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$L^{2,\lambda}$ Regularity of the Spatial Derivatives of the Solutions to Parabolic Systems in Divergence Form (*).

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Sunto. – *Si dimostra che, nelle ipotesi:* $f^i \in L^{2,\lambda}(Q, \mathbb{R}^N)$, $0 < \lambda < n+2$, $i = 0, 1, \dots, n$, $u \in L^{2,\lambda}(Q, \mathbb{R}^N) \cap \overset{*}{H}{}_{-T}^{0,1/2,(\lambda)}(Q, \mathbb{R}^N)$, $D_i u \in L^{2,\lambda}(Q, \mathbb{R}^N)$, $i = 1, 2, \dots, n$, *la soluzione* $v: Q \to \mathbb{R}^N$ *del problema di Cauchy-Dirichlet:*

$$\begin{cases} -\sum_{ij=1}^{n} D_i(A_{ij}(X) D_j v) + \frac{\partial v}{\partial t} = -\sum_{i=1}^{n} D_i f^i + f^0 & \textit{in } Q, \\ v = u \ \textit{sulla frontiera parabolica } \Gamma_Q \textit{ di } Q, \end{cases}$$

ha derivate spaziali $D_i v$ *appartenenti a* $L^{2,\lambda}(Q, \mathbb{R}^N)$ *e che sussiste la maggiorazione:*

$$\sum_{i=1}^{n} \|D_i v\|^2_{L^{2,\lambda}(Q, \mathbb{R}^N)} \le c \left\{ \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q, \mathbb{R}^N)} + \|u\|^2_{L^{2,\lambda}(Q, \mathbb{R}^N)} + [u]^2_{\overset{*}{H}{}_{-T}^{0,1/2,(\lambda)}(Q, \mathbb{R}^N)} + \sum_{i=0}^{n} \|f^i\|^2_{L^{2,\lambda}(Q, \mathbb{R}^N)} \right\}.$$

1. – Introduction.

Let Ω be a bounded open set of $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$ (for instance of class C^2), T a real positive number and Q the cylinder $\Omega \times (-T, 0)$. If $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n_x$ and $t \in \mathbb{R}_t$, we set $X = (x, t)$. By $I(X^0, \sigma)$, $Q(X^0, \sigma)$, $Q^+(X^0, \sigma)$, $M(X^0, \sigma)$, $M^+(X^0, \sigma)$ we denote the cylinders of $\mathbb{R}^n_x \times \mathbb{R}_t$:

$$I(X^0, \sigma) = B(x^0, \sigma) \times (t^0 - \sigma^2, t^0 + \sigma^2),$$

$$Q(X^0, \sigma) = B(x^0, \sigma) \times (t^0 - \sigma^2, t^0),$$

$$Q^+(X^0, \sigma) = B^+(x^0, \sigma) \times (t^0 - \sigma^2, t^0),$$

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$$M(X^0, \sigma) = B(x^0, \sigma) \times (t^0, t^0 + \sigma^2),$$

$$M^+(X^0, \sigma) = B^+(x^0, \sigma) \times (t^0, t^0 + \sigma^2),$$

where $X^0 = (x^0, t^0) \in \mathbb{R}^n_x \times \mathbb{R}_t$, $\sigma > 0$, $B(x^0, \sigma) = \{x \in \mathbb{R}^n : \|x - x^0\| < \sigma\}$, $B^+(x^0, \sigma) = B(x^0, \sigma) \cap \{x \in \mathbb{R}^n : x_n > x_n^0\}$ (1).

Let $H^{1/2}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N))$ (N integer > 1) $[H_0^{1/2}(-T, 0, L^2(\Omega, \mathbb{R}^N))]$ be the Banach space of those functions $u \in H^{1/2}(-T, 0, L^2(\Omega, \mathbb{R}^N))$ such that

$$\int_{-T}^{0} dt \int_{\Omega} \frac{\|u(X)\|^2}{t+T} dx < +\infty \left[\int_{-T}^{0} dt \int_{\Omega} \frac{\|u(X)\|^2}{t} dx < +\infty \right],$$

with norm

$$\|u\|_{H^{1/2}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N))} = \left\{ \|u\|^2_{H^{1/2}(-T, 0, L^2(\Omega, \mathbb{R}^N))} + \int_{-T}^{0} dt \int_{\Omega} \frac{\|u(X)\|^2}{t+T} dx \right\}^{1/2},$$

$$\left[\|u\|_{H_0^{1/2}(-T, 0, L^2(\Omega, \mathbb{R}^N))} = \left\{ \|u\|^2_{H^{1/2}(-T, 0, L^2(\Omega, \mathbb{R}^N))} + \int_{-T}^{0} dt \int_{\Omega} \frac{\|u(X)\|^2}{t} dx \right\}^{1/2} \right].$$

Let us denote by $H^{*1/2}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N))$ the set of those $u \in H^{1/2}(-T, 0, L^2(\Omega, \mathbb{R}^N))$ for which there exist $h \in L^2(\Omega, \mathbb{R}^N)$ such that $u - h \in H^{1/2}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N))$ (2).

