Annali di Matematica pura ed applicata
(IV), Vol. CLXIV (1993), pp. 365-399

Classifying Spectra for Generalized Homology Theories (*).

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Abstract. – *The well-known fact that each generalized homology theory h_* on the category of CW spaces has a classifying spectrum* $\boldsymbol{E}$ *which is unique up to an isomorphism in the Boardman (homotopy) category is proved by using the fact that each such h_* comes from a chain functor (cf. [1] or § 9). The proof does not use S-duality nor E. H. Brown's representation theorem.*

0. – Introduction.

The classical existence proof of a classisying spectrum $\boldsymbol{E}$ for a generalized homology theory h_* (defined on the category of *CW* spaces) involves S-duality and E. H. Brown's representation theorem [8], [9], [10]. Simultaneously with the discovery of this important result the question came up quite naturally whether there exists a direct proof, which eventually provides us with a canonical construction of a classifying spectrum. The proof of Theorem 1.1 settles this question in the affirmative. The basic tool which has to be involved is the existence of a chain functor $\boldsymbol{C}_*(\,)$, related to the given homology theory (cf. [1] resp. § 9 of the present paper). We get $\boldsymbol{E}$ almost immediately from the existence of a chain functor $\boldsymbol{C}_*$, determining the given homology theory h_* (which is guaranteed by Theorem 9.4). The present paper can in fact be regarded as a biproduct of this existence theorem for $\boldsymbol{C}_*$ (being originally designed for the purpose of establishing a *strong homology theory* on a strong shape category).

In § 3, § 4 we establish a natural isomorphism between $\boldsymbol{E}_*$ and the given homology theory h_*, while § 5 is devoted to a verification of the fact that a homology theory $\boldsymbol{E}_*$ (which is already given by means of a spectrum $\boldsymbol{E}$) is associated with a spectrum $\tilde{\boldsymbol{E}}$ (using the construction displayed in § 2) which is in the Boardman homotopy category isomorphic to $\boldsymbol{E}$. In § 6 we complete the proof of the main Theorem 1.1. The ensuing Sections § 7, § 8 contain a series of assertion which are needed in the previous parts of the paper, in particular Theorem 8.1, asserting that the spectrum $\boldsymbol{E}$ constructed in § 2 is a Kan spectrum. The final Section § 9 gathers together some facts

(*) Entrata in Redazione il 22 marzo 1991.

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about chain functors (without proofs). Lemmas 9.7, 9.8 are immediate consequences of the explicit construction of a chain functor C_* given in [1]. A detailed account of this material can be found in [1]. In § 1 we state the main theorem and discuss some of its consequences, some remarks about different categories of spectra are included.

The general reference for simplicial sets is [3], [7], while the theory of simplicial spectra is sufficiently well treated in [4]-[6].

1. – The formulation of the main theorem.

Let $\boldsymbol{CW}_{oh}$ be the homotopy category of based *CW* spaces. A (reduced) homology theory $h_*: \boldsymbol{CW}_{oh} \to \boldsymbol{AB}^{\mathbb{Z}}$ is 1) a sequence of functors $h_n: \boldsymbol{CW}_{oh} \to \boldsymbol{AB}$, $n \in \mathbb{Z}$, 2) a sequence of natural isomorphisms $\gamma = \gamma_n: h_n(X) \approx h_{n+1}(\Sigma X)$ (Σ = reduced suspension), satisfying an *exactness axiom* and an *axiom of compact carrier*, expressing the fact, that the natural transformation

$$\varinjlim_{K \subset X} h_n(K) \to h_n(X),$$

K being compact, is an isomorphism. We have the concept of a *natural transformation* $\varphi: h_* \to h'_*$ between such homology theories and therefore that of a category $\mathfrak{H}_*$ of homology theories on $\boldsymbol{CW}_{oh}$.

There is a well-known functor (being essentially due to G. W. WHITEHEAD [9])

$$W: \mathfrak{B}_h \to \mathfrak{H}_*,$$

assigning to each object in the Boardman category (i.e. to a spectrum or prespectrum $\boldsymbol{E} = \{E_k\}$, cf. [2] for further references) a homology theory $\boldsymbol{E}_*(\,) \in \mathfrak{H}_*$, defined by

$$\boldsymbol{E}_n(X) = \pi_n(X \wedge \boldsymbol{E}).$$

The spectrum $\boldsymbol{E}$ is called the *classifying spectrum* of this homology theory $W(\boldsymbol{E})_* = \boldsymbol{E}_*$.

On the other hand we take over from [1] the concept of a chain functor $C_*: \boldsymbol{K} \to \boldsymbol{ch}$ (= category of chain complexes), Definition 9.3, which can be defined on any category of topological spaces, in particular on $\boldsymbol{CW}$, the category of (unbased) *CW* spaces. There is an obvious functor (cf. Proposition 9.2):

$$H_*: \mathfrak{C}_* \to \mathfrak{H}_*$$

from the category of chain functors into the category $\mathfrak{H}_*$, which is simply defined by forming the homology theory related to $C_* \in \mathfrak{C}_*$ i.e. one considers $H_*(C_*)(\,)$ (and then going over to reduced homology by a well-known procedure). The morphisms of $\mathfrak{C}_*$ are transformations of chain functors, (cf. Definition 9.5) i.e. natural transformations being compatible with the additional structure of a chain functor, cf. Definition 9.3. Moreover we assume (unlike to the situation in [1]) that all chain functors in $\mathfrak{C}_*$ have compact carriers (which is motivated by the fact that all homology theories are supposed to have this property). For the convenience of the reader this material on chain functors is briefly recorded in § 9. The main objective of the present paper is embodied in the proof of the existence of a functor (now for $\boldsymbol{K} = \boldsymbol{CW}$)

$$\mathbf{Cl}\colon \mathfrak{C}_* \to \mathfrak{B}_h \tag{1}$$

which behaves as it is expected:

1.1. THEOREM. – *There exists* 1) *a functor* (1), 2) *a natural isomorphism*

$$\alpha\colon W\cdot \mathbf{Cl} \approx H_*$$

and 3) *to each* $\boldsymbol{E} \in \mathfrak{B}_h$ *and each* $\boldsymbol{C}_* \in \mathfrak{C}$ *satisfying* $H_*(\boldsymbol{C}_*)(\,) \approx \boldsymbol{E}_*(\,)$ *an equivalence in* $\mathfrak{B}_h$

$$\beta\colon \mathbf{Cl}(\boldsymbol{C}_*) \approx \boldsymbol{E}\,.$$

This theorem will be proved in the course of this paper. A stronger statement, asserting that W is in fact an equivalence of categories is well-known to be false. It occurs that different mappings in $\mathfrak{B}_h$ give rise to the same natural transformation between homology theories in $\mathfrak{H}_*$ (cf. [8] Theorem 17, p. 63). More precisely the category $\mathfrak{H}_*$ turns out to be equivalent to the category $\mathfrak{B}_h/ph$, where *homotopy* is replaced by *weak homotopy* (or alternatively: by taking classes of mappings in $\mathfrak{B}_h$, modulo some kind of *phantom maps*), cf. [8] Proposition 14, p. 75 for the details.

Theorem 1.1 implies immediately:

COROLLARY 1.2. – *Every homology theory* $h_* \in \mathfrak{H}_*$ *admits a classifying spectrum* $\boldsymbol{E}$, *i.e. one has* $h_* \approx \boldsymbol{E}_*$. *Moreover this spectrum is uniquely determined by* h_* *up to isomorphisms in* $\mathfrak{B}_h$.

So Theorem 1.1 appears as a reasonable way of providing us with classifying spectra in a functorial way. The functor H_* does again not admit an inverse. The quite complicated relationship between chain functors $\boldsymbol{C}_*, \boldsymbol{C}'_*$, giving rise to isomorphic homology theories $H_*(\boldsymbol{C}_*)(\,) \approx H_*(\boldsymbol{C}'_*)(\,)$ involves the concept of an ∞-functor and is not needed in the present paper.

In order to deduce Corollary 1.2 from Theorem 1.1 we have to apply Theorem 8.1 in [1] (or Theorem 9.4 in this paper) ensuring the existence of a chain functor for a given homology theory h_*. This result in conjunction with Theorem 1.1, guarantees, that every homology theory admits a classifying spectrum $\boldsymbol{E}$, which, according to Theorem 1.1. 3) is unique (within $\mathfrak{B}_h$).

Neither the existence nor the uniqueness of a classifying spectrum for a given homology theory are new (cf. [9] Theorem 14.35, p. 329; [8], chapter 5). Therefore full emphasis is laid upon the method of construction of $\mathbf{Cl}(\boldsymbol{C}_*(\,)) = \boldsymbol{E}$, resp. of the mapping β in § 5 which do *not* involve S-duality nor E. Brown's representation theorem for cohomology theories. The detection of such an independent, direct proof of the existence and uniqueness of a classifying spectrum has been an open question.

The proof of Theorem 1.1 uses at different occasions different descriptions of stable categories which are (at least on the homotopy level) all equivalent. In particular the category $\boldsymbol{Sp}$ of simplicial spectra ([4] Definition 4.1) is crucial four our purposes. There is a functor F: $\boldsymbol{Sp} \to \boldsymbol{Sp}_G$ into the category of group spectra (by forming the free groups, generated by the simplexes of a given $\boldsymbol{E} \in \boldsymbol{Sp}$). Since $\boldsymbol{G} \in \boldsymbol{Sp}_G$ (in particular $F(\boldsymbol{E})$, for any $\boldsymbol{E} \in \boldsymbol{Sp}$) is a Kan spectrum, admitting the concept of a homotopy, we define as usual homotopy groups in $\boldsymbol{Sp}$ by

$$\pi_n(\boldsymbol{E}) = \pi_n(F(\boldsymbol{E})).$$

So in many questions concerning homotopies we agree to work in $\boldsymbol{Sp}_G$ rather than in $\boldsymbol{Sp}$ (implying that we consider $F(\boldsymbol{E})$ instead of $\boldsymbol{E}$). The natural mapping $(\boldsymbol{E} \to \to F(\boldsymbol{E})) \in \boldsymbol{Sp}$ turns out to be a weak homotopy equivalence.

There are other descriptions of a stable category by means of so-called prespectra (cf. [2], [4], [5]) which can be defined *topologically* (leading to the Boardman category $\mathfrak{B}$ resp. to its homotopy category $\mathfrak{B}_h$) or *simplicially*: A prespectrum $\boldsymbol{E} = \{E_k, \Theta_k \colon \Sigma E_k \to E_{k+1}, k \in \mathbb{Z}\}$ is a family of based simplicial sets a with simplicial mappings $\Theta = \Theta_k$. Concerning morphisms between prespectra we refer to § 5. Because all these categories are equivalent on the homotopy level, it turns out to be a mere matter if convenience, which category one is using in dealing with a particular problem. In our case, the functor **Cl** is constructed in § 2 simplicially, while the natural transformation α will be established for simplicial prespectra.

The definition of a homotopy group $\pi_n(\boldsymbol{E})$ of a *Kan spectrum* $\boldsymbol{E}$ can be expressed in terms of the associated prespectrum $\{E_k\}$ by

$$\pi_n(\boldsymbol{E}) = \varinjlim_k \pi_{n+k}(E_k).$$

2. – The functor Cl.

We consider p-simplexes $\Delta^p = (a_0, \ldots, a_p)$, $a_i \in \mathbb{N}$, $a_0 \leq \ldots \leq a_p$ and define

$$\partial_i \Delta^p = (a_0, \ldots, \hat{a}_1, \ldots, a_p), \qquad 0 \leq i \leq p,$$

$$s_i \Delta^p = (a_0, \ldots, a_i, a_i \ldots, a_p), \qquad 0 \leq i \leq p.$$

The set of all these simplexes is denoted by $\boldsymbol{\Delta}$. With each non-degenerate simplex $\Delta^p \in \boldsymbol{\Delta}$ we associate a geometrical p-simplex, again denoted by Δ^p. Let $\boldsymbol{C}_* \in \mathfrak{C}_*$ be a chain functor (cf. § 9) and $\Delta^p \in \boldsymbol{\Delta}$ non-degenerate, then we have $\boldsymbol{Z}_n(\Delta^p)$ (cf. § 7) and define for each $\zeta_n = \zeta \in \boldsymbol{Z}_n(\Delta^p)$ the element $\partial_i \zeta \in \boldsymbol{Z}_{n-1}(\Delta^{p-1})$ by:

$$(1) \qquad (\partial_i \zeta)(\Delta^q) = \begin{cases} \zeta_n(\Delta^q),\ \Delta^q \subset \partial_i \Delta^p, & 0 \leqslant i \leqslant p, \\ 0 \ldots, & i > p. \end{cases}$$

Let again Δ^p be non-degenerate, $\zeta \in \boldsymbol{Z}_n(\Delta^p)$, then we define formally a $s_i \zeta$ being defined on the degenerate $s_i \Delta^p \in \boldsymbol{\Delta}$ as follows:

$$(2) \qquad s_i \zeta(\Delta^q) = \begin{cases} \zeta(\Delta^q) \ldots \Delta^q \subset \partial_i s_i \Delta^p = \Delta^p = \partial_{i+1} s_i \Delta^p, & 0 \leqslant i \leqslant p \\ 0 \ldots \quad \text{otherwise}. \end{cases}$$

In particular we have always $\overline{s_i \zeta} = 0$ and $s_i \zeta = 0$ whenever $i > p$. Now we define $\boldsymbol{Z}_{n+1}(s_i \Delta^p) = \{s_i \zeta \mid \zeta \in \boldsymbol{Z}_n(\Delta^p)\}$. This procedure can be iterated providing us with arbitrary degeneracies $\boldsymbol{Z}_n(s_{i_1} \ldots s_{i_k} \Delta^p)$. Moreover the definition of boundaries (1) can be transferred to these new degeneracies.

As a result we obtain boundaries ∂_i and degeneracies s_i for each $\zeta \in \boldsymbol{Z}_n(\Delta^p)$ (where now $\Delta^p \in \boldsymbol{\Delta}$ is arbitrary) being defined for all $i \in \mathbb{N}$.

Let $n \in \mathbb{Z}$ be an integer, then we define

$$E_{(n)} = \bigcup_{\Delta^p \in \boldsymbol{\Delta}} \boldsymbol{Z}_n(\Delta^p)/\sim$$

where the $\sim$-relation is generated by the following identifications:

E1) All zero elements $0 \in \boldsymbol{Z}_n((\Delta^p))$ $(\Delta^p \in \boldsymbol{\Delta})$ are identified to one single element $*$, the basepoint of $E_{(n)}$.

Let v: $\Delta^p = (a_0, \ldots, a_p) \subset \Delta^{p+1} = (a_0, \ldots, a_p, a_{p+1})$ be the inclusion of standard simplexes, then we have for each $\zeta \in \boldsymbol{Z}_n^{(p)}(\Delta^p)$, an induced $v_{\#} \zeta \in \boldsymbol{Z}_n(\Delta^{p+1})$ (cf. Lemma 7.8) and require:

E2) We identify $\zeta \in \boldsymbol{Z}_n(\Delta^p)$ with $v_{\#} \zeta \in \boldsymbol{Z}_n(\Delta^{p+1})$. By an abuse of notation we still write $\zeta = \zeta_n \in E_{(n)}$ instead of $[\zeta_n]$.

The formation of boundaries and degeneracies is compatible with these equivalence relations: E1), E2): $\zeta \sim \tilde{\zeta} \Rightarrow \partial_i \tilde{\zeta}$, $s_i \zeta \sim s_i \tilde{\zeta}$. So we are allowed to formulate:

2.1. PROPOSITION. – $\boldsymbol{E} = \{E_{(n)}, \partial_i, s_i\}$ *is a simplicial spectrum (in the sense of* [5] *Definition* 2.1, *p.* 240).

