

- $\zeta \subseteq \eta \Rightarrow T(\zeta) \subseteq T(\eta)$;
- $\zeta | \eta \Rightarrow T(\zeta) | T(\eta)$.

Given a tree T, we denote by $[T]$ the collection of infinite paths through T.

A *partial* tree is a partial function $T: 2^{<\omega} \to 2^{<\omega}$ which can be extended to a tree.

Let T_1, T_2 be partial trees and let ζ, A be a string and a set, respectively. Then write $T_1 \subseteq T_2$ if range$(T_1) \subseteq$ range(T_2); and write $\zeta \in T_1$ if $\zeta \in$ range(T_1).

If $S \subseteq 2^{<\omega}$ and T is a tree, then $T\restriction_S$ denotes the restriction of T to S. Let $S_n = \{\zeta \in 2^{<\omega}: |\zeta| < n\}$. We write $T\restriction_n$ for $T\restriction_{S_n}$. A partial tree T is said to be *saturated* if domain$(T) = S_n$, for some number n, or domain$(T) = 2^{<\omega}$. Given any number n, let $\{\zeta_i^{(n)}: i \leqslant 2^n - 1\}$ be the enumeration in order of magnitude (with respect to $\leqslant$) of the set $\{\zeta: |\zeta| = n\}$.

We now define several operations on trees.

1) If ζ is a string and T is a tree then $\zeta * T$ denotes the following tree: for all η,

$$\zeta * T(\eta) = \zeta * (T(\eta)) .$$

2) Let T^* be a partial tree such that domain$(T^*) = S_n$, for some n, and let T be a tree such that $T^* \subseteq T$. Define $R(T^*, T)$ to be the subtree of T defined as follows:

for all ζ,

$$R(T^*, T)(\zeta) = \begin{cases} T^*(\zeta) & \text{if } |\zeta| < n, \\ T(\bar{\zeta} * \zeta_{\restriction \geq n-1}) & \text{if } |\zeta| \geq n. \end{cases}$$

where, if $|\zeta| \geq n$, then $\bar{\zeta}$ is such that $T(\bar{\zeta}) = T^*(\zeta_{\restriction n})$.

3) Given $\zeta \in 2^{<\omega}$ and T a tree, let $\mathrm{Ext}(T, \zeta)$ be the tree defined by $\mathrm{Ext}(T, \zeta)(\eta) = T(\zeta * \eta)$.

e-total trees and e-splitting trees. We recall that a tree T is *e-total* if $(\forall \zeta)(\forall x)(\exists \eta \supseteq \zeta)[\phi_e^{T(\eta)}(x) \downarrow]$. A sufficient condition for a tree T to be e-total is the following

$$(1) \qquad (\forall n)(\forall \zeta)[|\zeta| = n \Rightarrow \varphi\phi_e^{T(\zeta)}(n) \downarrow].$$

We will often construct e-total trees, by making our trees satisfy property 1.

Given a tree T, we say that a number λ is the *level of e-totality of T at stage s* if

$$(\forall x \leq \lambda)(\forall \zeta)[|\zeta| = \lambda \Rightarrow \varphi_{e,s}^{T(\zeta)}(x) \downarrow].$$

A tree T is an *e-splitting* tree if $(\forall \zeta)(\exists x)[\varphi_e^{T(\zeta * 0)}(x) \downarrow \neq \varphi_e^{T(\zeta * 1)}(x) \downarrow]$. A pair $(T(\zeta), T(\eta))$ is called an *e-splitting* if, for some x, $\varphi_e^{T(\zeta)}(x) \downarrow \neq \varphi_e^{T(\eta)}(x) \downarrow$; if θ is such that $\theta \subseteq \zeta, \eta$, and $(T(\zeta), T(\eta))$ is an *e-splitting* then we say that $(T(\zeta), T(\eta))$ is an *e-splitting of* $T(\theta)$.

We say that a number ε is the *level of e-splitting of T* at stage s if

$$(\forall \zeta)[|\zeta| < \varepsilon \Rightarrow (\exists x)[\varphi_{e,s}^{T(\zeta * 0)}(x) \downarrow \neq \varphi_{e,s}^{T(\zeta * 1)}(x) \downarrow].$$

Similarly, one defines the notion of *level of e-totality* and *level of e-splitting* for a tree, i.e. replacing $\varphi_{e,s}$ with φ_e in the above definitions.

The following definitions are due essentially to Lachlan ([6]), on which we base our proof of Theorem 1.6. For notations and terminology relative to uniform trees, we follow closely [8].

DEFINITION 2.1. – 1) A tree T is *uniform* if for every n there exist strings η_n^0, η_n^1 such that $|\eta_n^0| = |\eta_n^1|$ and

$$(\forall \zeta)[|\zeta| = n \Rightarrow T(\zeta * 0) = T(\zeta) * \eta_n^0 \text{ and } T(\zeta * 1) = T(\zeta) * \eta_n^1].$$

2) A tree T is *strongly uniform* (*s.u.*) if T is uniform, and for every n, η_n^0 and η_n^1 differ on exactly one place (η_n^0, η_n^1 being as before). If $|\zeta| = n$, then the number on which η_n^0, η_n^1 differ will be called the *n-level branching value of T*.

The above definition extends naturally to saturated partial trees. We note

LEMMA 2.2. – *The identity tree* $\mathrm{Id}\colon 2^{<\omega} \to 2^{<\omega}$ *is s.u. Moreover, if* T^* *is a s.u. saturated partial tree,* T *is a s.u. tree, and* $T^* \subseteq T$, *then* $R(T^*, T)$ *is s.u. If* T *is s.u., then* $\eta * T$ *is s.u.*

PROOF. – Immediate. ■

Finally, we recall the following two definitions. See [8].

DEFINITION 2.3. – Let T be a tree and let Z be a subset of $2^{<\omega}$ such that there is an $m \in \omega$ such that each $\zeta \in Z$ has length m. Let (ζ, η) be such that $|\zeta|, |\eta| > 0$. We say that the pair (ζ, η) *induces a simultaneous e-splitting for Z relative to T* if

$$(\forall \theta \in Z)(\exists x)[\phi_e^{T(\theta * \zeta)}(x) \downarrow \neq \phi_e^{T(\theta * \eta)}(x) \downarrow].$$

If T is strongly uniform, then a simultaneous e-splitting as before is called *strongly uniform* if $|\zeta| = |\eta|$ and ζ, η differ on exactly one place.

DEFINITION 2.4. – Let T be a s.u. tree and let $Z \subseteq 2^{<\omega}$ be such that, for some number m, for all $\zeta \in Z$, $|\zeta| = m$. We say that T is *right e-splitting for Z* if, for every $\theta \in Z$, and for every n, the pair $(T(\theta * 0^{[n]} * 0), T(\theta * 0^{[n]} * 1))$ is an e-splitting.

The following result, due to Lachlan ([6]), is fundamental for the construction below. We follow the exposition of [8].

LEMMA 2.5. – *Let T be a s.u. recursive e-total tree. Let $Z \subseteq 2^{<\omega}$, $\beta \in 2^{<\omega}$ be such that, for all $\zeta \in Z$, $|\zeta| = |\beta|$. Suppose that T is right e-splitting for Z, but with no s.u. simultaneous e-splitting for $Z \cup \{\beta\}$. Then there are no e-splittings in* $\mathrm{Ext}(T, \beta)$.

PROOF. – See [8, Lemma VI.5.14]. ■

2.2. *Requirements and strategies.*

In order to prove Theorem 1.6, it is enough to construct a strongly uniform tree G, a set A, and an e-operator Φ such that, for all $M \in [G]$,

- M has minimal T-degree;
- A has quasi-minimal e-degree;
- A is uniformly e-coded by Φ relative to any branch of the tree G, i.e. $A = \Phi(M)$, for all $M \in [G]$.

Since there are $2^{\aleph_0}$ paths $M \in [G]$, it follows that the set A, together with the family of T-degrees $\{[M]_T: M \in [G]\}$, satisfies the theorem.

The way we guarantee the last item is by defining Φ so that

$$(\forall M, M' \in [G])[\Phi(M) = \Phi(M')]:$$

then letting $A = \Phi(M)$, for any $M \in [G]$.

The construction aims at satisfying the following requirements, for all e, k, i and all branches $M \in [G]$:

$$M_e\colon \varphi_e^M \text{is total} \Rightarrow (\varphi_e^M \text{is recursive} \vee M \leqslant_T \varphi_e^M),$$
$$N_k\colon A \neq W_k\,,$$
$$P_i\colon \{\zeta\colon |z| \leqslant i\} \subseteq \text{domain}(G),$$

where $A = \Phi(M)$, for any $M \in [G]$, and Φ is a suitable e-operator. In fact, for every axiom $\langle x, D\rangle \in \Phi$, card$(D) \leqslant 1)$ so Φ will be an s-operator. The construction will also ensures that $\Phi(M) = \Phi(M')$, for every $M, M' \in [G]$.

We note that satisfaction of all requirements M_e's and P_i's guarantees the construction of a tree G such that for each branch M of G, M has minimal T-degree. By Theorem 1.5, satisfaction of all N_k's makes $[A]_e$ a quasi-minimal e-degree. The fact that the tree is strongly uniform is fundamental for constructing an e-operator Φ such that, for *all* $M, M' \in [G]$, we have that $\Phi(M) = \Phi(M')$.