$H^{*1/2}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N))$ is a Banach space with norm

$$\|u\|_{H^{*1/2}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N))} = \left\{ \|u\|^2_{H^{1/2}(-T, 0, L^2(\Omega, \mathbb{R}^N))} + \int_{-T}^{0} dt \int_{\Omega} \frac{\|u(X) - u(x, -T)\|^2}{t+T} dx \right\}^{1/2},$$

where $u(x, -T) = h(x)$.

Let us consider in Q the following Cauchy-Dirichlet problem (with non homogeneous data)

$$-\sum_{ij=1}^{n} D_i(A_{ij}(X) D_j v) + \frac{\partial v}{\partial t} = -\sum_{i=1}^{n} D_i f^i + f^0 \quad \text{in } Q, \tag{1.1}$$

$$v = u \quad \text{on the parabolic boundary } \Gamma_Q \text{ of } Q\,(3), \tag{1.2}$$

(1) When $x_0 = 0$ we set, for the sake of brevity, $B^+(\sigma) = B^+(o, \sigma)$.

(2) If there exists $h \in L^2(\Omega, \mathbb{R}^N)$ such that $u - h \in H^{1/2}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N))$, it is uniquely determined by u; moreover, if there exists (in $L^2(\Omega, \mathbb{R}^N)$) the $\lim_{t \to -T^+} u(t)$, it results $h = \lim_{t \to -T^+} u(t)$.

(3) $\Gamma_Q = [\Omega \times \{-T\}] \cup [\partial\Omega \times (-T, 0)]$.

where

$$u \in L^2(-T, 0, H^1(\Omega, \mathbb{R}^N)) \cap H^{*1/2}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N)), \tag{1.3}$$

$$f^i \in L^2(Q, \mathbb{R}^N), \qquad i = 0, 1, \dots, n, \tag{1.4}$$

and $A_{ij}(X)$, $i, j = 1, 2, \dots, n$, are $N \times N$ matrices satisfying the conditions

$$A_{ij} \in C^0(\overline{Q}, \mathbb{R}^{N^2}), \qquad i, j = 1, 2, \dots, n, \tag{1.5}$$

(1.6) there exists $\nu > 0$ such that:

$$\sum_{ij=1}^{n} (A_{ij}(X) p^j | p^i) \geq \nu \sum_{i=1}^{n} \|p^i\|^2, \qquad \forall (X, p) \in \overline{Q} \times \mathbb{R}^{nN} \, (^4).$$

The following result is well known (see S. KAPLAN [4], Theorem 2)

THEOREM 1.1. - *If assumptions* (1.3)-(1.6) *hold, the problem* (1.1), (1.2) *admits a unique solution* v *in* $L^2(-T, 0, H^1(\Omega, \mathbb{R}^N)) \cap H^{*1/2}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N))$, *in the sense that* $w = u - v \in L^2(-T, 0, H^1_0(\Omega, \mathbb{R}^N)) \cap H^{1/2}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N))$ *and*

$$\int_Q \sum_{ij=1}^{n} (A_{ij} D_j w | D_i \varphi) \, dX + \tilde{B}\langle w, \varphi \rangle = \tag{1.7}$$

$$= \tilde{B}\langle u, \varphi \rangle + \int_Q \sum_{i=1}^{n} \left(\sum_{j=1}^{n} A_{ij} D_j u - f^i \,\Big|\, D_i \varphi \right) dX - \int_Q (f^0 | \varphi) \, dX,$$

$\forall \varphi \in L^2(-T, 0, H^1_0(\Omega, \mathbb{R}^N)) \cap H^{1/2}_0(-T, 0, L^2(\Omega, \mathbb{R}^N))$ (5).

For such a solution the following estimate holds

$$\int_Q \sum_{i=1}^{n} \|D_i v\|^2 \, dX + \|v\|^2_{H^{*1/2}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N))} \leq \tag{1.8}$$

$$\leq c \left\{ \|u\|^2_{H^{*1/2}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N))} + \int_Q \sum_{i=1}^{n} \|D_i u\|^2 \, dX + \int_Q \sum_{i=0}^{n} \|f^i\|^2 \, dX \right\}.$$

In this paper we study the $L^{2,\lambda}$-regularity of the spatial derivatives of the solution v to the problem (1.1), (1.2) (see Theorem 4.1), that has interest in itself and is preliminary to reach the boundedness of v (see [6]).

The boundedness of the solution v to the problem (1.1), (1.2) plays an important role in the study of the partial Hölder-continuity of the solutions u to nonlinear parabolic systems with quadratic growth (see [5], Section 5).

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(4) $p = (p^1 | \dots | p^n)$, $p^j \in \mathbb{R}^N$, denotes a vector of $\mathbb{R}^{nN}$.

