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Critical p-Laplacian Problems in $\boldsymbol{R}^N$ (*).

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Summary. – *The main theorem establishes the existence of a positive decaying solution $u \in D_0^{1,p}(\boldsymbol{R}^N)$ of a quasilinear elliptic problem involving the p-Laplacian operator and the critical Sobolev exponent $pN/(N-p)$, $1 < p < N$. The conclusion depends on the existence of a lowest eigenvalue of a related quasilinear eigenvalue problem. A preliminary result yields a Palais-Smale compactness condition for an associated functional via concentration-compactness methods of P. L. Lions.*

1. – Introduction.

Our objective is to prove the existence of a positive solution $u(x)$ of the quasilinear elliptic problem

$$(1.1)\qquad \begin{cases} -\Delta_p u = \lambda a(x) u^{p-1} + f(x) u^{p^*-1} + g(x) u^q, & x \in \boldsymbol{R}^N \\ u \in D_0^{1,p}(\boldsymbol{R}^N) \cap C_{\text{loc}}^{1,\alpha}(\boldsymbol{R}^N), & \lim\limits_{|x| \to \infty} u(x) = 0 \end{cases}$$

for all λ in some interval $[0, \lambda_0)$. In (1.1) $\Delta_p = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian and $p^* = pN/(N-p)$ denotes the critical Sobolev exponent, $1 < p < N$. As usual, $D_0^{1,p}(\boldsymbol{R}^N)$ denotes the completion of $C_0^\infty(\boldsymbol{R}^N)$ in the norm $\|\nabla u\|_p$, where $\|\cdot\|_p$ is the standard $L^p(\boldsymbol{R}^N)$-norm.

Hypotheses for (1.1).

$p - 1 < q < p^* - 1$ and a, f, g are nontrivial nonnegative bounded functions in $\boldsymbol{R}^N$

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such that

$$a \in L^{N/p}(\boldsymbol{R}^N), \qquad f \in C^0_{\mathrm{loc}}(\boldsymbol{R}^N), \qquad g \in L^Q(\boldsymbol{R}^N) \quad \text{for } Q = pN[pN - (q+1)(N-p)]^{-1}; \tag{1.2}$$

$$f(0) = \sup_{x \in \boldsymbol{R}^N} f(x) \equiv \|f\|_\infty; \tag{1.3}$$

and

$$f(x) = f(0) + o(|x|^\delta), \qquad g(x) \geqslant g_0 > 0 \tag{1.4}$$

in some neighborhood of $x = 0$, where

$$\delta = \begin{cases} \dfrac{N}{Q} & \text{if } N \geqslant \dfrac{p(q+1)}{q-p+2}, \\ \dfrac{N(Q-1)}{Q(p-1)} & \text{if } N < \dfrac{p(q+1)}{q-p+2}. \end{cases}$$

THEOREM 1.1. – *Under these conditions, there exists* $\lambda_0 > 0$ *such that problem* (1.1) *has a positive solution* u_λ *for all* λ *in* $0 \leqslant \lambda < \lambda_0$.

The proof will be given in § 4 on the basis of the compactness result in Theorem 3.1, where λ_0 is defined to be the lowest eigenvalue of the problem

$$\begin{cases} -\Delta_p u = \lambda a(x) u^{p-1}, & x \in \boldsymbol{R}^N, \\ u \in D_0^{1,p}(\boldsymbol{R}^N) \cap C^{1,\alpha}_{\mathrm{loc}}(\boldsymbol{R}^N), & \lim\limits_{|x| \to \infty} u(x) = 0. \end{cases} \tag{1.5}$$

The existence of λ_0 and an associated weak positive solution of (1.5) will be established in Theorem 2.1 via the constrained variational problem

$$\lambda_0 = \inf\{\|\nabla u\|_p^p : \|u\|_{p,a} = 1, \ u \in D_0^{1,p}(\boldsymbol{R}^N)\}, \tag{1.6}$$

where

$$\|u\|_{p,a}^p = \int_{\boldsymbol{R}^N} |u(x)|^p a(x)\, dx.$$

The necessity of the condition $\lambda < \lambda_0$ in Theorem 1.1 can be shown from an adaptation of Egnell's proof [7, Theorem 8], given for a case of (1.1) in a bounded domain Ω. If $g(x) \equiv 0$, positive solutions of (1.1) do not exist in general for $\lambda \leqslant 0$ by Pohožaev-type identities, e.g., no positive solution exists for $\lambda = 0$ if $g(x) \equiv 0$, $f(x)$ is nonconstant, and $x \cdot (\nabla f)(x)$ is either nonnegative or nonpositive throughout $\boldsymbol{R}^N$. However, a positive solution can exist for $\lambda \leqslant 0$ if $g(x) \equiv 0$ and $f(x)$ is a constant, as demonstrated by BENCI and CERAMI [3] in the case $p = 2$, $\lambda = -1$, $f(x) \equiv 1$, and $\|a\|_{N/2}$ sufficiently small. It is well known that the equation $-\Delta_p u = u^{p^*-1}$, with $\lambda = 0$ and $f(x) \equiv 1$, has solutions

$$u_\varepsilon(x) = K\left(\frac{\varepsilon^{1/(p-1)}}{\varepsilon^{p/(p-1)} + |x|^{p/(p-1)}}\right)^{(N-p)/p}, \qquad x \in \boldsymbol{R}^N \tag{1.7}$$

for any $\varepsilon > 0$ and a suitable normalization constant $K > 0$, as well as all translations of $u_\varepsilon(x)$. This fact is crucial in the theory of critical p-Laplacian problems.

Problems of type (1.1), usually with $g(x) \equiv 0$, in *bounded domains* have been studied in depth by AZORERO and ALONSO [1], BENCI and CERAMI [3], BREZIS and NIRENBERG [5], EGNELL [7], GUEDDA and VERON [8], and KNAAP and PELETIER [10]. Surprisingly the requirement $N \geqslant p^2$ for these results is not needed here. As far as we are aware, only NI and SERRIN [14], NOUSSAIR *et al.* [15], and ZHU and YANG [20] considered p-Laplacian equations in unbounded domains; however, the nonlinear structure, objectives, and/or methods differ from those presented here.

§ 2 contains notation, definitions, and an existence theorem for (1.5). A Palais-Smale compactness condition is proved in § 3, as required for the proof of the main Theorem 1.1 in § 4.

2. - Preliminaries.

