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Quasimonotonicity, Regularity and Duality for Nonlinear Systems of Partial Differential Equations (*).

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Summary. – *We prove partial regularity for the vector-valued differential forms solving the system* $\delta(A(x, \omega)) = 0$, $d\omega = 0$, *and for the gradient of the vector-valued functions solving the system* $\operatorname{div} A(x, Du) = B(x, u, Du)$. *Here the mapping A, with* $A(x, \omega) \approx (1 + |\omega|^2)^{(p-2)/2}\omega$ $(p \geq 2)$, *satisfies a* quasimonotonicity *condition which, when applied to the gradient* $A(x, \omega) = D_\omega f(x, \omega)$ *of a real-valued function f, is analogous to but stronger than quasiconvexity for f. The case* $1 < p < 2$ *for* monotone *A is reduced to the case* $p \geq 2$ *by a duality technique.*

1. – Introduction.

We prove partial regularity for the vector-valued differential m-forms $\omega = (\omega^1, \ldots, \omega^N): \Omega \to \Lambda_m$ solving the system of equations

$$\delta(A(x, \omega)) = 0 \quad \text{and} \quad d\omega = 0, \tag{1.1}$$

for a given mapping $A: \overline{\Omega} \times \Lambda_m \to \Lambda_m$. We call such solutions *A-harmonic differential forms.* Here $\Lambda_m \equiv \Lambda_m(\mathbb{R}^n, \mathbb{R}^N)$, with $n \geq 2$, $N \geq 1$, denotes the space of $\mathbb{R}^N$-valued alternating m-linear forms on $\mathbb{R}^n$, furnished with the standard inner product $\langle \cdot, \cdot \rangle$, d and δ are the exterior derivative and the exterior co-derivative respectively and Ω is a bounded open subset of $\mathbb{R}^n$. To avoid trivialities we assume $m \geq 1$.

The system (1.1) includes the important case $\omega = du$ for $u: \Omega \to \mathbb{R}^N$, reducing to

$$\operatorname{div} A(x, Du) \equiv \delta(A(x, du)) = 0, \tag{1.2}$$

or, in components, to

$$D_\alpha[A_i^\alpha(x, Du)] = 0 \quad \text{for } i = 1, \ldots, N.$$

Here and in the sequel we identify $\Lambda_0 \cong \mathbb{R}^N$ and $\Lambda_1 \simeq \mathbb{R}^{N \times n}$. Thus we do not distin-

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guish between the differential $du(x) \in \Lambda_1$ and the gradient $Du(x) = (D_\alpha u^i(x)) \in \mathbb{R}^{N \times n}$ of a function $u: \Omega \to \mathbb{R}^N$. One reason for studying the system (1.2) in the context of differential forms is a simple duality principle which relates an A-harmonic m-form with sublinear growth of A to an a-harmonic $(n-m)$-form with superlinear growth of a.

DEFINITION 1.1. – We say that $\omega \in L^1_{\mathrm{loc}}(\Omega, \Lambda_m)$ is a *weak solution of the system* (1.1) if $A(x, \omega) \in L^1_{\mathrm{loc}}(\Omega, \Lambda_m)$ and

$$\int_\Omega \langle A(x, \omega), d\varphi \rangle \, dx = 0 \qquad \text{for all } \varphi \in C_0^\infty(\Omega, \Lambda_{m-1}), \tag{1.3}$$

$$\int_\Omega \langle \omega, \delta\psi \rangle \, dx = 0 \qquad \text{for all } \psi \in C_0^\infty(\Omega, \Lambda_{m+1}). \tag{1.4}$$

The regularity of solutions $\omega \in L^p(\Omega, \Lambda_m)$, with $1 < p < \infty$, of the system (1.1) or (1.2) has been widely studied under the assumption that the mapping $A: \bar{\Omega} \times \Lambda_m \to \Lambda_m$ be *uniformly strictly monotone* and locally Lipschitz continuous, i.e.

$$\langle A(x, \omega) - A(x, \eta), \omega - \eta \rangle \geq \lambda (1 + |\omega|^2 + |\eta|^2)^{(p-2)/2} |\omega - \eta|^2, \tag{1.5}$$

$$|A(x, \omega) - A(x, \eta)| \leq L (1 + |\omega|^2 + |\eta|^2)^{(p-2)/2} |\omega - \eta|, \tag{1.6}$$

for every $\omega, \eta \in \Lambda_m$ and $x \in \bar{\Omega}$, with positive constants λ and L. (1.5) and (1.6) imply *uniform ellipticity* and a growth condition for A_ω

$$\langle A_\omega(x, \omega) \cdot \xi, \xi \rangle \geq \lambda (1 + |\omega|^2)^{(p-2)/2} |\xi|^2, \tag{1.7}$$

$$|A_\omega(x, \omega)| \leq L (1 + |\omega|^2)^{(p-2)/2}, \tag{1.8}$$

for every $\xi \in \Lambda_m$, and for every $x \in \bar{\Omega}$ and almost every $\omega \in \Lambda_m$. Moreover, if $A(x, \cdot)$ is of class C^1, then (1.5) and (1.6) are easily seen to be equivalent to (1.7) and (1.8), cf. [A-F, Lemma 2.1].

In general we expect only partial regularity for the solutions $\omega \in L^p(\Omega, \Lambda_m)$ of the system (1.1); by this we mean Hölder continuity outside a set of Lebesgue measure zero. The first result in this direction was obtained simultaneously by M. Giaquinta and G. Modica [G-M1] and by P.-A. Ivert [I] for the following monotone system with $p = 2$:

$$\operatorname{div} A(x, u, Du) = B(x, u, Du). \tag{1.9}$$

Partial regularity for the minimizers of the variational integral $\mathfrak{F}(u) = \int_\Omega f(Du)\, dx$, whose integrand f satisfies the *quasiconvexity* condition of C. B. Morrey [M, Definition 4.4.3], was proved by L. C. Evans [E] using a blow-up technique. A direct proof based on a reverse Hölder inequality and L^p-estimates was later supplied by M. Giaquinta and G. Modica [G-M2].

Quasiconvexity of f is equivalent to the sequential weak lower semicontinuity of $\mathfrak{F}$

on $W^{1,p}(\Omega, \mathbb{R}^N)$. It can be regarded as an integral version of the pointwise inequality

$$f(\omega_0 + D\varphi) \geq f(\omega_0) + f'(\omega_0) \cdot D\varphi ,$$

valid for any differentiable convex function f.

DEFINITION 1.2. – We call $f\colon \mathbb{R}^{N \times n} \to \mathbb{R}$ *uniformly strictly quasiconvex* if, for $\lambda > 0$,

$$\int_{\mathbb{R}^n} [f(\omega_0 + D\varphi) - f(\omega_0)]\, dx \geq \lambda \int_{\mathbb{R}^n} (1 + |\omega_0|^2 + |D\varphi|^2)^{(p-2)/2} |D\varphi|^2 dx \tag{1.10}$$

for every $\omega_0 \in \mathbb{R}^{N \times n}$ and $\varphi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^N)$.

In attempting to extend the theorem of L. C. Evans from minimizers to stationary points and, more generally, to solutions of the system (1.1) we are led to the concept of *quasimonotonicity*. Quasimonotonicity is obtained in the spirit of quasiconvexity by integrating the following inequality arising from (1.5)

$$\langle A(x_0, \omega_0 + d\varphi) - A(x_0, \omega_0), d\varphi \rangle \geq \lambda (1 + |\omega_0|^2 + |d\varphi|^2)^{(p-2)/2} |d\varphi|^2 .$$

DEFINITION 1.3. – We call $A\colon \overline{\Omega} \times \Lambda_m \to \Lambda_m$ *uniformly strictly quasimonotone* if, for $\lambda > 0$,

$$\int_{\mathbb{R}^n} \langle A(x_0, \omega_0 + d\varphi), d\varphi \rangle\, dx \geq \lambda \int_{\mathbb{R}^n} (1 + |\omega_0|^2 + |d\varphi|^2)^{(p-2)/2} |d\varphi|^2 dx \tag{1.11}$$

for every $x_0 \in \overline{\Omega}$, $\omega_0 \in \Lambda_m$ and $\varphi \in C_0^\infty(\mathbb{R}^n, \Lambda_{m-1})$.

Quasimonotonicity has been introduced independently by M. Fuchs [F], by Zhang Ke-Wei [Z], and by the author in the present work. Zhang Ke-Wei [Z] proved the existence of a weak solution for the quasimonotone system (1.9).

We next discuss some properties of quasimonotonicity. We have already seen that monotonicity is sufficient for quasimonotonicity. We find a necessary condition for quasimonotonicity of mappings A of class C^1 by writing the left-hand side of (1.11) as (we omit the variable x_0)

$$\int_{\mathbb{R}^n} \int_0^1 \langle A'(\omega_0 + t\, d\varphi) \cdot d\varphi, d\varphi \rangle\, dt\, dx .$$

Rescaling φ to $\varepsilon\varphi$ and letting $\varepsilon \to 0$, we obtain

$$\int_{\mathbb{R}^n} \langle A'(\omega_0)\cdot d\varphi, d\varphi\rangle \, dx \geq \lambda \int_{\mathbb{R}^n} (1 + |\omega_0|^2)^{(p-2)/2} |d\varphi|^2 \, dx \tag{1.12}$$

for every $\omega_0 \in \Lambda_m$ and $\varphi \in C_0^\infty(\mathbb{R}^n, \Lambda_{m-1})$. As in the case of quasiconvexity we can show by using Fourier transforms and the Plancherel formula that (1.12) is equivalent to the *condition of Legendre-Hadamard*, i.e. (1.7) holds for all decomposable $\xi = \eta \wedge \zeta$, where $\eta \in \Lambda_1(\mathbb{R}^n, \mathbb{R})$ is a scalar 1-form and $\zeta \in \Lambda_{m-1}(\mathbb{R}^n, \mathbb{R}^N)$ is a vector-valued $(m-1)$-form. For $m = 1$ the corresponding matrices $\xi \in \mathbb{R}^{N\times n}$ are characterized by rank $\xi \leq 1$.

The latter restriction on ξ being vacuous for $m = N = 1$, we conclude that monotonicity, quasimonotonicity and the Legendre-Hadamard condition are all equivalent in this case. Also, for linear $A(\omega)$, it is clear that quasimonotonicity (1.11) coincides with (1.12) and hence with the condition of Legendre-Hadamard. In general the condition of Legendre-Hadamard is strictly weaker than quasimonotonicity, and quasimonotonicity is strictly weaker than monotonicity.

An example of a nonlinear quasimonotone function $A\colon \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ is

$$A(\omega) = |\omega|^2\omega + 4 \det\omega \operatorname{cof}\omega\,, \tag{1.13}$$

where $\operatorname{cof}\omega$ denotes the matrix of co-factors of ω it will be discussed in the Appendix. To obtain a uniformly strictly quasimonotone function we add $\varepsilon(1 + |\omega|^2)\omega$, with $\varepsilon > 0$.

The function A, as given by (1.13), is just the gradient of the real-valued function

$$f(\omega) = \frac{1}{4}|\omega|^4 + 2(\det\omega)^2,$$

which is quasiconvex. This is not surprising in view of the simple fact that quasiconvexity of a function $f\colon \mathbb{R}^{N\times n} \to \mathbb{R}$ is implied by quasimonotonicity of its gradient Df.

In the nonlinear theory of elasticity, the equilibrium equations for a non-homogeneous elastic body with reference configuration $\Omega \subset \mathbb{R}^3$, deformation $u\colon \Omega \to \mathbb{R}^3$, Piola-Kirchhoff stress tensor $T(x, Du(x))$, density $\rho(x)$, and external body forces per unit mass $b(u(x))$ are given by the system

$$\operatorname{div} T(x, Du) + \rho(x)\, b(u) = 0\,.$$

For a hyperelastic material, quasiconvexity is the constitutive restriction which one imposes on the stored energy function $\mathcal{W}(x, \omega)$ which determines the stress tensor by the equation $T(x, \omega) = D_\omega \mathcal{W}(x, \omega)$, cf. [B], [Ci]. In this light, quasimonotonicity is a possible candidate for a constitutive restriction to be imposed directly on the stress tensor $T(x, \omega)$ for a non-hyperelastic material which does not possess a stored energy function. What is missing for applications, however, is a suitable sufficient condition for quasimonotonicity that would correspond to J. Ball's polyconvexity in the case of quasiconvexity.

We list the hypotheses of our regularity theorems as they apply to a function $A: \bar{\Omega} \times \Lambda_m \to \Lambda_m$; for a function $A: \bar{\Omega} \times \mathbb{R}^{N\times n} \to \mathbb{R}^{N\times n}$ they should be read replacing Λ_m and Λ_{m-1} by $\mathbb{R}^{N\times n}$ and $\mathbb{R}^N$ respectively.

HYPOTHESIS H1. – A is uniformly strictly quasimonotone, i.e. (1.11) holds.

HYPOTHESIS H1*. – A is uniformly strictly monotone, i.e. (1.5) holds.

HYPOTHESIS H2. – $A(x, \cdot)$ is of class C^1, uniformly with respect to $x \in \bar{\Omega}$, and satisfies (1.8), which implies that

$$|A_\omega(x, \omega) - A_\omega(x, \eta)| \leq (1 + |\omega|^2 + |\eta|^2)^{(p-2)/2} l(|\omega|, |\omega - \eta|) \tag{1.14}$$

for a continuous, bounded function $l(t, s)$, with $l(t, 0) = 0$, which is increasing in t for fixed s and increasing and concave in s for fixed t.

HYPOTHESIS H3. – $(1 + |\omega|^2)^{-(p-1)/2} A(\cdot, \omega)$ is a Hölder continuous function on $\bar{\Omega}$ uniformly with respect to $\omega \in \Lambda_m$, i.e.

$$|A(x, \omega) - A(y, \omega)| \leq L(1 + |\omega|^2)^{(p-1)/2} |x - y|^\delta \tag{1.15}$$

with some exponent $\delta \in]0, 1[$.