The strategy for the requirement M_e is the usual strategy for getting minimal T-degrees as adapted by [6] to strongly uniform trees: given a strongly uniform tree T, find a strongly uniform subtree T' of T such that if T' is e-total then *either* T' is an e-splitting tree—ensuring, for all M in $[G]$, if ϕ_e^M is total, then $\varphi_e^M \geqslant_T M$—*or* T' has no e-
splitting—ensuring that if φ_e^M is total then φ_e^M is recursive.

The strategy for N_k consists as usual in appointing a suitable witness n such that $n \in A \Leftrightarrow n \notin W_k$, i.e. for all $M \in [G]$, $n \in \Phi(M) \Leftrightarrow n \notin W_k$.

Finally, the strategy for P_i should make sure that the tree thinning activity performed on behalf of the M-requirements does not prevent us from eventually getting a tree, i.e., for every i ensure that $\{\zeta\colon |\zeta| \leqslant i\} \subseteq \text{domain}(G)$. Lower priority requirements are satisfied in the environments (i.e. trees) provided by higher priority requirements.

The infinite priority nature of the construction (amounting to a $0''$-priority argument) is due essentially to the fact that we may need to perform an infinite amount of activity to select, by finite approximations, the right subtree T' (e.g. if the eventual T' is going to be an e-splitting tree, by adding at infinitely many steps new levels of e-splittings) of a given tree T.

We give below a more detailed account of how the strategies work.

The requirement M_e. (The strategy acts inside some given s.u. tree T.) Given any s.u. tree T, find a s.u. subtree $F \subseteq T$, such that every $M \in [F]$ satisfies the requirement M_e. We want however $F\restriction_{m+1} = T\restriction_{m+1}$, where $m = 2e$, i.e. we want to preserve the first m levels of the tree on behalf of higher priority N- and P-requirements (namely, N_k, P_i, with $i, k < e$).

Our action will have certain «outcomes», or, better, «component outcomes», since the outcome may be broken into two components, when our action aimed at satisfying the requirement consists of two moves, each one with its own outcome.

Here is how F is defined. Notice that there are 2^m strings such that $|\zeta| = m$: for every $i < 2^m$, let $\alpha_i = T(\zeta_i^{(m)})$. Construct by induction a sequence of s.u. trees $T_{-1}, T_0, \ldots, T_{2^m-1}$, starting with T_{-1} being the s.u. tree defined by $\alpha_0 * T_{-1}(\zeta) = T(\zeta_0^{(m)} * \zeta)$. Having defined T_{i-1}, define T_i as follows. First look for a s.u. saturated (possibly partial) subtree $T^* \subseteq T_{i-1}$ (with maximal domain) such that $\alpha_i * T^*$ satisfies property (2.1) relative to all strings ζ such that $|\zeta| < k$, where domain$(T^*) = S_k$. Let $\widehat{T} = R(T^*, T_{i-1})$.

1) Failure of extending T^* to a total tree yields a string α such that, for some x, if $M \in [\alpha_i * \mathrm{Ext}(\widehat{T}, \alpha)]$, then $\varphi_e^M(x) \uparrow$. If this is the case then the outcome of our action aiming at satisfying M_e is a number which measures the level of e-totality of $\alpha_i * \mathrm{Ext}(\widehat{T}, \alpha)$. Define $T_i = \mathrm{Ext}(\widehat{T}, \alpha)$.

2) Otherwise, the first «component outcome» is ω, and we succeed in finding a s.u. subtree $\widehat{T} \subseteq T_{i-1}$ such that $\alpha_i * \widehat{T}$ is e-total (notice that in this case eventually $\widehat{T} = T^*$, since in this case T^* grows up to a total tree). We proceed our action as follows. We look for a s.u. subtree $\overline{T} \subseteq \widehat{T}$ with the aim of obtaining eventually an e-splitting tree $\alpha_i * \overline{T}$. This is done by employing Lachlan's technique in [6] for constructing minimal T-degrees.

- Failure of finding an e-splitting tree will result in getting a set of strings Z and a string β (such that, for all $\zeta \in Z$, $|\zeta| = |\beta|$), such that the tree $\alpha_i * T'$ (where $T' = R(\overline{T}, \widehat{T})$) is right e-splitting for Z, but there are no s.u. simultaneous e-splitting for $Z \cup \{\beta\}$, and, thus, by Lemma 2.5, there are no e-splittings in $\alpha_i * \mathrm{Ext}(T', \beta)$. In this case we define $T_i = \mathrm{Ext}(T', \beta)$. The second «component outcome» in this case is the string β.

- If $\alpha_i * \overline{T}$ eventually grows to an e-splitting tree, then the second «component outcome» is ω, and $T_i = \overline{T}$.

Having defined T_{2^m-1} we define the tree F as follows:

$$F(\zeta) = \begin{cases} T(\zeta) & \text{if } |\zeta| \leq m\,, \\ T(\zeta\restriction_m) * T_{2^m-1}(\zeta\restriction_{\geq m}) & \text{otherwise}\,. \end{cases}$$

The requirements N_k and P_i. Let F be a s.u. tree satisfying all the M-requirements. For every k, call $x(k)$ the $2k$-branching value of F. Construct an e-operator Φ such that, for every k there are numbers $n(k)$ such that $\langle n(k), \{x(k)\}\rangle \in \Phi$, and ensure that, for every set $M \in [F]$, we have that $n(k) \in \Phi(M) \Leftrightarrow x(k) \in M$. Consider the subtree $G \subseteq F$ defined by induction as follows:

- $G(\emptyset) = F(\emptyset)$;

- suppose that $G(\zeta)$, with $|\zeta| = k$, is defined, and, say, $G(\zeta) = F(\zeta')$: then, for every $j \in \{0, 1\}$, let $G(\zeta * j) = F(\zeta' * h * j)$ where h is chosen such that

$$F(\zeta' * h * j)(x(k)) = 1 \Leftrightarrow n(k) \notin W_k\,.$$

The tree G clearly enables us to meet all the requirements N_k, P_i. In particular notice that, for every k, $n(k) \in A \Leftrightarrow n(k) \notin W_k$.

DEFINITION 2.6. – If a tree G is defined from a tree F as above, we say that G is the *N-reduced* subtree of F.

2.3. *The tree of strategies.*

Let $B = \omega \cup \{\omega * \omega, \omega * \beta : \beta \in 2^{<\omega}\}$. Given $\beta, \beta' \in 2^{<\omega}$, let $\beta \leqslant_B \beta'$ if and only if $[|\beta| < |\beta'| \vee [|\beta| = |\beta'|$ and $\beta' \leqslant \beta]]$.

Extend $\leqslant_B$ to B as follows: given any $\varrho, \varrho' \in B$, let $\varrho \leqslant_B \varrho'$ if and only if

$$[\varrho, \varrho' \in \omega \text{ and } \varrho \geqslant \varrho'] \text{ or } [\varrho \in \alpha_M \text{ and } \varrho' \in \omega]$$

$$\text{or } [\varrho, \varrho' \in \alpha_M \text{ and } (\exists \beta, \beta')[\varrho = \omega * \beta \text{ and } \varrho' = \omega * \beta' \text{ and } \beta \leqslant_B \beta'] \text{ or } \varrho = \omega * \omega],$$

where $\alpha_M = \{\omega * \omega, \omega * \beta : \beta \in 2^{<\omega}\}$. The *tree of strategies* (*or tree of outcomes*) Σ is the smallest set consisting of B-strings satisfying the following clauses. Together with the nodes of Σ, we simultaneously define the *requirement assignment* R_σ.

1) $\emptyset \in \Sigma$; $\emptyset$ is an M-node; $R_\emptyset = M_0$;

2) if $\sigma \in \Sigma$ then

- if σ is a P-node then $\sigma * \omega \in \Sigma$ and $\sigma * \omega$ is an M-node, called an *originating* M-node; if $R_\sigma = P_i$ then $R_{\sigma * \omega} = M_{i+1}$; ω is the *outcome at* σ;

- if σ is an N-node then $\sigma * \omega \in \Sigma$ and $\sigma * \omega$ is a P-node. If $R_\sigma = N_k$, then $R_{\sigma * \omega} = P_k$; ω is the *outcome at* σ;

- if $\sigma * \omega$ is an originating M-node and $R_{\sigma * \omega} = M_e$, then σ is followed by 2^{2e} M-nodes $\sigma_0, \ldots, \sigma_{2^{2e}-1}$, with $\sigma_0 = \sigma * \omega$. For all $j < 2^{2e}$, let $R_{\sigma_j} = M_e$. For all $j < 2^{2e}$, $\sigma_j * \varrho \in \Sigma$, where $\varrho \in B$. If σ is a terminal M-node i.e. $\sigma = \sigma_{2^{2e}-1}$, where $R_\sigma = M_e$, then $\sigma * \varrho$ (with $\varrho \in B$) is an N-node; in this case, if $R_\sigma = M_e$ then $R_{\sigma * \varrho} = N_e$. The elements of B are the possible *outcomes* at σ.

REMARK 2.7. – Notice that there are no distinct outcomes for σ, if σ is an N-node or a P-node. The inclusion of nodes for N- and P-requirements in the tree of outcomes has only the purpose of recording some activity we do on behalf of the P- and N-requirements at those nodes.

The order relation $\leqslant_B$ of B extends to Σ, as we have already discussed in Subsection 2.1. We thus obtain an ordering $\leqslant_B$ on Σ.