PROOF. – By the preceding remark we have

$$\partial_i \colon E_{(n)} \to E_{(n-1)},$$

$$s_i \colon E_{(n)} \to E_{(n+1)},$$

with $\partial_i \zeta = *$ for sufficiently large i. More precisely: $\zeta \in \boldsymbol{Z}_n(\Delta^p)$ implies $\partial_i \zeta = 0$ for $i > p$. The well-known relations between compositions of different ∂_i's and s_j's are easily verified. ∎

2.2. DEFINITION. – We define $\boldsymbol{E} = \mathrm{Cl}(\boldsymbol{C}_*) \in \mathfrak{B}$ as the classifying spectrum associated with the chain functor $\boldsymbol{C}_* \in \mathfrak{C}$.

REMARKS. – 1) Let $\Delta^p \in \boldsymbol{\Delta}$ be non-degenerate, then the element $\bar{\zeta} = \zeta(\Delta^p) \in Z_n'(\Delta^p, \mathrm{bd}\,\Delta^p)$ does neither determine the element $\zeta \in \boldsymbol{Z}_n(\Delta^p)$ nor the simplex $\zeta \in E_{(n)}$: Let $\beta \neq 0$, $\beta \in h_{n-1}(\Delta^{p-2})$ be a non-trivial element, $b \in Z_{n-1}'(\Delta^{p-2}, \mathrm{bd}\,\Delta^{p-2})$ be such that $\psi p_*[b] = \beta$ (p: $C_*' \to C_*''$ is the projection, cf. § 9 concerning the terminology). Then we might start in establishing a $\zeta_n \in \boldsymbol{Z}_n(\Delta^p)$ by setting $\zeta_n(\Delta^p) = 0$,

$$\zeta_n(\partial_i \Delta^p) = \begin{cases} b \ldots i = 0, 1, \\ 0 \ldots \text{elsewhere}, \end{cases}$$

and continuing with any partition of b.

2) Let $h_*(\ ) = H_*(\ ; G)$ be an ordinary homology theory with coefficients in an abelian group G, $\boldsymbol{C}_*$ the ordinary simplicial chain complex associated with h_* (being defined e.g. on a category of triangulated polyhedra). We can use $C_*(X)$ as $C_*'(X, A)$ observing that φ, κ are becoming natural isomorphisms. So we have $Z_n'(\Delta^p, \mathrm{bd}\,\Delta^p) = Z_n(\Delta^p, \mathrm{bd}\,\Delta^p) = \{c \in C_n(\Delta^p) \mid dc \in \mathrm{im}(C_{n-1}(\mathrm{bd}\,\Delta^p) \to C_{n-1}(\Delta^p))\}$ in the classical sense. The relative cycles $c \in Z_n(\Delta^p, \mathrm{bd}\,\Delta^p)$ carry automatically the structure of elements of $\boldsymbol{Z}_n(\Delta^p)$: We can easily decompose $dc = \sum_{i=0}^{p} (-1)^i c_i$, $c_i \in Z_{n-1}(\partial_i \Delta^p, \mathrm{bd}\,\partial_i \Delta^p)$ and so on, endowing c with the structure of an element of $\boldsymbol{Z}_n(\Delta^p)$. On the other hand, any $\zeta_n \in \boldsymbol{Z}_n(\Delta^p)$, $\Delta^p \in \boldsymbol{\Delta}$ non-degenerate, can easily be associated with such a particular c in such a way that the related subspectrum of these c's turns out be homotopy equivalent to the original $\boldsymbol{E}$.

Let λ: $\boldsymbol{C}_* \to \tilde{\boldsymbol{C}}_*$ be a natural transformation of chain functors (cf. § 9) and $\boldsymbol{E} = \mathrm{Cl}(\boldsymbol{C}_*)$, $\tilde{\boldsymbol{E}} = \mathrm{Cl}(\tilde{\boldsymbol{C}}_*)$ the spectra corresponding to $\boldsymbol{C}_*$ resp. $\tilde{\boldsymbol{C}}_*$. The transformation λ induces a λ: $\boldsymbol{Z}_n(\Delta^p) \to \tilde{\boldsymbol{Z}}_n(\Delta^p)$. Since λ respects the equivalence relations $\boldsymbol{E}$ 1), $\boldsymbol{E}$ 2), we have a

$$\lambda_\# = \lambda_{\#(n)} : E_{(n)} \to \tilde{E}_{(n)} .$$

Obviously $\lambda_\#$ commutes with ∂_i and s_i, and one has $\lambda_\#(*) = *$, hence

$$\mathrm{Cl}(\lambda) = \lambda_\# : \boldsymbol{E} \to \tilde{\boldsymbol{E}}$$

is a mapping of spectra. This asignment is clearly functorial so that we are able so summarize.

2.3. PROPOSITION. – *The assignment*

$$\mathrm{Cl}\colon \mathfrak{C}_* \to \mathfrak{B}_h$$

($\mathfrak{B}_h$ *being the homotopy category of simplicial spectra) is a functor.*

For many applications it turns out to be more convenient to deal with simplicial prespectra $\{E_k, \theta_k\colon \Sigma E_k \to E_{k+1}\}$ (E_k being a simplicial set) rather than with the spectrum $\mathbf{Cl}(\boldsymbol{C}_*) = \boldsymbol{E}$. Following [5] Definition 2.4, p. 241, a q-simple $\zeta^q \in (E_k)_q$ is a $\zeta_n \in E_{(n)}$ (hence a $\zeta_n \in \boldsymbol{Z}_n(\Delta^p)$ for suitable p) such that

$$n = q - k,$$
$$\partial_j \zeta_n = * \;\ldots j > q,$$
$$\partial_0 \ldots \partial_q \zeta_n = *$$

holds. This implies in particular that if $\Delta^p = (a_0, \ldots, a_p)$ and $p \geqslant q$, then $\partial_i \zeta_n = *$ for $i > p$.

REMARK. – In the special case of an *ordinary* homology theory with coefficients in an abelian group G one has an Eilenberg-MacLane spectrum $\boldsymbol{K}(\boldsymbol{G})$ with spaces $K(G, k) = \boldsymbol{K}(\boldsymbol{G})_k$, which can be described simplicially (cf. [7], p. 101) as

$$(\boldsymbol{K}(\boldsymbol{G})_k)_q = K(G, k)_q = Z^k(\Delta^q; G),$$

with $Z^*(\ldots)$ denoting reduced cochains. Although Corollary 1.2 implies immediately that $\boldsymbol{K}(\boldsymbol{G})$ and $\tilde{\boldsymbol{E}} = \mathbf{Cl}(\boldsymbol{C}_*)$, $\boldsymbol{C}_* = a$ chain functor related to simplicial homology with coefficients in G, are of the same homotopy type, it might be valuable to indicate explicitely the existence of a homotopy equivalence $f\colon \tilde{\boldsymbol{E}} \to \boldsymbol{K}(\boldsymbol{G})$, based on the construction of $\tilde{\boldsymbol{E}}$ resp. of $\boldsymbol{K}(\boldsymbol{G})$:

1) Let to this end $\boldsymbol{C}_*$ be the simplicial chain functor with coeficients in G on the category of polyhedra. In view of Remark 2) following 2.2, we can set $C'_*(X, A) = C_*(X)$, φ, κ becoming isomorphisms and $Z'_n(X, A) = Z_n(X, A) = \{c \in C_n(X) \mid dc \in \operatorname{im}(C_{n-1}(A) \to C_{n-1}(X))\}$. Moreover $Z'_n(\Delta^p, \operatorname{bd}\Delta^p)$ carries in a natural way the structure of $\boldsymbol{Z}_n(\Delta^p)$. So we simply identify $(\tilde{\boldsymbol{E}}_k)_q$ with $Z_n(\Delta^q, \operatorname{bd}\Delta^q)$ (which is possible up to a homotopy equivalence).

2) There is (for $n > 0$) a well-known isomorphism $\alpha\colon Z_{n-1}(\operatorname{bd}\Delta^q) \approx Z^k(\Delta^q)$, $n = q - k$: With each simplex $\Delta^{n-1} = (a_{i_0}, \ldots, a_{i_{n-1}}) \subset \Delta^q = (a_0, \ldots, a_q)$ we associate the complementary $\Delta^k = (a_{j_0}, \ldots, a_{j_k})$ (having the remaining $a_j \neq a_{i_0}, \ldots, a_{i_{n-1}}$ as vertices). The assignment $g\Delta^{n-1} \mapsto \varepsilon g \Delta^k$, $\varepsilon = \pm 1$ a suitable sign, $g \in G$, establishes the isomorphism α for $n > 0$. In order to have α also available for $n = 0$ we set formally $Z_{-1}(\operatorname{bd}\Delta^q) = G$.

3) The boundary $d\colon Z_n'(\Delta^q, \mathrm{bd}\,\Delta^q) \to Z_{n-1}(\mathrm{bd}\,\Delta^q)$ (with $d(g\Delta^0) = g \in Z_{-1}(\mathrm{bd}\,\Delta^0)$ for $n = 0$) is only an epimorphism. However two $z_1, z_2 \in \boldsymbol{Z}_n(\Delta^q)$ with $dz_1 = dz_2$ are always homologous in $C_*'(\Delta^q, \mathrm{bd}\,\Delta^q)$. Hence, in terms of simplexes in $\tilde{\boldsymbol{E}}$, there exists a cell $\zeta^{q+1} \in (\tilde{E}_k)_{q+1}$ satisfying $\partial_0 \zeta = z_1$, $\partial_1 \zeta = z_2$, $\partial_i \zeta = *$, $i > 1$.

4) Composing α and d provides us with a mapping

$$f_q\colon (\tilde{E}_k)_q \to (K(G)_k)_q = Z^k(\Delta^q; G),$$

which is easily seen to be simplicial (due to the definition of boundaries and degeneracies in $\tilde{\boldsymbol{E}}$ as well as in $\boldsymbol{K}(\boldsymbol{G})$).

Moreover f_q turns out to be compatible with suspensions, yielding a mapping $f\colon \tilde{\boldsymbol{E}} \to \boldsymbol{K}(\boldsymbol{G})$ of spectra.

5) Finally, the fact that f is in fact a homotopy equivalence follows from 3). ■

3. – A natural transformation $\alpha\colon W\,\mathrm{Cl} \to h_*$.

Let $\boldsymbol{C}_* \in \mathfrak{C}$ be a given chain functor, $h_* = H_*(\boldsymbol{C}_*)$ the related homology theory and $\boldsymbol{E} = \mathrm{Cl}(\boldsymbol{C}_*)$ the spectrum of § 2 (we are now considering the related prespectrum $\{E_k\}$)). We are going to construct a transformation of homology theories:

$$\alpha = \alpha_{C_*}\colon \boldsymbol{E}_*(\,) \to h_*(\,),$$

which in the next section turns out to be an isomorphism.

Instead of the space X we deal with its singular complex $S(X)$, determining the prespectrum $\{S(X) \wedge E_k\} = \{\boldsymbol{E}(X)_k\}$ with $\wedge$-product as in [4]. Hence a q-simplex $\sigma^q \wedge \zeta^q \in (\boldsymbol{E}(X)_k)_q$ consists of a pair with $\sigma^q \in S(X)_q$, $\zeta^q \in \boldsymbol{Z}_n(\Delta^p) \subset (E_k)_q$ subject to the identification

$$\sigma^q \wedge 0^q = *, \qquad \sigma^q \in S(X),$$

$0^q \in \boldsymbol{Z}_n(\Delta^p)$ being the zero element, serving as basepoint of E_k. Boundaries and degeneracies are defined for each factor separately.

We have $\boldsymbol{E}_n(X) = \varinjlim_k \pi_{n+k}(S(X)) \wedge E_k) = \varinjlim_k \pi_{n+k}(\boldsymbol{E}(X)_k) = \pi_n(\boldsymbol{E}(X))$. Suppose we have a q-sphere $a = \sigma^q \wedge \zeta^q \in (\boldsymbol{E}(X)_k)_q$, $\zeta^q \in \boldsymbol{Z}_n(\Delta^p)$, then we consider three cases:

1) $p < q$: We have the inclusion $v\colon \Delta^p = (a_0, \dots, a_p) \subset \Delta^q = (b_0, \dots, b_q)$ (= standard q-simplex), $v(a_i) = b_i$, $i = 0, \dots, p$ and set $\tilde{a} = \sigma^q \wedge v_\# \zeta^q$, $v_\# \zeta^q \in \boldsymbol{Z}_n(\Delta^q)$ (cf. Lemma 7.8 and E2) in the definition of $\boldsymbol{E}$).

2) $q < p$: We form $\tilde{\sigma}^p = s_{p-1} \dots s_q \sigma^q$, consider $\zeta^q \in \boldsymbol{Z}_n(\Delta^p)$ now as an element of $(E_{k+p-q})_p$ (calling it $\tilde{\zeta}^p$, which amounts to applying the $(p-q)$-fold bonding map in the prespectrum $\{E_k\}$ to ζ^q) and come to $\tilde{a} = \tilde{\sigma}^p \wedge \tilde{\zeta}^p \in (\boldsymbol{E}(X)_{k+p-q})_p$.

3) $p = q$: We set $\tilde{a} = a$.

We have:

3.1. LEMMA. – *In all three cases a and $\tilde{a}$ are stably homotopic, hence they determine the same element $\{a\} = \{\tilde{a}\} \in \boldsymbol{E}_n(X)$.*

The proof is standard. ∎

As a result we can assume that (up to stable homotopy) $p = q$ and define

$$\alpha(a) = \sigma^q_{\#}\varphi_{\#}\bar{\zeta}^q, \tag{1}$$

where $\sigma^q: \Delta^q \to (X, *)$ denotes the related continuous mapping and $\varphi_{\#}: C'_n(\Delta^q, \mathrm{bd}\,\Delta^q) \to C_n(\Delta^q)$ is the (non-natural) transformation coming together with the chain functor $\boldsymbol{C}_*$ (cf. § 9). This assignment α has several properties:

1) We have for $\boldsymbol{E} = \mathrm{Cl}(\boldsymbol{C}_*)$ with the corresponding notions of suspension γ, Σ_* (to be recalled below) the diagram

$$\begin{array}{ccc} \boldsymbol{E}_n(X) & \xrightarrow{\alpha} & h_n(X) \\ \gamma\big\downarrow & & \big\downarrow\Sigma_* \\ \boldsymbol{E}_{n+1}(\Sigma X) & \xrightarrow{\alpha} & h_{n+1}(\Sigma X) \end{array} \tag{2}$$

$(\alpha(\{a\}) = \{\alpha\ (a)\})$ which turns out to be commutative.