(5) $\tilde{B}\langle w, \varphi \rangle$ and $\tilde{B}\langle u, \varphi \rangle$ have the meaning explained in Section 2 of [5].

2. - Preliminary lemmas.

In this section we show some lemmas concerning the solutions to the system

$$
(2.1) \qquad -\sum_{ij=1}^{n} D_i(A_{ij}^0 D_j w) + \frac{\partial w}{\partial t} = \frac{\partial u}{\partial t} - \sum_{i=1}^{n} D_i F^i + F^0,
$$

with A_{ij}^0, $i,j = 1,2,\dots,n$, $N \times N$ constant matrices satisfying the strong ellipticity condition

(2.2) there exists $\nu > 0$ such that:

$$
\sum_{ij=1}^{n} (A_{ij}^0 p^j | p^i) \geq \nu \sum_{i=1}^{n} \|p^i\|^2, \qquad \forall p \in \mathbb{R}^{nN}.
$$

LEMMA 2.1. - *If* $u \in L^2(-T,0,H^1(B(x^0,\sigma),\mathbb{R}^N)) \cap H_{-T}^{*1/2}(-T,0,L^2(B(x^0,\sigma),\mathbb{R}^N))$ *and* $F^i \in L^2(B(x^0,\sigma)\times(-T,0),\mathbb{R}^N)$, $i=0,1,\dots,n$, *if* $w \in L^2(-T,0,H^1(B(x^0,\sigma),\mathbb{R}^N)) \cap \cap H_{-T}^{1/2}(-T,0,L^2(B(x^0,\sigma),\mathbb{R}^N))$ *is solution in* $Q(X^0,\sigma) \subset B(x^0,\sigma)\times(-T,0)$, $X^0 = (x^0,t^0)$, *to the system* (2.1), *in the sense that*

$$
(2.3) \qquad \int_{Q(X^0,\sigma)} \sum_{ij=1}^{n} (A_{ij}^0 D_j w | D_i \varphi)\, dX + \tilde{B}\langle w,\varphi\rangle = \tilde{B}\langle u,\varphi\rangle + \int_{Q(X^0,\sigma)} \left\{ \sum_{i=1}^{n} (F^i | D_i\varphi) + (F^0|\varphi) \right\} dX,
$$

$\forall \varphi \in L^2(t^0-\sigma^2, t^0, H_0^1(B(x^0,\sigma),\mathbb{R}^N)) \cap [H_{t^0-\sigma^2}^{1/2} \cap H_{t^0}^{1/2}](t^0-\sigma^2, t^0, L^2(B(x^0,\sigma),\mathbb{R}^N))$,

then there exists a positive constant $c = c(\nu)$ *such that* $\forall \rho \in (0,\sigma)$ *it results*

$$
(2.4) \qquad \int_{Q(X^0,\rho)} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq
$$

$$
\leq c(\nu) \left\{ \left(\frac{\rho}{\sigma}\right)^{n+2} \int_{Q(X^0,\sigma)} \sum_{i=1}^{n} \|D_i w\|^2 dX + \int_{Q(X^0,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2 dX + \int_{Q(X^0,\sigma)} \sum_{i=0}^{n} \|F^i\|^2 dX \right\}.
$$

PROOF. - In virtue of the assumptions, the vector $v = u - w$ belongs to $L^2(t^0-\sigma^2, t^0, H^1(B(x^0,\sigma),\mathbb{R}^N))$ and is weak solution (in the usual sense) in $Q(X^0,\sigma)$ to the system

$$
(2.5) \qquad -\sum_{ij=1}^{n} D_i(A_{ij}^0 D_j v) + \frac{\partial v}{\partial t} = -\sum_{i=1}^{n} D_i f^i + f^0
$$

with

$$
f^i = -F^i + \sum_{j=1}^{n} A_{ij}^0 D_j u \in L^2(Q(X^0,\sigma),\mathbb{R}^N), \quad i = 1,\dots,n, \quad f^0 = -F^0 \in L^2(Q(X^0,\sigma),\mathbb{R}^N).
$$

Then we may use a well known estimate (see [2], Lemma 2.II) that assures $\forall \rho \in (0, \sigma)$

$$(2.6) \qquad \int_{Q(X^0,\rho)} \sum_{i=1}^{n} \|D_i v\|^2 dX \leq c(\nu) \left\{ \left(\frac{\rho}{\sigma}\right)^{n+2} \int_{Q(X^0,\sigma)} \sum_{i=1}^{n} \|D_i v\|^2 dX + \int_{Q(X^0,\sigma)} \sum_{i=0}^{n} \|f^i\|^2 dX \right\} \text{(6)},$$

from which, being $w = u - v$, $f^i = -F^i + \sum_{j=1}^{n} A_{ij}^0 D_j u$, $f^0 = -F^0$, we get

$$\int_{Q(X^0,\rho)} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq 2 \int_{Q(X^0,\rho)} \sum_{i=1}^{n} \|D_i u\|^2 dX + 2 \int_{Q(X^0,\rho)} \sum_{i=1}^{n} \|D_i v\|^2 dX \leq$$

$$\leq c(\nu) \left\{ \left(\frac{\rho}{\sigma}\right)^{n+2} \int_{Q(X^0,\sigma)} \sum_{i=1}^{n} \|D_i v\|^2 dX + \int_{Q(X^0,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2 dX + \int_{Q(X^0,\sigma)} \sum_{i=0}^{n} \|F^i\|^2 dX \right\}$$

and, thus, the assert.