HYPOTHESIS H4. – $B: \bar{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to \mathbb{R}^N$ is a Carathéodory function with *natural growth*, i.e. there exist constants a, b and M such that

$$|B(x, u, \omega)| \leq a|\omega|^p + b \quad \text{for every } |u| \leq M, \ x \in \bar{\Omega} \text{ and } \omega \in \mathbb{R}^{N\times n}. \tag{1.16}$$

In the following all constants may depend on the data n, N, m, p, λ, L, δ, a, b, M and we shall indicate dependence only on other parameters.

We define the mean value and the excess of a differential m-form $\omega: \Omega \to \Lambda_m$ on a ball $B_r(x) = \{y \in \mathbb{R}^n: |y - x| < r\} \subset \Omega$ by

$$\omega_{x,r} = \fint_{B_r(x)} \omega\, dy = |B_r(x)|^{-1} \int_{B_r(x)} \omega\, dy \quad \text{and} \quad \Phi(x, r) = \fint_{B_r(x)} |V(\omega) - V(\omega)_{x,r}|^2 dy$$

respectively, where we have used the auxiliary function

$$V(\omega) = (1 + |\omega|^2)^{(p-2)/4} \omega . \tag{1.17}$$

We also define the regular and the singular set of ω by

$$\mathrm{Reg}[\omega] = \{x_0 \in \Omega\colon \omega \text{ is continuous in a neighbourhood of } x_0\},$$

$$\mathrm{Sing}[\omega] = \Omega \setminus \mathrm{Reg}[\omega].$$

Obviously the regular set is open. We then have

THEOREM 1.1. – *Let $p \geq 2$ and suppose that $A\colon \bar{\Omega} \times \Lambda_m \to \Lambda_m$ satisfies Hypotheses* H1, H2 *and* H3.

Then any weak solution $\omega \in L^p_{\mathrm{loc}}(\Omega, \Lambda_m)$ of the system (1.1) *is locally Hölder continuous with exponent $\delta \in]0, 1[$ on the regular set* $\mathrm{Reg}[\omega]$, *for δ the exponent of Hypothesis* H3, *and the singular set has Lebesgue measure* $|\mathrm{Sing}[\omega]| = 0$.

We moreover have that

$$\mathrm{Reg}[\omega] = \left\{x_0 \in \Omega\colon \sup_{r>0} |\omega_{x_0, r}| < \infty \quad \textit{and} \quad \liminf_{r \to 0+} \Phi(x_0, r) = 0\right\}. \tag{1.18}$$

THEOREM 1.2. – *Let $p \geq 2$ and suppose hat $A(x, \omega)$ and $B(x, u, \omega)$ satisfy Hypotheses* H1, H2, H3 *and* H4. *Let $u \in W^{1,p}_{\mathrm{loc}} \cap L^\infty(\Omega, \mathbb{R}^N)$ be a weak solution of the system*

$$\operatorname{div} A(x, Du) = B(x, u, Du), \tag{1.19}$$

satisfying the condition

$$\|u\|_{L^\infty} \leq M \quad \textit{with} \quad 2aM < \lambda, \tag{1.20}$$

for λ the quasimonotonicity constant in (1.11).

Then the gradient Du is locally Hölder continuous with exponent $\delta \in]0, 1[$ on the regular set $\mathrm{Reg}[Du]$, *for δ the exponent of Hypothesis* H3, *and the singular set* $\mathrm{Sing}[Du]$ *has vanishing Lebesgue measure. Moreover, setting $\omega = du$, the regular set is given by* (1.18).

For the case $1 < p < 2$ and monotone A, we rely on the *duality principle for A-harmonic forms* of Section 2, thereby reducing the problem to the case $p \geq 2$. Since a duality principle does not hold if A is quasimonotone, we shall consider only *monotone A* in the case $1 < p < 2$. Then we obtain

THEOREM 1.3. – *Let $1 < p < 2$ and suppose that $A\colon \bar{\Omega} \times \Lambda_m \to \Lambda_m$ satisfies Hypotheses* H1*, H2 *and* H3.

Then any weak solution $\omega \in L^p_{\mathrm{loc}}(\Omega, \Lambda_m)$ of the system (1.1) *is locally Hölder continuous with exponent $\delta \in]0, 1[$ on the regular set* $\mathrm{Reg}[\omega]$, *for δ the exponent of Hypothesis* H3, *and the singular set has Lebesgue measure* $|\mathrm{Sing}[\omega]| = 0$.

Theorem 1.2 has been obtained by M. Fuchs [F] in the special case that $A\colon \mathbb{R}^{N \times n} \to \mathbb{R}^{N \times n}$ has linear growth ($p = 2$) and no dependence on the variable x. His proof uses an indirect blow-up argument. In the present paper we employ a direct method in proving the general case with $p \geq 2$. Theorems 1.2 and 1.3 are also partial generalizations of the result of [G-M1] and [I] for the monotone system (1.9) to $p \geq 2$ and $1 < p < 2$ respectively.

We here outline the direct method used in the proof of Theorems 1.1 and 1.2 which is based on the work of M. Giaquinta and G. Modica [G-M2]. Our aim is a decay estimate of the form $\Phi(x_0, \rho) \leq c\rho^{2\delta}$ implying regularity for $V(\omega)$ by virtue of Campanato's integral characterization of Hölder continuity. This is accomplished by comparing the local potential ξ, which satisfies $d\xi = \omega - \omega_0$ for some $\omega_0 \in \Lambda_m$ in a ball B around x_0, with the solution ζ of the coercive system with constant coefficients

$$\delta(A_\omega(x_0, \omega_0) \cdot d\zeta) + c\, d\delta\zeta = 0 \quad \text{in } B$$

and boundary condition $\zeta = \xi$ on ∂B. From the L^2- and L^p-estimates satisfied by ζ, we obtain a decay estimate for $\fint_{B_\rho} |V(\omega_0 + d\zeta) - V(\omega_0 + d\zeta)_{x_0, \rho}|^2 dx$, which yields an analogous decay estimate for $\Phi(x_0, \rho)$ modulo an error term. We control the error term, which is essentially $\int |V(\omega_0 + d\varphi) - V(\omega_0)|^2 dx$ with $\varphi = \zeta - \xi$, by using the quasimonotonicity (1.11) of A and a reverse Hölder inequality with increasing supports. For $x_0 \in \Omega$ with $\sup_{r>0} |\omega_{x_0, r}| < \infty$ and $\liminf_{r \to 0+} \Phi(x_0, r) = 0$, it turns out that the error term can be neglected and the desired decay estimate for $\Phi(x_0, \rho)$ results. This proves that ω is Hölder continuous in a neighbourhood of x_0.

In Section 3 we construct the local potential ξ, which satisfies $d\xi = \omega - \omega_0$ in a ball B, on the basis of the *plus-potential* of C. B. Morrey in the class of Neumann forms. We then prove the interior L^p-estimate $\|D\xi\|_{L^p} \leq c\|\omega - \omega_0\|_{L^p}$ for $p > 1$. To this end we first establish L^p-estimates up to the boundary of B for the plus-potential with $p \geq 2$, and thence derive the L^p-estimate with $1 < p < 2$ by duality. For the case of Theorem 1.2, however, all requirements for the local potential are trivially met by $\omega = du$ and $\xi = u - \omega_0 \cdot (x - x_0)$.

The reverse Hölder inequality with increasing supports needed in the proof of Theorems 1.1 and 1.2 is

$$\left(\fint_{B_{R/2}(x_0)} |V(\omega) - V(\omega_0)|^q dx \right)^{2/q} \leq c \fint_{B_R(x_0)} |V(\omega) - V(\omega_0)|^2 dx + H(|\omega_0|) R^{2\delta}, \tag{1.21}$$

for some exponent $q > 2$ and a nondecreasing function H. For the case of monotone A, (1.21) is easily derived from Caccioppoli's first inequality, for which we provide a proof in Section 4. Under the weaker assumption that A be only quasimonotone, we can still prove (1.21) provided that $p \geq 2$. This is a consequence of Caccioppoli's second inequality, which has been proved for minimizers of quasiconvex variational integrals by L. C. Evans [E]. In Section 4 we adapt his proof to the situation of A-harmonic forms and quasimonotonicity of A. We then deduce (1.21) with the help of a higher integrability theorem of F. W. Gehring, M. Giaquinta and G. Modica.

We do not know whether Theorem 1.2 extends to the general quasimonotone system (1.9). Both the direct and the indirect proof of the theorem do not work in this case since they rely heavily on the Caccioppoli inequality which, as we have seen (cf. (1.21)), implies higher integrability for the gradient, i.e. $Du \in L^{p+\varepsilon}_{\mathrm{loc}}$ for some $\varepsilon > 0$. That higher integrability as well as differentiability of the gradient of a solution of such a system can fail is illustrated by the following example taken from [G-S]. We note, however, that the solution in this example is partially regular.

EXAMPLE. – We let $\Omega \subset \mathbb{R}^3$ be the ball with radius e^{-2} centred at the origin and we define the function

$$u(x) = \frac{x}{|x|^{3/2}\log|x|} \in W^{1,2}(\Omega, \mathbb{R}^3)$$

which does not belong to any $W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^3)$ with $p > 2$ nor to $W^{2,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^3)$. Let $\eta(s)$ be a smooth bounded function such that $\eta(s) = s^{-1}$ for $s \geq 1$. Since $|u| \geq 1$, we have $\eta(|u|^2) = |u|^{-2}$, and it is not difficult to check that u is a solution of the quasilinear system

$$\operatorname{div}[A(x, u)\cdot Du] = 0$$

with

$$A^{\alpha\beta}_{ij}(x, u) = \delta_{ij}\delta_{\alpha\beta} + \left(\log^{-2}|x| - \frac{9}{8}\right)\eta(|u|^2)\,\varepsilon_{ijk}\,\varepsilon_{\alpha\beta\gamma}u^k u^\gamma,$$

ε_{ijk} being the completely antisymmetric tensor. $A(x, u)$ is clearly bounded, of class C^1 in x and smooth in u, and it satisfies a Legendre-Hadamard condition which implies that $A(x, u)\cdot\omega$ is quasimonotone in ω.

2. – The duality principle.

For a proof of the following lemma and other important properties of the diffeomorphism $V: \Lambda_m \to \Lambda_m$, defined for $p > 1$ by (1.17), we refer to [G-M2] and [H].

LEMMA 2.1. – *There exist two positive constants c_1 and c_2 such that, for all $\omega, \eta \in \Lambda_m$,*

$$c_1 \leq \frac{|V(\omega) - V(\eta)|}{(1 + |\omega|^2 + |\eta|^2)^{(p-2)/4}|\omega - \eta|} \leq c_2. \tag{2.1}$$

PROPOSITION 2.1. (Duality principle for A-harmonic forms). – *Suppose that $\omega \in L^p_{\mathrm{loc}}(\Omega, \Lambda_m)$ is an A-harmonic differential form and that $A: \overline{\Omega} \times \Lambda_m \to \Lambda_m$ satis-*

fies (1.5), (1.6) *and* (1.15) *for* $p > 1$. *Define the* $(n-m)$*-form* $\tau \in L^{p'}_{\mathrm{loc}}(\Omega, \Lambda_{n-m})$ *by the formula*

$$* \tau = A(x, \omega), \tag{2.2}$$

where $*$ *denotes Hodge duality, and where* p' *is the conjugate exponent of* p *defined by* $p' + p = pp'$.

Then $A(x, \cdot): \Lambda_m \to \Lambda_m$ *is a bijection for every* $x \in \overline{\Omega}$, *and* τ *is an* a*-harmonic differential form for the map* $a: \overline{\Omega} \times \Lambda_{n-m} \to \Lambda_{n-m}$ *defined by* $a(x, \tau) = * A(x, \cdot)^{-1}(* \tau)$.

Moreover, a *satisfies the same estimates* (1.5), (1.6) *and* (1.15) *with* p *replaced by* p'; *and if* A *satisfies* (1.14) *for* $1 < p < 2$, *then* a *satisfies*

$$|a_\tau(x, \tau) - a_\tau(x, \sigma)| \leq c(1 + |\tau|^2 + |\sigma|^2)^{(p'-2)/2} \cdot l\big(c(1 + |\sigma|^2)^{(p'-1)/2}, c(1 + |\sigma|^2)^{(p'-2)/2}|\tau - \sigma| + c|\tau - \sigma|^{p'-1}\big). \tag{2.3}$$

PROOF. – We first prove that $A(x, \cdot): \Lambda_m \to \Lambda_m$ is bijective. From (1.5) we deduce that

$$(1 + |\omega|^2 + |\eta|^2)^{(p-2)/2}|\omega - \eta| \leq c|A(x, \omega) - A(x, \eta)|, \tag{2.4}$$

which implies injectivity of $A(x, \cdot)$. Setting $\eta = 0$ in (2.4) and noting that $A(x, 0)$ is bounded on $\overline{\Omega}$ we obtain the first half of

$$c(1 + |\omega|^2)^{p-1} \leq 1 + |A(x, \omega)|^2 \leq c(1 + |\omega|^2)^{p-1}, \tag{2.5}$$

the second half following directly from (1.6).