We have (cf. Lemma 7.4)

$$\Sigma_*\{\alpha(a)\} = \Sigma_*\{\sigma^q_{\#}\varphi_{\#}(\bar{\zeta}^q)\} = \{(\Sigma\sigma^q)_{\#}\varphi_{\#}\Sigma\bar{\zeta}^q\}.$$

Let on the other hand $a = \sigma^q \wedge \zeta^q \in (\boldsymbol{E}(X)_k)_q$ be a sphere, $\gamma(\sigma^q \wedge \zeta^q) = \tau^{q+1} \wedge \eta^{q+1}$ a representative in the suspended $\{a\}$, then according to Lemma 8.3 we can take 1) for τ^{q+1} the simplex which is formed by applying the Kan-construction (of adding simplexes in a Kan complex) to two copies $C_{\pm}\sigma^q$ of the cone over σ^q, establishing one single simplex τ^{q+1}; 2) for η^{q+1} the corresponding addition of two copies $C_{\pm}\zeta^q$ of $C\zeta^q$ (cf. § 7, § 8 (3)), yielding $\Sigma\zeta^q$. As a result we have:

$$\alpha\{\gamma(\sigma^q \wedge \zeta^q)\} = \{\tau^{q+1}_{\#}\varphi_{\#}\overline{\Sigma\zeta^q}\} = \{(\Sigma\sigma^q)_{\#}\varphi_{\#}\Sigma\bar{\zeta}^q\} = \Sigma_*\{\alpha(a)\}.$$

2) Let $b \in (\boldsymbol{E}(X)_k)_{q+1}$ be a $q+1$ simplex satisfying

$$\partial_i b = \begin{cases} a \ \ldots i = 0, \\ a' \ldots i = 1, \\ * \ \ldots i > 1, \end{cases}$$

a, a', being spheres in $(\boldsymbol{E}(X)_k)_q$. Then we define $\alpha(b)$ as in (1) and conclude

$$d\alpha(b) = \alpha(a) - \alpha(a').$$

Hence we are allowed to talk about $\alpha[a] = \{\mathrm{a}(a)\}$ and, due to 1), about $\alpha\{a\} = \{\alpha(a)\}$, denoting by $[a] \in \pi_q(\boldsymbol{E}(X)_k)$ (resp. $\{a\} \in \boldsymbol{E}_n(X)$) the (stable) homotopy class of a. By $\{\alpha(a)\} \in h_n(X) = H_n(\boldsymbol{C}_*)(X)$ we denote the homology class of $\alpha(a)$.

3) Suppose $b \in (\boldsymbol{E}(X)_k)_{q+1}$ is a simplex satisfying

$$\partial_i b = \begin{cases} a \ \ldots i = 0, \\ c \ \ldots i = 1, \\ a' \ldots i = 2, \\ * \ \ldots i > 2, \end{cases}$$

a, a' (and therefore also c) being spheres, then we have

$$\alpha([a] + [a']) = \alpha[c] = [\alpha(c)],$$

$$d\alpha(b) = \alpha(a) + \alpha(a') - \alpha(c).$$

This works for $q \geqslant 2$, hence (because $q = n + k$) for all n and sufficiently large k. So α defines an additive, natural transformation

$$\alpha\colon W\mathrm{Cl}_*(\,) \to h_*(\,)$$

between homology theories.

4. – The transformation α is an isomorphism.

We will in several steps establish some kind of inverse $\bar{\alpha}$ of the transformation α:

1) Suppose at first that $\sigma^q \in S(X)_q$ is a *regular* simplex (i.e. the mapping $\sigma^q\colon \Delta^q \to \sigma^q(\Delta^q)$ is a homeomorphism):

1*a*) Suppose furthermore that $z \in \omega \in h_n(X, *)$, $z \in Z_n(C_*(X, *))$ is a cycle lying in the image of $\sigma^q_{\#}\colon Z_n(C_*(\sigma^q)) \to Z_n(C_*(X, *))$, $\sigma^q_{\#} z' = z$. Then we associate with z' the element $\zeta^q \in \boldsymbol{Z}_n(\Delta^q)$ satisfying $\bar{\zeta}^q = \kappa_{\#} z'(\kappa_{\#}\colon C_n(\sigma^q) \to C'_n(\sigma^q, \mathrm{bd}\,\sigma^q))$ resp. $\partial_i \zeta^q = *$, $i \geqslant 0$ (cf. Lemma 7.7).

We set

$$\bar{\alpha}(z) = \sigma^q \wedge \zeta^q \in (\boldsymbol{E}(X)_k)_q$$

and deduce immediately

$$\alpha\bar{\alpha}(z) = \sigma^q_\# \varphi_\# \kappa_\# z' \sim \sigma^q_\# z' = z,$$

taking into account the chain homotopy $\varphi_\# \kappa_\# \simeq 1$ (cf. § 9, D1)).

1*b*) Suppose $z \in \omega$ is lying in $\operatorname{im}(\sigma^q_\#: Z_n(C_*(\sigma^q, *)) \to Z_n(C_*(X, *)))$, $\sigma^q_\# z_1 = z$, then we find (because $\varphi'_\# i' \simeq i_\#$, $\varphi'_\#: C'_n(\sigma^q, *) \to C_n(\sigma^q)$) and this chain homotopy is for cycles $C_*(*)$ already lying in $\operatorname{im}(C_*(*) \to C_*(\sigma^q))$ (cf. § 9 Remark 3 following D3)) a $c \in \operatorname{im}(C_*(*) \to C_*(\sigma^q))$ such that

$$d\varphi'_\# z_2 = dc + i_\# dz_2,$$

$z_2 \in Z'_n(\sigma^q, *)$ being a relative cycle such that $z_2 \sim z_1$ (cf. Lemma 7.1).

We define $z' = \varphi'_\# z_2 - c \in C_n(\sigma^q)$, observing that $\kappa_\# z' \in Z'_n(\sigma^q, \operatorname{bd}\sigma^q)$. Now we proceed as before, establishing a $\zeta^q \in \boldsymbol{Z}_n(\Delta^q)$ $(\Delta^q \overset{\sigma^q}{\approx} \sigma^q)$ by setting $\bar{\zeta}^q = \kappa_\# z'$, $\partial_i \zeta^q = *$, $i > 0$, $\overline{\partial_i \zeta^q} = d\kappa_\# z'$. Defining again $\bar{\alpha}(z) = \sigma^q \wedge \zeta^q$, we obtain

$$\alpha\bar{\alpha}(z) = \sigma^q_\# \varphi_\# \kappa_\# (\varphi'_\# z_2 - c) \sim \sigma^q_\# (\varphi'_\# z_2 - c)$$

but because $\sigma_\# \varphi'_\#$ factorizes over $l': Z'_n(\sigma^q, *) \to C_n(\sigma^q, *)$, and $j'_\#(\varphi'_\# z_2 - c) \sim$ $\sim j'_\# \sigma'_\# z_2 \sim z_1$, $j': C_*(\sigma^q) \to C_*(\sigma^q, *)$, we deduce again

$$\alpha\bar{\alpha}(z) \sim z. \tag{1}$$

Suppose now that $\sigma^q \in S(X)_q$ is *any* singular simplex, $c: \Delta^q \to |\sigma^q| \subset |S(X)|$ the characteristic map and that $z \in \omega \in h_n(X, *)$ is as in case 1*a*). We take a concentric simplex $\Delta'^q \subset \Delta^q$ and find according to Lemma 7.5 a partition of $\kappa_\# z'$ (formed as in the first case) $\{\tilde{z}, z_1\}$, $\tilde{z} \in Z_n(\Delta'^q, \operatorname{bd}\Delta'^q)\, z_1 \in Z'_n(\sigma^q(R), \operatorname{bd}\sigma^q(R))$, $R = \overline{\Delta^q \setminus \Delta'^q}$ with $d\tilde{z} = -dz_1$. By employing Lemma 7.7 we detect a $\zeta^q \in \boldsymbol{Z}_n(\Delta'^q)$ with $\bar{\zeta}^q = \tilde{z}$. Let $\Delta'^q =$ $= (a'_0, \ldots, a'_q)$, $\Delta^q = (a_0, \ldots, a_q)$ be the two q-simplexes, then we have the projection $p: (\Delta^{q'}, \operatorname{bd}\Delta^{q'}) \to (\Delta^q, \operatorname{bd}\Delta^q)$, $p(a'_i) = a_i$, $i = 0, \ldots, q$. We obtain the q-simplex $\sigma'^q =$ $= \sigma^q p \in S(X)_q$ satisfying $\partial_i \sigma'^q = \partial_i \sigma^q$, $i \geqslant 0$, and find a chain

$$c = \alpha(\sigma'^q \wedge \zeta^q) = \sigma'^q_\# \varphi_\# \zeta^q \in C_n(X, *),$$

$dc \in \operatorname{im}(C_{n-1}(*) \to C_{n-1}(X, *))$. So we obtain a cycle $z'' \in Z_n(X, *)$ with $z'' = c + s$, $s \in C_n(*, *)$, $ds = -dc$, $z'' \sim z$.

The simplex $\sigma'^q \wedge \zeta^q$ is not a sphere. In order to repair this, we need a $(q+1)$-simplex $\Delta^{q+1} = (a'_0, \ldots, a'_q, b_{q+1}, \ldots, b_{q+1})$ such that $|\sigma^q| = \sigma^q(R) \cup \Delta'^q$ can be embedded in Δ^{q+1}, $v: |\sigma^q| \subset \Delta^{q+1}$ in such a way that $v(a'_i) = a'_i$ and $\sigma^q(R) \cap \operatorname{bd}\Delta^{q+1} = \operatorname{bd}\Delta^q$. We obtain by Lemma 7.7 a $\zeta^q \in \boldsymbol{Z}_n(\Delta^{q+1})$ by setting $\bar{\zeta}^q = v_\# \kappa_\# z'$ (which is now a cycle) and $\partial_i \zeta^q = *$, $i \geqslant 0$.

Calculation of $\alpha(\sigma'^q \wedge \zeta^q)$ (as defined in § 3) leads to a cycle $\tilde{z} \in Z_n(X, *)$ which is homologous to the previous z', hence to the original z. Moreover a simple cylinder argument assures us that $\sigma'^q \wedge \zeta^q \simeq \sigma^q \wedge \zeta^q$ in $\boldsymbol{E}(X)_k$. So we are enabled to define

$$\bar{\alpha}(z) = \sigma^q \wedge \zeta^q.$$

such that again (1) holds.

The case in which $z \in \omega$ behaves as in case 1*b*) is treated analogously. This allows us to define in any case where $z \in \omega \in h_n(X, *)$ is a cycle lying on a single singular simplex σ^q a sphere $\bar{\alpha}(z) \in (\boldsymbol{E}(X)_k)_q$ satisfying (1).

Suppose on the other hand that $a = \sigma^q \wedge \zeta^q \in (\boldsymbol{E}(X)_k)_q$ is a sphere, then we can assume without loss of generality (using simple mapping cyclinder techniques) that case 1*a*), *b*) prevails.

We have

$$\bar{\alpha}\alpha(a) = \bar{\alpha}(\sigma^q_\# \varphi_\# \bar{\zeta}^q) = \sigma^q \wedge \zeta'^q \tag{2}$$

with new $\zeta'^q \in \boldsymbol{Z}_n(\Delta^q)$ which is defined by

$$\bar{\zeta}'^q = \kappa_\# \varphi_\# \bar{\zeta}^q,$$

$$\partial_i \zeta'^q = *, \qquad i \geqslant 0.$$

Since $l\bar{\zeta}^q \sim l\bar{\zeta}'^q$ in $C_*(\Delta^q, \operatorname{bd}\Delta^q)$ we detect due to § 9 D3), chains $x \in C'_{n+1}(\Delta^q, \operatorname{bd}\Delta^q)$, $c \in C_n(\operatorname{bd}\Delta^q)$ such that

$$dx = i'(c) + \bar{\zeta}^q - \bar{\zeta}'^q$$

holds. We embed Δ^q into Δ^{q+1} (= cone over Δ^q with top vertex $*$) and inclusion $w\colon \Delta^q \subset \Delta^{q+1}$. According to 7.3 there exists a $y \in C'_{n+1}(\Delta^{q+1}, *)$ such that $dy = w_\# i'(c) - s$, $s \in C'_n(*)$. We set $\bar{\xi}^{q+1} = w_\# x - y$ and calculate (omitting inclusions from our notation)

$$d\bar{\xi}^{q+1} = \bar{\zeta}^q - \bar{\zeta}'^q + s.$$

Since s is bounding in $C_*(*, *)$, we can assume without loss of generality (by eventually changing $\bar{\xi}^{q+1}$) that $s = 0$, allowing us to define $\xi^{q+1} \in \boldsymbol{Z}_{n+1}(\Delta^{q+1})$ (cf. Lemma 7.7) by $\partial_0 \xi^{q+1} = \zeta^q$, $\partial_1 \xi^{q+1} = \zeta'^q$, $\partial_i \xi^{q+1} = *$, $i > 1$. So we deduce

$$\bar{\alpha}\alpha(a) = \sigma^q \wedge \zeta'^q \approx \sigma^q \wedge \zeta^q = a. \tag{3}$$

2) Suppose we have $z \in \omega \in h_n(X, *)$, $z \in Z_n(X', *)$, where $X' = \sigma_1^q \cup \sigma_2^q$, $\partial_0 \sigma_1^q = \partial_1 \sigma_2^q$, then we apply Lemma 7.5 obtaining a partition $\{z_i \in Z'_n(\sigma_i^q, \operatorname{bd}\sigma_i^q),\ i = 1, 2\}$ of z, $dz_1 = -dz_2$ and establish $\zeta_i^q \in \boldsymbol{Z}_n(\Delta^q)$ by setting $\bar{\zeta}_i^q = z_i$, $\overline{\partial_0 \zeta_1^q} = -\overline{\partial_1 \zeta_2^q} = dz_1 = -dz_2$, $\partial_i \zeta_1^q = *$, $i > 0$ and $\partial_i \zeta_2^q = *$, $i \neq 1$. All $\zeta_i^q(\Delta^m)$, $m < n - 1$ are trivial.

According to Theorem 8.1 $\boldsymbol{E}$ is a Kan spectrum; therefore $\boldsymbol{E}(X)_k = S(X) \wedge E_k$ is also a Kan complex for all k. So we apply the Kan extension property to the effect that $a_1 = \sigma_1^q \wedge \zeta_1^q$, $a_2 = \sigma_2^q \wedge \zeta_2^q$ are replaced by one single $\sigma^q \wedge \zeta^q = a$ (establishing a $\tau^{q+1} \wedge \xi^{q+1}$ with appropriate faces a_1, a_2, a) where $a \in (\boldsymbol{E}(X)_k)_q$ is a sphere.

We define $\bar{\alpha}(z)$ by $\bar{\alpha}(\bar{\zeta}^q) = a$. Since $z \sim \bar{\zeta}^q$ we observe

$$\alpha\bar{\alpha}(z) \sim z\,. \tag{1}$$

3) Suppose that $X' = \bigcup_{i=1}^{m} \sigma_i^q$ and assume that σ_i^q has a boundary (say $\partial_0\sigma_1^q$) in common with $\bigcup_{i=2}^{m} \sigma_1^q$. Then we proceed inductively: Assume that we have a partition $\{z_i \in Z_n'(\sigma_i^q, \operatorname{bd}\sigma_i^q) \,|\, i = 2, \ldots, m\}$ which has been employed to obtain a $\bar{\sigma}^q \wedge \bar{\zeta}^q = \bar{a}$, $\partial_1\bar{\sigma}^q = \partial_0\sigma_1^q$, $\{\bar{\zeta}^q, z_1\}$ being a partition of z, then we apply 2) in order to define $\bar{\alpha}(z)$ satisfying (1).

4) Suppose that $X' = \sigma_1^q \cup \sigma_2^q$ as in the second case, but that σ_1^q, σ_2^q have now a lower dimensional simplex γ^p in common. Take a partition $\{z_1, z_2\}$ of z and turn z_1, z_2 into elements $\zeta_i^q \in \boldsymbol{Z}_n(\sigma_i^q)$ (i.e. one has $\bar{\zeta}_i^q = z_i$)) in such a way that some iterated boundary of $\sigma_1^q \wedge \zeta_1^q$ (resp. of $\sigma_2^q \wedge \zeta_2^q$) is of the form $\gamma^p \wedge \eta^p$ (for some $\eta^p \in \boldsymbol{Z}_{n-q+p}(\Delta^p)$).