Let $B_\rho(x) = \{y \in \boldsymbol{R}^N\colon |y-x| < \rho\}$, $B_\rho = B_\rho(0)$, and $B'_\rho = \boldsymbol{R}^N \setminus B_\rho$ for $\rho > 0$, $x \in \boldsymbol{R}^N$. The standard norm in the weighted Lebesgue space $L^\sigma(\Omega, a)$ will be denoted by

$$\|u\|_{\sigma, a, \Omega} = \left[\int_\Omega |u(x)|^\sigma a(x)\, dx\right]^{1/\sigma}, \qquad \sigma \geqslant 1,\ \Omega \subseteq \boldsymbol{R}^N,$$

and we set $\|u\|_{\sigma, a} = \|u\|_{\sigma, a, \boldsymbol{R}^N}$, $\|u\|_\sigma = \|u\|_{\sigma, 1}$. The space $E = D_0^{1,p}(\boldsymbol{R}^N)$ is the completion of $C_0^\infty(\boldsymbol{R}^N)$ in the norm $\|\nabla u\|_p$. The norm in E is sometimes denoted by $\|u\|_E$.

THEOREM 2.1. - *The infimum* $\lambda_0 > 0$ *in* (1.6) *is attained by a positive weak solution* u_0 *of* (1.5).

PROOF. - Clearly $\lambda_0 > 0$ since

$$\|u\|_{p, a}^p \leqslant C\|a\|_{N/p}\|\nabla u\|_p^p \tag{2.1}$$

for all $u \in E$ by Hölder's inequality and the continuity of the embedding $E \hookrightarrow L^{p^*}(\boldsymbol{R}^N)$, where C is the embedding constant. The boundedness of a minimizing sequence $\{u_n\}$ for (1.6) implies that $\{u_n\}$ has a weakly convergent subsequence (also denoted by $\{u_n\}$) with weak limit $u_0 \in E$. The procedure for (2.1) yields the estimate

$$\|u_n - u_0\|_{p, a}^p \leqslant \|a\|_\infty \|u_n - u_0\|_{p, 1, B_k}^p + C\|a\|_{N/p, 1, B'_k}(\|\nabla u_n\|_p^p + \|\nabla u_0\|_p^p). \tag{2.2}$$

Since $a \in L^{N/p}(\boldsymbol{R}^N)$ by (1.2), $\|a\|_{N/p,1,B_k'} \to 0$ as $k \to \infty$, and hence the compactness of the embedding $W^{1,p}(B_k) \hookrightarrow L^p(B_k)$ implies that $\{u_n\}$ has a subsequence, denoted the same way, such that $\|u_n\|_{p,a} \to \|u_0\|_{p,a}$, as $n \to \infty$. Therefore $\|u_0\|_{p,a} = 1$, $\|\nabla u_0\|_p^p = \lambda_0$, i.e., u_0 attains the infimum in (1.6), and consequently u_0 is a weak solution of (1.5) by the Euler-Lagrange principle. Since $|u_0|$ also attains the infimum in (1.6), it can be assumed that $u_0 \geq 0$. The positivity of u_0 then follows from a Harnack-type inequality of SERRIN [16, Theorem 5]; see also [7, Proposition A3].

REMARK 2.2. - The method in [6] shows that λ_0 is a principal eigenvalue of (1.5), even if $a(x)$ changes sign in $\boldsymbol{R}^N$.

Solutions of (1.1) will be obtained as critical points of the functional J defined by

$$J(u) = \int_{\boldsymbol{R}^N} \left[\frac{1}{p} |\nabla u|^p - \frac{\lambda a}{p} u_+^p - \frac{f}{p^*} u_+^{p^*} - \frac{g}{q+1} u_+^{q+1} \right] dx, \qquad u \in E, \tag{2.3}$$

where $u_+(x) = \max\{u(x), 0\}$. On account of the continuity of the embedding $E \hookrightarrow L^{p^*}(\boldsymbol{R}^N)$ and estimates of type (2.2), standard procedure from the Calculus of Variations shows that $J(u)$ is well-defined on E and has a continuous Fréchet derivative given by

$$J'(u)v = \int_{\boldsymbol{R}^N} [\,|\nabla u|^{p-2} \nabla u \cdot \nabla v - \lambda a u_+^{p-1} v - f u_+^{p^*-1} v - g u_+^q v]\, dx, \qquad u, v \in E. \tag{2.4}$$

Furthermore, any critical point u of the variational problem in § 4 for $J(u)$ satisfies $J'(u) = 0$ in E^*, meaning that u is a weak solution of the equation

$$-\nabla_p u = \lambda a u_+^{p-1} + f u_+^{p^*-1} + g u_+^q, \qquad u \in E. \tag{2.5}$$

We use the notation $S = C_p^{-p}$, where C_p is the best (= minimum possible) constant for the Sobolev inequality

$$\|u\|_{p^*} \leq C_p \|\nabla u\|_p, \qquad u \in E.$$

It is known [12] that C_p is attained by the function u_ε in (1.7), i.e.,

$$S = \|\nabla u_\varepsilon\|_p^p / \|u_\varepsilon\|_{p^*}^p = \inf_{0 \neq u \in E} [\|\nabla u\|_p^p / \|u\|_{p^*}^p]. \tag{2.6}$$

Since u_ε solves $-\Delta_p u_\varepsilon = u_\varepsilon^{p^*-1}$, as already mentioned, integration by parts yields $\|\nabla u_\varepsilon\|_p^p = \|u_\varepsilon\|_{p^*}^{p^*}$ implying in view of (2.6) that

$$S = \left[\int_{\boldsymbol{R}^N} u_\varepsilon^{p^*}(x)\, dx \right]^{p/N}. \tag{2.7}$$

3. – The palais-smale compactness condition.

The functional J on E is said to satisfy the $(PS)_c$-condition if and only if every sequence $\{u_n\}$ in E for which $J(u_n) \to c$ and $J'(u_n) \to 0$ in E^*, as $n \to \infty$, has a convergent subsequence in the E-norm.