REMARK 2.8. – If σ is an M-node, then we also allow ourselves to write $\sigma * \omega \subseteq \tau$ (or even $\sigma * \omega \subseteq f$, if f is a branch of Σ), to mean that, for some $\varrho \in \alpha_M$, $\sigma * \varrho \subseteq \tau$ and $\varrho(0) = \omega$ (or $(\exists \varrho \in \alpha_M)[\sigma * \varrho \subseteq f$ and $\varrho(0) = \omega]$ if f is a branch in Σ).

REMARK 2.9. – Henceforth we will follow the convention of using $\varrho, \sigma, \tau, \ldots$ as variables for elements of Σ, we will use the lower case Greek letters $\zeta, \eta, \theta, \ldots$ as variables for elements of $2^{<\omega}$. We will happen to write $M_\sigma (N_\sigma, P_\sigma)$ if $R_\sigma = M_e (R_\sigma = N_k, R_\sigma = P_i)$, respectively, for some e, k, i. We may also happen to write W_σ for W_k, if $R_\sigma = N_k$.

2.4. *The construction.*

At step s of the construction, we define a string $\delta_s \in \Sigma$ of length s. We define also the values of several parameters, for each $\sigma \in \Sigma$. We define also parameters $\lambda(\sigma * \omega, s)$, $\beta(\sigma * \omega, s)$, if σ is an M-node. Here is a list of the parameters.

- A s.u. tree $T_{\sigma, s}$.

- If σ is an M-node, then we will define elements $\lambda(\sigma * \omega, s)$, $\varepsilon(\sigma * \varrho, s)$ (with $\varrho \in \alpha_M$) of $\omega \cup \{\infty\}$. We briefly describe the meaning of these parameters. Suppose that σ an M-node, $R_\sigma = M_e$, say. Let $\sigma = \sigma_i$, where $\sigma_0 \subset \ldots \subset \sigma_{i-1}$ are the consecutive M_e-nodes preceding σ (σ_0 being the originating M-node preceding σ).

Let $m = 2e$, and, for every $\zeta_j^{(m)}$ ($j \leq 2^m - 1$) let $\alpha_j = T_{\sigma, s}(\zeta_j^{(m)})$.

At step s we define a string σ^+ (with $\sigma^+ = \sigma * \varrho$, for some $\varrho \in B$) and a s.u. subtree $T_{\sigma^+, s} \subseteq T_{\sigma, s}$.

If $\varrho(0) = \omega$, the parameter $\lambda(\sigma * \omega, s)$ measures the level of e-totality at stage s of a tree $\alpha_i * \widehat{T}$, which arises from our attempts to build an e-total subtree of $\alpha_i * T$, where T is given by $\alpha_i * T(\zeta) = T_{\sigma, s}(\zeta_i^{(m)} * \zeta)$, needed to define $T_{\sigma^+, s}$: the parameter $\lambda(\sigma * \omega, s)$ gives a restraint to lower priority requirements, which are supposed to be respectful of this restraint if the construction suggests that the final tree T_{σ^+} is such that $\mathrm{Ext}(T_{\sigma^+}, \zeta_i^{(m)})$ is e-total. In a similar way, the parameter $\varepsilon(\sigma * \varrho, s)$, measures the level of e-splitting at stage s of the tree $\mathrm{Ext}(T_{\sigma^+}, \zeta_i^{(m)})$.

- If σ is an M-node, then we will define an element $\chi(\sigma * \varrho, s)$ of $\omega \cup \{\infty\}$, a string $\alpha(\sigma * \varrho, s)$, a string $\beta(\sigma * \omega, s)$ and a corresponding set of strings $Z(\beta(\sigma * \omega))$ (this set is uniquely determined by the string $\beta(\sigma * \omega, s)$); we use the parameter $\alpha(\sigma * \varrho, s)$ to approximate a string α which is a witness, when $\varrho \in \omega$, of the fact that the construction is suggesting that the final tree $\mathrm{Ext}(T_{\sigma^+}, \zeta_i^{(m)})$ will not be e-total: in this case we have

$$T_{\sigma^+, s}(\zeta) = \begin{cases} T_{\sigma, s}(\zeta) & \text{if } |\zeta| \leq m, \\ T_{\sigma, s}(\zeta \restriction_m) * \mathrm{Ext}(\widehat{T}, \alpha)(\zeta \restriction_{\geq m}) & \text{if } |\zeta| > m, \end{cases}$$

We use the parameters $\beta(\sigma * \omega, s)$ and $Z(\beta(\sigma * \omega, s)$ to denote a string β and a set of strings Z, respectively, such that

$$T_{\sigma^+, s}(\zeta) = \begin{cases} T_{\sigma, s}(\zeta) & \text{if } |\zeta| \leq m, \\ T_{\sigma, s}(\zeta \restriction_m) * \mathrm{Ext}(T', \beta)(\zeta \restriction_{\geq m}) & \text{if } |\zeta| > m, \end{cases}$$

for some tree T' of the form $T' = R(\overline{T}, \widehat{T})$, when there is evidence that T' will eventually end up growing to a tree T' such that $\alpha_i * T'$ is right e-splitting for Z but with no s.u. simultaneous e-splitting for $Z \cup \{\beta\}$ (for some β and Z: at step s, the parameters $\beta(\sigma * \omega, s)$ and $Z(\beta(\sigma * \omega, s))$ are approximations to the final values β and Z.)

Without loss of generality, we will assume that

$$Z(\beta(\sigma * \omega, s)) = \{\zeta\colon |\zeta| = |\beta(\sigma * \omega, s)| \text{ and } \zeta < \beta(\sigma * \omega, s)\}.$$

The parameter $\chi(\sigma * \varrho, s)$ measures the level of right e-splitting of $\alpha_i * T'$ with respect to $Z(\beta)$, when $\varrho = \omega * \beta$, for some $\beta \in 2^{<\omega}$.

- If σ is an N-node then we define numbers $n(\sigma, s)$, $x(\sigma, s)$ and a set $\omega(\sigma, s)$. The parameter $n(\sigma, s)$ denotes the current witness of N_σ and $x(\sigma, s)$ is some number such that $\langle n(\sigma, s), \{x(\sigma, s)\}\rangle \in \Phi^s$.

Finally, let $\omega(\sigma, s) = \{x(\sigma, t)\colon t \leq s\}$.

• If σ is a P-node, $R_\sigma = P_i$ say, then we will define a parameter $y(\sigma, s)$ to record the branching value, of the appropriate level, which is reserved for satisfying the requirement P_σ.

At step s, let also

$$\nu(\sigma, s) = \text{least}\,\{\lambda(\tau, s), \varepsilon(\tau, s), \chi(\tau, s) : \tau \subseteq \sigma\},$$

and, for every σ, let $H(\sigma, s) = \{t < s : \sigma \subseteq \delta_t\}$.

Assume that $\{\xi_\sigma : \sigma \in 2^{<\omega}\}$ is a recursive partition of ω into infinite recursive sets.

We assume also that $x < \infty$, for all $x \in \omega$, and $\infty - m = \infty$ for every $m \in \omega$.

If we need at step s to define values of some parameters, then we do this in such a way to preserve already defined values of the same parameters, if these still work. More precisely, suppose we are given some recursively enumerable relation $\Gamma \subseteq \omega^{n+2}$, for some n. We will always implicitly refer to some suitable selector function for Γ, i.e. a partial recursive function c_Γ such that, for every $\overline{x} \in \omega^n$,

- $c_\Gamma(\overline{x}) \downarrow \Leftrightarrow (\exists y)(\exists s)\, \Gamma(\overline{x}, y, s)$;
- $c_\Gamma(\overline{x}) \downarrow \Rightarrow \Gamma(\overline{x}, (c_\Gamma(\overline{x}))_0, (c_\Gamma(\overline{x}))_1)$.

If $(\exists y)(\exists s)\, \Gamma(\overline{x}, y, s)$, then we say that $(c_\Gamma(\overline{x}))_0$ is *chosen consistently*. Thus, if at stage s, given some parameters $\overline{x}$, we need to choose some y such that $\Gamma(\overline{x}, y, s)$, then we choose y consistently. In applications of this criterion, for simplicity we will always omit to specify the particular relation Γ which we will be dealing with, this relation being clear from the context.

All the parameters retain their values until they are assigned a new value at some later stage of the construction. Notice that, given any M-node σ, at step s the construction may change values of $\lambda(\sigma * \varrho, s)$ only if $\varrho = \omega$, and of $\varepsilon(\sigma * \varrho, s)$ only when $\varrho = (\omega, \omega)$.

We now proceed with the construction.

STEP 0. – Define $\delta_0 = \emptyset$. For all $\sigma \in \Sigma$, and for all σ of the form $\sigma = \tau * \omega$, where τ is an M-node, define:

- $\lambda(\sigma, 0) = \varepsilon(\sigma, 0) = \chi(\sigma, 0) = \infty$;
- $n(\sigma, 0) = x(\sigma, 0) = y(\sigma, 0) = \uparrow$ and $\omega(\sigma, 0) = \emptyset$;
- $\alpha(\sigma, 0) = \beta(\sigma, 0) = \emptyset$;
- $T_{\sigma, 0} = \mathrm{Id}$.

STEP $s+1$. – We define δ_{s+1} by induction as follows. Let $\delta_{s+1}\restriction 0 = \emptyset$. Suppose that for every $m \leq n$ we have already defined $\delta_{s+1}\restriction m$, together with the values of the corresponding parameters. Suppose also that each $T_{\delta_{s+1}\restriction m, s+1}$ is a s.u. recursive tree.