Now we apply 2) to two adjacent $(p+1)$-simplexes which have $\gamma^p \wedge \eta^p$ in common and which are subsimplexes of $\sigma_1^q \wedge \zeta_1^q$ resp. $\sigma_2^q \wedge \zeta_2^q$. By this procedure we are able to raise the dimension p by one to $p+1$. This can be done until we reach the situation of case 3).

If there is no such γ^p, then we go over to ΣX observing, that now $\{\Sigma_* z\} \in h_{n+1}(\Sigma X)$ is lying on simplexes which have at least a 0-dimensional simplex in common. This establishes $\bar{a}$ for ΣX in this case.

5) Suppose that $X' = \sigma_1^{q_1} \cup \sigma_2^{q_2}$ with eventually different q_1, q_2. Assume that $q_1 < q_2$ and take instead of $\sigma_1^{q_1}$ a suitable degeneracy, yielding finally new simplexes σ_1, σ_2 satisfying the assumptions of the previous case.

6) Since each $z \in Z_n(C_*(X, *))$ comes already from a $z' \in Z_n(C_*(X', *))$, $X' = \bigcup_{i=1}^{m} \sigma_i^{q_i}$ being compact, the general case can now be settled inductively using 1)-5), however not necessarily for X but for some suspension of X.

In other words we establish to any $z \in Z_n(C_*(X, *))$ a sphere $\bar{\alpha}(\Sigma^t z) = a \in (\boldsymbol{E}(\Sigma^t X)_k)_q$, $\partial_i a = *$, $i \geqslant 0$ such that

$$\alpha\bar{\alpha}(\Sigma^t z) \sim \Sigma^t z \tag{4}$$

holds.

In order to be able to establish an inverse to α, we still need:

a) Suppose that $z \sim 0$ in $C_*(X, *)$, then we have $\bar{\alpha}(\Sigma^t z) \approx *$ for a sufficiently high suspension.

b) Suppose that $z = 0$, the zero cycle, then $\bar{\alpha}(0) = *$.

PROOF. – *Ad a*): Suppose $dx = z$, $x \in C_{n+1}(X', *)$ then Lemma 7.1 and § 9 D3) provide us with a $x' \in C_{n+1}'(X', *)$, $z' \in Z_n'(X' *)$, $s \in C_n(*)$, for suitable compact $X' \subset X$ (because $\boldsymbol{C}_*$ has compact carrier) such that $dx' = z' + s$, $\{z'\} =$

$= \{z\} \in h_n(X', *)$. Replacing z' by $z' + s \in Z_n(C'_*(X, *))$ we can assume that $dx' = z'$.

Now we repeat the six steps of the construction of $\bar{\alpha}(z)$, this time replacing the cycle z the chain x'. The necessary adaptations are easily accomplished:

$\bar{1}$) Let z be as in case 1) and suppose that $x' \in C'_{n+1}(\sigma^q)$ with $dx' = z'$. Then we embed v: $\Delta^q \subset \Delta^{q+1}$ and define ξ^{q+1} by $\bar{\xi}^{q+1} = v_{\#} x' \in Z'_{n+1}(\Delta^{q+1}, \mathrm{bd}\,\Delta^{q+1})$, $\partial_0 \xi = \zeta$ (with $\bar{\zeta} = z$, $\partial_i \zeta = *$, $i \geq 0$), $\partial_i \xi = *$, $i > 0$. This settles case 1a). The remaining cases are treated analogously.

$\bar{2}$) Concerning case 2) above, we take a partition $\{x_1, x_2\}$ of x' with $dx_1 + dx_2 = z'$, $dx_i = dy_i + z_i$, $dy_1 = -dy_2$, $\{z_1, z_2\}$ being a partition of z' and define ξ_1^{q+1}, ξ_2^{q+1} by $\bar{\xi}_i^{q+1} = \bar{x}_i$, $\overline{\partial_0 \xi_1^{q+1}} = \overline{-\partial_1 \xi_2^{q+1}} = dy_1 = -dy_2$, $\overline{\partial_2 \xi_i^{q+1}} = z_i$ (hence $\partial_2 \xi_i^{q+1} = \zeta_i$ as in 2)) and correspondingly for the $q-1$ dimensional faces. Now the argument proceeds as in 2).

$\bar{3}$-$\bar{6}$) The remaining cases are treated as for a cycle z in 3)-6). This furnishes us with a $\bar{\alpha}(x')$ satisfying $\partial_0 \bar{\alpha}(x') = \bar{\alpha}(\Sigma^t z)$, $\partial_i \bar{\alpha}(x') = *$, $i > 0$, for an appropriate suspension level t. So we confirm $\bar{\alpha}(\Sigma^t z) \approx *$.

Ad b): This is a trivial.

One main part of Theorem 1.1 is embodied in

4.1. THEOREM. – *The transformation of homology theories*

$$\alpha\colon W\mathrm{Cl}_*(\,) \to h_*(\,)$$

is an isomorphism of homology theories.

We deduce the theorem from the following:

4.2. LEMMA. – I) *Let* $\{a\} \in W\mathrm{Cl}_n(X, *)$ *be such that* $\alpha(\{a\})) = \{*\}$, *then there exists a suspension level* t *such that* $(\gamma)^t \{a\} = 0$.

II) *Let* $a \in h_n(X, *)$ *be any element, then there exists a suspension* $\Sigma^t X$ *such that* $\Sigma^t_* \omega$ *has a counterimage* $\{a\} \in W\mathrm{Cl}_{n+t}(\Sigma^t X, *)$:

$$\alpha\{a\} = \Sigma^t_* \omega .$$

PROOF OF LEMMA 4.2. – Suppose that $\alpha\{a\} = \{*\}$, then we infer from (3), *a*) *b*) and § 3 (2)

$$\bar{\alpha}\Sigma^t \alpha(\{a\}) \approx \bar{\alpha}\alpha(\gamma^t\{a\}) \approx \gamma^t(\{a\})$$

for all t and on the other hand for suitable t: $0 = \{*\} = \bar{\alpha}(\Sigma^t_* \alpha\{a\})$. This confirms I). ■

Suppose that $z \in \omega \in h_n(X, *)$, then we find according to (5) a suspension level t

such that

$$\alpha(\{\bar{\alpha}(\Sigma^t z)\}) = \Sigma^t \omega .$$

This confirms II) and completes the proof of the lemma. ■

PROOF OF THEOREM 4.1. – The theorem is a consequence of Lemma 4.2 in view of the fact that γ and Σ_* are isomorphisms, commuting with α (cf. § 3 (2)).

5. – A mapping between two classifying spectra.

Let $\boldsymbol{E} \in \mathfrak{B}$ be any spectrum, $\boldsymbol{C}_*$ any chain functor related to $W(\boldsymbol{E})_* = \boldsymbol{E}_*$ (i.e. there exists an isomorphism of homology theories $\boldsymbol{E}_* \approx H_*(\boldsymbol{C}_*)(\,))$. Then we are going to establish a homotopy equivalence in $\mathfrak{B}_h$

$$\beta\colon \mathrm{Cl}(\boldsymbol{C}_*) \to \boldsymbol{E}.$$

In the present section we are constructing β, in the next one, we prove that $\pi_*(\beta)$ is an isomorphism.

The category which turns out to be most convenient for our purpose is in this case the (simplicial) Boardman category of prespectra $\boldsymbol{E} = \{E_k, \Theta\colon \Sigma E_k \to E_{k+1}, k \in \mathbb{Z}\}$, E_k being a based simplicial set. A *function* between prespectra $\boldsymbol{f}\colon \boldsymbol{E} \to \boldsymbol{E}'$ is a family of simplicial mappings $f_k\colon E_k \to E'_k$ being compatible with the resp. bonding maps Θ. We encounter in particular the *full embeddings* or *cofinal subspectra* $\boldsymbol{i}\colon \boldsymbol{A} \subset \boldsymbol{B}$ which are characterized by the following property:

Let $K \subset B_k$ be a finite subcomplex, then there exists a natural number m and a $L \subset A_{m+k}$ such that $i_{m+k}(L) = \Sigma^m K$.

In other words: $\boldsymbol{i}$ itself needs not to be an equivalence but every finite subcomplex $K \subset B_k$ can be retained in $\boldsymbol{A}$ after suspending sufficiently long. Denoting by $\mathfrak{D}$ the category of prespectra with functions as morphisms, then the Boardman category $\mathfrak{B} = \mathfrak{D}/\{i\}$ is a quotient category where $\{i\}$ denotes the class of all full embeddings (i.e. all full embeddings are forced to become isomorphisms in $\mathfrak{B}$, cf. [2] for further references). The related homotopy category $\mathfrak{B}_h$ is well-known to be equivalent to the ordinary (i.e. topological) Boardman category as well as the homotopy category of simplicial Kan spectra $\boldsymbol{Sp}_{Eh}$.

In this section it turns out to be more convenient to deal with this model of a Boardman category, resp. its homotopy analogue $\mathfrak{B}_h$. The required mapping $\beta = [\boldsymbol{rgi}^{-1}]$ is a homotopy class of a mapping in $\mathfrak{B}$, where $\boldsymbol{g}\colon \boldsymbol{A} \to F(\boldsymbol{E})$ is a function of spectra (which will be constructed below), $\boldsymbol{i}\colon \boldsymbol{A} \subset \mathrm{Cl}(\boldsymbol{C}_*) = \tilde{\boldsymbol{E}}$ a suitable full embedding and $\boldsymbol{r}\colon F(\boldsymbol{E}) \xrightarrow{\simeq} \boldsymbol{E}$ the well-known, classical homotopy equivalence in $\mathfrak{B}$ between the free group spectrum $F(\boldsymbol{E})$ and $\boldsymbol{E}$ itself. We could replaced $F(\boldsymbol{E})$ by any Kan spectrum which is equivalent to $\boldsymbol{E}$ in $\mathfrak{B}_h$ resp. assume that we are dealing with a Kan spectrum $\boldsymbol{E}$ from the beginning.

The homology groups $\boldsymbol{E}_n(\Delta^p, \mathrm{bd}\,\Delta^p)$ are stable homotopy classes of simplexes $b^q \in (S(\Delta^p) \wedge F(\boldsymbol{E})_k)_q$ with $\partial_i b^q \in (S(\mathrm{bd}\,\Delta^p) \wedge F(\boldsymbol{E})_k)_{q-1}$, $0 \leqslant i \leqslant q$. Let us denote all these b^q by $(S(\Delta^p, \mathrm{bd}\,\Delta^p) \wedge F(E)_k)_q$.

As customary, we are using the previously mentioned equivalences of stable categories quite freely, denoting by $\boldsymbol{E}$ sometimes a spectrum in $\boldsymbol{Sp}$, sometimes its related analogue in $\mathfrak{B}$ or in $\mathfrak{B}_h$.

We have the mapping η_i which is deduced from the diagram

$$\begin{tikzcd} \boldsymbol{E}_{n-1}(\mathrm{bd}\,\Delta^p) \arrow[r] \arrow[dr, "\eta_i"'] & \boldsymbol{E}_{n-1}(\mathrm{bd}\,\Delta^p, \bigcup_{j\neq i} \partial_j \Delta^p) \arrow[d, "\int\!\!\int"] \\ & \boldsymbol{E}_{n-1}(\partial_i \Delta^p, \mathrm{bd}\,\partial_i \Delta^p) \end{tikzcd}$$

and after composing with the boundary $\partial\colon \boldsymbol{E}_n(\Delta^p, \mathrm{bd}\,\Delta^p) \to \boldsymbol{E}_{n-1}(\mathrm{bd}\,\Delta^p)$ a mapping

$$\eta_i \partial\colon \boldsymbol{E}_n(\Delta^p, \mathrm{bd}\,\Delta^p) \to \boldsymbol{E}_{n-1}(\partial_i \Delta^p, \mathrm{bd}\,\partial_i \Delta^p).$$

On the other hand we have for each $\zeta^q \in (\tilde{E}_k)_q$, $\zeta^q \in \boldsymbol{Z}_n(\Delta^p)$, the class

$$\gamma(\overline{\partial_i \zeta^q}) \in \boldsymbol{E}_{n-1}(\partial_i \Delta^p, \mathrm{bd}\,\partial_i \Delta^p),$$

$$\text{resp.}\quad \gamma(\bar{\zeta}^q) \in \boldsymbol{E}_n(\Delta^p, \mathrm{bd}\,\Delta^p)$$

being determined by $\partial_i \zeta^q$ resp. ζ^q by means of Lemma 7.1.

5.1. LEMMA. – *One has*

$$\gamma(\overline{\partial_i \zeta^q}) = \eta_i \partial \gamma(\bar{\zeta}^q).$$

The proof is in view of the construction of ∂_i (in § 2) resp. of partitions in 7.5 immediate. ∎

Now we construct a subspectrum $\boldsymbol{A} = \{A_k\} \subset \tilde{\boldsymbol{E}}$ being determined by the following conditions:

The q-skeleton $(A_k)_q \subset (\tilde{E}_k)_q$ consists of all those $\zeta^q \in \boldsymbol{Z}_n(\Delta^p)$ satisfying:

A1) $p \leqslant q$.

A2) To each subsimplex $\Delta^m \subset \Delta^p$ there exists a

$$a(\Delta^m) = \sigma^{q-p+m} \wedge \tau^{q-p+m} \in (S(\Delta^m, \mathrm{bd}\,\Delta^m) \wedge F(\boldsymbol{E})_k)_{q-p+m}$$

with $\{a(\Delta^m)\} = \gamma(\zeta^q(\Delta^m)) \in \boldsymbol{E}_{n-p+m}(\Delta^m, \mathrm{bd}\,\Delta^m)$ satisfying

$$\begin{cases} \partial_i a(\Delta^m) = a(\partial_i \Delta^m), \\ \{\partial_i a(\Delta^m)\} = \eta_i \partial \{a(\Delta^m)\}. \end{cases} \tag{1}$$

Since each $A_k \subset \tilde{E}_k$ is easily recognized as a simplicial set, we have the subspectrum $\boldsymbol{A} = \{A_k\} \subset \tilde{\boldsymbol{E}} = \{\tilde{E}_k\}$ and claim:

5.2. LEMMA. – *The inclusion* $\boldsymbol{i}$: $\boldsymbol{A} \subset \tilde{\boldsymbol{E}}$ *is a full embedding.*

PROOF. – Observe that a $\zeta^q \in \boldsymbol{Z}_n(\Delta^p)$ can be considered as a $\zeta^q \in (\tilde{E}_k)_q$ as well as a $\tilde{\zeta}^{q+r} \in (\tilde{E}_{k+r})_{q+r}$, $r \geq 0$ because $n = q - k = (q + r) - (k + r)$. So we can assume, by going over to a sufficiently high k (i.e. by eventually suspending), that $p \leq q$. We have a with ζ^q and $\Delta^m \subset \Delta^p$ associated

$$a(\Delta^m) = \sigma^{q(m)} \wedge \tau^{q(m)} \in \big((S(\Delta^m, \mathrm{bd}\,\Delta^m) \wedge F(\boldsymbol{E}))_{l(m)}\big)_{q(m)}$$

such that

$$\{a(\Delta^m)\} = \gamma\,\overline{(\zeta^q(\Delta^m))} \in \boldsymbol{E}_{n-p+m}(\Delta^m, \mathrm{bd}\,\Delta^m)$$

$$n - p + m = q(m) - l(m), \qquad m = 0, \ldots, p$$

and that

$$\eta_i \partial\{a(\Delta^m)\} = \{a(\partial_i \Delta^m)\}, \qquad i = 0, \ldots, m.$$