We fix $x \in \overline{\Omega}$ and we write $A = A(x, \cdot): \Lambda_m \to \Lambda_m$ and $B_r = B_r(0) \subset \Lambda_m$. (2.5) implies that $|A(\omega)| \geq R = [c(1 + r^2)^{p-1} - 1]^{1/2}$ for $|\omega| = r$. Thus $B_R \cap A(\partial B_r) = \emptyset$, and the mapping degree $\deg(\omega, A, B_r)$ is defined and constant for $\omega \in B_R$. Now suppose that $\omega_0 \notin A(\Lambda_m)$ for some $\omega_0 \in \Lambda_m$. Then for $R > |\omega_0|$ we have $\deg(\omega_0, A, B_r) = 0$, and hence $\deg(\omega, A, B_r) = 0$ for every $\omega \in B_R$. Since $A: \Lambda_m \to \Lambda_m$ is injective, it follows that $B_R \cap A(B_r) = \emptyset$. As $R \to \infty$ for $r \to \infty$, we obtain a contradiction. Thus we have shown that $A: \Lambda_m \to \Lambda_m$ is surjective.

Consider now the $(n-m)$-form τ and the m-form ω which are related by (2.2)

$$A(x, \omega) = * \tau \Leftrightarrow a(x, \tau) = * \omega.$$

It follows that

$$\delta(A(x, \omega)) = \pm * d\tau \quad \text{and} \quad \delta(a(x, \tau)) = \pm * d\omega,$$

and we see that τ is a-harmonic if and only if ω is A-harmonic.

(2.5) and the identity $(p-1)(p'-2) = 2 - p$ yield

$$c(1 + |\omega|^2 + |\eta|^2)^{(2-p)/2} \leq (1 + |A(x, \omega)|^2 + |A(x, \eta)|^2)^{(p'-2)/2} \leq \leq c(1 + |\omega|^2 + |\eta|^2)^{(2-p)/2}. \tag{2.6}$$

On account of (2.6), (1.6) and (1.5) give

(2.7) $$(1+|A(x,\omega)|^2+|A(x,\eta)|^2)^{(p'-2)/2}|A(x,\omega)-A(x,\eta)|^2 \leq$$
$$\leq c(1+|\omega|^2+|\eta|^2)^{(p-2)/2}|\omega-\eta|^2 \leq c\langle A(x,\omega)-A(x,\eta),\omega-\eta\rangle,$$

while (2.4) gives

(2.8) $$|\omega-\eta| \leq c(1+|\omega|^2+|\eta|^2)^{(2-p)/2}|A(x,\omega)-A(x,\eta)| \leq$$
$$\leq c(1+|A(x,\omega)|^2+|A(x,\eta)|^2)^{(p'-2)/2}|A(x,\omega)-A(x,\eta)|\,.$$

(2.7) and (2.8) correspond to (1.5) and (1.6) with A replace by a and p replaced by p'.

For future reference we note here that (2.4) multiplied by (2.8) furnishes the first and (2.7) furnishes the second of the inequalities

$$c|V_p(\omega)-V_p(\eta)|^2 \leq |V_{p'}(A(x,\omega))-V_{p'}(A(x,\eta))|^2 \leq c|V_p(\omega)-V_p(\eta)|^2.$$

Thus, since by (2.5) and (1.15)

$$|V_{p'}(A(x,\omega))-V_{p'}(A(y,\omega))|^2 \leq c(1+|\omega|^2)^{p/2}|x-y|^{2\delta},$$

we obtain

(2.9) $$c|V_p(\omega)-V_p(\eta)|^2-c(1+|\omega|^2)^{p/2}|x-y|^{2\delta} \leq |V_{p'}(A(x,\omega))-V_{p'}(A(y,\eta))|^2 \leq$$
$$\leq c|V_p(\omega)-V_p(\eta)|^2+c(1+|\omega|^2)^{p/2}|x-y|^{2\delta}.$$

In order to prove the analogue of (1.15) for a, we let $x, y \in \bar{\Omega}$ and $\tau \in \Lambda_{n-m}$, and we choose $\omega, \eta \in \Lambda_m$ satisfying $*\,\tau = A(x,\omega) = A(y,\eta)$. Then, on account of (2.8), (1.15) and (2.5), we have for $p' \geq 2$

$$|a(x,\tau)-a(y,\tau)| =$$
$$= |\omega-\eta| \leq c(1+|A(x,\omega)|^2+|A(x,\eta)|^2)^{(p'-2)/2}|A(y,\eta)-A(x,\eta)| \leq$$
$$\leq c(1+|\tau|^2)^{(p'-2)/2}|A(y,\eta)-A(x,\eta)|+c|A(y,\eta)-A(x,\eta)|^{p'-1} \leq$$
$$\leq c(1+|\tau|^2)^{(p'-2)/2}(1+|\eta|^2)^{(p-1)/2}|x-y|^{\delta}+c(1+|\eta|^2)^{1/2}|x-y|^{\delta(p'-1)} \leq$$
$$\leq c(1+|\tau|^2)^{(p'-1)/2}|x-y|^{\delta}.$$

For the case $1 < p' < 2$, we omit the second terms in the above estimate and obtain the same result.

We finally examine how the continuity condition (1.14) on A_ω determines a similar condition on a_τ for $p' \geq 2$. For $\tau, \sigma \in \Lambda_{n-m}$ we let $\omega, \eta \in \Lambda_m$ satisfy $A(x,\omega) = *\,\tau$ and $A(x,\eta) = *\,\sigma$. From the definition of a, we have $a_\tau(x,\tau) = *\,A_\omega(x,\omega)^{-1}\,*$ etc., and, using the estimate for a corresponding to (1.8) as well as (1.14), (2.6), (2.8) and (2.5), we obtain

$$|a_\tau(x,\tau)-a_\tau(x,\sigma)| = |a_\tau(x,\sigma)[a_\tau(x,\sigma)^{-1}-a_\tau(x,\tau)^{-1}]a_\tau(x,\tau)| \leqslant$$
$$\leqslant |a_\tau(x,\sigma)|\,|a_\tau(x,\tau)|\,|A_\omega(x,\omega)-A_\omega(x,\eta)| \leqslant$$
$$\leqslant c(1+|\sigma|^2)^{(p'-2)/2}(1+|\tau|^2)^{(p'-2)/2}(1+|\omega|^2+|\eta|^2)^{(p-2)/2}l(|\eta|,|\omega-\eta|) \leqslant$$
$$\leqslant c(1+|\tau|^2+|\sigma|^2)^{(p'-2)/2}l(c(1+|\sigma|^2)^{(p'-1)/2}, c(1+|\tau|^2+|\sigma|^2)^{(p'-2)/2}|\tau-\sigma|),$$

thus proving the estimate (2.3). ■

3. – L^p-estimates for the local potential.

The following proposition is proved in [H] for $p \geqslant 2$, but here we also need the estimate (3.2) for $1 < p < 2$. We note that for the special case of the system (1.2) or (1.19), whose solution is given as a differential $\omega = du$ with $u \in W^{1,2}(\Omega, \mathbb{R}^N)$, Proposition 3.1 is trivially satisfied by $\xi = u - \omega_0 \cdot (x - x_0)$. However, when we invoke the duality principle for (1.2), as in the proof of Theorem 1.3, we end up with the $(n-1)$-form $\tau = \pm * A(x, du)$, for which Proposition 3.1 is essential.

PROPOSITION 3.1. – *Let $B_R(x_0)$ be an open ball in $\mathbb{R}^n$, and let $\omega \in L^2(B_R(x_0), \Lambda_m) \cap \ker d$ and $\omega_0 \in \Lambda_m$, for $m \geqslant 1$. Then there exists $\xi \in W^{1,2}(B_R(x_0), \Lambda_{m-1})$ satisfying*

$$d\xi = \omega - \omega_0 \quad \text{and} \quad \delta\xi = 0\,. \tag{3.1}$$

In addition we have for each p, with $1 < p < \infty$, and for each ρ, with $0 < \rho < R$, the estimate

$$\|D\xi\|_{L^p(B_\rho(x_0))} \leqslant c\|\omega - \omega_0\|_{L^p(B_R(x_0))}, \tag{3.2}$$

where the constant c depends only on n, N, m, p, and ρ/R.

In the proof of Proposition 3.1 we use reflection in the unit sphere $S(x) = r^{-2}x$, where $r = |x| > 0$. We write $B = B_1(0)$ for the unit ball, and we summarize some properties of S in

LEMMA 3.1. – *Let $g = \langle\cdot,\cdot\rangle = dx^i \otimes dx^i$ and $v_g = dx = dx^1 \wedge \ldots \wedge dx^n$ be the standard metric and volume form on $\mathbb{R}^n$ respectively.*

Let $\tilde g = S^* g$, with associated norm $\tilde{|}\cdot|$, volume form $v_{\tilde g}$, Hodge dual $\tilde *$ and exterior co-derivative $\tilde\delta$. Then, for any m-form Ω, we have

(a) $\tilde g = r^{-4} g$,

(b) $v_{\tilde g} = r^{-2n} v_g = -S^* v_g$,

(c) $\tilde * \,\Omega = r^{4m-2n} * \Omega$, (*d*) $\tilde{\delta}\Omega = r^4 \delta\Omega + (4m - 2n) r^2 i_x \Omega$,

(*e*) $S^*(* \Omega) = - \tilde{*} S^* \Omega$,

(*f*) $S^* \delta\Omega = \tilde{\delta} S^* \Omega$,

(*g*) $(S^*\Omega)^{\top} = \Omega^{\top}$ *and* $(S^*\Omega)^{\perp} = -\Omega^{\perp}$ *on* ∂B,

(*h*) $S^*\Omega = \Omega$ *on* ∂B *if and only if* $\Omega^{\perp} = 0$ *on* ∂B.

PROOF. – From the equation

$$S^* dx^i = d(r^{-2} x^i) = r^{-2} dx^i - 2r^{-4} x^i x^j dx^j, \tag{3.3}$$

we find that $S^* g = (S^* dx^i) \otimes (S^* dx^i) = r^{-4} g$, which is (*a*). The formulas $v_{\tilde{g}} = [\det(\tilde{g}_{ij})]^{1/2} dx$ and $S^* dx = \det DS\, dx$, with $DS = r^{-2}(I - 2r^{-2} x \otimes x)$, give (*b*). (*c*) is obtained by comparing the following definitions of the Hodge duals $* \Omega$ and $\tilde{*} \Omega$, for vectors $X_{m+1}, \ldots, X_n$, using (*a*) and (*b*)

$$(* \Omega)(X_{m+1}, \ldots, X_n) v_g = \Omega \wedge g(X_{m+1}, \cdot) \wedge \ldots \wedge g(X_n, \cdot), \tag{3.4}$$

$$(\tilde{*} \Omega)(X_{m+1}, \ldots, X_n) v_{\tilde{g}} = \Omega \wedge \tilde{g}(X_{m+1}, \cdot) \wedge \ldots \wedge \tilde{g}(X_n, \cdot). \tag{3.5}$$

(*d*) follows from the definition $\tilde{\delta}\Omega = (-1)^{nm+n} \tilde{*}\, d\, \tilde{*}\, \Omega$, (*c*) and the identity $*[\Omega \wedge g(X, \cdot)] = i_X * \Omega$. In order to show (*e*), we operate with S^* on both sides of (3.4), where we set $X_i = S_* Y_i$. By virtue of (*b*), this yields

$$-(* \Omega)(S_* Y_{m+1}, \ldots, S_* Y_n) v_{\tilde{g}} = S^* \Omega \wedge \tilde{g}(Y_{m+1}, \cdot) \wedge \ldots \wedge \tilde{g}(Y_n, \cdot).$$

Therefore, from (3.5),

$$-S^*(* \Omega)(Y_{m+1}, \ldots, Y_n) v_{\tilde{g}} = \tilde{*} (S^* \Omega)(Y_{m+1}, \ldots, Y_n) v_{\tilde{g}},$$

and we obtain (*e*). (*f*) follows directly from (*e*). By symmetry it suffices to check (*g*) at the point $(0, \ldots, 0, 1) \in \partial B$, which is obvious from (3.3). (*h*) is implied by (*g*). ■

We define the subspace of Neumann forms in $W^{1,2}(B, \Lambda_m)$

$$\mathfrak{P}^+(B, \Lambda_m) = \{\Omega \in W^{1,2}(B, \Lambda_m) \colon \Omega^{\perp} = 0 \text{ on } \partial B\}.$$

We note that the space of harmonic Neumann m-forms on B is, for $m \geq 1$, trivial

$$\mathfrak{H}^+(B, \Lambda_m) = \{\Omega \in L^2(B, \Lambda_m) \colon \delta\Omega = d\Omega = 0 \text{ in } B \text{ and } \Omega^{\perp} = 0 \text{ on } \partial B\} = 0,$$

by the fundamental result of Duff and Spencer [D-S] that its dimension equals the m-th Betti number of B.

REMARK 3.1. – The equations defining $\mathfrak{H}^+$ are understood in the weak sense as

$$\int_B \langle \Omega, d\varphi \rangle dx = \int_B \langle \Omega, \delta\psi \rangle dx = 0 \quad \text{for every } \varphi \in C^\infty(\overline{B}, \Lambda_{m-1}) \text{ and } \psi \in C_0^\infty(B, \Lambda_{m+1}).$$

We shall use the following notation: we extend a given m-form $\Omega: B \to \Lambda_m$ from the unit ball B to $\mathbb{R}^n$ by

$$\widehat{\Omega}(x) = \begin{cases} \Omega(x) & \text{for } x \in B, \\ S^*\Omega(x) & \text{for } x \in B^c. \end{cases} \tag{3.6}$$

We let $\mathfrak{D}(\Omega)$ denote the Dirichlet integral of Ω over B

$$\mathfrak{D}(\Omega) = \int_B (|d\Omega|^2 + |\delta\Omega|^2)\, dx,$$

and we start by proving the following inequalities valid for $\mathfrak{D}$ on $\mathfrak{P}^+$.