Let $\sigma = \delta_{s+1}\restriction n$. We will define a string σ^+ such that $\sigma \subset \sigma^+$ and $|\sigma^+| = n+1$. Eventually we will let $\delta_{s+1}\restriction n+1 = \sigma^+$.

We distinguish three cases, according whether σ is an M-node, or σ is an N-node, or σ is a P-node.

Case 1). σ is an M-node, i.e. $R_\sigma = M_e$, for some e. Let $m = 2e$. We recall that there are 2^m consecutive M-nodes σ_i such that $R_{\sigma_i} = M_e$. Let $\sigma = \sigma_i$, for some such σ_i.

Let $T_{\sigma, s+1}(\zeta_i^{(m)}) = \alpha_i$ and let T be the tree defined by the following equation, for each ζ:

$$\alpha_i * T(\zeta) = T_{\sigma, s+1}(\zeta_i^{(m)} * \zeta).$$

That such a tree T exists follows easily from the fact that $T_{\sigma, s+1}$ is s.u.

We first define a saturated partial tree $T^* \subseteq T$ so that $T^* \subseteq T_{\restriction \nu(\sigma, s+1) - m}$. The restraint $\nu(\sigma, s+1) - m$ is meant to ensure that our action at σ will be respectful of the levels of e-totality, e-splitting and right e-splitting previously determined for higher priority M-requirements.

The idea is to define T^* such that $\alpha_i * T^*$ satisfies property (2.1), on all strings of its domain.

Let $T^*(\emptyset) = \emptyset$. Assume by induction that we have already defined T^* on S_k (with $0 < k < s$), and consider the sequence $\zeta_0^{(k-1)}, \ldots, \zeta_{2^{k-1}-1}^{(k-1)}$ of strings of length $k-1$.

We define by induction a sequence $\xi_{-1}, \xi_0, \ldots, \xi_j$ (for some $j \leqslant 2^{k-1} - 1$). Assume $\xi_{-1} = \emptyset$, and let us suppose we have already defined $\xi_0, \ldots, \xi_{j-1}$.

See whether there exists ξ such that

- $T^*(\zeta_j^{(k-1)}) * \xi \in T_{\restriction \nu(\sigma, s+1) - m - 1}$;
- $\xi_{j-1} \subseteq \xi$;
- $\varphi_i^\alpha * T_{e, s+1}^{*(\zeta_{j+1}^{(k-1)}) * \xi}(k) \downarrow$.

Choose consistently some such ξ_j.

If we successfully define $\xi_{2^{k-1}-1}$, then we extend the partial tree T^* to S_{k+1} in the following way. Let η be such that $T(\eta) = T^*(\zeta_{2^{k-1}-1}^{(k-1)}) * \xi_{2^{k-1}-1}$. Then let

$$T^*(\zeta_j^{(k)}) = T^*((\zeta_j^{(k)})_{\restriction k-1} * b) = T((\zeta_j^{(k)})_{\restriction k-1} * \eta_{\restriction \geqslant k-1} * b).$$

We note that $(\forall x \leqslant k)(\forall \zeta)[\,|\zeta| = k \Rightarrow \varphi_{e, s+1}^{\alpha_i * T^*(\zeta)}(x) \downarrow]$. Also notice that $T^*_{\restriction k+1} \subseteq$ $\subseteq T_{\restriction \nu(\sigma+1) - m}$.

Next, try to extend T^* to strings of length $k+2$. We say that *the process stops at* $k+1$ if T^* is defined on S_k, but we can not extend T^* to S_{k+1}: in this case domain$(T^*) = S_k$.

Finally, let

$$\widehat{T} = R(T^*, T).$$

We now look at what progress we have made towards obtaining e-totality of $\alpha_i * \widehat{T}$.

Assume that the process stops at k. We have the following possibilities:

Case a) towards obtaining a s.u. tree that is not e-total. If

$$k \leqslant \max\{\lambda(\sigma * \omega, t) \colon t \in H(\sigma * \omega, s+1)\},$$

then let j be such that the above sequence $\xi_{-1}, \xi_0, \ldots, \xi_{j-1}$ can not be further extended.

Let $\alpha(\sigma, s+1)$ be such that $\widehat{T}(\alpha(\sigma, s+1)) = T^*(\zeta_j^{(k-1)}) * \xi_{j-1}$, and let

- $\sigma^+ = \delta_{s+1}\upharpoonright_{n+1} = \sigma \cup \{(n, k)\}$;
- $\lambda(\sigma^+, s+1) = \infty$;
- finally, let

$$T_{\sigma^+, s+1}(\zeta) = \begin{cases} T_{\sigma, s+1}(\zeta), & \text{if } |\zeta| \leq m, \\ T_{\sigma, s+1}(\zeta\upharpoonright_m) * \operatorname{Ext}(\widehat{T}, \alpha(\sigma, s+1))(\zeta\upharpoonright_{\geq m}), & \text{if } |\zeta| > m. \end{cases}$$

Case b) progress towards a s.u. e-total tree. If

$$k > \max\{\lambda(\sigma * \omega, t): t \in H(\sigma * \omega, s+1)\},$$

we say that $s+1$ is an *e-total σ-expansionary step*. Then the outcome at σ will be of the form $\varrho \in \alpha_M$, with $\varrho(0) = \omega$, i.e. the first component outcome is ω. Let $\lambda(\sigma^+, s+1) = m + k$, and proceed as follows.

As before we first define a saturated partial subtree $\overline{T} \subseteq \widehat{T}$, such that $\overline{T} \subseteq \widehat{T}\upharpoonright_{\lambda(\sigma * \omega, s+1) - m}$ (notice that $T\upharpoonright_{\lambda(\sigma * \omega, s+1) - m} = T^*$).

We argue by induction on $t \leq s$ as follows, see [8, Proposition VI.5.15]:

SUBSTEP 0. – Define

- $Z_0 = S_1 = \{\emptyset\}$;
- $\beta_0 = \uparrow$;
- $\overline{T}(\emptyset) = \emptyset$, hence domain$(\overline{T}) = S_1$.

SUBSTEP $t+1$). – See if (a) and (b) below are true.
Let $V_t = \{\delta : (\exists\zeta)[\widehat{T}(\delta) = \overline{T}(\zeta) \text{ and } \zeta \in Z_t]\}$:

(a) There exist strings ζ, η such that

1) $(\forall \delta \in V_t)[\,|\delta * \zeta| = |\delta * \eta| \text{ and } \delta * \zeta, \delta * \xi \in \operatorname{domain}(T^*)]$;

2) (ζ, η) induces a s.u. simultaneous e-splitting for the set V_t (relative to the tree $\alpha_i * \widehat{T}$).

(b) $Z_t = S_{t+1}$.

Subcase 1) If (a) is true, then choose consistently ζ^0_{t+1}, ζ^1_{t+1} satisfying (a).
Extend $\overline{T}$ to strings of length $t+1$ as follows. Given ζ such that $|\zeta| = t$, let δ be such that $\widehat{T}(\delta) = \overline{T}(\zeta)$ and for all $j \in \{0, 1\}$, let

$$\overline{T}(\zeta * j) = \widehat{T}(\delta * \zeta^j_{t+1}).$$

Subcase 1.1) If (b) is true, let $Z_{t+1} = \{\zeta : |\zeta| = t+1\}$.

Subcase 1.2) If (b) is false, then let $Z_{t+1} = \{\zeta * 0 : \zeta \in Z_t\}$. In both cases, let $\beta_{t+1} = \uparrow$.

Subcase 2) Case (a) is false. Then there are $Z \subsetneq Z_t$ and $\xi \in Z_t - Z$ such that (a) holds with Z in place of Z_t, but (a) fails for $Z \cup \{\xi\}$ in place of Z_t.

Choose consistently such a string ξ. Notice that we can assume that the corresponding Z is given by $Z = \{\zeta\colon |\zeta| = \xi| \text{ and } \zeta < \xi\}$. The set Z is called the Z-set corresponding to ξ and we define $Z(\xi) = Z$.

If $Z = \emptyset$ then stop defining $\overline{T}$, thus domain$(\overline{T}) = S_{t+1}$. Do not proceed to substep $t+2$.

If $Z \neq \emptyset$, then repeat the above procedure and extend the definition of $\overline{T}$ in a similar way, with Z in place of Z_t. Let now

- $\beta_{t+1} = \xi$;
- $Z_{t+1} = \{\sigma * 0\colon \sigma \in Z\}$.

Notice that we get in this case a sequence of distinct strings $\beta_{t_1}, \ldots, \beta_{t_n}$ with $t_1 \leqslant \leqslant t_2 \leqslant \ldots t_n$, and $n \geqslant 1$ such that (a) fails with $Z(\beta_{t_j}) \cup \{\beta_{t_j}\}$ in place of Z_t above.