THEOREM 3.1. – *Let λ_0, S be the numbers in* (1.6), (2.6), *respectively. If $0 \leq \lambda < \lambda_0$, then $J = J_\lambda$ satisfies the* $(PS)_c$*-condition for every c in the interval*

$$0 < c < N^{-1} S^{N/p} \|f\|_\infty^{(p-N)/p} . \tag{3.1}$$

PROOF. – Let $\{u_n\}$ be a sequence in E such that $J(u_n) \to c$ and $J'(u_n) \to 0$ in E^*. By (2.3) and (2.4) this means that

$$\int_{R^N} \left[\frac{1}{p} |\nabla u_n|^p - \frac{\lambda a}{p} u_{n+}^p - \frac{f}{p^*} u_{n+}^{p^*} - \frac{g}{q+1} u_{n+}^{q+1} \right] dx = c + o(1) \tag{3.2}$$

and

$$\int_{R^N} [\, |\nabla u_n|^p - \lambda a u_{n+}^p - f u_{n+}^{p^*} - g u_{n+}^{q+1}] \, dx = o(1) \|u_n\|_E , \tag{3.3}$$

as $n \to \infty$, implying

$$\left(1 - \frac{p}{p^*}\right) \int_{R^N} f u_{n+}^{p^*} dx + \left(1 - \frac{p}{q+1}\right) \int_{R^N} g u_{n+}^{q+1} dx = cp + o(1) + o(\beta_n),$$

where $\beta_n = \|u_n\|_E$. Since $p < q + 1 < p^*$ it follows that

$$\int_{R^N} f u_{n+}^{p^*} dx = 0(1) + o(\beta_n) \tag{3.4}$$

and

$$\int_{R^N} g u_{n+}^{q+1} dx = 0(1) + o(\beta_n). \tag{3.5}$$

Combining (3.3)-(3.5) we obtain

$$\beta_n^p - \lambda \|u_{n+}\|_{p,a}^p = 0(1) + o(\beta_n).$$

Then the definition of λ_0 in (1.6) yields

$$0 < \left(1 - \frac{\lambda}{\lambda_0}\right) \beta_n^p \leq 0(1) + o(\beta_n),$$

implying that $\{\beta_n\}$ is a bounded sequence. It follows that $\{u_n\}$ has a subsequence, also denoted by $\{u_n\}$, which converges weakly in E to a weak limit $u \in E$. Also the sequence of norms $\{\|\nabla u_n\|_p\}$ has a convergent subsequence (denoted the same way) whose limit must be positive as a simple consequence of (3.2):

$$L \equiv \lim_{n\to\infty} \|\nabla u_n\|_p^p > 0 . \tag{3.6}$$

The weak lower semicontinuity of the functionals, as described in § 2, implies that

$$\lim_{n\to\infty} \int_{R^N} (\lambda a u_{n+}^p + g u_{n+}^{q+1})\, dx = \int_{R^N} (\lambda a u_+^p + g u_+^{q+1})\, dx . \tag{3.7}$$

We now verify that

$$H \equiv \int_{R^N} (\lambda a u_+^p + g u_+^{q+1})\, dx > 0 . \tag{3.8}$$

If $H = 0$, (3.3) would imply

$$\int_{R^N} |\nabla u_n|^p\, dx = \int_{R^N} f u_{n+}^{p^*}\, dx + o(1) + o(1)\|u_n\|_E \tag{3.9}$$

as $n \to \infty$. In view of (3.6), it follows from (3.9) in the limit $n \to \infty$ that

$$L = \lim_{n\to\infty} \int_{R^N} f u_{n+}^{p^*}\, dx > 0 , \tag{3.10}$$

and consequently (3.2) yields

$$c = \frac{L}{p} - \frac{L}{p^*} = \frac{L}{N} ,$$

i.e., by (3.1),

$$L = Nc < S^{N/p} \|f\|_\infty^{(p-N)/p} . \tag{3.11}$$

However, by the definition of S in (2.6),

$$L = \lim_{n\to\infty} \|\nabla u_n\|_p^p \geq S \lim_{n\to\infty} \|u_n\|_{p^*}^p \geq S \|f\|_\infty^{(p-N)/N} \lim_{n\to\infty} \left[\int_{R^N} f u_{n+}^{p^*}\, dx \right]^{(N-p)/N} =$$

$$= S \|f\|_\infty^{(p-N)/N} L^{(N-p)/N} ,$$

equivalent to $L \geq S^{N/p} \|f\|_\infty^{(p-N)/p}$, contradicting (3.11). This completes the proof of (3.8). On account of (3.8), it cannot be that

$$\lim_{n\to\infty} \sup_{y\in R^N} \int_{B_R(y)} (\lambda a u_{n+}^p + g u_{n+}^{q+1})\, dx = 0 \tag{3.12}$$

for all $R \in (0, \infty)$, i.e., «vanishing» of the sequence $\{\lambda a u_{n+}^p + g u_{n+}^{q+1}\}$ cannot occur [11, p. 115]. We next show that the sequence $\{z_n\}$ defined by

(3.13)
$$z_n = |\nabla u_n|^p + |u_n|^{p^*} + \lambda a u_{n+}^p + g u_{n+}^{q+1}$$

is «tight», as defined by LIONS [11, p. 115]. Note first from (3.6) that

(3.14)
$$L_0 \equiv \lim_{n\to\infty} \int_{R^N} z_n(x)\,dx > 0\,,$$

passing to a subsequence if necessary. Since «vanishing» of $\{z_n\}$ cannot occur, it follows from the proof of LIONS [11, pp. 116-117] that $\{z_n\}$ is tight unless, for arbitrary $\varepsilon > 0$, there exist $R > 0$, $\Lambda \in (0, L_0)$ and sequences $R_n \uparrow +\infty$, $y_n \in \boldsymbol{R}^N$ such that

(3.15)
$$\begin{cases} \left| \displaystyle\int_{B_R(y_n)} z_n(x)\,dx - \Lambda \right| < \varepsilon\,, \\ \left| \displaystyle\int_{B_{R_n}(y_n)} z_n(x)\,dx - L_0 + \Lambda \right| < \varepsilon\,, \\ \left| \displaystyle\int_{B_{R_n}(y_n)\setminus B_R(y_n)} z_n(x)\,dx \right| < \varepsilon\,, \end{cases}$$

for all $n \geq n_0(R)$. We introduce functions $\phi^i \in C_0^\infty(\boldsymbol{R}^N)$ such that $0 \leq \phi^i(x) \leq 1$, $i = 1, 2$, and

$$\phi^1(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| \geq 2, \end{cases} \qquad \phi^2(x) = \begin{cases} 0 & \text{if } |x| \leq 1, \\ 1 & \text{if } |x| \geq 2, \end{cases}$$

and we define $u_n^i = \phi_n^i u_n$, $i = 1, 2$, $n = 1, 2, \ldots$, where $\phi_n^1(x) = \phi^1((x - y_n)/R)$, $\phi_n^2(x) = \phi^2((x - y_n)/R_n)$. Then $\operatorname{supp} u_n^1$ and $\operatorname{supp} u_n^2$ are disjoint sets for every $n = 1, 2, \ldots$. Use of (3.15) in (3.2) and (2.4) (taking $v = u_n^i$), respectively, gives

(3.16)
$$\sum_{i=1}^{2} \int_{R^N} \left[\frac{1}{p} |\nabla u_n^i|^p - \frac{\lambda a}{p} (u_{n+}^i)^p - \frac{f}{p^*} (u_{n+}^i)^{p^*} - \frac{g}{q+1} (u_{n+}^i)^{q+1} \right] dx = c + o_n(1) + o_\varepsilon(1)$$

and

(3.17)
$$\int_{R^N} [\,|\nabla u_n^i|^p - \lambda a(u_{n+}^i)^p - f(u_{n+}^i)^{p^*} - g(u_{n+}^i)^{q+1}]\,dx = o_n(1)\|u_n\|_E + o_\varepsilon(1)\,,$$

where $o_\varepsilon(1) \to 0$ as $\varepsilon \to 0+$.