LEMMA 3.2. [M, Thm. 7.7.2]. – *For $\Omega \in \mathfrak{P}^+(B, \Lambda_m)$, we have $\widehat{\Omega} \in W^{1,2}_{\text{loc}}(\mathbb{R}^n, \Lambda_m)$ and*

$$\|\Omega\|^2_{L^2(B)} \leq c\mathfrak{D}(\Omega), \tag{3.7}$$

$$\|\widehat{\Omega}\|^2_{W^{1,2}(B_2(0))} \leq c\mathfrak{D}(\Omega). \tag{3.8}$$

PROOF. – $\Omega \in W^{1,2}(B, \Lambda_m)$ implies $S^*\Omega \in W^{1,2}_{\text{loc}}(B^c, \Lambda_m)$. Further, by Lemma 3.1 (*h*), $\Omega^\perp = 0$ implies $S^*\Omega = \Omega$ on ∂B, and we conclude that $\widehat{\Omega} \in W^{1,2}_{\text{loc}}(\mathbb{R}^n, \Lambda_m)$.

Let $\eta \in C_0^\infty(B_3(0))$ be a cut-off function with $0 \leq \eta \leq 1$ and $\eta = 1$ on $B_2(0)$. Using the identity $D_i D_i = \Delta = \delta d + d\delta$, as well as formulas following from Lemma 3.1(*b, f*) such as $\bar{|}\tilde{\delta} S^*\Omega|^2 v_{\bar{g}} = -S^*(|\delta\Omega|^2 v_g)$, we obtain the estimate

$$\int_{B_3(0)} |D(\eta\widehat{\Omega})|^2 dx = \int_{B_3(0)} (|d(\eta\widehat{\Omega})|^2 + |\delta(\eta\widehat{\Omega})|^2)\, dx \leq$$

$$\leq c\int_B (|\Omega|^2 + |d\Omega|^2 + |\delta\Omega|^2)\, dx + c \int_{B_3(0)\setminus B} (\bar{|}S^*\Omega|^2 + \bar{|}dS^*\Omega|^2 + \bar{|}\tilde{\delta}S^*\Omega|^2)\, v_{\bar{g}} \leq$$

$$\leq c\int_B (|\Omega|^2 + |d\Omega|^2 + |\delta\Omega|^2)\, dx,$$

and we conclude that

$$\|\widehat{\Omega}\|^2_{W^{1,2}(B_2(0))} \leq c\|\Omega\|^2_{L^2(B)} + c\mathfrak{D}(\Omega). \tag{3.9}$$

If (3.7) were not true, then there would exist a sequence $\{\Omega_k\} \subset \mathfrak{P}^+$ such that

$$\|\Omega_k\|_{L^2(B)} = 1 \quad \text{and} \quad \mathfrak{D}(\Omega_k) \to 0 . \tag{3.10}$$

By (3.9), $\{\Omega_k\}$ is bounded in $W^{1,2}$, which implies that for a subsequence and for some $\Omega \in \mathfrak{P}^+$

$$\Omega_{k_i} \to \Omega \quad \text{weakly in } W^{1,2} \text{ and strongly in } L^2. \tag{3.11}$$

By the sequential weak lower semicontinuity of the Dirichlet integral, we infer from (3.10) and (3.11) that $\mathfrak{D}(\Omega) = 0$ and hence that $\Omega \in \mathfrak{H}^+ = 0$. This contradicts $\|\Omega\|_{L^2(B)} = 1$.

(3.8) is obtained by combining (3.9) and (3.7). ∎

DEFINITION 3.1. [M, Thm. 7.7.3]. – $\Omega \in \mathfrak{P}^+(B, \Lambda_m)$ is called the *plus-potential* of $\omega \in L^2(B, \Lambda_m)$ if

$$\mathfrak{D}(\Omega, \psi) = \int_B (\langle d\Omega, d\psi\rangle + \langle \delta\Omega, \delta\psi\rangle)\, dx = \int_B \langle \omega, \psi\rangle\, dx \quad \text{for every } \psi \in \mathfrak{P}^+(B, \Lambda_m). \tag{3.12}$$

REMARK 3.2. – The bilinear form $\mathfrak{D}(\Omega, \psi)$ associated with the Dirichlet integral is coercive on $\mathfrak{P}^+$ by Lemma 3.2, and the plus-potential is furnished by the Theorem of Lax-Milgram [G-T, Theorem 5.8].

REMARK 3.3. – By [M, Thm. 7.7.4(i)] we have $d\Omega, \delta\Omega \in \mathfrak{P}^+$. From (3.12) we then infer the pointwise equation

$$\delta d\Omega + d\delta\Omega = -\omega . \tag{3.13}$$

LEMMA 3.3. – *Let $\Omega \in \mathfrak{P}^+(B, \Lambda_m)$ be the plus-potential of $\omega \in L^2(B, \Lambda_m)$. Then the following L^p-estimate holds for $1 < p < \infty$*

$$\|d\Omega\|_{L^p(B)} + \|\delta\Omega\|_{L^p(B)} \leq c(p)\|\omega\|_{L^p(B)} . \tag{3.14}$$

PROOF. – We define the operator $T: L^2(B, \Lambda_m) \to L^2(B, (\Lambda_m)^n)$ by $T\omega = D\Omega = (D_1\Omega, \ldots, D_n\Omega)$, where $\Omega \in \mathfrak{P}^+(B, \Lambda_m)$ is the plus-potential of $\omega \in L^2(B, \Lambda_m)$. We show T to be continuous.

We let $\hat{\omega}$ and $\hat{\Omega}$ be the extensions to $\mathbb{R}^n$ of the forms ω and Ω as in (3.6). By substituting $\psi = \Omega$ in (3.12) and using (3.7) and (3.8), it follows that

$$\|\hat{\Omega}\|_{W^{1,2}(B_2(\dot{0}))} \leq c\|\omega\|_{L^2(B)} . \tag{3.15}$$

This proves that $T: L^2 \to L^2$ is continuous. The next step is to show that T is also continuous as an operator between the spaces $T: L^\infty \to C^{0,\sigma}$ for $0 < \sigma < 1$.

We define a Lipschitz metric $\hat{g}$ on $\mathbb{R}^n$ by

$$\hat{g}(x) = \begin{cases} g(x) & \text{for } x \in B\,, \\ \tilde{g}(x) = r^{-4} g(x) & \text{for } x \in B^c\,, \end{cases}$$

with associated exterior co-derivative $\hat{\delta}$. We claim that $\hat{\delta}\, d\hat{\Omega} + d\,\hat{\delta}\hat{\Omega} = -\,\hat{\omega}$, i.e. that for every $\varphi \in C_0^\infty(\mathbb{R}^n, \Lambda_m)$

$$\int_{\mathbb{R}^n} [\hat{g}(d\hat{\Omega}, d\varphi) + \hat{g}(\hat{\delta}\hat{\Omega}, \hat{\delta}\varphi)]\, v_{\hat{g}} = \int_{\mathbb{R}^n} \hat{g}(\hat{\omega}, \varphi)\, v_{\hat{g}}. \tag{3.16}$$

To see this we note that Lemma 3.1(g) asserts that $\psi = \varphi + S^* \varphi \in \mathfrak{P}^+(B, \Lambda_m)$, so we can insert it into (3.12):

$$\int_B [g(d\Omega, d\varphi) + g(\delta\Omega, \delta\varphi)]\, v_g + \int_B [g(d\Omega, dS^*\varphi) + g(\delta\Omega, \delta S^*\varphi)]\, v_g =$$

$$= \int_B g(\omega, \varphi)\, v_g + \int_B g(\omega, S^*\varphi)\, v_g\,.$$

We then convert the integrals involving $S^*\varphi$ into integrals over B^c by using Lemma 3.1(b, f) and the equation $S \circ S = id$, for example

$$\int_B g(\delta\Omega, \delta S^*\varphi)\, v_g = -\int_{B^c} S^*\{g(\delta\Omega, \delta S^*\varphi)\, v_g\} = \int_{B^c} \tilde{g}(\tilde{\delta} S^*\Omega, \tilde{\delta}\varphi)\, v_{\tilde{g}} = \int_{B^c} \hat{g}(\hat{\delta}\hat{\Omega}, \hat{\delta}\varphi)\, v_{\hat{g}}\,.$$

In this manner we obtain (3.16).

We write $B_t = B_t(y_0)$ for $y_0 \in B$ and $0 < t \leq 1$, and we let $\Sigma \in W^{1,2}(B_t, \Lambda_m)$ be the solution of the Dirichlet problem $\Delta\Sigma = 0$ in B_t, $\Sigma = \hat{\Omega}$ on ∂B_t. Then $\varphi = \hat{\Omega} - \Sigma \in W_0^{1,2}(B_t, \Lambda_m)$ and we obtain

$$\int_{B_t} |D\varphi|^2\, dx = \int_{B_t} \langle D\hat{\Omega} - D\Sigma, D\varphi\rangle\, dx = \int_{B_t} (\langle d\hat{\Omega}, d\varphi\rangle + \langle \delta\hat{\Omega}, \delta\varphi\rangle)\, dx\,.$$

Subtracting (3.16) and using Lemma 3.1(a, b, d), the fact that $|1 - r^k| \leq c(k)\, t$ on $B_t \cap B^c$ for $t \leq 1$ and the Poincaré inequality for φ, we have

$$\int_{B_t} |D\varphi|^2\, dx = \int_{B_t} \hat{g}(\hat{\omega}, \varphi)\, v_{\hat{g}} + \int_{B_t \cap B^c} ((1 - r^{4(m+1)-2n})(\langle d\hat{\Omega}, d\varphi\rangle + \langle \delta\hat{\Omega}, \delta\varphi\rangle) -$$

$$- (4m - 2n)\, r^{4m+2-2n}(\langle \delta\hat{\Omega}, i_x\varphi\rangle + \langle i_x\hat{\Omega}, \delta\varphi\rangle) - (4m - 2n)^2 r^{4m-2n}\langle i_x\hat{\Omega}, i_x\varphi\rangle)\, dx \leq$$

$$\leq c\int_{B_t} |\hat{\omega}|\, |\varphi|\, dx + c\int_{B_t \cap B^c} (t|D\hat{\Omega}| + |i_x\hat{\Omega}|)(|D\varphi| + t^{-1}|\varphi|)\, dx \leq$$

$$\leq ct^2 \int_{B_t} |\hat{\omega}|^2\, dx + ct^2 \int_{B_t} |D\hat{\Omega}|^2\, dx + c\int_{B_t \cap B^c} |i_x\hat{\Omega}|^2\, dx + \frac{1}{2}\int_{B_t} |D\varphi|^2\, dx\,.$$

The formula $\Omega^{\perp} = \langle x, \cdot\rangle \wedge i_x \Omega$ holds on ∂B, whose unit normal is x. We define the radial component $\widehat{\Omega}_{\mathrm{rad}} = r^{-2}\langle x, \cdot\rangle \wedge i_x \widehat{\Omega}$, which satisfies $i_x \widehat{\Omega} = i_x \widehat{\Omega}_{\mathrm{rad}}$. Now $\widehat{\Omega}_{\mathrm{rad}} = \Omega^{\perp} = 0$ on ∂B, so we can apply the Poincaré inequality on $B_t \cap B^c$

$$\int_{B_t \cap B^c} |i_x \widehat{\Omega}|^2 dx \leq c \int_{B_t \cap B^c} |\widehat{\Omega}_{\mathrm{rad}}|^2 dx \leq ct^2 \int_{B_t} (|\widehat{\Omega}|^2 + |D\widehat{\Omega}|^2)\, dx .$$

We thus obtain

$$\int_{B_t} |D\varphi|^2 dx \leq c\|\omega\|^2_{L^\infty(B)} t^{n+2} + ct^2 \int_{B_t} (|\widehat{\Omega}|^2 + |D\widehat{\Omega}|^2)\, dx . \tag{3.17}$$

The Campanato estimates satisfied by Σ for $s < t$ (see [G, Thm. 2.1 and Remark 2.3, pp. 78-79]) are

$$\int_{B_s} (|\Sigma|^2 + |D\Sigma|^2)\, dx \leq c(s/t)^n \int_{B_t} (|\Sigma|^2 + |D\Sigma|^2)\, dx , \tag{3.18}$$

$$\int_{B_s} |D\Sigma - D\Sigma_{y_0, s}|^2 dx \leq c(s/t)^{n+2} \int_{B_t} |D\Sigma - D\Sigma_{y_0, t}|^2 dx . \tag{3.19}$$

(3.18) translates, by virtue of (3.17) and the Poincaré inequality for φ, into an estimate for $\widehat{\Omega}$

$$\int_{B_s} (|\widehat{\Omega}|^2 + |D\widehat{\Omega}|^2)\, dx \leq c[(s/t)^n + t^2] \int_{B_t} (|\widehat{\Omega}|^2 + |D\widehat{\Omega}|^2)\, dx + c\|\omega\|^2_{L^\infty(B)} t^{n+2}.$$

By applying Lemma 3.4 and (3.15) we obtain, for $0 < \sigma < 1$ and $s < t \leq t_0(\sigma) \leq 1$,

$$\int_{B_s} (|\widehat{\Omega}|^2 + |D\widehat{\Omega}|^2)\, dx \leq c(s/t)^{n-2+2\sigma} \int_{B_t} (|\widehat{\Omega}|^2 + |D\widehat{\Omega}|^2)\, dx + c\|\omega\|^2_{L^\infty(B)} s^{n-2+2\sigma} \leq \leq c(t)\|\omega\|^2_{L^\infty(B)} s^{n-2+2\sigma}. \tag{3.20}$$

Inserting (3.20) (with t in place of s) into (3.17), we infer from the Campanato estimate (3.19) that

$$\int_{B_s} |D\widehat{\Omega} - D\widehat{\Omega}_{y_0, s}|^2 dx \leq c(s/t)^{n+2} \int_{B_t} |D\widehat{\Omega} - D\widehat{\Omega}_{y_0, t}|^2 dx + c\|\omega\|^2_{L^\infty(B)} t^{n+2\sigma}.$$

A further application of Lemma 3.4 and (3.15) gives

$$\int_{B_s} |D\widehat{\Omega} - D\widehat{\Omega}_{y_0, s}|^2 dx \leq c(s/t)^{n+2\sigma} \int_{B_t} |D\widehat{\Omega} - D\widehat{\Omega}_{y_0, t}|^2 dx + c\|\omega\|^2_{L^\infty(B)} s^{n+2\sigma} \leq \leq c(t)\|\omega\|^2_{L^\infty(B)} s^{n+2\sigma}.$$

Using Theorem 3.1 and (3.15), we conclude that

$$\|D\Omega\|_{C^{0,\sigma}(\bar{B})} \leq c(\|D\Omega\|_{L^2(B)} + [D\Omega]_{2,\sigma;B}) \leq c\|\omega\|_{L^\infty(B)}.$$

We have shown that $T: L^2 \to L^2$ and $T: L^\infty \to C^{0,\sigma}$ are continuous. Hence, by the interpolation theorem of Stampacchia [G, Thm. 1.4, p. 75], T is continuous as an operator $T: L^p \to L^p$ for all p with $2 \leq p < \infty$. In particular, we obtain (3.14) for $p \geq 2$.