LEMMA 2.2. - *If* $u \in L^2(-T, 0, H^1(B(x^0,\sigma), \mathbb{R}^N)) \cap H_{-T}^{*1/2}(-T, 0, L^2(B(x^0,\sigma), \mathbb{R}^N))$ *and* $F^i \in L^2(B(x^0,\sigma) \times (-T, 0), \mathbb{R}^N)$, $i = 0, 1, \ldots, n$, *if* $w \in L^2(-T, 0, H^1(B(x^0,\sigma), \mathbb{R}^N)) \cap H_{-T}^{1/2}(-T, 0, L^2(B(x^0,\sigma), \mathbb{R}^N))$ *is solution in* $M(X^0,\sigma) \subset B(x^0,\sigma) \times (-T, 0)$, $X^0 = (x^0, t^0)$, $t^0 = -T$, *to the system* (2.1) (7), *then there exists a positive constant* $c = c(\nu)$ *such that* $\forall \rho \in (0, \sigma)$

$$(2.7) \qquad \int_{M(X^0,\rho)} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq c(\nu) \left\{ \left(\frac{\rho}{\sigma}\right)^{n+2} \int_{M(X^0,\sigma)} \sum_{i=1}^{n} \|D_i w\|^2 dX + \right.$$

$$\left. + \|u\|^2_{H_{t^0}^{*1/2}(t^0, t^0+\sigma^2, L^2(B(x^0,\sigma),\mathbb{R}^N))} + \int_{M(X^0,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2 dX + \int_{M(X^0,\sigma)} \sum_{i=0}^{n} \|F^i\|^2 dX \right\}.$$

PROOF. - Let $v_1 \in L^2(t^0, t^0+\sigma^2, H^1(B(x^0,\sigma), \mathbb{R}^N)) \cap H_{t^0}^{*1/2}(t^0, t^0+\sigma^2, L^2(B(x^0,\sigma), \mathbb{R}^N))$ be the solution (8) to the C.D. problem

$$(2.8) \quad -\sum_{ij=1}^{n} D_i(A_{ij}^0 D_j v_1) + \frac{\partial v_1}{\partial t} = -\sum_{i=1}^{n} D_i \left(\sum_{j=1}^{n} A_{ij}^0 D_j u - F^i \right) - F^0 \quad \text{in } M(X^0,\sigma)$$

$$(2.9) \qquad v_1 = u \quad \text{on } \Gamma_{M(X^0,\sigma)}.$$

(6) This estimate in [2] is shown when $f^0 = 0$, but it can be extended to the weak solutions to the system (2.5).

(7) In the sense considered in the statement of Lemma 2.1.

(8) In the sense indicated in Section 1.

In $M(X^0, \sigma)$ let us decompose w in the sum $w_1 + w_2$ where

$$w_1 = u - v_1 \in L^2(t^0, t^0 + \sigma^2, H_0^1(B(x^0, \sigma), \mathbb{R}^N)) \cap H_{t^0}^{1/2}(t^0, t^0 + \sigma^2, L^2(B(x^0, \sigma), \mathbb{R}^N)),$$

while

$$w_2 \in L^2(t^0, t^0 + \sigma^2, H^1(B(x^0, \sigma), \mathbb{R}^N)) \cap H_{t^0}^{1/2}(t^0, t^0 + \sigma^2, L^2(B(x^0, \sigma), \mathbb{R}^N)),$$

is solution (7) in $M(X^0, \sigma)$ to the system

$$- \sum_{ij=1}^{n} D_i(A_{ij}^0 D_j w_2) + \frac{\partial w_2}{\partial t} = 0. \tag{2.10}$$

Fixed $\rho \in (0, \sigma)$ and set $X^* = (x^0, t^0 + \rho^2)$, let us consider the extension of the function w_2 to the cylinder $B(x^0, \sigma) \times (t^0 + \rho^2 - \sigma^2, t^2 + \sigma^2)$ obtained by means of the position:

$$W_2(x, t) = \begin{cases} w_2(x, t) & \text{in } M(X^0, \sigma), \\ 0 & \text{in } B(x^0, \sigma) \times (t^0 + \rho^2 - \sigma^2, t^0). \end{cases} \tag{2.11}$$