Now, for any β in the sequence, define

$$\chi(\sigma * \omega * \beta, s+1) =$$

$$= \begin{cases} \max\{h\colon (\forall \zeta \in Z(\beta))(\forall k < h)[\overline{T}(\zeta * 0^{[k]} * 0), (\overline{T}(\zeta * 0^{[k]} * 1)\ e-\text{split}]\} & \text{if } Z \neq \emptyset\,, \\ \infty & \text{otherwise}\,. \end{cases}$$

We say that $s+1$ is *right e-splitting* (σ, β)*-expansionary* if

$$\chi(\sigma * \omega * \beta, s+1) > \max\{\chi(\sigma * \omega * \beta, t)\colon t \in H(\sigma * \omega * \beta, s+1)\}\,.$$

At this point, let

$$\beta(\sigma * \omega, s+1) = \begin{cases} \leqslant_B\text{-least}\{\beta_{t_j}\colon s+1 \text{ is right } e\text{-splitting } (\sigma, \beta_{t_j})\text{-expansionary}\} \\ \text{if any } \beta_{t_n} \text{ if } s+1 \text{ is not } (\sigma, \beta_{t_j})\text{-expansionary, for any } j\,. \end{cases}$$

We now measure what progress we have made towaeds an e-splitting tree.
Let ε be maximal such that $\alpha_i * \overline{T}\restriction_\varepsilon$ is e-splitting.

Towards a s.u. tree with no e-splitting. If

$$\varepsilon \leqslant \max\{\varepsilon(\sigma * \omega * \omega, t)\colon t \in H(\sigma * \omega * \omega, s+1)\}\,,$$

then define

- $\varrho = \omega * \beta(\sigma * \omega, s+1)$;
- $T' = \mathrm{Ext}(R(\overline{T}, \widehat{T}), \beta(\sigma * \omega, s+1))$;
- $\varepsilon(\sigma * \varrho, s+1) = \infty$.

Progress towards a s.u. e-splitting tree. If

$$\varepsilon > \max\{\varepsilon(\sigma * \omega * \omega, t)\colon t \in H(\sigma * \omega * \omega, s+1)\}\,,$$

then we call $s+1$ an *e-splitting σ-expansionary step.* In this case we let

- $T' = R((\overline{T})_\varepsilon, \widehat{T})$;
- $\varrho = \omega * \omega$ is the outcome at σ at $s+1$;
- $\varepsilon(\sigma * \varrho, s+1) = m + \varepsilon$.

Finally, let $\sigma^+ = \delta_{s+1}\restriction_{n+1} = \sigma \cup \{(n, \varrho)\}$, and let

$$T_{\sigma^+, s+1} = \begin{cases} T_{\sigma, s+1}(\zeta), & \text{if } |\zeta| \leqslant m, \\ T_{\sigma, s+1}(\zeta\restriction_m) * T'(\zeta\restriction_{\geqslant m}), & \text{if } |\zeta| > m. \end{cases}$$

This concludes the case of σ being an M-node.

REMARK 2.10. – We notice that the partial trees T^*, $\widehat{T}$, $\overline{T}$, T' defined at step $s+1$ are in fact parameters depending on σ and s, i.e. it would more correct to write $T^*_{\sigma, s}$, $\widehat{T}_{\sigma, s}$, $\overline{T}_{\sigma * \omega, s}$, $T'_{\sigma * \omega, s}$. We have omitted the subscripts for simplicity, and will continue to do so as long as the context makes superfluous mentioning σ and s.

Case 2). σ is an N-node. Then $\sigma^+ = \sigma * \omega$. If some requirement P_τ, with $\tau \subseteq \sigma$, is not satisfied at $s+1$, then do nothing. Otherwise, suppose that $R_\sigma = N_k$, and let $m = 2k$.

If $m \geqslant \nu(\sigma, s+1)$ then do nothing. Otherwise, let x be the m-level branching value, i.e. let $x > |T_{\sigma, s+1}(\zeta)|$ be such that $T_{\sigma, s+1}(\zeta * 0)(x) \neq T_{\sigma, s+1}(\zeta * 1)(x)$ (where ζ is any string such that $|\zeta| = m$.) Let n be the least number in ξ_σ such that $\langle n, \emptyset\rangle \notin \Phi^s$. Let

$$n(\sigma, s+1) = n, \qquad x(\sigma, s+1) = x.$$

Let also $x(\sigma, s+1) \in \omega(\sigma, s+1)$, and $T_{\sigma^+, s+1} = T_{\sigma, s}$.

Case 3). σ is a P-node. Then $\sigma^+ = \sigma * \omega$. Let $R_\sigma = P_i$. Let $m = 2i + 1$. If some requirement P_{τ},, with $\tau \subset \sigma$, is not satisfied at $s+1$, then do nothing.

Otherwise, let η be the least string such that

$$T_{\sigma, s+1}(\zeta * \eta * 0),\ T_{\sigma, s+1}(\zeta * \eta * 1) \in (T_{\sigma, s+1})\restriction_{\nu(\sigma, s+1)}$$

(where ζ is any string such that $|\zeta| = m$), and the $m + |\eta|$-level branching value y of $T_{\sigma, s+1}$ is such that $y \notin \bigcup \{\omega(\sigma', s+1): \sigma' <_B \sigma\}$. For every ζ, let

$$T_{\sigma^+, s+1}(\zeta) = \begin{cases} T_{\sigma, s+1}(\zeta) & \text{if } |\zeta| \leqslant m, \\ T_{\sigma, s+1}(\zeta\restriction_m * \eta * j * \zeta\restriction_{\geqslant m+1}) & \text{if } |\zeta| \geqslant \mu \text{ and } \zeta(m) = j. \end{cases}$$

If no such η exists, then do nothing. If such a string η exists then we say that P_σ is *satisfied* at $s+1$. Otherwise we say that P_σ is *not satisfied* at $s+1$.

Updating Φ. Update the e-operator Φ as follows:

1) for all $\sigma \subseteq \delta_{s+1}$, let $\langle n(\sigma, s+1), \{x(\sigma, s+1)\}\rangle \in \Phi^{s+1}$;

2) for every y and n, if $\langle n, \{y\}\rangle \in \Phi^s$ and $y \in \{y(\sigma, s+1): \sigma \subseteq \delta_{s+1}\}$, then let $\langle n, \emptyset\rangle \in \Phi^{s+1}$.

2.5. *Proof that the construction works.*

The proof that the construction works consists of several lemmas. Let $H(\sigma) = \{s\colon \sigma \subseteq \delta_s\}$, and, similarly, define $H(\sigma * \omega) = \{s\colon \sigma * \omega \subseteq \delta_s\}$.

LEMMA 2.11. – *For every n, the following hold*

1) $\liminf_s \delta_{s \upharpoonright n}$ *exists* (*where the* lim inf *is taken with respect to the ordering relation* $\leqslant_B$ of Σ;)

2) *If* $\sigma = \liminf_s \delta_{s\upharpoonright n}$ *then*

a) $\lim_s \nu(\sigma, s) = \infty$;

b) *there exists t such that*

- *the sequence* $\{\nu(\sigma, s)\colon s \geqslant t\}$ *is nondecreasing*;
- *for all* $s \geqslant t$, *if* $s \in H(\sigma)$ *then*

$$(\forall u \geqslant s)[(T_{\sigma, s})_{\upharpoonright \nu(\sigma, s)} = (T_{\sigma, u})_{\upharpoonright \nu(\sigma, s)}];$$

- *for all* ζ, $\lim_s T_{\sigma, s}(\zeta)$ *exists*;

c) *if* $\sigma = \tau * \varrho$, τ *is an M-node and* $\varrho = \omega * \beta$, *then* $\liminf_s \beta(\tau * \omega, s) = \beta$, *where the* lim inf *is taken with respect to* $\leqslant_B$;

d) *there exists t such that* $(\forall s \geqslant t)[\alpha(\sigma, s) = \alpha(\sigma, t)]$;

e) *there exists a stage t such that for all* $s \geqslant t$, $\omega(\sigma, s) = \omega(\sigma, t)$, $n(\sigma, s) = n(\sigma, t)$, $x(\sigma, s) = x(\sigma, t)$, *hence* $\lim_s \omega(\sigma, s)$, $\lim_s n(\sigma, s)$ *and* $\lim_s x(\sigma, s)$ *exist.*

f) *there exists t such that, for all* $s \geqslant t$, $y(\sigma, s) = y(\sigma, t)$; *moreover, if σ is a P-node, then P_σ is satisfied at any* $s \geqslant t$.

PROOF. – The proof is by induction on n.

For $n = 0$ the claim is trivial.

Assume that the lemma is true of n: let $\sigma = \liminf_s \delta_{s\upharpoonright n}$. By inductive assumptions, for each $\sigma' \leqslant_B \sigma$, let $\lim_s T_{\sigma', s}(\zeta) = T_{\sigma'}(\zeta)$: this defines a tree $T_{\sigma'} = \lim_s T_{\sigma', s}$; let $\lim_s \omega(\sigma', s) = \omega(\sigma')$, $\lim_s n(\sigma', s) = n(\sigma')$, $\lim_s x(\sigma', s) = x(\sigma')$; let $\lim_s y(\sigma', s) = y(\sigma')$; let $\lim_s \alpha(\sigma', s) = \alpha(\sigma')$. The above limits exist, with the understanding that we allow ourselves also the case when the parameter under consideration is eventually undefined.

Let t_σ be a stage such that t_σ satisfies, for *all* $\tau \subseteq \sigma$ and for $s \geqslant t_\sigma$, the statements of the lemma, and such that, for all $\sigma' \leqslant_B \sigma$ and $\sigma' \nsubseteq \sigma$, for all $s \geqslant t_\sigma$, we have that $\sigma' \nsubseteq \delta_s$.