As in (3.6), passing to subsequences if necessary, there exist nonnegative limits α_i, β_i, $i = 1, 2$, defined by

$$\alpha_i = \lim_{n \to \infty} \int_{R^N} [\lambda a (u_{n+}^i)^p + g(u_{n+}^i)^{q+1}]\, dx, \tag{3.18}$$

$$\beta_i = \lim_{n \to \infty} \int_{R^N} f(u_{n+}^i)^{p^*}\, dx, \qquad i = 1, 2. \tag{3.19}$$

It follows from (3.17) that

$$\int_{R^N} |\nabla u_n^i|^p\, dx = \alpha_i + \beta_i + o_n(1) \|u_n\|_E + o_\varepsilon(1). \tag{3.20}$$

Since $q + 1 > p$ by hypothesis, substitution of (3.18)-(3.20) into (3.16) leads to

$$c \geq \sum_{i=1}^{2} \left(\frac{\alpha_i + \beta_i}{p} - \frac{\beta_i}{p^*} - \frac{\alpha_i}{p} \right) + o_\varepsilon(1),$$

equivalent to

$$c \geq \frac{\beta_1 + \beta_2}{N} + o_\varepsilon(1). \tag{3.21}$$

It can be verified easily from (3.15), similarly to (3.16) and (3.17), that either

$$\lim_{n \to \infty} \int_{R^N} [\lambda a (u_{n+}^2)^p + g(u_{n+}^2)^{q+1}]\, dx = 0 \tag{3.22}$$

or

$$\lim_{n \to \infty} \int_{R^N} [\lambda a (u_{n+}^1)^p + g(u_{n+}^1)^{q+1}]\, dx = 0 \tag{3.23}$$

according as $\{y_n\}$ is bounded or $|y_n| \to \infty$, respectively. In the case (3.22), let $n \to \infty$ in (3.17) to obtain, in view of (2.6) and (3.19),

$$\beta_2 + o_\varepsilon(1) = \lim_{n \to \infty} \|\nabla u_n^2\|_p^p \geq S \lim_{n \to \infty} \|u_{n+}^2\|_{p^*}^p \geq S \|f\|_\infty^{(p-N)/N} \beta_2^{(N-p)/N}.$$

By (3.21) this implies that

$$Nc \geq \beta_2 + o_\varepsilon(1) \geq S^{N/p} \|f\|_\infty^{(p-N)/p} + o_\varepsilon(1),$$

contrary to hypothesis (3.1) since ε is arbitrary. Virtually the same procedure also leads to a contradiction in the case (3.23). Accordingly, (3.15) is impossible and hence the sequence $\{z_n\}$ in (3.13) is tight, i.e., there exists a sequence $\{y_n\}$ in $\boldsymbol{R}^N$ such that, for arbitrary $\varepsilon > 0$ there exists $R \in (0, \infty)$ with

$$\int_{B_R^c(y_n)} z_n(x)\,dx < \varepsilon . \tag{3.24}$$

It must be that $\{y_n\}$ is bounded, for otherwise (3.24) would imply, in the limit $n \to \infty$,

$$\int_{R^N} (\lambda a u_+^p + g u_+^{q+1})\,dx \leqslant C\varepsilon^b$$

for some positive constants b and C, independent of ε, contrary to (3.8). Thus we can replace y_n by 0 in (3.24) to obtain

$$\int_{B_R^c} |u_n(x)|^{p^*} dx \leqslant \int_{B_R^c} z_n(x)\,dx < \varepsilon ,$$

showing that $\{|u_{n+}|^{p^*}\}$ is tight.

It follows from the foregoing that there exist bounded nonnegative measures μ, ν on $\boldsymbol{R}^N$ such that $|\nabla u_n|^p \to \mu$ weakly and $|u_n|^{p^*} \to \nu$ tightly as $n \to \infty$ [12, p. 158], and likewise $|\nabla u_{n+}|^p \to \mu_+$ weakly, $|u_{n+}|^{p^*} \to \nu_+$ tightly. Lemma I.1 of LIONS [12] states that sequences $\{x_j\} \subset \boldsymbol{R}^N$, $\{\mu_j\}$, $\{\nu_j\} \subset (0, \infty)$ exist such that $\mu_j \geqslant S\nu_j^{p/p^*}$ and

$$\begin{cases} \mu \geqslant |\nabla u|^p + \sum_{j \in I} \mu_j \delta_{x_j} , \\ \nu = |u|^{p^*} + \sum_{j \in I} \nu_j \delta_{x_j} , \\ \nu_+ = |u_+|^{p^*} + \sum_{j \in I_+} \nu_{j+} \delta_{x_j} , \end{cases} \tag{3.25}$$

where δ_{x_j} denotes a Dirac measure, $j \in I$, $I_+ \subseteq I$. In the limit $n \to \infty$, (3.2) and (3.3) then yield, respectively,

$$\frac{1}{p} \int_{R^N} d\mu = c + \int_{R^N} \left[\frac{\lambda a}{p} u_+^p + \frac{f}{p^*} u_+^{p^*} + \frac{1}{p^*} \sum_{j \in I_+} \nu_{j+} f(x_j) + \frac{g}{q+1} u_+^{q+1} \right] dx , \tag{3.26}$$

$$\int_{R^N} d\mu = \int_{R^N} \left[\lambda a u_+^p + f u_+^{p^*} + \sum_{j \in I_+} \nu_{j+} f(x_j) + g u_+^{q+1} \right] dx . \tag{3.27}$$

Since $1 - p/p^* = p/N$ and $q + 1 > p$, multiplication of (3.26) by p and subtraction of the result from (3.27) gives

$$c \geqslant \frac{1}{N} \int_{R^N} \left[f u_+^{p^*} + \sum_{j \in I_+} \nu_{j+} f(x_j) \right] dx . \tag{3.28}$$