W_2 belongs to $L^2(t^0 + \rho^2 - \sigma^2, t^2 + \sigma^2, H^1(B(x^0, \sigma), \mathbb{R}^N)) \cap H_{t^0 + \rho^2 - \sigma^2}^{1/2}(t^0 + \rho^2 - \sigma^2, t^0 + + \sigma^2, L^2(B(x^0, \sigma), \mathbb{R}^N))$ and is weak solution (in the usual sense) in $Q(X^*, \sigma)$ to the system

$$- \sum_{ij=1}^{n} D_i(A_{ij}^0 D_j W_2) + \frac{\partial W_2}{\partial t} = 0. \tag{2.12}$$

Then, by means of the estimate (2.4) of [2], we get

$$\int_{Q(X^*, \rho)} \sum_{i=1}^{n} \|D_i W_2\|^2 \, dX \leq c(\nu) \left(\frac{\rho}{\sigma}\right)^{n+2} \int_{Q(X^*, \sigma)} \sum_{i=1}^{n} \|D_i W_2\|^2 \, dX \tag{2.13}$$

from which, in virtue of (2.11) and being $Q(X^*, \rho) = M(X^0, \rho)$, it follows

$$\int_{M(X^0, \rho)} \sum_{i=1}^{n} \|D_i w_2\|^2 \, dX \leq c(\nu) \left(\frac{\rho}{\sigma}\right)^{n+2} \int_{B(x^0, \sigma) \times (t^0, t^0 + \rho^2)} \sum_{i=1}^{n} \|D_i w_2\|^2 \, dX. \tag{2.14}$$

From (2.14) and from the inclusion

$$B(x^0, \sigma) \times (t^0, t^0 + \rho^2) \subset M(X^0, \sigma),$$

we obtain $\forall \rho \in (0, \sigma)$

$$\int_{M(X^0, \rho)} \sum_{i=1}^{n} \|D_i w_2\|^2 \, dX \leq c(\nu) \left(\frac{\rho}{\sigma}\right)^{n+2} \int_{M(X^0, \sigma)} \sum_{i=1}^{n} \|D_i w_2\|^2 \, dX. \tag{2.15}$$

On the other hand, in virtue of Theorem 1.1, v_1 satisfies the estimate

$$\int_{M(X^0,\sigma)} \sum_{i=1}^{n} \|D_i v_1\|^2 dX \leqslant$$

$$\leqslant c \left\{ \|u\|^2_{H_{t^0}^{*1/2}(t^0,\, t^0+\sigma^2,\, L^2(B(x^0,\sigma),\, \mathbb{R}^N))} + \int_{M(X^0,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2 dX + \int_{M(X^0,\sigma)} \sum_{i=0}^{n} \|F^i\|^2 dX \right\},$$

and hence, being $w_1 = u - v_1$,

$$\int_{M(X^0,\sigma)} \sum_{i=1}^{n} \|D_i w_1\|^2 dX \leqslant \tag{2.16}$$

$$\leqslant c \left\{ \|u\|^2_{H_{t^0}^{*1/2}(t^0,\, t^0+\sigma^2,\, L^2(B(x^0,\sigma),\, \mathbb{R}^N))} + \int_{M(X^0,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2 dX + \int_{M(X^0,\sigma)} \sum_{i=0}^{n} \|F^i\|^2 dX \right\}.$$

Since $w = w_1 + w_2$ in $M(X^0, \sigma)$, from (2.15) and (2.16) the assert follows in a standard way.

LEMMA 2.3. - *If* $u \in L^2(-T, 0, H^1(B^+(x^0,\sigma), \mathbb{R}^N)) \cap H_{-T}^{*1/2}(-T, 0, L^2(B^+(x^0,\sigma), \mathbb{R}^N))$ *and* $F^i \in L^2(B^+(x^0,\sigma) \times (-T, 0), \mathbb{R}^N)$, $i = 0, 1, \ldots, n$, *if*

$$w \in L^2(-T, 0, \tilde{H}_0^1(B^+(x^0,\sigma), \mathbb{R}^N)) \cap H_{-T}^{1/2}(-T, 0, L^2(B^+(x^0,\sigma), \mathbb{R}^N))\ (^9)$$

is solution in $Q^+(X^0,\sigma) \subset B^+(x^0,\sigma) \times (-T, 0)$, $X^0 = (x^0, t^0)$, *to the system* (2.1)(7), *then there exists a positive constant* $c = c(\nu)$ *such that* $\forall \rho \in (0, \sigma)$

$$\int_{Q^+(X^0,\rho)} \sum_{i=1}^{n} \|D_i w\|^2 dX \leqslant c(\nu) \left\{ \left(\frac{\rho}{\sigma}\right)^{n+2} \int_{Q^+(X^0,\sigma)} \sum_{i=1}^{n} \|D_i w\|^2 dX + \right. \tag{2.17}$$

$$\left. + \int_{Q^+(X^0,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2 dX + \int_{Q^+(X^0,\sigma)} \sum_{i=0}^{n} \|F^i\|^2 dX \right\}.$$

PROOF. - Let us consider the functions W, f^i, $i = 0, 1, \ldots, n$, and U defined in

(9) $\tilde{H}_0^1(B^+(x^0,\sigma), \mathbb{R}^N)$ is the class of the functions $u \in H^1(B^+(x^0,\sigma), \mathbb{R}^N)$ with trace on the iperplane $x_n = x_n^0$ zero.