Now let $\omega \in L^2(B, \Lambda_m)$ and $\psi \in C_0^\infty(B, \Lambda_{m+1})$, with plus-potentials Ω and Ψ respectively, and let $1 < p < 2$. Then (3.14) holds for Ψ and ψ with the conjugate exponent $p' > 2$. Therefore, since $d\Psi^\perp = \delta\Omega^\perp = d\Omega^\perp = 0$ on ∂B,

$$\int_B \langle \delta\Omega, \psi\rangle\, dx = -\int_B (\langle \delta\Omega, \delta d\Psi\rangle + \langle \delta\Omega, d\delta\Psi\rangle)\, dx = \int_B \langle d\delta\Omega, d\Psi\rangle\, dx =$$

$$= -\int_B \langle \omega + \delta d\Omega, d\Psi\rangle\, dx = -\int_B \langle \omega, d\Psi\rangle\, dx \leq \|\omega\|_{L^p(B)} \|d\Psi\|_{L^{p'}(B)} \leq c(p')\|\omega\|_{L^p(B)} \|\psi\|_{L^{p'}(B)}.$$

As $C_0^\infty(B)$ is dense in $L^{p'}(B)$, we infer that

$$\|\delta\Omega\|_{L^p(B)} \leq c(p')\|\omega\|_{L^p(B)}$$

for $1 < p < 2$. The L^p-estimate of $d\Omega$ for $1 < p < 2$ is obtained similarly. ■

PROOF OF PROPOSITION 3.1. – We prove the proposition for $\omega_0 = 0$, $x_0 = 0$ and $R = 1$, the general result following by a homothety argument.

We let $\Omega \in \mathfrak{P}^+(B, \Lambda_m)$ be the plus-potential of $\omega \in L^2(B, \Lambda_m) \cap \ker d$. We observe from (3.13) that $\delta d\Omega = -\omega - d\delta\Omega \in \ker d$, while $d\Omega \in \mathfrak{P}^+$ implies that $(\delta d\Omega)^\perp = 0$ on ∂B. Thus $\delta d\Omega \in \mathfrak{H}^+$ and we conclude that $\delta d\Omega = 0$. This in turn implies that $d\Omega \in \mathfrak{H}^+ = 0$, and (3.13) reduces to $d\delta\Omega = -\omega$. This last equation becomes $d\xi = \omega$, where $\xi = -\delta\Omega$. With this choice we also have $\delta\xi = 0$ and hence (3.1) is satisfied.

Next, (3.1) implies that ξ is a weak solution of $\Delta\xi = \delta\omega$ in B. Therefore the L^p-theory for elliptic equations with constant coefficients furnishes the interior estimate, for $1 < p < \infty$,

$$\|D\xi\|_{L^p(B_\rho)} \leq c(p, \rho)(\|\omega\|_{L^p(B)} + \|\xi\|_{L^p(B)}).$$

Combining this, for $\xi = -\delta\Omega$, with (3.14) yields (3.2). ■

In this section we have used the following lemma and theorem, which are proved in [G, Lemma 2.1, p. 86] and in [G, Thm. 1.2, p. 70] respectively.

LEMMA 3.4. – *Let $f(t)$ be a nonnegative and nondecreasing function. Suppose that*

$$f(s) \leq A[(s/t)^\alpha + \varepsilon] f(t) + Bt^\beta$$

for all $0 \leq s \leq t \leq T$, *with* $A, B \geq 0$ *and* $0 \leq \beta < \alpha$. *Then there exist constants* ε_0 *and* c *depending only on* A, α *and* β *such that, for* $\varepsilon < \varepsilon_0$ *and* $0 \leq s \leq t \leq T$, *we have*

$$f(s) \leq c[(s/t)^\beta f(t) + Bs^\beta].$$

THEOREM 3.1. (Campanato's characterization of Hölder continuity). – *Let the bounded open set* Ω *in* $\mathbb{R}^n$ *satisfy* $|\Omega_\rho(x_0)| \geq A\rho^n$ *for some positive constant* A *and for every* $x_0 \in \Omega$ *and* $0 < \rho < \operatorname{diam}\Omega$, *where* $\Omega_\rho(x_0) = B_\rho(x_0) \cap \Omega$. *Then, for* $\sigma > 0$ *and* $p \geq 1$, $u \in C^{0,\sigma}(\overline{\Omega})$ *if and only if* $u \in L^p(\Omega)$ *and*

$$[u]_{p,\sigma;\Omega} = \sup_{\substack{x_0 \in \Omega \\ 0 < \rho < \operatorname{diam}\Omega}} \rho^{-\sigma} \left(\fint_{\Omega_\rho(x_0)} |u - u_{x_0,\rho}|^p dx \right)^{1/p} < \infty,$$

where $u_{x_0,\rho} = \fint_{\Omega_\rho(x_0)} u\,dx$. *Moreover, an equivalent norm on* $C^{0,\sigma}(\overline{\Omega})$ *is defined by* $\|u\|_{p,\sigma} = \|u\|_{L^p(\Omega)} + [u]_{p,\sigma;\Omega}$.

4. – The Caccioppoli inequalities.

We recall Caccioppoli's first inequality which applies to the case of *monotone A*, and we present Caccioppoli's second inequality for the case of *quasimonotone A*. They both lead to a reverse Hölder inequality with increasing supports.

Proofs of Caccioppoli's first inequality for $m = 1$ can be found in [G], [C1], [C2] and [N-W]. As a first application of the duality principle for the case $1 < p < 2$, we here give a proof valid for arbitrary m and $p > 1$. We note in particular that the restriction [C2, (1.5)] is superfluous.

PROPOSITION 4.1. (Caccioppoli's first inequality). – *Let* $1 < p < \infty$ *and suppose that* $A: \overline{\Omega} \times \Lambda_m \to \Lambda_m$ *satisfies* (1.5), (1.6), *and* (1.15) *with* $\delta = 1$, *i.e.* $A(\cdot, \omega)$ *is Lipschitz continuous. Let* $\omega \in L^p_{\mathrm{loc}}(\Omega, \Lambda_m)$ *be an A-harmonic differential form.*

Then $V(\omega) \in W^{1,2}_{\mathrm{loc}}(\Omega, \Lambda_m)$, *and for every ball* $B_R(x_0) \subset\subset \Omega$ *and for every* $\omega_0 \in \Lambda_m$ *we have*

$$\int_{B_{R/2}(x_0)} |D[V(\omega)]|^2 dx \leq cR^{-2} \int_{B_R(x_0)} |V(\omega) - V(\omega_0)|^2 dx + c \int_{B_R(x_0)} (1 + |\omega|^2)^{p/2} dx. \tag{4.1}$$

PROOF. – We first assume that $p \geq 2$. We use the notation $D^h_\alpha \psi(x) = h^{-1}[\psi(x + he_\alpha) - \psi(x)]$ for the difference quotient. Inserting $\varphi = D^{-h}_\alpha \psi$, for $\psi \in C^\infty_0(\Omega, \Lambda_{m-1})$

and $h < \operatorname{dist}(\operatorname{supp}\psi, \partial\Omega)$ in (1.3), writing $\omega_h(x) = \omega(x + he_\alpha)$ and using (1.15), gives

$$h^{-1}\int_\Omega \langle A(x + he_\alpha, \omega_h) - A(x + he_\alpha, \omega), d\psi\rangle\, dx \leq c\int_\Omega (1 + |\omega|^2)^{(p-1)/2}|d\psi|\, dx. \tag{4.2}$$

For $B_R(x_0) \subset\subset \Omega$ and $\omega_0 \in \Lambda_m$, we let $\xi \in W^{1,p}(B_R(x_0), \Lambda_{m-1})$ be the $(m-1)$-form of Proposition 3.1. We choose a cut-off function $\eta \in C_0^\infty(\Omega)$ such that $0 \leq \eta \leq 1$, $\operatorname{supp}\eta \subset B_{3R/4}(x_0)$, $\eta \equiv 1$ on $B_{R/2}(x_0)$ and $|D\eta| \leq cR^{-1}$. We then set $\psi = D_\alpha^h \xi \eta^2$ in (4.2), for which

$$d\psi = D_\alpha^h \omega \eta^2 + 2\eta\, d\eta \wedge D_\alpha^h \xi,$$

and we deduce, using (1.5) and (1.6), that

$$\lambda\int_\Omega (1 + |\omega_h|^2 + |\omega|^2)^{(p-2)/2}|D_\alpha^h \omega|^2 \eta^2\, dx \leq$$

$$\leq c\int_\Omega (1 + |\omega_h|^2 + |\omega|^2)^{(p-2)/2}|D_\alpha^h\omega|\,\eta\,|D\eta|\,|D_\alpha^h\xi|\, dx +$$

$$+ c\int_\Omega (1 + |\omega|^2)^{(p-1)/2}(|D_\alpha^h\omega|\eta^2 + \eta|D\eta|\,|D_\alpha^h\xi|)\, dx \leq$$

$$\leq \varepsilon\int_\Omega (1 + |\omega_h|^2 + |\omega|^2)^{(p-2)/2}|D_\alpha^h\omega|^2\eta^2\, dx +$$

$$+ c(\varepsilon)\int_\Omega (1 + |\omega_h|^2 + |\omega|^2)^{(p-2)/2}|D\eta|^2|D_\alpha^h\xi|^2\, dx + c(\varepsilon)\int_\Omega (1 + |\omega|^2)^{p/2}\eta^2\, dx,$$

and hence

$$\int_{B_{R/2}} |D_\alpha^h[V(\omega)]|^2\, dx \leq$$

$$\leq cR^{-2}\int_{B_{3R/4}} (1 + |\omega_h|^2 + |\omega|^2)^{(p-2)/2}|D_\alpha^h\xi|^2\, dx + c\int_{B_R} (1 + |\omega|^2)^{p/2}\, dx.$$

Since $\omega_h \to \omega$ and $D_\alpha^h\xi \to D_\alpha\xi$ in L_{loc}^p as $h \to 0$, we infer that $D_\alpha[V(\omega)]$ exists in $L_{\mathrm{loc}}^2(\Omega, \Lambda_m)$ and that

$$\int_{B_{R/2}} |D[V(\omega)]|^2\, dx \leq cR^{-2}\int_{B_{3R/4}} (1 + |\omega|^2)^{(p-2)/2}|D\xi|^2\, dx + c\int_{B_R} (1 + |\omega|^2)^{p/2}\, dx.$$

Further, with the aid of Young's inequality and (3.2), we estimate the term

$$\int_{B_{3R/4}} (1 + |\omega|^2)^{(p-2)/2} |D\xi|^2 dx \leq$$

$$\leq c \int_{B_{3R/4}} [(1 + |\omega_0|^2)^{(p-2)/2} |D\xi|^2 + |\omega - \omega_0|^p + |D\xi|^p] dx \leq$$

$$\leq c \int_{B_R} [(1 + |\omega_0|^2)^{(p-2)/2} |\omega - \omega_0|^2 + |\omega - \omega_0|^p] dx \leq c \int_{B_R} |V(\omega) - V(\omega_0)|^2 dx ,$$

and we obtain (4.1) for $p \geq 2$.

In the case $1 < p < 2$ we invoke the duality principle, Proposition 2.1, thus obtaining (4.1) for the forms $\tau = \pm * A(x, \omega)$ and $\tau_0 = \pm * A(x_0, \omega_0)$:

$$\text{(4.3)} \quad \int_{B_{3R/4}} |D[V_{p'}(A(x, \omega))]|^2 dx \leq$$

$$\leq cR^{-2} \int_{B_R} |V_{p'}(A(x, \omega)) - V_{p'}(A(x_0, \omega_0))|^2 dx + c \int_{B_R} (1 + |A(x, \omega)|^2)^{p'/2} dx \leq$$

$$\leq cR^{-2} \int_{B_R} |V_p(\omega) - V_p(\omega_0)|^2 dx + c \int_{B_R} (1 + |\omega|^2)^{p/2} dx .$$

The last inequality is due to (2.9) with $\delta = 1$ and (2.5). From (2.9) also we have

$$|D_\alpha^h [V_p(\omega)]|^2 \leq c |D_\alpha^h [V_{p'}(A(x, \omega))]|^2 + c(1 + |\omega|^2)^{p/2},$$

which, by (4.3), gives

$$\int_{B_{R/2}} |D_\alpha^h [V_p(\omega)]|^2 dx \leq cR^{-2} \int_{B_R} |V_p(\omega) - V_p(\omega_0)|^2 dx + c \int_{B_R} (1 + |\omega|^2)^{p/2} dx .$$

We conclude that $V_p(\omega) \in W^{1,2}_{\text{loc}}(\Omega, \Lambda_m)$ and letting $h \to 0$ we recover (4.1) for the case $1 < p < 2$. ∎

By an application of the Poincaré-Sobolev inequality and [H, Lemma 2.3], we derive from (4.1) a reverse Hölder inequality with increasing supports

$$\left(\fint_{B_{R/2}(x_0)} |V(\omega) - V(\omega)_{x_0, R/2}|^q dx \right)^{2/q} \leq c \fint_{B_R(x_0)} |V(\omega) - V(\omega_0)|^2 dx + cR^2 (1 + |\omega_0|^2)^{p/2},$$

$q = 2^* = 2n/(n-2) > 2$ being the Sobolev exponent associated with the embedding $W^{1,2} \subset L^{2^*}$. For quasimonotone A and $p \geq 2$ a similar reverse Hölder inequality follows from

PROPOSITION 4.2. (Caccioppoli's second inequality). – *Let* $p \geq 2$ *and suppose that* $A(x, \omega)$ *and* $B(x, u, \omega)$ *satisfy* (1.6) *and Hypotheses* H1, H3 *and* H4. *Let* $\omega \in L^p_{\text{loc}}(\Omega, \Lambda_m)$ *be a weak solution of the system* (1.1) *or let* $u \in W^{1,p}_{\text{loc}} \cap L^\infty(\Omega, \mathbb{R}^N)$ *be a weak solution of the system* (1.19) *satisfying condition* (1.20) *and set* $\omega = du$.