For nonnegative $\phi \in C_0^\infty(\boldsymbol{R}^N)$, it follows as in GUEDDA and VERON [8, p. 898] that

$$\int_{\boldsymbol{R}^N} \phi\, d\mu \leqslant \int_{\boldsymbol{R}^N} \phi V \cdot \nabla u\, dx + \int_{\boldsymbol{R}^N} \phi f\, d\nu_+ ,$$

where $V \in L^{p'}(\boldsymbol{R}^N)^N$ is the weak limit of $|\nabla u_n|^{p-2}\nabla u_n$. If ϕ is concentrated on the sequence $\{x_j\}$, $j \in I_+$, this reduces to $\mu_j \leqslant \nu_{j+} f(x_j)$. Since also $S\nu_j^{p/p^*} \leqslant \mu_j$ from (3.25), it follows that

$$\nu_{j+} \geqslant S^{N/p}[f(x_j)]^{-N/p} .$$

If I_+ is nonempty, (3.28) would imply that

$$c \geqslant \frac{1}{N} S^{N/p} \sum_{j \in I_+} [f(x_j)]^{(p-N)/p} \geqslant \frac{1}{N} S^{N/p} \|f\|_\infty^{(p-N)/p} ,$$

contrary to (3.1). Therefore I_+ is empty, and (3.25) shows that $\|u_{n+}\|_{p^*} \to \|u_+\|_{p^*}$ as $n \to \infty$. By a lemma of BREZIS and LIEB [4], $u_{n+} \to u_+$ in the norm $\|\cdot\|_{p^*}$.

In conjunction with (2.4), we use the notation

$$J_0'(u)v = \int_{\boldsymbol{R}^N} [\lambda a u_+^{p-1} v + f u_+^{p^*-1} v + g u_+^q v]\, dx$$

for $u, v \in E$. An inequality of THELIN [17] (see also KICHENASSAMY and VERON [9]) yields

$$(3.29) \qquad |\nabla u_m - \nabla u_n|^p \leqslant (|\nabla u_m|^{p-2}\nabla u_m - |\nabla u_n|^{p-2}\nabla u_n)\cdot(\nabla u_m - \nabla u_n), \qquad p \geqslant 2,\ m, n = 1, 2, \ldots$$

and

$$(3.30) \qquad |\nabla u_m - \nabla u_n|^p \leqslant [(|\nabla u_m|^{p-2}\nabla u_m - |\nabla u_n|^{p-2}\nabla u_n)\cdot(\nabla u_m - \nabla u_n)]^{p/2} \cdot [|\nabla u_m|^p + |\nabla u_n|^p]^{(2-p)/2}, \qquad 1 < p \leqslant 2,\ m, n = 1, 2, \ldots .$$

In the case $p \geqslant 2$, it follows from (3.29) that

$$\|u_m - u_n\|_E \leqslant |J'(u_m)(u_m - u_n)| + |J'(u_n)(u_m - u_n)| + |J_0'(u_m)(u_m - u_n) - J_0'(u_n)(u_m - u_n)| .$$

This together with the convergence of $\{u_{n+}\}$ in $L^{p^*}(\boldsymbol{R}^N)$ implies the convergence of $\{u_n\}$ in the E-norm. The argument via (3.30) in the case $1 < p \leqslant 2$ is virtually the same, completing the proof of Theorem 3.1.

4. – Proof of Theorem 1.1.

In order to apply the mountain pass theorem [1] to the functional J in (2.3), we first show there exists a function $u_\varepsilon \in E$ of type (1.7) such that $J(t_0 u_\varepsilon) < 0$ for sufficiently large $t_0 > 0$ and sufficiently small $\varepsilon > 0$, and furthermore $\sup_{t \geq 0} J(tu_\varepsilon) = c$ is in the interval (3.1).

LEMMA 4.1. – *Under the stated conditions for* (1.1), *there exist positive numbers* ε *and* t_0 *such that* $J(t_0 u_\varepsilon) < 0$ *and*

$$0 < \sup_{t \geq 0} J(tu_\varepsilon) < \frac{1}{N} S^{N/p} \|f\|_\infty^{(p-N)/p} . \tag{4.1}$$

PROOF. – For $0 < \lambda < \lambda_0$, Theorem 2.1 shows that

$$\|\nabla u\|_p^p - \lambda \|u\|_{p,a}^p \geq c \|\nabla u\|_p^p , \qquad u \in E \tag{4.2}$$

for some $c > 0$, independent of u. On account of (1.2), (2.3), and (4.1), an estimate of type (2.1) for $\|u\|_{q+1,g}^{q+1}$ and the continuity of the embedding $E \hookrightarrow L^{p^*}(\mathbf{R}^N)$ imply the existence of a constant C, independent of u, such that

$$J(u) \geq \frac{c}{p} \|u\|_E^p - C(\|u\|_E^{p^*} + \|u\|_E^{q+1}) , \qquad u \in E .$$

Since $p - 1 < q < p^* - 1$ by assumption, a sufficiently small positive number ρ can be found for which

$$J(u) \geq \frac{c\rho^p}{2p} \equiv c_0 \qquad \text{for all } u \text{ with } \|u\|_E = \rho . \tag{4.3}$$

With u_ε as in (1.7), $\varepsilon > 0$, it is clear from (2.3) that $\lim_{t \to \infty} J(tu_\varepsilon) = -\infty$ for all $\varepsilon > 0$, and hence $\sup_{t \geq 0} J(tu_\varepsilon)$ is attained at some number t_ε ($t_\varepsilon > 0$ by an estimate of type (4.3)). It is an easy consequence of $J'(t_\varepsilon u_\varepsilon) = 0$ and (2.4) (with $u = t_\varepsilon u_\varepsilon$, $v = u_\varepsilon$) that

$$t_\varepsilon \leq \left[\int_{\mathbf{R}^N} |\nabla u_\varepsilon|^p \, dx \Big/ \int_{\mathbf{R}^N} f(x) u_\varepsilon^{p^*} dx \right]^{(N-p)/p^2} . \tag{4.4}$$

By the change of variable $x = \varepsilon y$ it is a consequence of (1.7) that

$$\int_{\mathbf{R}^N} |\nabla u_\varepsilon(x)|^p \, dx = \int_{\mathbf{R}^N} |\nabla u_1(y)|^p \, dy$$

and

$$\int_{\mathbf{R}^N} f(x) u_\varepsilon^{p^*}(x) \, dx = \int_{\mathbf{R}^N} f(\varepsilon y) u_1^{p^*}(y) \, dy .$$

The continuity of f at 0 together with (1.3) and (4.4) show that there exists $R > 0$ such that

$$t_\varepsilon \leqslant \left[2 \int_{R^N} |\nabla u_1|^p \, dy \Big/ f(0) \int_{B_R} u_1^{p^*} \, dy \right]^{(N-p)/p^2}. \tag{4.5}$$