$B(x^0, \sigma) \times (-T, 0)$ in the following way:

$$W(x, t) = \begin{cases} w(x, t) & \text{if } x_n \geq x_n^0, \\ w(2x^0 - x, t) & \text{if } x_n < x_n^0, \end{cases} \tag{2.18}$$

$$\begin{gathered} f^0(x, t) = \begin{cases} F^0(x, t) & \text{if } x_n \geq x_n^0, \\ F^0(2x^0 - x, t) & \text{if } x_n < x_n^0, \end{cases} \\ f^i(x, t) = \begin{cases} F^i(x, t) & \text{if } x_n \geq x_n^0, \\ -F^i(2x^0 - x, t) & \text{if } x_n < x_n^0, \end{cases} \quad i = 1, \dots, n, \end{gathered} \tag{2.19}$$

$$U(x, t) = \begin{cases} u(x, t) & \text{if } x_n \geq x_n^0, \\ u(2x^0 - x, t) & \text{if } x_n < x_n^0, \end{cases} \tag{2.20}$$

where $2x^0 - x = (2x_1^0 - x_1, 2x_2^0 - x_2, \dots, 2x_n^0 - x_n)$.

The functions f^i, $i = 0, 1, \dots, n$, belong to the space $L^2(B(x^0, \sigma) \times (-T, 0), \mathbb{R}^N)$, U belongs to $L^2(-T, 0, H^1(B(x^0, \sigma), \mathbb{R}^N)) \cap H_{-T}^{*1/2}(-T, 0, L^2(B(x^0, \sigma), \mathbb{R}^N))$, while W belongs to $L^2(-T, 0, H^1(B(x^0, \sigma), \mathbb{R}^N)) \cap H_{-T}^{1/2}(-T, 0, L^2(B(x^0, \sigma), \mathbb{R}^N))$ and is solution in $Q(X^0, \sigma)$ to the system

$$-\sum_{ij=1}^{n} D_i(A_{ij}^0 D_j W) + \frac{\partial W}{\partial t} = \frac{\partial U}{\partial t} - \sum_{i=1}^{n} D_i f^i + f^0.$$

Then W verifies the assumptions of Lemma 2.1 and hence the estimate (2.17) follows from (2.4), written for W, f^i and U, taking into account that $\forall r \in (0, \sigma]$ it results:

$$\int_{Q(X^0, r)} \sum_{i=1}^{n} \|D_i W\|^2 dX = 2 \int_{Q^+(X^0, r)} \sum_{i=1}^{n} \|D_i w\|^2 dX,$$

$$\int_{Q(X^0, r)} \sum_{i=0}^{n} \|f^i\|^2 dX = 2 \int_{Q^+(X^0, r)} \sum_{i=0}^{n} \|F^i\|^2 dX,$$

$$\int_{Q(X^0, r)} \sum_{i=1}^{n} \|D_i U\|^2 dX = 2 \int_{Q^+(X^0, r)} \sum_{i=1}^{n} \|D_i u\|^2 dX.$$

LEMMA 2.4. - *If* $u \in L^2(-T, 0, H^1(B^+(x^0, \sigma), \mathbb{R}^N)) \cap H_{-T}^{*1/2}(-T, 0, L^2(B^+(x^0, \sigma), \mathbb{R}^N))$ *and* $F^i \in L^2(B^+(x^0, \sigma) \times (-T, 0), \mathbb{R}^N)$, $i = 0, 1, \dots, n$, *if* $w \in L^2(-T, 0, \tilde{H}_0^1(B^+(x^0, \sigma), \mathbb{R}^N)) \cap H_{-T}^{1/2}(-T, 0, L^2(B^+(x^0, \sigma), \mathbb{R}^N))$ *is solution in* $M^+(X^0, \sigma) \subset B^+(x^0, \sigma) \times (-T, 0)$, $X^0 = (x^0, t^0)$, $t^0 = -T$, *to the system* (2.1)(7), *then there exists a positive constant* $c = c(\nu)$ *such that* $\forall \rho \in (0, \sigma)$

$$\int_{M^+(X^0,\rho)} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq c(\nu) \left\{ \left(\frac{\rho}{\sigma} \right)^{n+2} \int_{M^+(X^0,\sigma)} \sum_{i=1}^{n} \|D_i w\|^2 dX + \right. \tag{2.21}$$

$$\left. + \|u\|^2_{H^{*1/2}_{t^0}(t^0, t^0+\sigma^2, L^2(B^+(x^0,\sigma), \mathbb{R}^N))} + \int_{M^+(X^0,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2 dX + \int_{M^+(X^0,\sigma)} \sum_{i=0}^{n} \|F^i\|^2 dX \right\}.$$