Then, for $M_0 > 0$, *there exist positive constants* $R_0(M_0)$ *and* c *such that, for every ball* $B_R(x_0) \subset\subset \Omega$ *with* $R \leq R_0$ *and for every* $\omega_0 \in \Lambda_m$ *with* $|\omega_0| \leq M_0$, *we have*

$$(4.4) \qquad \fint_{B_{R/2}(x_0)} |V(\omega) - V(\omega_0)|^2 dx \leq$$

$$\leq c\left(\fint_{B_R(x_0)} |V(\omega) - V(\omega_0)|^{2_*} dx\right)^{2/2_*} + cR^{2\delta}(1 + |\omega_0|^2)^{(p+2)/2},$$

where $2_* = 2n/(n+2) < 2$ *is the inverse Sobolev exponent.*

PROOF. – We treat the systems (1.1) and (1.19) together by considering the system

$$(4.5) \qquad \delta(A(x, \omega)) = B(x, \omega) \quad \text{and} \quad d\omega = 0,$$

with $|B(x, \omega)| \leq a|\omega|^p + b$. We suppose that we are given either (i) a solution ω of (4.5) with $a = 0$ or (ii) a solution $\omega = du$ of (4.5) for $u \in W^{1,p}_{\text{loc}} \cap L^\infty(\Omega, \mathbb{R}^N)$ satisfying (1.20). For $B_R(x_0) \subset\subset \Omega$ and $\omega_0 \in \Lambda_m$ with $|\omega_0| \leq M_0$, we let ξ be the $(m-1)$-form of Proposition 3.1 and we set $\xi_0 = \xi_{x_0, 3R/4}$. For the case (ii), we choose $\xi = u - \omega_0 \cdot (x - x_0)$ for which $\xi_0 = u_{x_0, 3R/4}$.

For $R/2 \leq s < t \leq 3R/4$, we let $\eta \in C_0^\infty(\Omega)$ be a cut-off function such that $0 \leq \eta \leq 1$, $\operatorname{supp} \eta \in B_t(x_0)$, $\eta \equiv 1$ on $B_s(x_0)$ and $|D\eta| \leq c(t-s)^{-1}$, and we define

$$\varphi = \eta(\xi - \xi_0) \quad \text{and} \quad \psi = (1 - \eta)(\xi - \xi_0).$$

By (3.1) and (1.20), we have, in either of the cases (i) or (ii),

$$(4.6) \qquad \omega = \omega_0 + d\varphi + d\psi, \qquad a|\varphi| \leq 2aM + aM_0 R \quad \text{and} \quad 2aM < \lambda.$$

We infer from the quasimonotonicity of A, (4.5), (4.6), (1.6), (1.15), Young's inequality, the estimate $(\alpha + \beta)^p \leq (1 + \varepsilon)\alpha^p + c(\varepsilon)\beta^p$ and the Poincaré inequality applied to φ on B_t that

$$\lambda \int_{B_t} (1 + |\omega_0|^2 + |d\varphi|^2)^{(p-2)/2} |d\varphi|^2 dx \leq \int_{B_t} \langle A(x_0, \omega_0 + d\varphi) - A(x_0, \omega), d\varphi \rangle dx +$$

$$+\int_{B_t}\langle A(x_0,\omega)-A(x,\omega),d\varphi\rangle\,dx-\int_{B_t}\langle B(x,\omega),\varphi\rangle\,dx\leq$$

$$\leq c\int_{B_t}(1+|\omega_0|^2+|d\varphi|^2+|d\psi|^2)^{(p-2)/2}|d\psi|\,|d\varphi|\,dx+$$

$$+cR^\delta\int_{B_t}(1+|\omega|^2)^{(p-1)/2}|d\varphi|\,dx+\int_{B_t}(a|\omega|^p+b)|\varphi|\,dx\leq$$

$$\leq c(1+|\omega_0|^2)^{(p-2)/2}\int_{B_t}|d\psi|\,|d\varphi|\,dx+[2(1+\varepsilon)aM+c(1+M_0)R^\delta+\varepsilon]\int_{B_t}|d\varphi|^p\,dx+$$

$$+c(\varepsilon)\int_{B_t}|d\psi|^p\,dx+c(\varepsilon)R^\delta(1+|\omega_0|^2)^{p/2}\int_{B_t}|D\varphi|\,dx\,.$$

By choosing ε and R small enough so that $2(1+\varepsilon)aM+c(1+M_0)R^\delta+\varepsilon<\lambda$, we conclude, since $\psi\equiv 0$ on B_s, that

$$\int_{B_s}|V(\omega)-V(\omega_0)|^2\,dx\leq\int_{B_t}|V(\omega_0+d\varphi)-V(\omega_0)|^2\,dx\leq$$

$$\leq c(t-s)^{-2}(1+|\omega_0|^2)^{(p-2)/2}\int_{B_{3R/4}}|\xi-\xi_0|^2\,dx+c(t-s)^{-p}\int_{B_{3R/4}}|\xi-\xi_0|^p\,dx+$$

$$+c_1\int_{B_t\setminus B_s}|V(\omega)-V(\omega_0)|^2\,dx+cR^\delta(1+|\omega_0|^2)^{p/2}\int_{B_{3R/4}}|D\xi|\,dx+$$

$$+cR^{n+2\delta}(1+|\omega_0|^2)^{(p+2)/2}.$$

We fill the hole, that is we add c_1 times the left hand side to both sides. We then divide through by $1+c_1$ and apply Lemma 4.1 with $\theta=c_1/(1+c_1)<1$. In this manner we arrive at

$$⨍_{B_{R/2}}|V(\omega)-V(\omega_0)|^2\,dx\leq cR^{-2}(1+|\omega_0|^2)^{(p-2)/2}⨍_{B_{3R/4}}|\xi-\xi_0|^2\,dx+$$

$$+cR^{-p}⨍_{B_{3R/4}}|\xi-\xi_0|^p\,dx+cR^\delta(1+|\omega_0|^2)^{p/2}⨍_{B_{3R/4}}|D\xi|\,dx+cR^{2\delta}(1+|\omega_0|^2)^{(p+2)/2}\leq$$

$$\leq c(1+|\omega_0|^2)^{(p-2)/2}\left(⨍_{B_{3R/4}}|D\xi|^{2_*}dx\right)^{2/2_*}+c\left(⨍_{B_{3R/4}}|D\xi|^{p_*}dx\right)^{p/p_*}+$$

$$+cR^{2\delta}(1+|\omega_0|^2)^{(p+2)/2} \leqslant c(1+|\omega_0|^2)^{(p-2)/2}\left(\fint_{B_R} |\omega-\omega_0|^{2_*}dx\right)^{2/2_*}+$$

$$+c\left(\fint_{B_R} |\omega-\omega_0|^{p_*}dx\right)^{p/p_*}+cR^{2\delta}(1+|\omega_0|^2)^{(p+2)/2} \leqslant$$

$$\leqslant c\left(\fint_{B_R} |V(\omega)-V(\omega_0)|^{2_*}dx\right)^{2/2_*}+cR^{2\delta}(1+|\omega_0|^2)^{(p+2)/2},$$

which proves (4.4). We have also used the Sobolev embeddings $W^{1,2_*}\subset L^2$, $W^{1,p_*}\subset L^p$ and the estimate (3.2). Note that, since $2_* = 2n/(n+2) < 2$, we require (3.2) for $1 < q < 2$. ∎

We fix $B_R(x_0) \subset\subset \Omega$ with $R \leqslant R_0$ and $\omega_0 \in \Lambda_m$ with $|\omega_0| \leqslant M_0$. We then apply Theorem 4.1 below to the function $g = |V(\omega) - V(\omega_0)|^2$ and the constant function $f = cR^{2\delta}(1+|\omega_0|^2)^{(p+2)/2}$ on the ball $B_R(x_0)$, for $s = 2_*/2 < 1$. In this manner we obtain the

COROLLARY 4.1. – *Let the assumptions of Proposition 4.2 be satisfied. Then, for $M_0 > 0$, there exist positive constants $R_0(M_0)$ and c and an exponent $q > 2$ such that*

$$\left(\fint_{B_{R/2}(x_0)} |V(\omega)-V(\omega_0)|^q dx\right)^{2/q} \leqslant \tag{4.7}$$

$$\leqslant c\fint_{B_R(x_0)} |V(\omega)-V(\omega_0)|^2 dx + cR^{2\delta}(1+|\omega_0|^2)^{(p+2)/2},$$

for every ball $B_R(x_0) \subset\subset \Omega$ with $R \leqslant R_0$ and for every $\omega_0 \in \Lambda_m$ with $|\omega_0| \leqslant M_0$.

In this section we have used the following lemma [E, Lemma 5.2] or [G-G, Lemma 1.1], and the following theorem [G, Prop. 1.1, p. 122].

LEMMA 4.1. – *Let $f:[R/2, 3R/4] \to [0, \infty[$ be bounded and satisfy*

$$f(s) \leqslant \theta f(t) + A(t-s)^{-2} + B(t-s)^{-p} + C,$$

for $R/2 \leqslant s < t \leqslant 3R/4$, where A, B, C are nonnegative constants and $0 \leqslant \theta < 1$. Then there exists a constant $c_0 = c_0(\theta, p)$ such that

$$f(R/2) \leqslant c_0(AR^{-2} + BR^{-p} + C).$$

THEOREM 4.1. (Higher integrability theorem). – *Let Ω be a bounded open set in $\mathbb{R}^n$, and let $g \in L^1_{\text{loc}}(\Omega)$ and $f \in L^t_{\text{loc}}(\Omega)$ be nonnegative functions with $0 < s < 1 < t < \infty$. Suppose we have*

$$\fint_{B_{R/2}(x_0)} g\,dx \leq b\left(\fint_{B_R(x_0)} g^s\,dx\right)^{1/s} + \fint_{B_R(x_0)} f\,dx$$

for every ball $B_R(x_0) \subset\subset \Omega$. Then $g \in L^{1+\varepsilon}_{\text{loc}}(\Omega)$ for $\varepsilon < \varepsilon_0$, and

$$\left(\fint_{B_\rho(x_0)} g^{1+\varepsilon}\,dx\right)^{1/(1+\varepsilon)} \leq c\fint_{B_R(x_0)} g\,dx + c\left(\fint_{B_R(x_0)} f^{1+\varepsilon}\,dx\right)^{1/(1+\varepsilon)}$$

for any ball $B_R(x_0) \subset\subset \Omega$ and $0 < \rho < R$, where $\varepsilon_0 = \varepsilon_0(n, s, t, b)$ and $c = c(n, s, t, b, \rho/R, \varepsilon)$.

5. – Partial regularity.

We first prove

PROPOSITION 5.1. – *Suppose that the hypotheses of Theorems 1.1 or 1.2 are satisfied. Set $\omega = du$ for the case of Theorem 1.2. Then, for $\varepsilon_0 > 0$ and $M_0 > 0$, there exist positive constants c_1, c_2, $c(\varepsilon_0)$, $H(M_0)$ and $R_0(M_0)$ such that, for every $B_R(x_0) \subset\subset \Omega$ and for every ρ with $0 < \rho < R$,*

$$\Phi(x_0, \rho) \leq c_1\big((\rho/R)^2 + (R/\rho)^n[\varepsilon_0 + c(\varepsilon_0)\chi(M_0, \Phi(x_0, R), R)]\big)\Phi(x_0, R) + c(\varepsilon_0)H(M_0)(R/\rho)^n R^{2\delta}, \tag{5.1}$$

provided that

$$R \leq R_0 \quad \text{and} \quad |\omega_{x_0, R}| + c_2\Phi(x_0, R)^{1/p} \leq M_0. \tag{5.2}$$

Here $\chi(M_0, \Phi, R)$ is a continuous nonnegative function with $\chi(M_0, 0, 0) = 0$.