On account of the definition of t_ε, it follows from (2.3) that

$$\sup_{t \geqslant 0} J(tu_\varepsilon) = J(t_\varepsilon u_\varepsilon) = F_1(\varepsilon) - F_2(\varepsilon) + F_3(\varepsilon), \tag{4.6}$$

where

$$F_1(\varepsilon) = \frac{1}{p} t_\varepsilon^p \int_{R^N} |\nabla u_1|^p \, dx - \frac{1}{p^*} t_\varepsilon^{p^*} f(0) \int_{R^N} u_1^{p^*} \, dx,$$

$$F_2(\varepsilon) = \int_{R^N} \left[\frac{\lambda}{p} t_\varepsilon^p a u_\varepsilon^p + \frac{1}{q+1} t_\varepsilon^{q+1} g u_\varepsilon^{q+1} \right] dx,$$

$$F_3(\varepsilon) = \frac{1}{p^*} t_\varepsilon^{p^*} \int_{R^N} [f(0) - f(\varepsilon y)] u_1^{p^*}(y) \, dy.$$

For positive numbers A and B, the maximum of $\phi(t) = Ap^{-1}t^p - B(p^*)^{-1}t^{p^*}$ for $t \geqslant 0$ is attained at $t = (A/B)^{(N-p)/p^2}$ from which (2.6) gives

$$F_1(\varepsilon) \leqslant \left(\frac{1}{p} - \frac{1}{p^*} \right) [f(0)]^{(p-N)/p} \|\nabla u_1\|_p^N \|u_1\|_{p^*}^{-N} = \frac{1}{N} S^{N/p} \|f\|_\infty^{(p-N)/p}. \tag{4.7}$$

It can be assumed without loss of generality that there exists a positive constant $\hat{t}$, independent of ε, such that $t_\varepsilon \geqslant \hat{t}$ for all ε in an interval $0 < \varepsilon \leqslant \varepsilon_0$, for otherwise there would be nothing to prove. In fact, if there exists a sequence $\{\varepsilon_n\}$ such that $t_{\varepsilon_n} \downarrow 0$, then by $\|\nabla u_\varepsilon\|_p^p = S^{N/p}$ for all $\varepsilon > 0$ from (2.6) and (2.7), it would follow that

$$\sup_{t \geqslant 0} J(tu_\varepsilon) \leqslant \frac{1}{p} t_\varepsilon^p \|\nabla u_\varepsilon\|_p^p < \frac{1}{N} S^{N/p} \|f\|_\infty^{(p-N)/p}$$

by a choice of $\varepsilon = \varepsilon_n$ for which

$$t_\varepsilon^p < \frac{p}{N} \|f\|_\infty^{(p-N)/p}.$$

Calculations using (1.7) show that there exists a positive constant C, independent of

ε, such that

$$(4.8)\qquad \begin{cases} F_2(\varepsilon) \geqslant C\varepsilon^{\delta} & \text{if } N \neq \dfrac{p(q+1)}{q-p+2}\,, \\ F_2(\varepsilon) \geqslant C\varepsilon^{\delta}\log\dfrac{1}{\varepsilon} & \text{if } N = \dfrac{p(q+1)}{q-p+2}\,, \\ F_3(\varepsilon) \leqslant \dfrac{C}{2}\varepsilon^{\delta} & \text{if } N \neq \dfrac{p(q+1)}{q-p+2}\,, \\ F_3(\varepsilon) \leqslant \dfrac{C}{2}\varepsilon^{\delta}\log\dfrac{1}{\varepsilon} & \text{if } N = \dfrac{p(q+1)}{q-p+2}\,, \end{cases}$$

in some interval $0 < \varepsilon \leqslant \varepsilon_0 < 1$ where δ is as in (1.4).

To verify the first two inequalities (4.8), we use the abbreviations

$$\gamma = \frac{(N-p)(q+1)}{p}\,, \qquad \zeta = N - 1 - \frac{p\gamma}{p-1}\,.$$

The definitions of Q and δ in (1.2) and (1.4) show that $N(Q-1)/Q = \gamma$ and

$$\begin{cases} \delta = N - \gamma & \text{if } N \geqslant \dfrac{p(q+1)}{q-p+2}\,, \\ \delta = N - \gamma - \zeta - 1 & \text{if } N < \dfrac{p(q+1)}{q-p+2}\,, \end{cases}$$

equivalent to

$$\delta = \begin{cases} N - \gamma & \text{if } \zeta \leqslant -1\,, \\ N - \gamma - \zeta - 1 & \text{if } \zeta > -1\,. \end{cases}$$

By assumption (1.4), $g(x) \geqslant g_0 > 0$ in some ball $B_\rho(0)$, $\rho > 0$. The definition (1.7) of $u_\varepsilon(x)$ shows that there exists a constant $K > 0$, independent of ε, such that

$$F_2(\varepsilon) \geqslant K \int_0^{\rho} \left(\frac{\varepsilon^{1/(p-1)}}{\varepsilon^{p/(p-1)} + r^{p/(p-1)}} \right)^{\gamma} r^{N-1}\, dr\,,$$

where $r = |x|$. For $\varepsilon > 0$ small enough that $\rho/\varepsilon > 1$, and $s = r/\varepsilon$, this implies

$$F_2(\varepsilon) \geqslant K 2^{-\gamma} \varepsilon^{N-\gamma} \int_1^{\rho/\varepsilon} s^{\zeta}\, ds\,.$$

Hence there exist positive constants K_1, K_2, K_3, independent of ε, such that

$$F_2(\varepsilon) \geq \begin{cases} K_1 \varepsilon^{N-\gamma} & \text{if } \zeta < -1, \\ K_2 \varepsilon^{N-\gamma} \log \frac{1}{\varepsilon} & \text{if } \zeta = -1, \\ K_3 \varepsilon^{N-\gamma-\zeta-1} & \text{if } \zeta > -1, \end{cases}$$

in some interval $0 < \varepsilon \leq \varepsilon_1 < 1$, proving the first two inequalities (4.8). The other inequalities (4.8) are established similarly. The conclusion of Lemma 4.1 then follows from (4.6)-(4.8).

We can now prove the following weak version of Theorem 1.1.

THEOREM 4.2. – *The differential equation in* (1.1) *has a nontrivial nonnegative weak solution* $u \in E$.