We want first to show that $\sigma^+ = \liminf_s \delta_{s\upharpoonright n+1}$ exists. Clearly we need only consider the case when σ is an M-node. If there are only finitely many stages s such that $\sigma * \omega \subseteq \delta_s$, then the construction ensures that, for some $n \in \omega$, $\sigma^+ = \sigma * n$, hence σ^+ exists. If there exist infinitely many s such that $\sigma * \omega \subseteq \delta_s$, then, when there exist infinitely many stages s such that $\sigma * \omega * \omega \subseteq \delta_s$, we obtain $\sigma^+ = \sigma * \omega * \omega$. Otherwise, as in the proof of [8, Proposition VI.5.15], we are able to select a set of strings Z, and a string

β such that, for all $\zeta \in Z$, $|\zeta| = |\beta|$, and we construct a tree which is right e-splitting for Z, but with no s.u. simultaneous e-splitting for $Z \cup \{\beta\}$, thus eventually obtaining a s.u. tree with no e-splitting. The argument to show that σ^+ exists can be sketched as follows. Let $t_0 \geq t_\sigma$ be a stage such that $(\forall s \geq t_0)[s \notin H(\sigma * \omega * \omega)]$. Then there exist a string β_1 and a stage $t_1 \geq t_0$ such that, for every $s \geq t_1$, if $s \in H(\sigma * \omega)$, then we get a sequence $\{\beta^1_j : 1 \leq j \leq n_1\}$ at s beginning with $\beta^1_1 = \beta_1$; if $H(\sigma * \omega * \beta_1)$ is not infinite, then there exist a string β_2 and a stage $t_2 \geq t_1$, with $\beta 1 <_B \beta_2$, such that, for every $s \geq t_2$, if $s \in H(\sigma * \omega)$, then we get a sequence $\{\beta^2_j : 1 \leq j \leq n_2\}$ at s beginning with $\beta^2_1 = \beta_1$ and $\beta^2_2 = \beta_2$. Notice also that $Z(\beta_2) = \{\zeta * 0^{[k]} : \zeta \in Z\}$, for some proper subset $Z \subset Z(\beta_1)$; hence $\mathrm{card}(Z(\beta_2)) < \mathrm{card}(Z(\beta_1))$. But, then, eventually we find a string β_n such that $H(\sigma * \beta_n)$ is infinite: otherwise we would obtain an infinite sequence $\beta_1 <_B \beta_2 <_B \ldots <_B \beta_n < \ldots$, and, for every n, $\mathrm{card}(Z(\beta_{n+1})) < \mathrm{card}(Z(\beta_n))$, a contradiction.

This shows that there exists the $\leq_B$-least string β for which $H(\sigma * \omega * \beta)$ is infinite.

Next, notice that clearly $\lim_s \nu(\sigma^+, s) = \infty$.

We know by induction that for every $\tau \subset \sigma^+$, for every ζ, $\lim_s T_{\tau, s}(\zeta)$ exists, say $\lim_s T_{\tau, s}(\zeta) = T_\tau(\zeta)$.

Case 1). First suppose that σ is an M-node, $R_\sigma = M_e$, say, and let $m = 2e$. Let $\sigma = \sigma_i$, i.e. assume that σ is preceded by consecutive M-nodes $\sigma_0 \subset \sigma_1 \subset \ldots \sigma_{i-1}$, where $R_{\sigma_j} = M_e$, for every $j \leq i-1$. Let also T be the tree defined by: $\alpha_i * T(\zeta) = T_\sigma(\zeta_i^{(m)'} * \zeta)$. Let ϱ be such that $\sigma^+ = \sigma * \varrho$. Let $t_1 \geq t_\sigma$ be such that for all $s \geq t_1$, $T_{\sigma, s \restriction m+1} = T_{\sigma, t_1 \restriction m+1}$.

1) $\varrho \in \omega$. Let $t_2 \geq t_1$ be such that, for all $s \geq t_2$,

- $\alpha(\sigma^+, s) = \alpha(\sigma^+, t_2)$: let $\alpha = \lim_s \alpha(\sigma^+, s)$;
- $(\widehat{T}_{\sigma, s})_{\restriction |\alpha| + 1} = (\widehat{T}_{\sigma, t_2})_{\restriction |\alpha| + 1}$.

Such a stage t_2 exists; in fact it is easy to see that $\lim_s T^*_{\sigma, s}$ exists. Let T^* be this limit (notice that domain(T^*) is finite): it follows that $\lim_s \widehat{T}_{\sigma, s} = R(T^*, T)$ exists, equal, say, to $\widehat{T}$. Thus $\lim_s T_{\sigma^+, s} = T_{\sigma^+}$ exists, where

$$T_{\sigma^+}(\zeta) = \begin{cases} T_\sigma(\zeta) & \text{if } |\zeta| \leq m, \\ T_\sigma(\zeta_{\restriction m}) * \mathrm{Ext}(\widehat{T}, \alpha)(\zeta_{\restriction > m}) & \text{otherwise}. \end{cases}$$

Moreover, for all $s \geq t_2$, if $s \in H(\sigma^+)$ then, since our choices of parameters are consistent,

$$(\forall u \geq s)[(T_{\sigma^+, s})_{\restriction \nu(\sigma^+, s)} = (T_{\sigma^+, u})_{\restriction \nu(\sigma^+, s)}].$$

2) $\varrho = \omega * \beta$. Since we have $\sigma * \varrho \subseteq \delta_s$ only at e-total expansionary stages, we have that there exists $t_2 \geq t_1$ such that the sequence

$$\{\lambda(\sigma * \omega, s) : s \geq t_2 \text{ and } s \in H(\sigma * \omega)\}$$

is strictly ascending.

Similarly, since we have $\sigma * \varrho \subseteq \delta_s$ only at stages that are right e-total (σ, β)-expansionary, we may assume also that the sequence

$$\{\chi(\sigma * \varrho, s): s \geq t_2 \text{ and } s \in H(\sigma * \varrho)\}$$

is strictly ascending; trivially, for all $s \geq t_2$, if $s \in H(\sigma * \varrho)$, we have that $\beta(\sigma * \omega, s) = \beta$. Let $Z = Z(\beta)$.

It is now easy to see that $\lim_s \overline{T}_{\sigma * \omega, s}$ and, consequently, $\lim_s T'_{\sigma * \omega, s}$ exist: thus, define $\overline{T} = \lim_s \overline{T}_{\sigma * \omega, s}$, and $T' = \lim_s T'_{\sigma * \omega, s}$ (clearly $T' = R(\overline{T}, \widehat{T})$: notice that $\overline{T}$ need not be partial, i.e. we may get $T' = \overline{T}$, if $Z \neq \emptyset$.)

Then $\lim_s T_{\sigma^+, s}$ exists, this limit being the following tree T_{σ^+}:

$$T_{\sigma^+}(\zeta) = \begin{cases} T_\sigma(\zeta) & \text{if } |\zeta| \leq m\,, \\ T_\sigma(\zeta\restriction_m) * \operatorname{Ext}(T', \beta)(\zeta\restriction_{\geq m}) & \text{otherwise}\,. \end{cases}$$

It also follows that, for all $s \geq t_2$, if $s \in H(\sigma^+)$ then, by consistent choices,

$$(\forall u \geq s)[(T_{\sigma^+, s})\restriction_{\nu(\sigma^+, s)} = (T_{\sigma^+, u})\restriction_{\nu(\sigma^+, s)}]\,.$$

3) $\varrho \in \alpha_M$ and $\varrho = \omega * \omega$. Let $\overline{T}$ be as in the previous case: Since $\sigma * \varrho \subseteq \delta_s$ only at e-expansionary stages, we have that there exists some $t_2 \geq t_1$ such that the sequence $\{\varepsilon(\sigma^+, s): s \geq t_2 \text{ and } s \in H(\sigma^+)\}$ is strictly increasing. Moreover, it is easy to see that $\lim_s T_{\sigma^+, s}$ exists, this limit being the tree defined as follows:

$$T_{\sigma^+}(\zeta) = \begin{cases} T_\sigma(\zeta) & \text{if } |\zeta| \leq m\,, \\ T_\sigma(\zeta\restriction_m) * \overline{T}(\zeta\restriction_{\geq m+1}) & \text{otherwise}\,. \end{cases}$$

Moreover, for all $s \geq t_2$, if $s \in H(\sigma^+)$ then, by consistent choices,

$$(\forall u \geq s)[(T_{\sigma^+, s})\restriction_{\nu(\sigma^+, s)} = (T_{\sigma^+, u}))\restriction_{\nu(\sigma^+, s)}]\,.$$

Case 2). Suppose that σ is an N-node. Let $R_\sigma = N_k$ and let $m = 2k$; wait for a stage $t_1 \geq t_\sigma$ such that all P_τ, with $\tau \subsetneq \sigma$, are satisfied at any $s \geq t_1$ and $(\forall s \geq t_1)[(T_{\sigma, s})\restriction_{m+1} = (T_\sigma)\restriction_{m+1}]$. Then, at any $t_2 \geq t_1$, we can choose a suitable $n(\sigma, t_2) \in \xi_\sigma$, and for all $s \geq t_2$, $n(\sigma, s) = n(\sigma, t_2)$ and $x(\sigma, s) = x(\sigma, t_2)$, because at subsequent stages, no action forces us to enumerate $\langle n, \emptyset\rangle \in \Phi$; finally notice that $\omega(\sigma, s) = \omega(\sigma, t_2)$, for all $s \geq t_2$.