Without loss of generality (by suspending a sufficient number of times) we find a universal l such that 1) $l(m) = l$ and therefore $q(m) = n - p + m + l$ and 2) $k \leq l$. By suspending ζ^q (i.e. by regarding ζ^q as a simplex of $\tilde{E}_l$) we obtain a new $\zeta^{q+r} \in (\tilde{E}_l)_{q+r}$ (being equal to the original ζ^q as an element of $\boldsymbol{Z}_n(\Delta^p)$) which determines a function $a(\,)$ which satisfies (1) *up to a stable homotopy*. However by again suspending a sufficient numbler of times we convert this *stable* homotopy into an *ordinary* one. Let $\partial_i a(\Delta^m)$ be inductively defined such that (1) holds for all dimensions $\leq m$ and $0 \leq i \leq m$, then we can add this homotopy to the given $a(\Delta^m)$ (in a well-known manner) in such a way that (1) holds (now strictly). So A2) is satisfied. As a result we find to each ζ^q an $s \geq 0$ such that $\Sigma^s \zeta^q \in A_{k+s}$. Hence the inclusion i: $\boldsymbol{A} \subset \tilde{\boldsymbol{E}}$ reveals itself as a full embedding. ■

We come now to the construction of a function $\boldsymbol{g}: = \{g_k : A_k \to F(E_k)\}$: Let $\zeta^q \in (A_k)_q$ be a given, then we have

$$a(\Delta^p) = \sigma^q \wedge \tau^q \in (S(\Delta^p, \mathrm{bd}\,\Delta^p) \wedge F(E)_k)_q$$

and set

$$g_k^q(\zeta^q) = \tau^q.$$

Condition A2) guarantees that g_k is a simplicial map. Moreover we have a commutative square

$$\begin{array}{ccc} \Sigma A_k & \longrightarrow & A_{k+1} \\ {\scriptstyle \Sigma g_k}\downarrow & & \downarrow{\scriptstyle g_{k+1}} \\ \Sigma F(E)_k & \longrightarrow & F(E)_{k+1}. \end{array}$$

Together with the well-known homotopy equivalence $\boldsymbol{r}: F(\boldsymbol{E}) \to \boldsymbol{E}$ we otain the desired $\beta_E = [\boldsymbol{rgi}^{-1}] \in \mathfrak{B}_h(\mathrm{Cl}(\boldsymbol{C}_*), \boldsymbol{E})$. ∎

6. – Proof of Theorem 1.1.

What remains to verify in order to complete the proof of Theorem 1.1 is the fact that the morphism β_E (established in § 5) is an equivalence in $\mathfrak{B}_h$.

To this end it suffices to observe that by construction β_E induces an isomorphism of homology groups

$$\pi_*(S^p \wedge \beta_E): \pi_*(S^p \wedge \tilde{\boldsymbol{E}}) \xrightarrow{\approx} \pi_*(S^p \wedge \boldsymbol{E})$$

for all p-spheres (because $\pi_*(S^p \wedge \tilde{\boldsymbol{E}}) = \tilde{\boldsymbol{E}}_*(\Delta^p, \mathrm{bd}\,\Delta^p)$, $\pi_*(S^p \wedge \boldsymbol{E}) = \boldsymbol{E}_*(\Delta^p, \mathrm{bd}\,\Delta^p)$ and $\boldsymbol{g}$ induces isomorphisms of these relative groups).

So we conclude that

$$\pi_*(\beta_E): \pi_*(\tilde{\boldsymbol{E}}) \approx \pi_*(\boldsymbol{E})$$

is an isomorphism.

In view of the Whitehead theorem in $\mathfrak{B}_h$, β_E turns out to be a homotopy equivalence.

Since the other parts of Theorem 1.1 have already been settled by verifying 4.1 the proof of Theorem 1.1 is complete. ∎

7. – The «relative cycles».

In this and in the following section we are establishing some results which are needed in the proof of the main Theorem 1.1.

Let $C_*: \boldsymbol{K} \to \boldsymbol{ch}$ be any functor on a category of topological spaces into the category of chain complexes, such that inclusions induce monomorphisms and suppose $(X, A) \in \boldsymbol{K}^2$ is a pair, then a relative cycle $z \in C_n(X)$ is a chain such that $dz \in \mathrm{im}\,(C_{n-1}(A) \to C_{n-1}(X))$.

In case of a chain functor $\boldsymbol{C}_*$ carrying the additional structure displayed in § 9 the situation is a little more subtle. We define

$$Z'_n(X, A) = \{z \in C'_n(X, A) \,|\, dz \in \mathrm{im}\,(i': C_{n-1}(A) \to C'_{n-1}(X, A))\}.$$

The following properties of $Z'_n(X, A)$ ensure that the terminology *relative cycle* for a $z \in Z'_n(X, A)$ is justified.

7.1. LEMMA. – 1) *Let $z \in Z_n(C_*(X, A))$ be a cycle, then there exists a $z' \in Z'_n(X, A)$ as well as a $a \in C_n(A, A)$ satisfying $z \sim lz' + q_\#(a)$ in $C_*(X, A)$.*

2) *Suppose that* $lz_1' + q_\#(a_1) \sim lz_2' + q_\#(a_2)$ *in* $C_*(X,A)$, *then we have* $i'^{-1}dz_1 \sim i'^{-1}dz_2$ *in* $C_*(A)$.

PROOF. – *Ad* 1): Follows because $\psi\colon H_*(C_*''(X,A)) \to H_*(C_*(X,A))$ is epic.

Ad 2): We have $[z_1' - z_2] \in \ker\psi$, hence $[d(z_1' - z_2')] \in \ker\bar\partial$, (cf. § 9, D3) whence $[i'^{-1}dz_1] = [i'^{-1}dz_2] \in H_*(C_*(A))$ follows. ∎

7.2. LEMMA. – *Let* (X_i, A_i), $i = 1, 2$, $A_i = B_i \cup A$, $X_i \cap X_2 = A$ *be a pairs and spaces in* $\boldsymbol{K}$, $z_i \in Z_n'(X_i, A_i)$, *satisfying* $i'^{-1}dz_i = b_i + a$. $b_i \in C_{n-1}(B_i)$, $a \in C_{n-1}(A)$, *then we have*

$$[lz_1 - lz_2] \in \operatorname{im}(H_n(C_*(X_1 \cup X_2, B_1 \cup B_2)) \to H_n(C_*(X_1 \cup X_2, A_1 \cup A_2))).$$

PROOF. – Assume $B_1 = B_2 = \emptyset$, $A = A_1 = A_2$, then we have $d(z_1 - z_2) = 0$, $j_*\varphi([z_1 - z_2]) = l_*[z_1 - z_2] = [lz_1 - lz_2]$, $j_*\colon H_n(C_*(X_1 \cup X_2)) \to H_n(C_*(X_1 \cup X_2, A))$. The general case follows similarly.

REMARKS. – 1) The preceding assertion represents a kind of «Mayer-Vietoris property» for chain functors. Because the boundary $\partial\colon H_n(C_*(X, A)) \to H_{n-1}(C_*(A))$ is established by means of the short exact sequence $(\bar 2)$ in § 9, i.e. by employing C_*' other than C_*, we are dealing with Z_n' (instead of e.g. $Z_n(C_*(X, A))$ as in the case of ordinary homology theories).

2) Let $z \in Z_n'(X, A)$ be a relative cycle, then Lemma 7.1 asserts the existence of an element $[l(z) + q_\#(a)] \in H_n(C_*(X, A))$ which depends solely on z (but not on a). By an abuse of notation we will sometimes write $[z] \in H_n(C_*(X, A))$ instead of $[l(z) + q(a)] \in H_n(C_*(X, A))$.

7.3. LEMMA. – *Let* $H\colon f_0 \simeq f_1\colon (X, A) \to (Y, B)$ *be a homotopy satisfying the assumptions of Lemma* 9.8 *and* $z \in Z_n'(X, A)$ *such that* $f_{i\#}(z) \in \operatorname{im}(C_n(B, B) \to C_n(Y, B))$, *then there exists a* $D'(z) \in Z_{n+1}'(Y, B)$, *satisfying* $dD'(z) = dD(z)$ *and* $D(z) - lD'(z) \sim 0$ *in* $C_*(Y, B)$.

PROOF. – We have $dD(z) + D(dz) = f_{0\#}(z) - f_{1\#}(z)$ and notice that $dz \in \operatorname{im}(i'\colon C_{n-1}(A) \to C_{n-1}'(X, A))$ implies $D(dz) \in \operatorname{im}(i'\colon C_n(B) \to C_n'(Y, B))$. By assumption $f_{i\#}(z) \in \operatorname{im}(C_n(B, B) \to C_n(Y, B))$ so that $dD(z) \in \operatorname{im}(C_n(B, B) \to C_n(Y, B))$, where it bounds because $C_n(B, B)$ is acyclic: $dD(z) = -dq_\#(b)$, $b \in C_{n+1}(B, B)$. Due to Lemma 7.1 we have 1) a $z' \in Z_{n+1}'(Y, B)$, 2) a $b' \in C_{n+1}(B, B)$ such that $l(z') + q_\#(b') \sim D(z) + q_\#(b)$ in $C_*(Y, B)$. As a result we obtain a chain $y \in C_{n+2}(Y, B)$ satisfying

$$dy = l(z') + q_\#(b' - b) - D(z).$$

Suppose $y = x + y_1$, $x \in C_{n+2}(Y, B) - C_{n+2}(B, B)$, $y_1 \in C_{n+2}(B, B)$, then Lemma 9.8

ensures that $D(z) \in \tilde{C}_{n+1}(Y, B)$ and we detect $x_2 = b' - b - dy_1 \in C_{n+1}(B, B)$ such that $dx = l(z') - D(z) + q(x_2)$, $-dx_2 = l\,dz' - dD(z)$ in $C_*(B, B)$.

According to Lemma 9.7 1) we have $dx \in \tilde{C}_{n+1}(Y, B)$. Since $l(z') - - D(z) \in \tilde{C}_{n+1}(Y, B)$ we deduce $q_{\#}(x_2) \in \tilde{C}_{n+1}(Y, B)$. Therefore the element x_2 (inclusions are not written down) can be expressed as $x_2 = c_1 + c_2$, $c_1 \in C'_{n+1}(Y, B)$, $c_2 \in C_{n+1}(Y, B) - C_{n+1}(B, B)$. On the other hand Lemma 9.7 3) implies $c_1 = c_1' + c_1''$, $c_1' \in (C_{n+1}(Y, B) - C_{n+1}(B, B)) \cap C'_{n+1}(Y, B) = C_{n+1}(B)$ (due to Lemma 9.7 2) or [1], Lemma 5.4). We infer that

$$c_1' + c_2 = x_2 - c_1'' \in (C_{n+1}(Y, B) - C_{n+1}(B, B)) \cap C_{n+1}(B, B) = \{0\}.$$

This implies $x_2 \in \operatorname{im}(C_{n+1}(B) \to C_{n+1}(B, B))$, allowing us to set

$$D'(z) = z' + x_2 \in Z'_{n+1}(Y, B).$$

As a result we have $dD'(z) = dD(z)$ and

$$D(z) - lD'(z) \sim 0 \quad \text{in } C_*(Y, B), \tag{1}$$

completing the proof of Lemma 7.3. ∎

Our main application consists in the case where $(Y, B) = C(X, A)$, the reduced cone over (X, A) with top vertex $* \in A$, while $f_0: (X, A) \subset C(X, A)$, $f_1 = 0$, $0(x) \equiv *$, are the inclusion resp. the projection.

We obtain

$$dD'(z) + D'(dz) = i_{\#}(z) - 0_{\#}(z),$$

$0_{\#}(z) \in C_n'(*, *)$. With each $\tilde{z} \in Z_n(C_*(X, A))$ there is associated 1) a $z \in Z_n'(X, A)$, 2) an $a \in C_n(A, A)$, satisfying $\tilde{z} \sim l(z) + q_{\#}(a)$ (cf. Lemma 7.1). Let $\Sigma(X, A) = = C_+(X, A) \cup C_-(X, A)$, $C_+ \cap C_- = (X, A)$ be the reduced suspension, then we have $0_{\#}^+(z) = 0_{\#}^-(z)$ and according to Lemma 7.2

$$\Sigma z = D_+'(z) - D_-'(z) \in Z'_{n+1}(\Sigma X, \Sigma A),$$

moreover

$$\Sigma\, dz = d\Sigma z \in \operatorname{im}(C_n(\Sigma A) \to C_n'(\Sigma X, \Sigma A)).$$

Lemma 7.1 yields a cycle $l\Sigma z + q_{\#}(\bar{a}) \in Z_n(C_*(\Sigma X, \Sigma A))$, $\bar{a} \in C_{n+1}(\Sigma A, \Sigma A)$ such that

$$[l\Sigma z + q_{\#}(\bar{a})] = \Sigma_*[\tilde{z}] \in H_{n+1}(\boldsymbol{C}_*)(\Sigma X, \Sigma, A), \tag{2}$$

where $\Sigma_*: H_n(\boldsymbol{C}_*)(X, A) \to H_{n+1}(\boldsymbol{C}_*)(\Sigma X, \Sigma A)$ is the suspension homomorphism on the homology level.

Let again $\tilde{z} \in Z_n(C_*(X, A))$ be a cycle, then we can repeat this construction with $D(z)$ resp. $D_+(z)$, $D_-(z) \in C_{n+1}(\Sigma X, \Sigma A)$ (now using the homotopy axiom Definition 9.3 C_* 2) for $\boldsymbol{C}_*$ itself, instead of 7.3.) obtaining $\Sigma\tilde{z} \in Z_{n+1}(C_*(\Sigma X, \Sigma A))$. We have again

$$\Sigma_*[\tilde{z}] = [\Sigma\tilde{z}].$$

We summarize:

7.4. LEMMA. – *The suspension* (2) *yields a homomorphism*

$$\Sigma\colon Z'_n(X, A) \to Z'_{n+1}(\Sigma X, \Sigma A) \tag{3}$$

inducing the homology suspension $\Sigma_*\colon H_*(\boldsymbol{C}_*)(\,) \to H_{*+1}(\boldsymbol{C}_*)(\Sigma\,)$. *This suspension can be chosen in such a way that for any relative cycle* $z \in Z'_n(X, A)$, $l(z) + q_\#(b) \in Z_n(C_*(X, A))$ (*cf. Lemma* 7.1) *one has*

$$\{\varphi_\#\Sigma z + i_\#(\Sigma b)\} = \Sigma_*\{\varphi_\# z + i_\# b\}.$$

In other words: Σ *commutes with* $\varphi_\#$ *as long as this makes sense.*

PROOF. – Only the last assertion needs proof. We have homologies

$$\Sigma lz + \Sigma q_\#(b) \sim l\Sigma z + \Sigma q_\#(b),$$

$$j_\#\Sigma\varphi_\# z + j_\# i_\#\Sigma b \sim j_\#\varphi_\#\Sigma z + j_\# i_\#\Sigma b,$$

in $C_*(\Sigma X, \Sigma A)$.

Hence there exist $y \in C_{n+1}(\Sigma X)$, $a \in C_n(\Sigma A)$ such that

$$dy = \varphi_\#\Sigma z - \Sigma\varphi_\# z + a.$$

On the other hand a is easily recognized as a cycle $a \in Z_n(\boldsymbol{C}_*(\Sigma A))$ (observe that $d\varphi_*\Sigma z = d\Sigma\varphi_* z$).