PROOF. – First note that ρ in (4.3) can be selected small enough that $\rho < \|t_0 u_\varepsilon\|_E$ as well as $J(\phi) \geq c_0 > 0$ for all $\phi \in E$ with $\|\phi\|_E = \rho$, where $J(t_0 u_\varepsilon) < 0$ and ε, t_0 are as in Lemma 4.1. We define

$$c = \inf_{\psi \in \Gamma} \max_{0 \leq t \leq 1} J(\psi(t)),$$

where Γ denotes the class of all continuous paths ψ in E joining $\boldsymbol{O}$ to $t_0 u_\varepsilon$. Lemma 4.1 implies that

$$0 < c < N^{-1} S^{N/p} \|f\|_\infty^{(p-N)/p}$$

and hence J satisfies the $(\mathrm{PS})_c$-condition by Theorem 3.1. Consequently the mountain pass theorem [1] can be applied to conclude that J has a critical point u with corresponding critical value c. As mentioned in § 2, u is a weak solution of equation (2.5), i.e., $J'(u) = 0$ in E^*. The choice $v = u_-$ in (2.4) shows that $u \geq 0$ a.e. in $\boldsymbol{R}^N$. Furthermore, u is nontrivial since $J(u) = c > 0$, completing the proof of Theorem 4.2.

To obtain the strict positivity, regularity, and asymptotic decay of this weak solution, we require the next lemma.

LEMMA 4.3. – *Let* u *be the weak solution in Theorem* 4.2. *Then* $u \in L^t(\boldsymbol{R}^N)$ *for all* $t \geq p^*$.

PROOF. – In view of (2.4), the equation $J'(u) = 0$ in E^* can be rewritten in the form

$$\int_{\boldsymbol{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx = \int_{\boldsymbol{R}^N} w u^{p-1} v \, dx, \tag{4.9}$$

for all $v \in E$, where

$$w = \lambda a + f u^{p^*-p} + g u^{q+1-p}.$$

Assumption (1.2) implies that $w \in L^{N/p}(\boldsymbol{R}^N)$; in fact,

$$\int_{\boldsymbol{R}^N} (fu^{p^*-p})^{N/p}\, dx \le \|f\|_\infty^{N/p} \|u\|_{p^*}^{p^*}$$

and by Hölder's inequality

$$\int_{\boldsymbol{R}^N} (gu^{q+1-p})^{N/p}\, dx \le \|g\|_Q^{N/p} \|u\|_{p^*}^{N(q+1-p)/p} .$$

Following the procedure of GUEDDA and VERON [8, p. 882] we introduce a test function $v = \phi_k(u)$ in (4.9) defined for $k > 0$, $t \ge p^*$ by

$$\phi_k(u) = \int_0^u [\eta_k'(s)]^p\, ds ,$$

where

$$\eta_k(s) = \begin{cases} s^{t/p} & \text{if } 0 \le s \le k , \\ k^{t/p} + \dfrac{t}{p} k^{(t-p)/p}(s-k) & \text{if } s \ge k . \end{cases}$$

It can be verified easily that

$$\begin{cases} 0 \le u^{p-1}\phi_k(u) \le C_t[\eta_k(u)]^p , \\ 0 \le \phi_k(u) \le C_t[\eta_k(u)]^{p(t+1-p)/t} , \end{cases} \tag{4.10}$$

for a constant C_t independent of k, and $\eta_k(u)$, $\phi_k(u) \in E = D_0^{1,p}(\boldsymbol{R}^N)$ for all $k > 0$. Substituting $v = \phi_k(u)$ in (4.9) we obtain

$$\int_{\boldsymbol{R}^N} |\nabla u|^p [\eta_k'(u)]^p\, dx = \int_{\boldsymbol{R}^N} wu^{p-1}\phi_k(u)\, dx . \tag{4.11}$$

Define

$$\Omega_m = \{x \in \boldsymbol{R}^N \colon w(x) > m\} , \qquad m > 0 .$$

Then (4.10) and Hölder's inequality yield the estimate

$$\begin{aligned} \int_{\boldsymbol{R}^N} wu^{p-1}\phi_k(u)\, dx &\le m \int_{\Omega_m'} u^{p-1}\phi_k(u)\, dx + \int_{\Omega_m} wu^{p-1}\phi_k(u)\, dx \le \\ &\le mC_t\|\eta_k(u)\|_p^p + C_t\|w\|_{N/p,\,\Omega_m}\|\eta_k(u)\|_{p^*}^p . \end{aligned} \tag{4.12}$$

However, the definition of S in (2.6) means that

$$\int_{\boldsymbol{R}^N} |\nabla u|^p [\eta_k'(u)]^p\, dx = \|\nabla \eta_k(u)\|_p^p \ge S\|\eta_k(u)\|_{p^*}^p . \tag{4.13}$$

Substitution of (4.12) and (4.13) into (4.11) yields

$$(S - C_t \|w\|_{N/p,\,\Omega_m})\|\eta_k(u)\|_{p^*}^p \leqslant mC_t\|\eta_k(u)\|_p^p .$$

For fixed m large enough that $\|w\|_{N/p,\,\Omega_m} \leqslant S/2C_t$, it follows that

$$\|\eta_k(u)\|_{p^*}^p \leqslant \frac{2m}{S} C_t \|\eta_k(u)\|_p^p , \qquad k > 0 . \tag{4.14}$$

By the definition of $\eta_k(s)$, there exists a constant C, independent of k, such that $\eta_k(s) \leqslant Cs^{t/p}$ for all $k \geqslant 0$, $s \geqslant 0$, and furthermore $\lim_{k\to\infty} \eta_k(s) = s^{t/p}$. Now choose $t = p^*$ and apply Fatou's lemma to obtain

$$\liminf_{k\to\infty} \|\eta_k(u)\|_{p^*}^p \geqslant \|u\|_{(p^*)^2/p}^{p^*} .$$

Together with (4.14) this implies that

$$\|u\|_{(p^*)^2/p}^{p^*} \leqslant \frac{2m}{S} C_t C^p \|u\|_{p^*}^{p^*} .$$

Therefore $u \in L^{(p^*)^2/p}(\boldsymbol{R}^N)$ and consequently $u \in L^t(\boldsymbol{R}^N)$ for $p^* \leqslant t \leqslant (p^*)^2/p$ by a standard interpolation theorem. Continuing this iteration with $t_i = p^*(p^*/p)^i$, $i = 1, 2, \ldots$ we conclude that $u \in L^t(\boldsymbol{R}^N)$ for all $t \geqslant p^*$.