PROOF. – Let us define W, f^i, $i = 0, 1, \ldots, n$, and U in $M(X^0, \sigma)$ by means of positions (2.18), (2.19) and (2.20). The functions f^i, $i = 0, 1, \ldots, n$, belong to the space $L^2(B(x^0, \sigma) \times (-T, 0), \mathbb{R}^N)$, U belongs to $L^2(-T, 0, H^1(B(x^0, \sigma), \mathbb{R}^N)) \cap \cap H^{*1/2}_{-T}(-T, 0, L^2(B(x^0, \sigma), \mathbb{R}^N))$, while W verifies in $M(X^0, \sigma)$ the assumptions of the Lemma 2.2 with f^i and U instead of F^i and u, respectively. The estimate (2.21) follows from (2.7), written for W, f^i and U, taking into account that $\forall r \in (0, \sigma]$ it results:

$$\int_{M(X^0,r)} \sum_{i=1}^{n} \|D_i W\|^2 dX = 2 \int_{M^+(X^0,r)} \sum_{i=1}^{n} \|D_i w\|^2 dX,$$

$$\int_{M(X^0,r)} \sum_{i=0}^{n} \|f^i\|^2 dX = 2 \int_{M^+(X^0,r)} \sum_{i=0}^{n} \|F^i\|^2 dX,$$

$$\|U\|^2_{H^{*1/2}_{t^0}(t^0, t^0+r^2, L^2(B(x^0,r), \mathbb{R}^N))} = 2\|u\|^2_{H^{*1/2}_{t^0}(t^0, t^0+r^2, L^2(B^+(x^0,r), \mathbb{R}^N))},$$

$$\int_{M(X^0,r)} \sum_{i=1}^{n} \|D_i U\|^2 dX = 2 \int_{M^+(X^0,r)} \sum_{i=1}^{n} \|D_i u\|^2 dX.$$

3. – $L^{2,\lambda}$-regularity results.

Let $L^{2,\lambda}(Q, \mathbb{R}^N)$, $0 < \lambda < n+2$, be the usual Morrey spaces related to parabolic metric

$$d(X, Y) = \max\{\|x - y\|, |t - \tau|^{1/2}\}, \qquad X = (x, t), \quad Y = (y, \tau).$$

Let us denote by $H^{*0,1/2,(\lambda)}_{-T}(Q, \mathbb{R}^N)$, $0 < \lambda < n+2$, the space of those functions

$u(X) \in H^{*1/2}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N))$ for which

$$[u]^2_{H^{*0,1/2,(\lambda)}_{-T}(Q,\mathbb{R}^N)} = \sup_{\substack{x^0 \in \bar{\Omega} \\ \sigma > 0}} \frac{1}{\sigma^\lambda} \left\{ \int_{-T}^{m_\sigma} dt \int_{-T}^{m_\sigma} d\xi \int_{B(x^0,\sigma)\cap\Omega} \frac{\|u(X) - u(x,\xi)\|^2}{|t-\xi|^2} dx + \right.$$

$$\left. + \int_{-T}^{m_\sigma} dt \int_{B(x^0,\sigma)\cap\Omega} \frac{\|u(X) - u(x,-T)\|^2}{t+T} dx \right\} < +\infty,$$

where $m_\sigma = \min(0, -T + \sigma^2)$.

Let $w \in L^2(-T, 0, H^1(\Omega, \mathbb{R}^N)) \cap H^{1/2}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N))$ a solution in Q to the system (7)

$$(3.1) \qquad -\sum_{ij=1}^{n} D_i(A_{ij}(X) D_j w) + \frac{\partial w}{\partial t} = \frac{\partial u}{\partial t} - \sum_{i=1}^{n} D_i F^i + F^0,$$

where, for a certain λ, $0 < \lambda < n+2$,

$$(3.2) \qquad F^i \in L^{2,\lambda}(Q, \mathbb{R}^N), \qquad i = 0, 1, \ldots, n,$$

while u and $A_{ij}(X)$ $i, j = 1, \ldots, n$, verify the conditions (1.3), (1.5) and (1.6).