We replace Σz by $\Sigma' z = \Sigma z + \kappa_* a$ to the effect that

$$\begin{aligned}\varphi_\#\Sigma' z - \Sigma\varphi_\# z &= \varphi_\#\Sigma z - \Sigma\varphi_\# z + a + \varphi_\#\kappa_\# a - a = \\ &= dy + \varphi_\#\kappa_\# a - a = dy + dy' = dy'', \qquad y'' \in C_{n+1}(\Sigma X)\end{aligned}$$

with $y' \in C_{n+1}(\Sigma X)$ originating from the chain homotopy $\varphi_\#\kappa_\# \simeq 1$.

This confirms on one hand

$$l\Sigma z + q_\#(b) \sim l\Sigma' z + q_\#(b)$$

in $C_*(\Sigma X, \Sigma A)$ and

$$\{\varphi_\#\Sigma' z + i_\#(b)\} = \{\Sigma\varphi_\# z + i_\#(b)\}$$

on the other. ∎

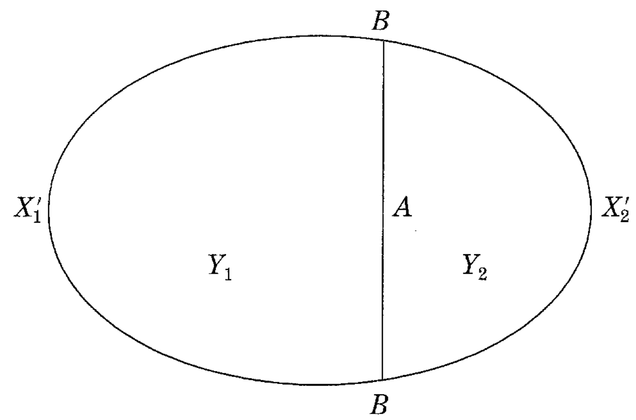

Figure 1.

REMARK. – This Σ in (3) is the suspension on the chain-level.

Let the following spaces in $\boldsymbol{K}$ be given: $Y = Y_1 \cup Y_2$, $Y_1 \cap Y_2 = A$, $X_i' \subset X_i \subset Y_i$, $X_1 \cap X_2 = A$, $X_1' \cap X_2' = B \subset A$, $X' = X_1' \cup X_2'$ (cf. fig. 1). Suppose that we have a relative cycle $z \in Z_n'(Y, X')$, then two relative cycles $z_i \in Z_n'(Y_i, X_i)$, $i = 1, 2$ are called a *partition of* z whenever we have:

1) $$\kappa'_{\#}\varphi_{\#}(z_1 + z_2) \in Z_n'(Y, X'),$$

$\varphi_{\#}: C_n'(Y, X) \to C_n(Y)$, $\kappa'_{\#}: C_n(Y) \to C_n'(Y, X')$, inclusions omitted from the notation, and

2) $$\{\kappa'_{\#}\varphi_{\#}(z_1 + z_2)\} = \{z\} \in h_n(Y, X')$$

for the associated homology classes.

Suppose on the other hand that $z_i \in Z_n'(Y_i, X_i)$, $i = 1, 2$ are two relative cycles such that

$$d\varphi_{\#}(z_1 + z_2) \in \operatorname{im}(C_n(X')) \to C_n(X)),$$

$z_1 + z_2 \in C_n'(Y, X)$, then we have $\kappa'_{\#}\varphi_{\#}(z_1 + z_2) \in Z_n'(Y, X')$ (cf. § 9 D1)) and we call $z = \kappa'_{\#}\varphi_{\#}(z_1 + z_2)$ the *composite* of $\{z_1, z_2\}$.

It is immediately clear that $\{z_1, z_2\}$ is a partition of z. If z admits a partition $\{z_1, z_2\}$, then z is not necessarily the composite z' of z_1, z_2 but z and z' determine the same homology class in $h_n(Y, X')$. So, from this point of view, both processes can be considered as inverses to each other.

7.5. LEMMA. – *Each* $z \in Z_n'(Y, X')$ *admits a partition* $\{z_1, z_2\}$.

PROOF. – 1) We start with the absolute case: $X' = \emptyset$, $z \in Z_n(C'_*(Y))$, $X = A$, $B = \emptyset$. Let $\{z\} = \zeta \in h_n(Y)$ be the homology class of z, then by taking inclusions resp. excisions, we obtain homology classes $\zeta_i \in h_n(Y, Y_j)$, $i \neq j$, $\zeta_i' \in h_n(Y_i, A)$ and relative cycles $z_i' \in Z_n'(Y, A)$, $z_i' \in \zeta_i'$. The sum $\zeta_1 + \zeta_2$ is equal to ζ after application of the inclusion $(Y, \emptyset) \subset (Y, A)$.

Therefore (omitting henceforth inclusions from our notation) $z_1' + z_2' - z \in Z_n'(Y, A)$ is a bounding relative cycle in $C_*(Y,A)$. Due to § 9, D3) $\bar{\partial}(z_1' + z_2' - z) \in C_{n-1}(A)$ is bounding, delivering a $b \in C_n(A)$ satisfying

$$db = dz_1' + dz_2' - dz.$$

We set $z_1'' = z_1' - b \in Z_n'(Y_i, A)$, $z_2'' = z_2'$ so that $z_1'' + z_2'' - z \in Z_n'(Y, A)$ satisfies $dz_1'' + dz_2'' = dz$. Hence $dz_1'' + dz_2'' = dz = 0$ and we conclude that

$$\kappa_\#' \varphi_\# (z_1'' + z_2'') \in Z_n'(Y),$$

$$\kappa_\#' : C_n(Y) \to C_n'(Y, X'), \qquad \varphi_\# : C_n'(Y, X) \to C_n(Y).$$

We consider the mappings (displayed already for the general case because of further need)

$$\varphi_\#' : C_n'(Y, X') \to C_n(Y), \qquad l' : C_n'(Y, X') \to C_n(Y, X'), \qquad \varphi_\#'' : C_n'(Y, B) \to C_n(Y),$$

$$j : Y \subset (Y, X), \quad j' : Y \subset (Y, X'), \quad k : (Y, X') \overset{\subset}{\to} (Y, X), \quad k' : (Y, B) \subset (Y, X'), \quad j'' : Y \subset (Y, B)$$

(hence $kj' = j$, $k'j'' = j'$ in the general case and $j' = j'' = k' = 1_Y$, $k = j$ in the special case).

Since $z_i'' \in \zeta_i'$, we conclude that

$$l(z_1'' + z_2'') - lj_\# z \sim j_\# \varphi_\#' (\kappa_\#' \varphi_\# (z_1'' + z_2'') - z) \in C_n(Y, A)$$

is bounding. Exactness of the homology sequence of the pair (Y,A) (in the general case: of the pair of pairs $((Y,X'), (X,X'))$ yields a $\bar{a} \in Z_n'(A)$ such that

$$\kappa_\#' \varphi_\# (z_1'' + z_2'') + \bar{a} - z \sim 0 \quad \text{in } h_n(Y). \tag{4}$$

We introduce $a = \kappa_\# \varphi_\#'' \bar{a} \in C_n'(Y, A)$ $(\kappa_\# : C_n(Y) \to C_n'(Y, A))$, observe that $da = \kappa_\# \varphi_\#'' d\bar{a} = 0$, so that $d(z_1'' + a) = dz_1''$. We define $z_1 = z_1'' + a$, $z_2 = z_2''$. There exists a chain homotopy $\varphi_\# \kappa_\# \simeq 1$, therefore $\varphi_\# a$ and $\varphi_\#'' \bar{a}$ are homologous in $C_*(Y)$. So we calculate

$$\kappa_\#' \varphi_\# (z_1 + z_2) - z = \kappa_\#' \varphi_\# (z_1'' + z_2'') + \kappa_\#' \varphi_\# a - z, \tag{5}$$

and (formulated for the general case)

$$j_\#' \varphi_\#'' = k_\#' j_\#'' \varphi_\#'' \sim k_\#' l'' = l' k_\#',$$

$l'' : C_n'(Y, B) \to C_n(Y, B)$, so that application of l' yields

$$l' \kappa_\#' \varphi_\# a \sim j_\#' \varphi_\# a = j_\#' (dy + \varphi_\#'' \bar{a}) \sim l' k_\#' \bar{a} \tag{6}$$

in $C_*(Y)$. As a result we infer from (4)-(6)

$$\{\kappa'_\# \varphi_\# (z_1 + z_2)\} = \{z\} \quad \text{in } h_n(Y).$$

This settles the absolute case $X' = \emptyset$. ∎

2) The general case is treated like the absolute case by taking into account the following adaptations, which we are now going to display schematically:

$$z_i' \in \zeta_i' \in h_n(Y_i, X_i), \qquad x_i' \in \xi_i' \in h_{n-1}(X_i', B) \approx h_{n-1}(X_i, A),$$

$$a_i \in \alpha_i \in h_{n-1}(A, B) \approx h_{n-1}(X_i, X_i')$$

(by excision) such that

$$\partial \zeta_i' = \xi_i' + \alpha_i \quad \text{in } h_{n-1}(X_i, B).$$

Let $t \in Z_n'(X')$ be such that $t \sim dz$ in $C_*(X')$, then $\{x_1', x_2'\}$ can be choosen (because of 1)) to be a partition of t; $dx_1' = -dx_2' = da_i \in C_{n-2}(B)$. Obviously $\alpha_1 = -\alpha_2 = \alpha$, hence $a_1 = -a_2 = a \in Z_{n-1}(A, B)$.

$\{x_i', (-1)^{i+1} a\}$ has a composite $t_i \sim dz_i'$. We find $z_i'' \sim z_1'$ such that dz_i'' is the composite of $\{x_i', (-1)^{i+1} a\}$. Hence $d\kappa'_\# \varphi_\# (z_1'' + z_2'')$ has a partition $\{x_1', x_2'\}$ so that

$$\kappa'_\# \varphi_\# (z_1'' + z_2'') \in Z_n'(Y, X').$$

We reach $\{\bar{a}\} \in h_n(X, X') \approx h_n(A, B)$ (by excision $\bar{a} \in Z_n'(A, B)$ now employing the exactness of the homology sequence of the pair of pairs $((Y,X'), (X,X'))$ such that (4) (with $h_n(Y, X')$ replacing $h_n(Y)$) holds.

We have again $a = \kappa_\# \varphi''_\# \bar{a} \in C_n'(Y, X)$ and

$$d\varphi_\# i' d\bar{a} = c + i_\# d\bar{a},$$

where according to Remark 3) following D3) in § 1, $c \in C_{n-1}(B)$ stems from the chain homotopy $\varphi_\# i' \simeq i$.

So (i' omitted from the notation) we get $\varphi_\# d\bar{a} = d\varphi_\# \bar{a} \in C_{n-1}(B)$. On the other hand, since $\kappa_\# i_\# = i'_\#$ (§ 9, D1)) we infer $da = \kappa_\# d\varphi''_\# \bar{a} \in C_{n-1}(B)$. As a result $d(z_1'' + a) \in C_{n-1}(X_1)$, allowing us to argue as in the absolute case. We obtain again (5) and (6) (replacing $C_n(Y)$ by $C_n(Y, X')$) so that

$$\{\kappa'_\# \varphi_\# (z_1 + z_2)\} = \{z\} \quad \text{in } h_n(Y, X')$$

follows.

This completes the proof of Lemma 7.5. ∎

For the sake of completeness we add:

7.6. LEMMA. - 1) *Suppose that* $z_i \in Z_n'(Y_i, X_i)$, $i = 1, 2$, *are two relative cycles, satisfying* $d\varphi_\# (z_1 + z_2) \in \operatorname{im}(C_n(X') \to C_n(X))$, *then* $z = k'_\# \varphi_\# (z_1 + z_2) \in Z_n'(Y, X')$ *and* $\{z_1, z_2\}$ *is a partition of* z.

2) *Let on the other hand* $\{z_1, z_2\}$ *be a partition of some* $z' \in Z'_n(Y, X')$, $z = \kappa'_{\#} \varphi_{\#}(z_1 + z_2)$ *the composite of* z_1, z_2, *then* $\{z\} = \{z'\}$ *in* $h_n(Y, X')$.

3) *Let* $z \in Z'_n(Y, X')$ *and* $\{z_1, z_2\}, \{z'_1, z_2\}$ *be two partition of* z; $z_1, z'_1 \in Z'_n(Y_1, X_1)$, $z_2 \in Z'_n(Y_2, X_2)$, *then we have* $lz_1 - lz'_1 \sim 0$ *in* $C_*(Y_1, X_1)$.

While 1) and 2) follow immediately, we observe concerning 3) that $z_1 - z'_1$ is a relative cycle in $Z_n(Y_1, X_1)$ whose homology class is mapped under the inclusion $(Y_1, X_1) \subset (Y, Y_2)$ into the class of $z - z = 0$. An excision argument yields the result. This confirm 3). ∎

REMARKS. – 1) Observe that $\{z_1, z_2\}$ determines (unlike the pair $\{\zeta'_1, \zeta'_2\}$) the element $\zeta = \{z\} \in h_n(Y, X')$ (rather than merely its image in $h_n(Y, X)$). 2) The process of partitioning can be iterated inductively from two to any finite number $\{z_1, \dots, z_n\}$ of relative cycles. The most frequent example will be the following. Let Δ^p be a p-simplex, $z \in Z'_n(\Delta^p, \mathrm{bd}\,\Delta^p)$ a relative cycle, $\{(-1)^i a_i \mid a_i \in Z'_{n-1}(\partial_i \Delta^p, \mathrm{bd}\,\partial_i \Delta^p), i = 0, \dots, p\}$ a partition of $dz \in Z'_{n-1}(\mathrm{bd}\,\Delta^p)$, (more precisely: of some $\hat{z} \in Z'_{n-1}(\mathrm{bd}\,\Delta^p)$, $\hat{z} \sim dz$ in $C_{n-1}(bd\,\Delta^p)$) the sign $(-1)^i$ is introduced for convenience. We infer from Lemma 7.6.3):

($*$) Let $\{a_i\}$, $\{a'_i\}$ be two partitions of dz, $a_i = a'_i$, $i \neq j$, then, we have $a_j \sim a'_j$ in $C_*(\partial_j \Delta^p, bd\,\partial_j \Delta^p)$.

($**$) Let $\{z_1, z_2\}$ be a partition of z as in Lemma 7.5, then there exists 1) a partition $\{b_i \in Z'_{n-1}(X'_i, B), i = 1, 2\}$ of dz and 2) an element $a \in Z'_{n-1}(A, B)$ such that $\{b_i, (-1)^i a\}$ is a partition of dz_i, $i = 1, 2$.

PROOF. – We find b_1, b_2 as in Lemma 7.5 as well as $a_i \in Z'_{n-1}(A, B)$ such that $\{b_1, b_2\}$ (resp. $\{b_1, b_2, a_2, -a_1\}$ is a partition of $dz \in Z'_{n-1}(X', B)$ (resp. of its image in $Z'_{n-1}(X, B)$). Since on the other hand $\{b_1, b_2, 0\}$ also has this property, we infer from Lemma 7.6. 3) that $a_1 - a_2 \sim 0$ in $C_*(A, B)$ enabling us to set $a = a_2$, $a_1 = -a$. ∎

Of course ($**$) is again some form of a Mayer-Vietoris property for chain complexes (cf. remark following Lemma 7.2).

This process of partitioning of a relative cycle is used at many different occasions.