PROOF. – We first assume the hypotheses of Theorem 1.1. We only have to prove (5.1) for $0 < \rho < R/4$. We choose $\omega_0 \in \Lambda_m$ in such a way that $V(\omega_0) = V(\omega)_{x_0, R}$. Then, since

$$|\omega_{x_0, R} - \omega_0| = \left|\fint_{B_R}(\omega - \omega_0)\,dx\right| \leq c_2\Phi(x_0, R)^{1/p},$$

assumption (5.2) implies that

$$|\omega_0| \leq |\omega_{x_0, R}| + c_2\Phi(x_0, R)^{1/p} \leq M_0. \tag{5.3}$$

By (1.12), the bilinear form

$$\mathfrak{B}(\varphi, \psi) = \int_{B_{R/4}} (\langle A_\omega(x_0, \omega_0)\cdot d\varphi, d\psi\rangle + (1 + |\omega_0|^2)^{(p-2)/2}\langle \delta\varphi, \delta\psi\rangle)\, dx$$

is coercive on the Hilbert space $W_0^{1,2}(B_{R/4}(x_0), \Lambda_{m-1})$. The Lax-Milgram theorem [G-T, Theorem 5.8] therefore provides a solution $\varphi \in W_0^{1,2}(B_{R/4}(x_0), \Lambda_{m-1})$ of the system:

$$\int_{B_{R/4}} (\langle A_\omega(x_0, \omega_0)\cdot(d\varphi + \omega - \omega_0), d\psi\rangle + (1 + |\omega_0|^2)^{(p-2)/2}\langle \delta\varphi, \delta\psi\rangle)\, dx = 0 \tag{5.4}$$

$$\text{for all } \psi \in W_0^{1,2}(B_{R/4}(x_0), \Lambda_{m-1}).$$

We set $\zeta = \varphi + \xi$, where ξ is the $(m-1)$-form on $B_{R/2}(x_0)$ defined in Proposition 3.1 which satisfies $d\xi = \omega - \omega_0$, $\delta\xi = 0$. Then we have

$$d\zeta = d\varphi + \omega - \omega_0, \qquad \delta\zeta = \delta\varphi, \tag{5.5}$$

and we recognize ζ as the solution of the Dirichlet problem

$$\begin{cases} \delta(A_\omega(x_0, \omega_0)\cdot d\zeta) + (1 + |\omega_0|^2)^{(p-2)/2} d\delta\zeta = 0 & \text{in } B_{R/4}(x_0), \\ \zeta = \xi \quad \text{on } \partial B_{R/4}(x_0). \end{cases} \tag{5.6}$$

As a coercive system with constant coefficients, it yields the following L^2- and L^p-estimates (see [G, ch. III]), for all $\rho < R/4$ and $2 \leq q < \infty$,

$$\fint_{B_\rho} |D\zeta|^2 dx \leq c \fint_{B_{R/4}} |D\zeta|^2 dx, \tag{5.7}$$

$$\fint_{B_\rho} |D\zeta - D\zeta_{x_0,\rho}|^q dx \leq c(q)(\rho/R)^q \fint_{B_{R/4}} |D\zeta - D\zeta_{x_0, R/4}|^q dx, \tag{5.8}$$

$$\int_{B_{R/4}} |D\zeta|^q dx \leq c(q) \int_{B_{R/4}} |D\xi|^q dx. \tag{5.9}$$

We note that the constants in (5.7) to (5.9) are independent of ω_0.

Since $\varphi = \zeta - \xi$, (5.9) together with (3.2) imply, for $q \geq 2$, that

$$\int_{B_{R/4}} |d\varphi|^q dx \leq c(q) \int_{B_{R/4}} |D\varphi|^q dx \leq c(q) \int_{B_{R/2}} |\omega - \omega_0|^q dx. \tag{5.10}$$

From (5.8), (5.9) and (3.2) it follows, for $\rho < R/4$, that

$$\fint_{B_\rho} |d\zeta - d\zeta_{x_0,\rho}|^q dx \leq c(q) \fint_{B_\rho} |D\zeta - D\zeta_{x_0,\rho}|^q dx \leq c(q)(\rho/R)^q \fint_{B_{R/4}} |D\zeta|^q dx \leq$$

$$\leq c(q)(\rho/R)^q \fint_{B_{R/2}} |\omega - \omega_0|^q dx .$$

Finally, by (5.7), (5.9) and (3.2), we have

$$|d\zeta_{x_0,\rho}|^2 = \left| \fint_{B_\rho} d\zeta \, dx \right|^2 \leq c \fint_{B_\rho} |D\zeta|^2 dx \leq c \fint_{B_{R/4}} |D\zeta|^2 dx \leq c \fint_{B_{R/2}} |\omega - \omega_0|^2 dx .$$

These estimates yield

$$\fint_{B_\rho} |V(\omega_0 + d\zeta) - V(\omega_0 + d\zeta_{x_0,\rho})|^2 dx \leq$$

$$\leq c(1 + |\omega_0|^2 + |d\zeta_{x_0,\rho}|^2)^{(p-2)/2} \fint_{B_\rho} |d\zeta - d\zeta_{x_0,\rho}|^2 dx + c \fint_{B_\rho} |d\zeta - d\zeta_{x_0,\rho}|^p dx \leq$$

$$\leq c(\rho/R)^2 \left((1 + |\omega_0|^2)^{(p-2)/2} \fint_{B_{R/2}} |\omega - \omega_0|^2 dx + \fint_{B_{R/2}} |\omega - \omega_0|^p dx \right) \leq$$

$$\leq c(\rho/R)^2 \fint_{B_R} |V(\omega) - V(\omega_0)|^2 dx = c(\rho/R)^2 \Phi(x_0, R) .$$

By (5.5), we thus obtain, for all $\rho < R/4$,

$$\Phi(x_0, \rho) = \fint_{B_\rho} |V(\omega) - V(\omega)_{x_0,\rho}|^2 dx \leq$$

$$\leq 2 \fint_{B_\rho} |V(\omega) - V(\omega_0 + d\zeta)|^2 dx + 2 \fint_{B_\rho} |V(\omega_0 + d\zeta) - V(\omega_0 + d\zeta_{x_0,\rho})|^2 dx \leq$$

$$\leq c(R/\rho)^n \fint_{B_{R/4}} |V(\omega) - V(\omega + d\varphi)|^2 dx + c(\rho/R)^2 \Phi(x_0, R) .$$

So it remains to estimate

$$\int_{B_{R/4}} |V(\omega)-V(\omega+d\varphi)|^2\,dx \le c\int_{B_{R/4}} (1+|\omega_0|^2+|\omega-\omega_0|^2+|d\varphi|^2)^{(p-2)/2}|d\varphi|^2\,dx \le$$

$$\le \varepsilon_0 \int_{B_{R/4}} |\omega-\omega_0|^p\,dx + c(\varepsilon_0)\int_{B_{R/4}} [(1+|\omega_0|^2)^{(p-2)/2}|d\varphi|^2+|d\varphi|^p]\,dx \le$$

$$\le \varepsilon_0 c\int_{B_R} |V(\omega)-V(\omega_0)|^2\,dx + c(\varepsilon_0)\int_{B_{R/4}} |V(\omega_0+d\varphi)-V(\omega_0)|^2\,dx\,.$$

Using the quasimonotonicity of A, as well as (5.4) with $\psi=\varphi$, (1.3), (1.15) and (1.14), we control

$$c\int_{B_{R/4}} |V(\omega_0+d\varphi)-V(\omega_0)|^2\,dx \le \int_{B_{R/4}} \langle A(x_0,\omega_0+d\varphi), d\varphi\rangle\,dx =$$

$$= \int_{B_{R/4}} \langle A(x_0,\omega_0+d\varphi)-A(x_0,\omega_0)-A_\omega(x_0,\omega_0)\cdot d\varphi, d\varphi\rangle\,dx +$$

$$+\int_{B_{R/4}} \langle A(x_0,\omega)-A(x_0,\omega_0)-A_\omega(x_0,\omega_0)\cdot(\omega-\omega_0), d\varphi\rangle\,dx +$$

$$+\int_{B_{R/4}} \langle A(x,\omega)-A(x_0,\omega), d\varphi\rangle\,dx - \int_{B_{R/4}} (1+|\omega_0|^2)^{(p-2)/2}\langle \delta\varphi, \delta\varphi\rangle\,dx \le$$

$$\le \int_{B_{R/4}}\int_0^1 \langle [A_\omega(x_0,\omega_0+t\,d\varphi)-A_\omega(x_0,\omega_0)]\cdot d\varphi, d\varphi\rangle\,dt\,dx +$$

$$+\int_{B_{R/4}}\int_0^1 \langle [A_\omega(x_0,t\omega+(1-t)\omega_0)-A_\omega(x_0,\omega_0)]\cdot(\omega-\omega_0), d\varphi\rangle\,dt\,dx +$$

$$+cR^\delta\int_{B_{R/4}} (1+|\omega|^2)^{(p-1)/2}|d\varphi|\,dx \le$$

$$\le c\int_{B_{R/4}} (1+|\omega_0|^2+|\omega_0+d\varphi|^2)^{(p-2)/2}|d\varphi|^2 l(|\omega_0|, |d\varphi|)\,dx +$$

$$+c\int_{B_{R/4}} (1+|\omega_0|^2+|\omega-\omega_0|^2)^{(p-2)/2}|\omega-\omega_0|\,|d\varphi|\,l(|\omega_0|, |\omega-\omega_0|)\,dx +$$

$$+\varepsilon \int_{B_{R/4}} [(1+|\omega_0|^2)^{(p-2)/2}|d\varphi|^2 + |d\varphi|^p]\,dx + c(\varepsilon) R^{\delta p'} \int_{B_{R/4}} |\omega - \omega_0|^p\,dx +$$

$$+c(\varepsilon) R^{n+2\delta}(1+|\omega_0|^2)^{p/2} = (A)+(B)+(C)+(D)+(E).$$

We estimate the second term with the help of Young's inequality and the boundedness of l

$$(B) \leqslant c \int_{B_{R/4}} [(1+|\omega_0|^2)^{(p-2)/2}|\omega-\omega_0| + |\omega-\omega_0|^{p-1}]|d\varphi|\, l(|\omega_0|, |\omega-\omega_0|)\,dx \leqslant$$

$$\leqslant \varepsilon \int_{B_{R/4}} [(1+|\omega_0|^2)^{(p-2)/2}|d\varphi|^2 + |d\varphi|^p]\,dx +$$

$$+c(\varepsilon) \int_{B_{R/4}} [(1+|\omega_0|^2)^{(p-2)/2}|\omega-\omega_0|^2 + |\omega-\omega_0|^p]\, l(|\omega_0|, |\omega-\omega_0|)\,dx.$$

By absorbing the two terms with coefficient ε in the left-hand side and using (5.3), we thus obtain

$$\int_{B_{R/4}} |V(\omega_0+d\varphi)-V(\omega_0)|^2\,dx \leqslant c \int_{B_{R/4}} |V(\omega_0+d\varphi)-V(\omega_0)|^2\, l(M_0, |d\varphi|)\,dx +$$

$$+c \int_{B_{R/4}} |V(\omega)-V(\omega_0)|^2\, l(M_0, |\omega-\omega_0|)\,dx + cR^{\delta p'} \int_{B_R} |V(\omega)-V(\omega_0)|^2\,dx +$$

$$+H(M_0) R^{n+2\delta} = (I)+(II)+(III)+(IV).$$

By virtue of the reverse Hölder inequality (4.7), and Jensen's inequality together with the concavity of l in its second argument, we estimate the second term by

$$(II) \leqslant c \left(\int_{B_{R/2}} |V(\omega)-V(\omega_0)|^q\,dx \right)^{2/q} \left(\int_{B_{R/4}} l(M_0, |\omega-\omega_0|)\,dx \right)^{1-2/q} \leqslant$$

$$\leqslant c \int_{B_R} |V(\omega)-V(\omega_0)|^2\,dx \left(\fint_{B_{R/4}} l(M_0, |\omega-\omega_0|)\,dx \right)^{1-2/q} + H(M_0) R^{n+2\delta} \leqslant$$

$$\leqslant c \int_{B_R} |V(\omega)-V(\omega_0)|^2\,dx\; l\left(M_0, \fint_{B_{R/4}} |\omega-\omega_0|\,dx \right)^{1-2/q} + H(M_0) R^{n+2\delta}.$$

(I) is estimated in a similar way by making use of (5.10) which yields

$$\int_{B_{R/4}} |V(\omega_0 + d\varphi) - V(\omega_0)|^q dx \le c \int_{B_{R/4}} [(1 + |\omega_0|^2)^{(p-2)/2} |d\varphi|^2 + |d\varphi|^p]^{q/2} dx \le$$

$$\le c \int_{B_{R/2}} [(1 + |\omega_0|^2)^{(p-2)/2} |\omega - \omega_0|^2 + |\omega - \omega_0|^p]^{q/2} dx \le c \int_{B_{R/2}} |V(\omega) - V(\omega_0)|^q dx .$$

This completes the proof of (5.1), under the hypotheses of Theorem 1.1, with

$$\chi(M_0, \Phi, R) = l(M_0, c\Phi^{1/p})^{1-2/q} + R^{\delta p'} . \quad \blacksquare$$

REMARK 5.1. – If we replace the estimate (1.14) in the hypotheses of Theorem 1.1 by the estimate corresponding to (2.3)

$$|A_\omega(x, \omega) - A_\omega(x, \eta)| \le c(1 + |\omega|^2 + |\eta|^2)^{(p-2)/2} \cdot$$

$$\cdot l(c(1 + |\omega|^2)^{(p-1)/2}, c(1 + |\omega|^2)^{(p-2)/2} |\omega - \eta| + c|\omega - \eta|^{p-1}),$$

we can still prove Proposition 1.1 with

$$\chi(M_0, \Phi, R) = l(H(M_0), H(M_0)\Phi^{1/p} + c\Phi^{1/p'})^{1-2/q} + R^{\delta p'} .$$