Case 3). Let t_σ be chosen as in the preceding cases. Let $R_\sigma = P_i$ and let $m = 2i + 1$; wait for a stage $t_1 \geq t_\sigma$, with $t_1 \in H(\sigma)$, such that all P_τ, with $\tau \subset \sigma$, are satisfied at any $s \geq t_1$ and

- $(\forall s \geq t_1)[(T_{\sigma, s})\restriction_{m+1} = (T_\sigma)\restriction_{m+1}]$;
- there is a string η such that

$$T_{\sigma, t_1}(\zeta_j^{(m)} * \eta * 0), \quad T_{\sigma, t_1}(\zeta_j^{(m)} * \eta * 1) \in (T_{\sigma, t_1})\restriction_{\nu(\sigma, t_1)}$$

and the strings $T_{\sigma, t_1}(\zeta_j^{(m)} * \eta * 0)$, $T_{\sigma, t_1}(\zeta_j^{(m)} * \eta * 1)$ differ on some y, such that $y \notin \bigcup\{\omega(\sigma'): \sigma' <_B \sigma\}$ and y is the $m + |\eta|$-level branching value of T_{σ, t_1}.

Then if $s \geqslant t_1$ is such that $\sigma \subseteq \delta_s$, it follows that $y(\sigma, s) = y(\sigma, t_1)$. So P_σ is eventually satisfied.

It follows that

$$T_{\sigma^+}(\zeta) = \begin{cases} T_\sigma(\zeta) & \text{if } |\zeta| \leqslant m\,, \\ T_\sigma(\zeta\restriction_m * \eta * i * \zeta\restriction_{\geqslant m+1}) & \text{if } |\zeta| > m \text{ and } \zeta(m) = i\,, \end{cases}$$

for a suitable string η as above. ■

By the previous lemma, Let f be the *true path*, i.e. $f\restriction_n = \liminf_s \delta_s\restriction_n$.

LEMMA 2.12. – *For every $\sigma \subset f$, the tree T_σ is s.u.*

PROOF. – Immediate since in the construction, at every node σ we start with a s.u. tree and we use tree operations that take s.u. trees to s.u. trees. ■

LEMMA 2.13. – 1) *For all $\sigma, \sigma' \subseteq f$, $\sigma \subseteq \sigma' \Rightarrow T_{\sigma'} \subseteq T_\sigma$;*

2) *for every ζ, let*

$$\sigma(\zeta) = \mathit{least}\,\{\tau \subset f\colon \tau \text{ is an } M\text{-node},\ R_\tau = M_e,\ \mathit{say},\ \mathit{and}\ |\zeta| \leqslant 2e)\}\,.$$

Then $(\forall\sigma)[\sigma(\zeta) \subseteq \sigma \subset f \Rightarrow T_\sigma(\zeta) = T_{\sigma(\zeta)}(\zeta)]$.

PROOF. – Part 1) is obvious.

Part 2) follows from the fact that at each node σ, and stage s, we are allowed to modify $T_{\sigma,s}$ only on strings η's such that $|\eta| > 2e$, if σ is an M-node and $R_\sigma = M_e$, and $|\eta| > 2i+1$, if σ is a P-node and $R_\sigma = N_i$. ■

Lemma 2.13 enables us to define a tree F as follows: for every ζ, let $F(\zeta) = T_{\sigma(\zeta)}(\zeta)$.

Finally, let G be the N-reduced subtree of F.

We now pass on to show that the requirements are satisfied.

LEMMA 2.14. – *For every $e \in \omega$ we have:*

$$(\forall M \in [F])[\varphi_e^M \ \mathit{total} \Rightarrow \varphi_e^M \ \mathit{recursive\ or}\ M \leqslant_T \varphi_e^M]\,.$$

PROOF. – Let $M \in [F]$. Let $e \in \omega$ be given and let $\sigma_0 \subseteq f$ be such that $R_{\sigma_0} = M_e$, and σ_0 is an originating M-node. Let $m = 2e$. By Lemma 2.11 and Lemma 2.13, there exists some i such that $i < 2^m - 1$ and $T_{\sigma_0}(\zeta_i^{(m)}) \subset M$. Let ϱ be the true outcome at σ_i. If ϕ_e^M is total then, by construction, $\varrho \in \alpha_M$. We have two cases.

Case 1). $\varrho = \omega * \beta$. By Lemma 2.5 there is no e-splitting in $\alpha_i * \mathrm{Ext}(T', \beta)$, where $\alpha_i = T_{\sigma_0}(\zeta_i^{(m)})$; notice that $\alpha_i * \mathrm{Ext}(T', \beta) = \mathrm{Ext}(T_{(\sigma_i)^+}, \zeta_i^{(m)})$. Since $M \in [\mathrm{Ext}(T_{(\sigma_i)^+}, \zeta_i^{(m)})]$, we conclude that φ_e^M is recursive, by the following procedure: in order to compute $\varphi_e^M(x)$, wait for s such that $s \geqslant t_{(\sigma_i)^+}$ and $s \in H((\sigma_i)^+)$ and, for some ζ such that $|\zeta| + m < \nu((\sigma_i)^+, s)$, we have that $\varphi_{e,s}^{\mathrm{Ext}(T_{(\sigma_i)^+,s},\, \zeta_i^{(m)})(\zeta)}(x)\downarrow$.

Then $\phi_e^M(x) = \phi_{e,s}^{\mathrm{Ext}(T_{(\sigma_i)^+},s,\,\zeta_i^{(m)}(\zeta))}(x)$. The procedure is clearly recursive by Lemma 2.11.

Case 2). If $\varrho = \omega * \omega$, then we conclude as in [8, Proposition VI.5.15] that $\alpha_i * T'$ is an e-splitting tree, thus $M \leqslant_T \phi_e^M$, by known arguments, using a recursive set of stages that, as in the previous case, are true stages in the approximation to $T_{(\sigma_i)^+}$. ■

Let $\Phi = \bigcup_{s \in \omega} \Phi^s$. Then

LEMMA 2.15. – *For every* $M, M' \in [G]$, $\Phi(M) = \Phi(M')$.

PROOF. – Let $M, M' \in [G]$ be given. It is enough to show that

$$(\forall n)[n \in \Phi(M) \Rightarrow n \in \Phi(M')].$$

Suppose that $n \in \Phi(M)$. If $\langle n, \emptyset \rangle \in \Phi$, then the claim is trivial. Otherwise the claim follows from updating of Φ, since if $M(x) \neq M'(x)$ and $\langle n, \{x\} \rangle \in \Phi$, then the construction forces $\langle n, \emptyset \rangle \in \Phi$ (since M, M' are allowed to differ only on numbers of the form $y(\tau)$, for some $\tau \subset f$). ■

By the previous lemma, let now $A = \Phi(M)$, for any $M \in [G]$.

LEMMA 2.16. – *For every* $k \in \omega$, *the requirement* N_k *is satisfied.*

PROOF. – Let k be given, and let $\sigma \subset f$ be such that $R_\sigma = N_k$. By Lemma 2.11 and by updating of Φ, it follows that

$$n(\sigma) \in A (\Leftrightarrow (\forall M \in [G])[x(\sigma) \in M]) \Leftrightarrow n(\sigma) \notin W_k. \quad ■$$

LEMMA 2.17. – *For every* $i \in \omega$, *the requirement* P_i *is satisfied.*

PROOF. – Immediate by Lemma 2.11 and Lemma 2.13. ■

REMARK 2.18. – Inspection of the proof shows that $A \in \Delta_3$.

3. – On a question of Solon.

In this section, we apply the properties of Lachlan minimal degrees to answer a question of [12].

DEFINITION 3.1 ([12]). – • If H is a set, let p_H be the principal function for H, that is p_H is the function which enumerates the elements of H in increasing order.

• A set H is *e-hyperimmune* if for every total function f such that $H \geqslant_e f$ there are infinitely many n such that $p_H(n) > f(n)$.

Solon was interested in characterizing the e-degrees of nontotal functions. In this direction, he proved the following theorem.

THEOREM 3.2 ([12]). – *For any set H, if H is e-hyperimmune then the e-degree of H is not total.*

Solon posed the question of whether the converse of Theorem 3.2 is also true. An affirmative answer would give a natural characterization of nontotality. However, we show that the converse fails.

One can produce a counterexample to Solon's proposal in more than one way. For example, Spector ([14]) produced a set M of minimal Turing degree such that every function recursive in M is bounded by a recursive function. Gutteridge ([4]) showed that there is no minimal nontrivial e-degree, so there is a nontotal enumeration degree X below M. By the same argument that we will give below, the e-degree of X cannot contain an e-hyperimmune set. The advantage to the proof below is that we can directly construct the required counterexample using just the techniques of the previous section.

THEOREM 3.3. – *Using Lachlan's construction, it is possible to construct a set M such that the enumeration degree of M is not total and M fails to be e-hyperimmune.*

PROOF. – Exploit Lachlan's techniques, to construct a sequence T_k of s.u. recursive trees such that, for every k, $T_{k+1} \subseteq T_k$, as follows. Begin with $T_0 = \mathrm{Id}$. A Step $3k+1$,

- if there is a s.u. recursive subtree $\widehat{T} \subseteq T_{3k}$ such that $\widehat{T}$ is not k-total, i.e. there exists some x such that for no path $M \in [\widehat{T}]$, do we have $\phi_k^M(x)\downarrow$, then let $T_{3k+1} = \widehat{T}$.