Here we need it for the introduction of the set $\boldsymbol{Z}(\Delta^p)$:

Let $\Delta^p = (a_0, \dots a_p)$ be a non-degenerate p-simplex with boundaries $\partial_j \Delta^p = (a_0, \dots, \hat{a}_j, \dots, a_p)$, $0 \leq j \leq p$. A simplex $\Delta^m = \partial_{i_1} \dots \partial_{i_{p-m}} \Delta^p$, $m \leq p$ is called a *subsimplex of* Δ^p.

The elements of $\boldsymbol{Z}_n(\Delta^p)$ are functions $\zeta = \zeta_n$, assigning to each subsimplex $\Delta^m \subset \Delta^p$ 1) a subsimplex $\Delta^k \subset \Delta^m$ (v: $\Delta^k \subset \Delta^m$ being the inclusion) and 2) a $\zeta(\Delta^m) \in$

$\in Z'_{n-p+m}(\Delta^k, \mathrm{bd}\,\Delta^k)$ such that

$$\{(-1)^i v_{i\#}\,\zeta(\partial_i \Delta^m) \mid v_i \colon \Delta^{k_i} \subset \partial_i \Delta^m\}$$

is a partition of $\overline{dv_\# \zeta(\Delta^m)}$, $v\colon \Delta^k \subset \Delta^m$.

We call $\overline{\zeta(\Delta^m)}$ the element $v_\# \zeta(\Delta^m) \in Z'_{n-p+m}(\Delta^m, \mathrm{bd}\,\Delta^m)$, $v\colon \Delta^k \subset \Delta^m$ being the inclusion.

We define $\partial_i \zeta \in \boldsymbol{Z}_{n-1}(\partial_i \Delta^p)$, where $\zeta(\partial_i \Delta^p) \in Z'_{n-1}(\Delta^k, \mathrm{bd}\,\Delta^k)$, by

$$(\partial_i \zeta)(\Delta^m) = \zeta(\Delta^m)\,, \qquad \Delta^m \subset \partial_i \Delta^p$$

observing that this defines in fact a $\partial_i \zeta \in \boldsymbol{Z}_{n-1}(\partial_i \Delta^p)$ and that we have

$$\partial_i \partial_j \zeta = \partial_{j-i} \partial_i \zeta \qquad i < j_j$$

degeneracies $s_j \zeta$ are treated in § 2.

7.7. Lemma. – *To each $z \in Z'_n(\Delta^p, \mathrm{bd}\,\Delta^p)$ there exists a $\zeta \in \boldsymbol{Z}_n(\Delta^p)$ such that $\bar{\zeta} = \overline{\zeta(\Delta^p)} = z$.*

Proof. – We start with a given partition $\{(-1)^j a_j \in Z'_{n-1}(\partial_j \Delta^p, \mathrm{bd}\,\partial_j \Delta^p)\}$ and proceed with partitions of each a_j (taking into account remark $(**)$ following Lemma 7.6) until we have established a $\zeta \in \boldsymbol{Z}_n(\Delta^p)$ with $\bar{\zeta} = z$. ∎

Suppose $\zeta \in \boldsymbol{Z}_n(\Delta^p)$, then we can use 7.3 to establish a $C\zeta \in \boldsymbol{Z}_{n+1}(\Delta^{p+1})$, $\Delta^{p+1} = C\Delta^p$, the cone over Δ^p. We define

$$(C\zeta)(\Delta^{p+1}) = D(\zeta(\Delta^p))\,,$$

$$(C\zeta)(\Delta^m) = \begin{cases} D(\zeta(\Delta^{m-1})) \ldots \Delta^m = C\Delta^{m-1}, & \Delta^{m-1} \subset \Delta^p \\ (\zeta(\Delta^m) \ldots \Delta^m) \subset \Delta^p \subset \Delta^{p+1} = C\Delta^p. \end{cases}$$

The construction of $D(\zeta(\ ))$ is taken from 7.3 using the inclusions resp. projection $\Delta^p \subset C\Delta^p = \Delta^{p+1} \leftarrow \Delta^p$ in the same way as we argued in the proof of 7.4. ∎

Let $v\colon \Delta^p = (a_0, \ldots, a_p) \subset \Delta^q = (b_0, \ldots, b_q)$ be an inclusion, $v(a_i) = b_i$, $0 \leq i \leq p$, $\zeta \in \boldsymbol{Z}_n(\Delta^p)$, then we call a subsimplex $\Delta^m \subset \Delta^q$ *full* whenever Δ^m has the form

$$\Delta^m = (v(a_{i_0}), \ldots, v(a_{i_l}))\,, \qquad b_{p+1}, \ldots, b_q)\,,$$

defining $\Delta^l = (b_{i_0}, \ldots, b_{i_l})$, $l = m + p - q$.

We set

$$(v_\# \zeta)(\Delta^m) = \begin{cases} \zeta(\Delta^l) \ldots \text{whenever } \Delta^m \text{ is full}, \\ 0 \ldots \text{otherwise}. \end{cases}$$

7.8. LEMMA. – *We have* 1) $v_{\#}\zeta \in \boldsymbol{Z}_n(\Delta^q)$; 2) $v_{i\#}\partial_i\zeta = \partial_i v_{\#}\zeta$, *where* $v_i\colon \partial_i\Delta^p \subset \partial_i\Delta^q$ *denotes the restriction of* v. *If in particular* ζ *is a sphere* (*i.e. one has* $\partial_i\zeta = *$, $i \geqslant 0$), *then* $v_{\#}\zeta$ *is also a sphere.*

The proof is immediate. ∎

8. – $\mathrm{Cl}(C_*)$ is a Kan spectrum.

This section is devoted to a proof of the following:

8.1. THEOREM. – *The spectrum* $\boldsymbol{E} = \mathrm{Cl}(\boldsymbol{C}_*)$ *is a Kan spectrum.*

Recall that a spectrum $\boldsymbol{E}$ is a *Kan spectrum* whenever all E_k are Kan complexes, where $\{E_k\}$ is the associated prespectrum. So the contention of Theorem 8.1 is equivalent to the following.

8.2. LEMMA. – *Let* $\zeta_i^q \in (E_k)_q$, $i = 0, \dots, k-1, k+1, \dots, q+1$ *be* $q+1$ *q-simplexes in* E_k, *satisfying*

$$(1) \qquad \partial_i\zeta_j^q = \partial_{j-1}\zeta_i^q, \qquad i<j, \ \ i, \ j \neq k,$$

then there exists a $\xi^{q+1} \in (E_k)_{q+1}$ *such that* $\partial_i\xi^{q+1} = \zeta_i^q$, $i \neq k$.

We have $\zeta_i^q \in \boldsymbol{Z}_n(\Delta_i^{p_i})$ and deal in a first step with the case that all p_i are equal to q.

Suppose that we have the same situation as in Lemma 7.5: $Y = Y_1 \cup Y_2$, $Y_1 \cap Y_2 = A$, $X_i' \subset X_i \subset Y_i$, $X_1 \cap X_2 = A$, $X_1' \cap X_2' = B$, $X = X_1 \cup X_2$, $X' = X_1' \cup X_2'$ (cf. § 7, fig. 1).

(*) Let $z_i \in Z_{n-1}'(X_i, B)$, $i = 1, 2$, be relative cycles admitting a partition $\{z_i', (-1)^i a\}$, $z_i' \in Z_{n-1}'(X_i', B)$, $a \in Z_{n-1}'(A, B)$, then $z_1' + z_2' \in Z_{n-1}'(X', B)$ is (in $h_{n-1}(X, B)$) homologous to $z_1 + z_2$.

PROOF. – Since $z_1 + z_2 \sim (z_1' - a) + (z_2' + a)$, the assertion follows (cf. also 7.2). ∎

(**) Let $y_i \in Z_n'(Y_i, X_i)$, $i = 1, 2$, be relative cycles such that $dy_i = z_i \in Z_{n-1}'(X_i, B)$ satisfies the assumptions of (*), then there exists a composite (cf. § 7) $y \in Z_n'(Y, X')$ of $\{y_1, y_2\}$.

PROOF. – Because of our assumptions and Remark 3) following D3) in § 9, Lemma 7.6 1) applies, ensuring the existence of a composite y, which is (up to homology) uniquely defined. ∎

These assertions are now applied to $q+1$ (instead of two) summands to the effect that we detect

$$c = \sum_{i \neq k} (-1)^i \bar{\zeta}_i^q \in Z_n' \left(\bigcup_{i \neq k} \Delta_i^q, \operatorname{bd} \Delta_k^q \right).$$

There is a homeomorphism between Δ_k^q and $\bigcup_{i \neq k} \Delta_i^q$ (both considered as subspaces of $\operatorname{bd} \Delta^{q+1} \subset \Delta^{q+1}$) so that we have a $c' \in Z_n'(\Delta_k^q, \operatorname{bd} \Delta_k^q)$ corresponding to c. According to 7.3 there exists a $b \in Z_n'(\Delta^{q+1}, \operatorname{bd} \Delta^{q+1})$, representing the homotopy between the homotopic mappings $\Delta^q \to \bigcup_{i \neq k} \Delta_i^q \subset \Delta^{q+1} \supset \Delta_k^q \leftarrow \Delta^q$ (on the boundary this homotopy is the identity). Hence we have $db = c - c'$.

We establish ξ^{q+1} (resp. ζ_k^q) by setting $\xi^{q+1} = b$ ($\bar{\zeta}_k^q = c'$) and $\partial_i \xi^{q+1} = \zeta_i^q$ ($\partial_i \zeta_k^q = \partial_{k-1} \zeta_i^q$, $i < k$, $\partial_j \zeta_k^q = \partial_k \zeta_{j+1}^q$, $k \leqslant j$). So ξ^{q+1} is a filling of the cone consisting of simplexes ζ_i^q, $i = 0, \ldots, k-1, k+1, \ldots, q+1$, settling the case where all p_i equal q. ■

Suppose that $p_i \leqslant q$, then we have inclusions $v_i \colon \Delta_i^{p_i} \subset \Delta_i^q$ and $\zeta_i'^q = v_{i\#} \zeta_i^q$. From (1) it follows (cf. Lemma 7.8) that

$$\partial_i \zeta_i'^q = \partial_{j-1} \zeta_i'^q, \qquad i < j, \quad i, j \neq k. \tag{1'}$$

So we find $\xi^{q+1} \in \boldsymbol{Z}_{n+1}'(\Delta^{q+1})$ as in the first case but now defining $\partial_i \xi^{q+1} = \zeta_i^q$.

If there is a $p_i > q$, we take $p = \max p_i > q$ and employ instead of ζ_i^q again $v_{i\#} \zeta_i^q \in \boldsymbol{Z}_n'(\Delta^p)$, $v_i \colon \Delta_i^{p_i} \subset \Delta^q$. The series of simplexes $\zeta_0'^q, \ldots, \zeta_{q+1}'^q$ is extended to a cone by requiring $\zeta_{q+2}'^q, \ldots, \zeta_{p+1}'^q$ to be degenerate simplexes: $\zeta_j'^q = *$, $j > q+1$. Now we proceed again as in the first case (with Δ^p replacing Δ^q), finding a $\xi^{q+1} \in \boldsymbol{Z}_{n+1}'(\Delta^{p+1})$ with $\partial_i \xi^{q+1} = \zeta_i^q$, $0 \leqslant i \leqslant q+1$, $i \neq k$, $\partial_i \xi^{q+1} = \zeta_i'^q = *$ for $i > q+1$.

This settles the last remaining case, completing the proof of Lemma 8.2 and therefore also of Theorem 8.1. ■

The Kan extension condition for $\boldsymbol{E}$ implies the same property for $\boldsymbol{E}(X)$ (cf. § 3). Let $\zeta = \zeta_n \in \boldsymbol{Z}_n(\Delta^p) \subset (E_k)_q$ be a q-simplex, then we construct a $\Sigma\zeta \in \boldsymbol{Z}_{n+1}(\Delta^{q+1})$ in the following way, resembling the construction of Σz, $z \in Z_n'(\Delta^p, \operatorname{bd} \Delta^p)$ in 7.4:

Let $C_\pm \zeta \in \boldsymbol{Z}_{n+1}(C_\pm \Delta^p)$ be two different copies of $C\zeta$ (being introduced at the end of § 7). These two simplexes can be regarded as a $(q+1)$-cone (using induction with regard to q, starting with two 1-simplexes which have a vertex in common, giving a 1-cone which can be filled up by the Kan condition yielding a 2-simplex etc.). The filling ξ^{q+1} of this cone has an extra face (not lying in the original cone) which is our $\Sigma\zeta \in (\boldsymbol{E}_k)_{q+1}$, $\Sigma\zeta \in \boldsymbol{Z}_{n+1}(\Delta^{p+1})$.

The degenerate simplexes of $\boldsymbol{E}$ (cf. § 2) are treated similarly.

The same procedure can be performed for $\boldsymbol{E}(X)$ (cf. § 3) instead of $\boldsymbol{E}$ (by doing it for each factor separately). So we obtain a simplicial mapping

$$\delta_k \colon \Sigma \boldsymbol{E}(X)_k \to \boldsymbol{E}(\Sigma X)_k$$

in the following way: A non-degenerate $(q+1)$-simplex τ of $\Sigma \boldsymbol{E}(X)_k$ is a q-simplex

$\sigma^q \wedge \zeta^q$ of $\boldsymbol{E}(X)_k$. Assign to τ the simplex $\Sigma\sigma \wedge \Sigma\zeta$ of $(\boldsymbol{E}(\Sigma X)_k)_{q+1}$. For the degenerate simplexes we proceed analogously, providing us with a simplicial δ_h.

We have (cf. § 3)

$$\boldsymbol{E}_n(X) = \pi_n(\boldsymbol{E}(X)) = \varinjlim_k \pi_{n+k}(\boldsymbol{E}(X)_k).$$

So the suspension isomorphism

$$\gamma\colon \boldsymbol{E}_n(X) \xrightarrow{\approx} \boldsymbol{E}_{n+1}(\Sigma X) \tag{2}$$

can (simplicially) be described by combining δ_k with the suspension:

$$\pi_{n+k}(\boldsymbol{E}(X)_k) \xrightarrow{\Sigma} \pi_{n+k+1}(\Sigma\boldsymbol{E}(X)_k) \xrightarrow{\delta_{k\#}} \pi_{n+1+k}(\boldsymbol{E}(\Sigma X)_k).$$

The proof of the fact that this assignment

$$\gamma\{a\} = \{\delta_{k\#}\Sigma a\} \tag{3}$$

yields the suspension isomorphism (2) is standard.

We summarize:

8.3. LEMMA. – *We have a simplicial mapping*

$$\delta_k\colon \Sigma\boldsymbol{E}(X)_k \to \boldsymbol{E}(\Sigma X)_k,$$

such that the suspension isomorphism (2) *is* (*in simplicial language*) *expressed by* (3).

9. – Chain functors.

Let $h_*\colon \boldsymbol{CW}^2 \to \boldsymbol{Ab}^{\mathbb{Z}}$ be a homology theory being defined on the category of CW pairs, then we are in [1] establishing a *chain functor* $\boldsymbol{C}_*\colon \boldsymbol{CW} \to \boldsymbol{ch}$ (= category of chain complexes) which determines h_* in a certain sense. This does *not* mean that we have to each inclusion $i\colon A \subset X$ an exact sequence of chain complexes

$$0 \to \boldsymbol{C}_*(A) \xrightarrow{i_\#} \boldsymbol{C}_*(X) \to \underset{\displaystyle \boldsymbol{C}_*(X)/\operatorname{im} i_\#}{\underset{\|}{\boldsymbol{C}_*(X, A)}} \to 0 \tag{1}$$

yielding $h_*(X,A)$ as derived homology $H_*(\boldsymbol{C}_*(X, A))$. It is known (cf. [1] for further reference) that the existence of such a functor $\boldsymbol{C}_* \to \boldsymbol{ch}$ implies that h_* is simply isomorphic to the direct sum of *ordinary* homology theories

$$h_*(X, A) \approx \sum_{n=-\infty}^{\infty} H_{*+n}(X, A; G_n), \qquad G_n \in Ab.$$

This leads to the concept of a functor $C_*: \boldsymbol{CW} \to \boldsymbol{ch}$ *with domination* or briefly a D-functor [1].