PROOF OF THEOREM 1.1. – The nontrivial nonnegative function $u \in E$ in Theorem 4.2 is a weak solution of the equation $-\Delta_p u = F \geqslant 0$, where

$$F(x) = \lambda a(x)[u(x)]^{p-1} + f(x)[u(x)]^{p^*-1} + g(x)[u(x)]^q , \qquad x \in \boldsymbol{R}^N .$$

Lemma 4.3 shows that $F \in L^\sigma(\boldsymbol{R}^N)$ for some $\sigma > N/p$. The uniform boundedness and asymptotic decay property $\lim_{|x|\to\infty} u(x) = 0$ of the solution then follow from Serrin's *a priori* estimate [16, Theorem 1] for $-\Delta_p u = F$ in $B_2(x)$, $x \in \boldsymbol{R}^N$:

$$\|u\|_{\infty,\,B_1(x)} \leqslant \text{Constant}\,[\,\|u\|_{p^*,\,B_2(x)} + \|F\|_{\sigma,\,B_2(x)}] .$$

The strict positivity of u is a consequence of a Harnack-type inequality of SERRIN [16, Theorem 5] applied to an arbitrary ball in $\boldsymbol{R}^N$. Tolksdorf's theorem [18, Theorem 1] implies the local $C^{1,\alpha}$-regularity of the solution.

REMARK 4.4. – An analogue of Theorem 1.1 can be proved for $0 < \lambda < \lambda_0$ by essentially the same procedure in the case that $g(x) \equiv 0$ provided we adjoin the conditions $a(x) \geqslant a_0 > 0$ and $f(x) \leqslant f(0)$ in some ball centred at the origin. Condition (1.4) is then replaced by the same condition with $q = p - 1$, i.e. $f(x) = f(0) + o(|x|^\delta)$, where

$$\delta = \begin{cases} p & \text{if } N \geqslant p^2 , \\ \dfrac{N-p}{p-1} & \text{if } N < p^2 . \end{cases}$$

If $N = p^2$, this can be weakened to $f(x) = f(0) + O(|x|^p)$.

REMARK 4.5. – If $N \geq p^2$, our method extends to more general equations

$$-\operatorname{div}\left[b(x)|\nabla u|^{p-2}\nabla u\right] = \lambda a(x)u^{p-1} + f(x)u^{p^*-1} + g(x)u^q, \qquad x \in \boldsymbol{R}^N$$

where $b(x) = b(0) + o(|x|^{N/Q})$ in some neighborhood of the origin. Such an extension requires an additional estimate for the function u_ε in (1.7), of the form

$$\int_{\boldsymbol{R}^N} |b(\varepsilon x) - b(0)|\,|\nabla u_\varepsilon(x)|^p\,dx = o(\varepsilon^{N/Q})$$

as $\varepsilon \to 0$. The details will be deleted. Of course the conclusions apply to all $N \geq (p(q+1))/(q-p+2)$, $p-1 < q < p^*-1$.

REMARK 4.6. – The function $g(x)u^q$ in (1.1) could be replaced by a more general function $g(x,u)$ with upper and lower majorants of type $g_1(x)u^{q_1}$, $g_2(x)u^{q_2}$ satisfying appropriate technical conditions.

REFERENCES

[1] A. AMBROSETTI - P. H. RABINOWITZ, *Dual variational methods in critical point theory and applications*, J. Funct. Anal., 14 (1973), pp. 349-381.

[2] J. P. G. AZORERO - I. P. ALONSO, *Existence and nonexistence for the p-Laplacian nonlinear eigenvalues*, Comm. PDE, **12** (1987), pp. 1389-1430.

[3] V. BENCI - G. CERAMI, *Existence of positive solutions of the equation* $-\Delta u + a(x)u = u^{(N+2)/(N-2)}$ *in* $\boldsymbol{R}^N$, J. Math. Anal. Appl., 88 (1990), pp. 90-117.

[4] H. BREZIS - E. H. LIEB, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc., 88 (1983), pp. 486-490.

[5] H. BREZIS - L. NIRENBERG, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., **36** (1983), pp. 437-477.

[6] K. J. BROWN - C. COSNER - J. FLECKINGER, *Principal eigenvalues for problems with indefinite weight function on* $\boldsymbol{R}^N$, Proc. Amer. Math. Soc., **109** (1990), pp. 147-155.

[7] H. EGNELL, *Existence and nonexistence results for m-Laplace equations involving critical Sobolev exponents*, Arch. Rational Mech. Anal., **104** (1988), pp. 57-77.

[8] M. GUEDDA - L. VERON, *Quasilinear elliptic equations involving critical Sobolev exponents*, Nonlinear Anal., **13** (1989), pp. 879-902.

[9] S. KICHENASSAMY - L. VERON, *Singular solutions of the p-Laplace equation*, Math. Ann., **275** (1986), pp. 599-615.

[10] M. C. KNAAP - L. A. PELETIER, *Quasilinear elliptic equations with nearly critical growth*, Comm. PDE, **14** (1989), pp. 1351-1383.

[11] P. L. LIONS, *The concentration-compactness principle in the Calculus of Variations. The locally compact case, Parts 1 and 2*, Ann. Inst. H. Poincaré: Anal. non linéaire, 1 (1984), pp. 109-145 and 223-283.

[12] P. L. LIONS, *The concentration-compactness principle in the Calculus of Variations. The limit case, Parts 1 and 2*, Revista Math. Iberoamericana, 1 (1985), pp. 145-201 and 1 (2) (1985), pp. 45-120.

[13] W.-M. NI, *On the elliptic equation* $\Delta u + K(x)\, u^{(n+2)/(n-2)} = 0$, *its generalizations, and applications in geometry*, Indiana Univ. Math. J., **31** (1982), pp. 493-529.

[14] W.-M. NI - J. SERRIN, *Existence and nonexistence theorems for ground states of quasilinear partial differential equations. The anomalous case*, Accad. Naz. Lincei, **77** (1986), pp. 231-257.

[15] E. S. NOUSSAIR - C. A. SWANSON - YANG JIANFU, *Quasilinear elliptic problems with critical exponents*, J. Nonlinear Anal., **20** (1993), pp. 285-301.

[16] J. SERRIN, *Local behavior of solutions of quasilinear equations*, Acta Math., **111** (1964), pp. 247-302.

[17] F. D. THELIN, *Local regularity properties for the solutions of a nonlinear partial differential equation*, Nonlinear Anal., **6** (1982), pp. 839-844.

[18] P. TOLKSDORF, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations, **51** (1984), pp. 126-150.

[19] J. L. VAZQUEZ, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim., **12** (1984), pp. 191-202.

[20] ZHU XIPING - YANG JIANFU, *The quasilinear elliptic equation on unbounded domain involving critical Sobolev exponent*, J. Partial Differential Equations, **2** (1989), pp. 53-64.