- Otherwise, let $\widehat{T}$ be a k-total s.u. recursive subtree of T_{3k}. Then let T_{3k+1} be *either* a k-splitting s.u. recursive subtree of $\widehat{T}$, *or* a s.u. recursive subtree of $\widehat{T}$ with no k-splitting, i.e., in the latter case, $T_{3k+1} = \mathrm{Ext}(\overline{T}, \beta)$, where β is a string such that, for some set Z of strings, all of the same of length as β, we have that $\overline{T}$ is a s.u. recursive subtree of $\widehat{T}$ and $\overline{T}$ is right k-splitting for Z but with no s.u. simultaneous k-splittings for $Z \cup \{\beta\}$.

At step $3k+2$, we construct a s.u. subtree T_{3k+2} of T_{3k+1} so as to satisfy the following requirement: for all $M \in [T_{3k+2}]$, $M \neq W_k$. For this, let x be the 0-level branching value of T_{3k+1}, and let $T_{3k+2} = \mathrm{Ext}(T_{3k+1}, h)$, where $h \in \{0, 1\}$ is the string of length 1 such that $T_{3k+1}(h)(x) = 1 \Leftrightarrow x \notin W_k$.

Finally, at step $3k+3$ we ensure quasi-minimality as follows. We contruct a s.u. subtree T_{3k+3} of T_{3k+2} such that

$$(\forall M \in [T_{3k+3}])[\Phi_k(M) \text{ total } \Rightarrow \Phi_k(M) \text{ recursive}]:$$

if there is a finite set D and a branch $M \in [T_{3k+2}]$ such that $D \subseteq M$ and $\Phi_k(D)$ is not single-valued, then it is easily seen that there is a s.u. recursive subtree $T_{3k+3} \subseteq T_{3k+2}$ such that $D \subseteq M$, for all $M \in [T_{3k+3}]$ (indeed, $T_{3k+3} = \mathrm{Ext}(T_{3k+2}, \eta)$, for a suitable string η).

Otherwise, there is no branch $M \in [T_{3k+2}]$ such that $\Phi_k(M)$ is not single-valued. In this latter case, take $T_{3k+3} = T_{3k+2}$, and use the fact that T_{3k+2} is recursive to show that, for all $M \in [T_{3k+3}]$, if $\Phi_k(M)$ is total then $\Phi_k(M)$ is recursive, being

$$\Phi_k(M) = \{x : (\exists \zeta)[x \in \Phi_k^{\{y:\, T_{3k+3}(\zeta)(y)=1\}}]\}.$$

Now, let $M \in [T_k]$, for all k. Clearly M has quasi-minimal e-degree. Then, consider M as a total object. (One can substitute the characteristic function of M for M in this context.) Any function ϕ_k^M which is Turing computable relative to M is forced to be total by T_{3k+1}. But, then, ϕ_k is total relative to any path through T_{3k+1}. But then, by compactness, for each n there is a bound on the values achieved by ϕ_k relative to paths in T_{3k+1} and this bound can be found recursively. Thus every function which is recursive in M is bounded by a recursive function.

Suppose that W is recursively enumerable in M. Then W has an infinite subset R which is recursive in M and p_R is recursive in M. Since $R \subseteq W$, for each n $p_R(n) \geq p_W(n)$; that is to say that the n-th element of R is greater than or equal to the n-th element of any superset of R. Thus the principal function of any set which is e-reducible to M is dominated by a function which is recursive in M and hence by a recursive function.

Thus, M has nontotal enumeration degree and each set which is enumeration reducible to M has a principal function which is dominated by a recursive function, as required. ∎

4. – Concluding remarks.

The notion of a quasi-minimal e-degree can be relativized as follows.

DEFINITION 4.1. – Given any $\mathfrak{J} \subseteq \mathcal{D}_e$, we say that an e-degree $\boldsymbol{a}$ is *$\mathfrak{J}$-quasi-minimal* if

- $(\forall \boldsymbol{c} \in \mathfrak{J})[\boldsymbol{c} < \boldsymbol{a}]$;
- $(\forall \text{ total } \boldsymbol{c})[\boldsymbol{c} \leq \boldsymbol{a} \Leftrightarrow (\exists \boldsymbol{b} \in \mathfrak{J})[\boldsymbol{c} \leq \boldsymbol{b}]$.

THEOREM 4.2. – *For every countable ideal $\mathfrak{J}$, $\mathfrak{J}$-quasi-minimal e-degrees exist.*

PROOF. – The proof is similar to the proof of the Exact Pair Theorem in [14]. Let $\mathfrak{J}$ be a countable ideal. Then, by known arguments, we can find a sequence of sets $B_{-1} \leq_e B_0 \leq_e \ldots \leq_e B_n \leq_e \ldots$, whose e-degrees generate $\mathfrak{J}$. Assume that $B_{-1} = \emptyset$. Let $B = \bigcup \{B_i : i \in \omega\}$: without loss of generality, we may also assume that B is coinfinite.

Construct a set A such that $(\forall n)[B_n \leq_e A]$ and satisfying, for all $i, e \in \omega$, the following requirements:

$$N_{\langle e,\, i\rangle} : A \neq \Phi_e(B_i)$$

$$P_k : \Phi_k(A) \text{ total} \Rightarrow \Phi_k(A) \leq_e B_{k-1}.$$

Given any set X, let $X^n = \{\langle x, y\rangle : x = n \text{ and } \langle x, y\rangle \in X\}$, and let $n \times X = \{\langle n, x\rangle : x \in X\}$. We construct A of the form $A = \bigcup \{\Theta_k : k \geq -1\}$, where, for every n, $A^n = (n \times B_n) \cup G$, for some finite set G (assume $\Theta_{-1} = \emptyset$.) This yields that $B_n \leq_e A$, for every n.

During the construction, define also by induction finite sets F_k, E_k, and values of parameters x_k. Eventually we will have that F_k is contained in A, whereas $E_k \cap A = \emptyset$, for every $k \in \omega$.

The set Θ_k will be defined at stage $2k+1$.

Step 2k. This step aims at satisfying the requirement N_k. Let $k = \langle e, i \rangle$. Choose $x_k \notin B$ such that $\langle k, x_k \rangle \notin F_{2k-1} \cup E_{2k-1}$ (where we assume $F_{-1} = E_{-1} = \emptyset$), Define

$$F_{2k} = \begin{cases} F_{2k-1} \cup \{\langle k, x_k \rangle\} & \text{if } \langle k, x_k \rangle \notin \Phi_e(B_i), \\ F_{2k} & \text{otherwise}, \end{cases}$$

and

$$E_{2k} = \begin{cases} E_{2k-1} \cup \{\langle k, x_k \rangle\} & \text{if } \langle k, x_k \rangle \in \Phi_e(B_i), \\ E_{2k} & \text{otherwise}. \end{cases}$$

Step $2k+1$ (Requirement P_k). Say that a finite set F is *compatible with* Θ_{k-1}, if $F \cap \bigcup \{\omega^i : i \leqslant k-1\} \subseteq \Theta_{k-1}$ (assume $\omega^{-1} = \emptyset$).

We distinguish the following two cases.

- $(\exists F)$ [F finite and $F \cap E_{2k} = \emptyset$ and $\Phi_k(F)$ not single-valued and F compatible with Θ_{k-1}].

In this case choose the least (e.g. with least canonical index) such set F, let $F_{2k+1} = F_{2k} \cup F$, $E_{2k+1} = E_{2k}$ and let $\Theta_k = \Theta_{k-1} \cup F_{2k+1} \cup (k \times B_k)$.

- Otherwise, let $F_{2k+1} = F_{2k}$, $E_{2k+1} = E_{2k}$ and $\Theta_k = \Theta_{k-1} \cup (k \times B_k)$.

To show that the requirement P_k is satisfied, assume for a contradiction that $\Phi_k(A)$ is total. Then it is not difficult to show that

$$\Phi_k(A) = \{x : (\exists F)[F \cap E_{2k} = \emptyset \text{ and } x \in \Phi_k(F) \text{ and } F \text{ compatible with } \Theta_{k-1}]\}.$$

This clearly implies that $\Phi_k(A) \leqslant_e B_{k-1}$, as desired. ■

In particular, if $\mathfrak{I}$ is a principal ideal, we are led to the following definition:

DEFINITION 4.3. – Given any $\boldsymbol{b}$, define an e-degree $\boldsymbol{a}$ to be $\boldsymbol{b}$-*quasi-minimal* if $\boldsymbol{a}$ is $\mathfrak{I}$-quasi-minimal, where $\mathfrak{I}$ is the principal ideal generated by $\boldsymbol{b}$, (i.e. $\boldsymbol{b} < \boldsymbol{a}$ and $(\forall \text{ total } \boldsymbol{c})[\boldsymbol{c} \leqslant \boldsymbol{a} \Rightarrow \boldsymbol{c} \leqslant \boldsymbol{b}]$.)

Theorem 1.6 can be relativized to any total e-degree $\boldsymbol{b}$:

THEOREM 4.4. – *For every total e-degree* $\boldsymbol{b}$ *there exist a* $\boldsymbol{b}$*-quasi-minimal e-degree* $\boldsymbol{a}$, *and a family* $\{\boldsymbol{b}_i : i \in I\}$ *of cardinality* $2^{\aleph_0}$ *such that, for all* $i \in I$,

- $\boldsymbol{b} < \boldsymbol{b}_i$ *and there is no total e-degree* $\boldsymbol{c}$ *such that* $\boldsymbol{b} < \boldsymbol{c} < \boldsymbol{b}_i$;
- $\boldsymbol{a} < \boldsymbol{b}_i$.

PROOF. – Given any total set B carry on the construction of Theorem 1.6, but using B-pointed trees (see [8, Section V.4]). ■

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