Let $C_*: \boldsymbol{CW}^2 \to \boldsymbol{ch}$ be a functor satisfying

$*$) All inclusions i: $(X_1, A_1) \subset (X_2, A_2)$ induce monomorphisms.

$*$) $C_*(X, X)$ is acyclic.

The sequence

$$C_*(A) \xrightarrow{i_*} C_*(X) \xrightarrow{j_*} C_*(X, A) \tag{2}$$

does not suffice for establishing a boundary ∂: $H_*(C_*(X, A)) \to H_{*-1}(C_*(A))$. So we describe what is meant by

$***$) (2) is naturally dominated by an exact sequence

$$0 \to C_*(A) \xrightarrow{i'} C'_*(X, A) \xrightarrow{p} C''_*(X, A) \to 0. \tag{$\bar{2}$}$$

At first we assume the existence of another functor $C'_*: \boldsymbol{CW}^2 \to \boldsymbol{ch}$ together with natural inclusions

$$C_*(A) \xrightarrow{i'} C'_*(X, A) \xrightarrow{l} C_*(X, A).$$

The chain complex $C''_*(X, A)$ has the form $C'_*(X, A)/\operatorname{im} i'$. We have the inclusions q: $(A, A) \subset (X, A)$, s: $A \subset (A, A)$ and construct a natural homomorphism

$$\psi = \psi_{(X, A)}: H_*(C''_*(X, A)) \to H_*(C_*(X, A))$$

in the following way.

To each $z \in Z_n(C''_*(X, A))$ we find *a* $b \in C'_*(X, A)$ with $db \in \operatorname{im} i'$. Since $C_*(A, A)$ is acyclic we detect *a* $\bar{a} \in C_*(A, A)$ such that $d\bar{a} = l(b)$. So we are allowed to set

$$\psi[z] = [l(b) - q_*(\bar{a})].$$

It is immediate that ψ is a well-defined, natural homomorphism satisfying

$$\psi p_* = l_*,$$

p: $C'_*(X, A) \to C''_*(X, A)$ denoting the projection. In addition to this we require the existence of chain mappings:

$$\varphi_\#: C'_*(X, A) \to C_*(X),$$

$$\kappa_\#: C_*(X) \to C'_*(X, A),$$

with induced homomorphisms $\varphi_{\#*} = \varphi$, $\kappa_{\#*} = \kappa$, between the related homology groups, as well as the following equations resp. chain homotopies:

$$\text{D1)} \qquad \begin{cases} \kappa_{\#} i_{\#} = i' , \\ j_{\#} \varphi \simeq l , \\ \varphi_{\#} \kappa_{\#} \simeq 1 , \end{cases}$$

satisfying

D2) ψ is an epimorphism; there exists a ρ: $\operatorname{im} j_* \to H_*(C''_*(X, A))$ satisfying $\psi\rho = 1$: $\operatorname{im} j_* \to \operatorname{im} j_*$ and $p_* \kappa = \rho j_*$.

Let $\bar\partial$: $H_*(C''_*(X, A)) \to H_{*-1}(C_*(A))$ be the boundary homomorphism related to the short exact sequence $(\bar 2)$, then we require

$$\text{D3)} \qquad \ker \bar\partial \supset \ker \psi .$$

The sequence $(\bar 2)$ is called a *natural domination* of (2). Observe that we do *not* expect ρ, φ or κ to be natural.

REMARK. – 1) Lemma 5.4 in [1] (which is the origin of D1)) asserts only a *chain homotopy* $\kappa_{\#} i \simeq i'$: However in the course of the proof of this assertion in [1], one establishes in fact an equality $\kappa_{\#} i = i'$.

2) Moreover the proof of [1] Lemma 5.4 yields the following additional property of the chain homotopy D: $\varphi_{\#} \kappa_{\#} \simeq 1$ (not explicitely stated in the formulation of [1] Lemma 5.4): Let $z \in Z_n(A)$ be a cycle, then $D(z) \in \operatorname{im}(C_{n+1}(A) \to C_{n+1}(X))$.

3) A chain homotopy $\tilde D$: $\varphi_{\#} i' \simeq i$ can be immediately deduced from $k_{\#} i = i'$ and D (in 2)). Therefore we have again

$$z \in Z_n(A) \Rightarrow \tilde D(z) \in \operatorname{im}(C_{n+1}(A) \to C_{n+1}(X)) .$$

We obtain a natural boundary operator

$$\partial = \partial_n : H_n(C_*(X, A)) \to H_{n-1}(C_*(A))$$

by setting

$$(3) \qquad \partial_n [z] = \bar\partial [z'] ,$$

$z' \in Z_n(C''_*(X, A))$, $\psi[z'] = [z]$. Condition D3) ensures that ∂ is well-defined.

4) The existence of ρ in D2) turns out to be equivalent to the condition: $\ker(j_* : H_*(C_*(X)) \to H_*(C_*(X, A))) \subset \ker p_* \kappa$.

In practice it appears to be more convenient to replace the functor $C_* : \boldsymbol{CW}^2 \to \boldsymbol{ch}$ by the functor $\boldsymbol{C}_* : \boldsymbol{CW} \to \boldsymbol{ch}$ being related to C_* by

$$(4) \qquad \boldsymbol{C}_*(X) = C_*(X, X) .$$

All chain complexes $C_*(A)$, $C_*(X,A)$, $C'_*(X,A)$, $C_*(X)$ appear now as subcomplexes of this single enveloping chain complex $\boldsymbol{C}_*(X)$.

We summarize:

9.1. DEFINITION ([1] Appendix). – A functor $C_*: \boldsymbol{CW}^2 \to \boldsymbol{ch}$ (resp. the related functor $\boldsymbol{C}_*: \boldsymbol{CW} \to \boldsymbol{ch}$) satisfying $*$), $**$) being endowed with the additional structure of a natural domination (hence satisfying D1)-D3)) is called a *functor with domination* or simply a *D-functor.*

We deduce immediately ([1] Proposition A.2):

9.2. PROPOSITION. – *The variety* $H_* = \{H_n(C_*), \partial_n, n \in \mathbb{Z}\}$ *of functors and natural transformations as described in* (3) *give rise to an exact homology sequence*

$$\ldots \to H_n(C_*(A)) \to H_n(C_*(X)) \to H_n(C_*(X,A)) \xrightarrow{\partial} H_{n-1}(C_*(A)) \to \ldots.$$

We agree to use the notation

$$H_*(\boldsymbol{C}_*)(X,A) = H_*(C_*(X,A))$$

whenever we talk about the homology groups of the D-functor $\boldsymbol{C}_*$.

9.3. DEFINITION (cf. [1] Definition 2.1). – A *chain functor* $\boldsymbol{C}_*: \boldsymbol{CW} \to \boldsymbol{ch}$ is a functor $\boldsymbol{C}_*$ satisfying

$\boldsymbol{C}_*1)$ $\boldsymbol{C}_*$ is a D-functor.

$C_*2)$ To each homotopy H: $f_0 \simeq f_1: (X,A) \to (Y,B)$ in $\boldsymbol{CW}^2$ there exists a chain homotopy $D(H)$: $C_n(X,A) \to C_{n+1}(Y,B)$ being natural in the following sense: Let

$$\begin{array}{ccc}(X,A)\times I & \xrightarrow{H} & (Y,B)\\ {\scriptstyle f\times 1}\downarrow & & \downarrow{\scriptstyle g}\\ (\tilde X,\tilde A)\times I & \xrightarrow{\tilde H} & (\tilde Y,\tilde B)\end{array}$$

be commutative in $\boldsymbol{CW}^2$, then the diagram

$$\begin{array}{ccc}C_n(X,A) & \xrightarrow{D(H)} & C_{n+1}(Y,B)\\ {\scriptstyle f_\#}\downarrow & & \downarrow{\scriptstyle g_\#}\\ C_n(\tilde X,\tilde A) & \xrightarrow{D(\tilde H)} & C_{n+1}(\tilde Y,\tilde B)\end{array}$$

(with e.g. $C_*(f) = f_\#$) is commutative.

$C_*3)$ one has $\boldsymbol{C}_*(\emptyset) = 0$ (= trivial chain complex).

$C_*4)$ (Axiom of *carrier*). To each $c \in C_n(X)$ there exists a space $\overline{X} \subset X$ (*not* necessarily a *CW* space) satisfying

1) To each *CW* space $\overline{X} \subset X' \subset X$ one has a $c' \in C_*(X')$ satisfying $(X' \subset X)_\# c' = c$.

2) For each $c' \in C_n(X')$, $X''X$ satisfying $(X' \subset X)_\#(c') = c$, one has $\overline{X} \subset X'$.

The definition of a chain functor is understood to include 1) a specific choice of a D-structure involving all particular items like e.g. dominating sequences, chain mappings $\varphi_\#$, $\kappa_\#$ etc. 2) chain homotopies as described in $C_*2)$ and 3) carriers of chain as described in $C_*4)$. So two functors ${}^1\boldsymbol{C}_*$, ${}^2\boldsymbol{C}_*: \boldsymbol{CW} \to \boldsymbol{ch}$ represent different *chain functors* whenever at least one of these three items differ although they might be equal as functors. The main result of [1] (Theorem 8.1) is the assertion:

9.4. THEOREM. – *To each homology theory* $h_* = \{h_n, \partial_n, n \in \mathbb{Z}\}\, h_*: \boldsymbol{CW}^2 \to \boldsymbol{Ab}^{\mathbb{Z}}$, *there exists a chain functor* $\boldsymbol{C}_*: \boldsymbol{CW} \to \boldsymbol{ch}$ *as well as a natural isomorphism of homology theories*:

$$\mu: h_*(\,) \approx H_*(\boldsymbol{C}_*)(\,).$$

We can assume that 1) all $\boldsymbol{C}_*(X)$, $X \in \boldsymbol{CW}$ are free chain complexes, 2) $C_n(A, A) \subset C_n(X, A)$ is, for all pairs $(X, A) \in \boldsymbol{K}^2$, a direct summand and 3) $\operatorname{im} l \subset C_n(X, A)$ is a direct summand.

In order to be able to define the category $\mathfrak{C}_*$ of chain functors, we need the concept of a natural transformation $\lambda: {}^1\boldsymbol{C}_* \to {}^2\boldsymbol{C}_*$ between chain functors. This is a natural transformation between the two functors respecting the additional structures:

9.5. DEFINITION. – Let ${}^1\boldsymbol{C}_*$, ${}^2\boldsymbol{C}_*: \boldsymbol{CW} \to \boldsymbol{ch}$ be two chain functors (cf. Definition 9.3) λ: ${}^1\boldsymbol{C}_* \to {}^2\boldsymbol{C}_*$, $\lambda: {}^1C_* \to {}^2C_*$, $\lambda': {}^1C'_* \to {}^2C'_*$ natural transformations such that

$$\lambda\, {}^1l_\# = {}^2l_\# \lambda', \qquad {}^il_\#: {}^iC'_* \to {}^iC_*,$$

$${}^2\varphi_\# \lambda' = \lambda\, {}^1\varphi_\#, \qquad {}^i\varphi_\#: {}^iC'_*(X, A) \to {}^iC_*(X),$$

$${}^2\kappa_\# \lambda = \lambda'\, {}^1\kappa_\#, \qquad {}^i\kappa_\#: {}^iC'_*(X) \to {}^iC'_*(X, A),$$

$$\boldsymbol{\lambda}_X = \lambda_{(X,X)},$$

and that for each homotopy $H: f_0 \simeq f_1: (X, A) \to (Y, B)$ one has

$$\lambda\, {}^1D(H) = {}^2D(H)\, \lambda,$$

with

$${}^iD(H): {}^iC_*(X, A) \to {}^iC_{*+1}(Y, B).$$

Then we call $\lambda: {}^1\boldsymbol{C}_* \to {}^2\boldsymbol{C}_*$ a transformation of chain functors.

9.6. PROPOSITION. – 1) *Any transformation of chain functors induces a natural transformation of homology theories*

$$\lambda_*: H_*({}^1\boldsymbol{C}_*) \to H_*({}^2\boldsymbol{C}_*).$$

2) *This class of chain functors together with transformations between chain functors form a category* $\mathfrak{C}_*$.

3) *There exists to each* $\lambda \in \mathfrak{C}_*({}^1\boldsymbol{C}_*, {}^2\boldsymbol{C}_*)$ *a natural* $\lambda''_*: H_*({}^1\boldsymbol{C}''_*) \to H_*({}^2\boldsymbol{C}''_*)$ *satisfying*

$${}^2\psi\lambda''_* = \lambda_*{}^1\psi,$$

with ${}^i\psi: H_*({}^iC''_*) \to H_*(C_*)$ *being the corresponding natural transformations.*

The proof is immediate.

REMARK. – Two chain functors ${}^1\boldsymbol{C}_*$, ${}^2\boldsymbol{C}_*$ which agree in all the additional structure with possible exception of the chain mapping $\rho_\#$ (inducing ρ in D2)) and the carriers are, according to Definition 9.5, equivalent in the category $\mathfrak{C}_*$.

In the present paper we are obliged to use two additional properties of a chain functor $\boldsymbol{C}_*$ which follow from the *construction* presented in [1] more or less immediately. Since we do not intend to repeat any details of this construction we simply state these assertions leaving the proofs to the reader of [1]:

Recall that $C_n(A, A)$ is a direct summand of the free abelian group $C_n(X, A)$; denote the complementary summand by $C_n(X, A) - C_n(A, A)$.

We denote by $\tilde{C}_n(X, A) \subset C_n(X, A)$ the subgroup generated by $C_n(X, A) - C_n(A, A)$ and $C'_n(X, A)$.

9.7. LEMMA. – 1) $x \in C_n(X, A) - C_n(A, A) \Rightarrow dx \in \tilde{C}_n(X, A)$.

2) $C'_n(X, A) \cap C_n(A, A) = C_n(A)$ (cf. [1] *Lemma* 5.4).

3) $C'_n(X, A) = ((C_n(X, A) - C_n(A, A)) \cap C'_n(X, A)) \oplus (C_n(A, A) \cap C'_n(X, A))$.

9.8. LEMMA. – *Let* H: $f_0 \simeq f_1: (X, A) \to (Y, B)$ *be a homotopy such that* $H(X \setminus A \times (0, 1)) \subset Y \setminus B$ *and* $z \in Z'_n(X, A)$, *then the chain homotopy* $D(H)(z) = D(z)$ *satisfies*

$$D(z) \in \tilde{C}_{n+1}(Y, B).$$

REMARKS. – 1) Lemma 9.7 1) assures us that a chain which lies on the boundary of generators in the complement of $C_n(A, A)$ cannot be contained «too deeply» in $C_*(A, A)$. This is of importance because the splitting of $C_n(X, A)$ into $C_n(A, A)$ and its complement is not a splitting of chain complexes but only of individual abelian groups.

2) Lemma 9.8 constitues a stronger version of the naturality property of the chain homotopy $D(H)$. It will be needed to repair the lack of a chain homotopy $D'(H)$: $C_n(X, A) \to C'_{n+1}(Y, B)$ (which does not exist), cf. Lemma 7.3. ■

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