For the case of Theorem 1.2, we set $\omega = du$ and $\xi = u - \omega_0 \cdot (x - x_0)$ in the above proof. Then, by applying a maximum estimate for coercive systems with constant coefficients [G, Prop. 2.3, p. 83] to (5.6), we infer, by (1.20) and (5.3), that

$$\|\zeta\|_{L^\infty} \le c\|\xi\|_{L^\infty} \le c(M + M_0 R),$$

and we see that $\varphi = \zeta - \xi$ is bounded. We have to take into account the extra term $\int \langle B(x, u, \omega), \varphi \rangle dx$ which we control with the help of (1.16), (5.10) and the Poincaré, Cauchy and Hölder inequalities

$$\fint_{B_{R/4}} \langle B(x, u, \omega), \varphi \rangle dx \le \fint_{B_{R/4}} (a|\omega|^p + b)|\varphi| dx \le$$

$$\le cR(1 + |\omega_0|^2)^{p/2} \fint_{B_{R/4}} |D\varphi| dx + c \fint_{B_{R/4}} |V(\omega) - V(\omega_0)|^2 |\varphi| dx \le$$

$$\le \varepsilon_0 \fint_{B_R} |V(\omega) - V(\omega_0)|^2 dx + c(\varepsilon_0) R^{2\delta} (1 + |\omega_0|^2)^{(p+2)/2} +$$

$$+ c\left(\fint_{B_{R/4}} |V(\omega) - V(\omega_0)|^q dx \right)^{2/q} \left(\fint_{B_{R/4}} |\varphi|^{q/(q-2)} dx \right)^{1-2/q} = (i) + (ii) + (iii).$$

Since we can assume that $q/(q-2) \geqslant pq/2$, i.e. that $2 < q \leqslant 2 + 2/p$, we estimate the last term, using (5.10), (4.7) and (5.3), by

$$(iii) \leqslant c \left(\fint_{B_{R/4}} |V(\omega) - V(\omega_0)|^q \, dx \right)^{2/q} \left(\fint_{B_{R/4}} |D\varphi|^{pq/2} \, dx \right)^{1-2/q} \leqslant$$

$$\leqslant c \fint_{B_{R/2}} |V(\omega) - V(\omega_0)|^q \, dx \leqslant c \left(\fint_{B_R} |V(\omega) - V(\omega_0)|^2 \, dx \right)^{q/2} + H(M_0) R^{2\delta}.$$

Setting

$$\chi(M_0, \Phi, R) = l(M_0, c\Phi^{1/p})^{1-2/q} + R^{\delta p'} + \Phi^{(1/2)q-1},$$

we obtain (5.1) for the case of Theorem 1.2. ∎

Theorems 1.1 and 1.2 follow in a standard way from Proposition 5.1, see [G, pp. 197-199]. For completeness we here repeat the argument. Let $M_1 > 0$ be given. We write $\rho = \tau R$, $\varepsilon_0 = \tau^{n+2}$ and $M_0 = 3M_1$ in (5.1) for $0 < \tau < 1$:

$$(5.11) \quad R \leqslant R_0(3M_1) \quad \text{and} \quad |\omega_{x_0, R}| + c_2 \Phi(x_0, R)^{1/p} \leqslant 3M_1 \Rightarrow$$

$$\Rightarrow \Phi(x_0, \tau R) \leqslant c_1 \tau^2 [2 + \tau^{-n-2} c(\tau^{n+2}) \chi(3M_1, \Phi(x_0, R), R)] \Phi(x_0, R) + H_0 R^{2\delta},$$

where we have set $H_0 = c(\tau^{n+2}) H(3M_1) \tau^{-n}$. We let $\sigma \in]\delta, 1[$, and we choose $\tau \in]0, 1[$ so as to satisfy $3c_1 \tau^{2-2\sigma} \leqslant 1$. We next choose $\varepsilon_1 > 0$ and $0 < R_1 \leqslant R_0(3M_1)$ sufficiently small that

$$(5.12) \quad R \leqslant R_1, \ |\omega_{x_0, R}| \leqslant 2M_1 \quad \text{and} \quad \Phi(x_0, R) < 2\varepsilon_1 \Rightarrow$$

$$\Rightarrow c_2 \Phi(x_0, R)^{1/p} \leqslant M_1 \quad \text{and} \quad \tau^{-n-2} c(\tau^{n+2}) \chi(3M_1, \Phi(x_0, R), R) \leqslant 1 \Rightarrow$$

$$\Rightarrow \Phi(x_0, \tau R) \leqslant \tau^{2\sigma} \Phi(x_0, R) + H_0 R^{2\delta},$$

by (5.11) and by our choice of τ. We also suppose that we have chosen R_1 small enough so that

$$\frac{H_0 R_1^{2\delta}}{\tau^{2\delta} - \tau^{2\sigma}} < \varepsilon_1.$$

LEMMA 5.1. – *With the above choices of* H_0, σ, τ, ε_1 *and* R_1 *we have, for all* $k \in \mathbb{N}_0$,

$$(5.13)_k \qquad R \leq R_1,\ |\omega_{x_0,R}| \leq M_1 \quad \textit{and} \quad \Phi(x_0, R) < \varepsilon_1 \Rightarrow$$

$$\Rightarrow \Phi(x_0, \tau^k R) \leq \tau^{2k\sigma}\Phi(x_0, R) + H_0(\tau^{k-1}R)^{2\delta}\sum_{s=0}^{k-1}(\tau^{2\sigma-2\delta})^s \leq$$

$$\leq \tau^{2k\delta}\left(\Phi(x_0, R) + \frac{H_0 R^{2\delta}}{\tau^{2\delta} - \tau^{2\sigma}}\right) < 2\varepsilon_1\tau^{2k\delta}.$$

PROOF BY INDUCTION. – The case $k = 0$ is trivial. Assume next that $(5.13)_l$ holds for $0 \leq l \leq k$ and that the hypotheses of $(5.13)_{k+1}$ are satisfied. We then have

$$|\omega_{x_0,\tau^{l+1}R} - \omega_{x_0,\tau^l R}| = \left| \ \fint_{B_{\tau^{l+1}R}} (\omega - \omega_{x_0,\tau^l R})\,dx \right| \leq$$

$$\leq \tau^{-n}\fint_{B_{\tau^l R}} |\omega - \omega_{x_0,\tau^l R}|\,dx \leq c_2\tau^{-n}\Phi(x_0, \tau^l R)^{1/p} < c_2\tau^{-n}[2\varepsilon_1]^{1/p}[\tau^{2\delta/p}]^l,$$

and it follows by summation that

$$(5.14) \qquad |\omega_{x_0,\tau^k R}| \leq |\omega_{x_0,R}| + \sum_{l=0}^{k-1}|\omega_{x_0,\tau^{l+1}R} - \omega_{x_0,\tau^l R}| \leq 2M_1,$$

provided that ε_1 has also been chosen so small as to make

$$c_2\tau^{-n}[2\varepsilon_1]^{1/p}\frac{1}{1 - \tau^{2\delta/p}} \leq M_1.$$

Further, from $(5.13)_k$,

$$(5.15) \qquad \Phi(x_0, \tau^k R) < 2\varepsilon_1.$$

Now, (5.14), (5.15), (5.12) and $(5.13)_k$ imply that

$$\Phi(x_0, \tau^{k+1}R) \leq \tau^{2\sigma}\Phi(x_0, \tau^k R) + H_0(\tau^k R)^{2\delta} \leq$$

$$\leq \tau^{2\sigma}\left(\tau^{2k\sigma}\Phi(x_0, R) + H_0(\tau^{k-1}R)^{2\delta}\sum_{s=0}^{k-1}(\tau^{2\sigma-2\delta})^s\right) + H_0(\tau^k R)^{2\delta} =$$

$$= \tau^{2(k+1)\sigma}\Phi(x_0, R) + H_0(\tau^k R)^{2\delta}\sum_{s=0}^{k}(\tau^{2\sigma-2\delta})^s,$$

and we have shown that the conclusion of $(5.13)_{k+1}$ holds. ■

The next lemma follows easily from Lemma 5.1, provided that $2\varepsilon_1 \tau^{-n-2\delta} \leq \leq 1$.

LEMMA 5.2. – *Given $M_1 > 0$, there exist positive constants $R_1(M_1)$ and $\varepsilon_1(M_1)$ such that, for every $B_R(x_0) \subset\subset \Omega$,*

$$(5.16) \qquad R \leq R_1, \ |\omega_{x_0, R}| < M_1 \quad \textit{and} \quad \Phi(x_0, R) < \varepsilon_1 \Rightarrow$$

$$\Rightarrow \Phi(x_0, \rho) \leq (\rho/R)^{2\delta} \quad \textit{for all } \rho \textit{ with } 0 < \rho < R.$$

If $x_0 \in \Omega$ with $\sup_{r>0} |\omega_{x_0, r}| < \infty$ and $\liminf_{r \to 0+} \Phi(x_0, r) = 0$, we can find $M_1 > 0$ and $R > 0$ such that the hypotheses of (5.16) are satisfied at x_0. By continuity of the functions $x \to |\omega_{x, R}|$ and $x \to \Phi(x, R)$, they are satisfied in a whole neighbourhood of x_0. Therefore, from the conclusion of (5.16) in combination with Theorem 3.1, we infer that $V(\omega)$ is Hölder continuous with exponent δ in this neighbourhood, and hence that $x_0 \in \mathrm{Reg}[\omega]$. We have thus shown that

$$\left\{x_0 \in \Omega \colon \sup_{r>0} |\omega_{x_0, r}| < \infty \quad \text{and} \quad \liminf_{r \to 0+} \Phi(x_0, r) = 0\right\} \subset \mathrm{Reg}[\omega],$$

while the reverse inclusion is trivial. Finally, $|\mathrm{Sing}[\omega]| = 0$ by Lebesgue's theorem. ■

THE PROOF OF THEOREM 1.3 is easily accomplished by applying the duality principle, Proposition 2.1. Recall that under the hypotheses of Theorem 1.3 the $(n-m)$-form τ, given by

$$(5.17) \qquad * \tau(x) = A(x, \omega(x)) \Leftrightarrow * \omega(x) = a(x, \tau(x)),$$

is a solution of the system

$$\delta(a(x, \tau)) = 0, \qquad d\tau = 0;$$

and that the mapping $a\colon \overline{\Omega} \times \Lambda_{n-m} \to \Lambda_{n-m}$, defined by $a(x, \tau) = * A(x, \cdot)^{-1}(* \tau)$, satisfies the hypotheses of Theorem 1.1 with conjugate exponent $p' > 2$ and with (1.14) replaced by (2.3). In light of Remark 5.1, we conclude from Theorem 1.1 that

$$\tau \in C^{0,\tau}(\mathrm{Reg}[\tau], \Lambda_{n-m}) \quad \text{and} \quad |\mathrm{Sing}[\tau]| = 0.$$

Now, by (1.6), (1.15) and Proposition 2.1, the functions $A(x, \omega)$ and $a(x, \tau)$ are Hölder continuous in x with exponent δ, and Lipschitz continuous in ω and τ respectively. From the relations (5.17) we thus infer that $\mathrm{Reg}[\omega] = \mathrm{Reg}[\tau]$ and $\omega \in C^{0,\delta}(\mathrm{Reg}[\omega], \Lambda_m)$. ■

Appendix.

We here prove quasimonotonicity of the function (1.13). For $\omega \in \mathbb{R}^{2\times 2}$ the reordered matrix $\overline{\omega} = \operatorname{cof}\omega$ satisfies $\langle \overline{\omega}, \omega\rangle = 2\det\omega$, $\langle \overline{\omega}, \xi\rangle = \langle \omega, \overline{\xi}\rangle$ and $|\overline{\omega}| = |\omega|$. We let $\varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^2)$. Bearing in mind that those integrals vanish whose integrands are linear in $\langle \overline{D\varphi}, D\varphi\rangle = 2\det D\varphi$ or in $D\varphi$, and estimating $|D\varphi|^2\langle D\varphi, \omega_0\rangle \geqslant -|D\varphi|^3|\omega_0|$ and $-(1/4)\langle\overline{D\varphi}, D\varphi\rangle^2 \geqslant -(1/4)|D\varphi|^4$, we obtain

$$\int_{\mathbb{R}^2}\langle A(\omega_0 + D\varphi), D\varphi\rangle\, dx =$$

$$= \int_{\mathbb{R}^2}\left(|\omega_0 + D\varphi|^2\langle\omega_0 + D\varphi, D\varphi\rangle + 2\langle\overline{\omega}_0 + \overline{D\varphi}, \omega_0 + D\varphi\rangle\langle\overline{\omega}_0 + \overline{D\varphi}, D\varphi\rangle\right) dx =$$

$$= \int_{\mathbb{R}^2}\big(|D\varphi|^4 + 3|D\varphi|^2\langle D\varphi, \omega_0\rangle + 2\langle D\varphi, \omega_0\rangle^2 + |D\varphi|^2|\omega_0|^2 +$$

$$+2\langle\overline{D\varphi}, D\varphi\rangle^2 + 6\langle\overline{D\varphi}, D\varphi\rangle\langle\overline{D\varphi}, \omega_0\rangle + 4\langle\overline{D\varphi}, \omega_0\rangle^2\big)\, dx \geqslant$$

$$\geqslant \int_{\mathbb{R}^2}\big(|D\varphi|^4 + 2|D\varphi|^2\langle D\varphi, \omega_0\rangle - |D\varphi|^3|\omega_0| + 2\langle D\varphi, \omega_0\rangle^2 + |D\varphi|^2|\omega_0|^2 +$$

$$+(9/4)\langle\overline{D\varphi}, D\varphi\rangle^2 - (1/4)|D\varphi|^4 + 6\langle\overline{D\varphi}, D\varphi\rangle\langle\overline{D\varphi}, \omega_0\rangle + 4\langle\overline{D\varphi}, \omega_0\rangle^2\big)\, dx =$$

$$= \int_{\mathbb{R}^2}\big(2((1/2)|D\varphi|^2 + \langle D\varphi, \omega_0\rangle)^2 + |D\varphi|^2((1/2)|D\varphi| - |\omega_0|)^2 +$$

$$+((3/2)\langle\overline{D\varphi}, D\varphi\rangle + 2\langle\overline{D\varphi}, \omega_0\rangle)^2\big)\, dx \geqslant 0. \quad \blacksquare$$

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