The following theorem holds:

THEOREM 3.1. – *If* $w \in L^2(-T, 0, H^1(\Omega, \mathbb{R}^N)) \cap H^{1/2}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N))$ *is solution in* Q *to the system* (3.1), *if the conditions* (1.3), (1.5), (1.6), (3.2) *are satisfied and if* $u \in L^{2,\lambda}(Q, \mathbb{R}^N) \cap H^{*0,1/2,(\lambda)}_{-T}(Q, \mathbb{R}^N)$, $D_i u \in L^{2,\lambda}(Q, \mathbb{R}^N)$, $0 < \lambda < n+2$, $i = 1, 2, \ldots, n$, *then, for every cylinder* $Q_0 = \Omega_0 \times (-T, 0)$ *with* $\Omega_0 \subset\subset \Omega$, *it results*

$$(3.3) \qquad D_i w \in L^{2,\lambda}(Q_0, \mathbb{R}^N), \qquad i = 1, 2, \ldots, n,$$

and we have

$$(3.4) \qquad \sum_{i=1}^{n} \|D_i w\|^2_{L^{2,\lambda}(Q_0,\mathbb{R}^N)} \le c(\nu, n, \lambda) \left\{ \frac{1}{\bar{\sigma}^\lambda} \sum_{i=1}^{n} \|D_i w\|^2_{L^2(Q,\mathbb{R}^N)} + \right.$$

$$\left. + \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} + \|u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} + [u]^2_{H^{*0,1/2,(\lambda)}_{-T}(Q,\mathbb{R}^N)} + \sum_{i=0}^{n} \|F^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} \right\}$$

where $\bar{\sigma}$ *is the positive number defined by* (3.14).

PROOF. – Let us fix $Q_0 = \Omega_0 \times (-T, 0)$ with $\Omega_0 \subset\subset \Omega$, let us denote by R_0 the euclidean distance of Ω_0 from $\partial\Omega$ and let us set

$$(3.5) \qquad \sigma_0 = \frac{1}{2} \min(R_0, \sqrt{T}).$$

Let $\omega(\sigma)$ be the non negative function defined for every $\sigma > 0$ in the following way

$$\omega(\sigma) = \sup_{\substack{X,\, Y \in \overline{Q} \\ d(X,\, Y) \leq \sigma}} \sum_{ij=1}^{n} \|A_{ij}(X) - A_{ij}(Y)\|^2 \tag{3.6}$$

and let us observe that, in virtue of the continuity in $\overline{Q}$ of the coefficients A_{ij}, $\omega(\sigma) \to 0$ when $\sigma \to 0$.

Theorem 3.1 will be achieved if we are able to determine a constant $\bar{\sigma} \in (0, \sigma_0]$ such that $\forall X^0 \in \overline{Q}_0$ and $\forall \rho \in (0, \bar{\sigma})$ it results (10):

$$\int_{I(X^0,\, \rho) \cap Q} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq c(\nu, n, \lambda) \rho^{\lambda} \left\{ \frac{1}{\bar{\sigma}^{\lambda}} \sum_{i=1}^{n} \|D_i w\|^2_{L^2(Q,\, \mathbb{R}^N)} + \right. \tag{3.7}$$

$$\left. + \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q,\, \mathbb{R}^N)} + \|u\|^2_{L^{2,\lambda}(Q,\, \mathbb{R}^N)} + [u]^2_{H^{*0,\, 1/2,\omega}_{-T}(Q,\, \mathbb{R}^N)} + \sum_{i=0}^{n} \|F^i\|^2_{L^{2,\lambda}(Q,\, \mathbb{R}^N)} \right\}.$$

Let us fix $X^0 = (x^0, t^0) \in \overline{Q}_0$; to achieve (3.7) we must consider two cases (see [1], Theorem 9.I for an analogous point)

a) $t^0 = -T$. For every $\sigma \in (0, \sigma_0]$ we have

$$I(X^0, \sigma) \cap Q = M(X^0, \sigma) \subset B(x^0, \sigma) \times (-T, 0) \subset Q; \tag{3.8}$$

the function $w \in L^2(-T, 0, H^1(B(x^0, \sigma), \mathbb{R}^N)) \cap H^{1/2}_{-T}(-T, 0, L^2(B(x^0, \sigma), \mathbb{R}^N))$ and is solution in $M(X^0, \sigma)$ to the system

$$-\sum_{ij=1}^{n} D_i(A_{ij}(X^0) D_j w) + \frac{\partial w}{\partial t} = \frac{\partial u}{\partial t} - \sum_{i=1}^{n} D_i f^i + F^0 \tag{3.9}$$

with $f^i = F^i + \sum_{j=1}^{n} (A_{ij}(X^0) - A_{ij}(X)) D_j w \in L^2(B(x^0, \sigma) \times (-T, 0), \mathbb{R}^N)$, $\quad i = 1, 2, \ldots, n$.

The Lemma 2.2 assures that $\forall \rho \in (0, \sigma)$

$$\int_{M(X^0,\, \rho)} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq c(\nu) \left\{ \left(\frac{\rho}{\sigma}\right)^{n+2} \int_{M(X^0,\, \sigma)} \sum_{i=1}^{n} \|D_i w\|^2 dX + \right.$$

$$\left. + \|u\|^2_{H^{*1/2}_{t^0}(t^0,\, t^0 + \sigma^2,\, L^2(B(x^0,\, \sigma),\, \mathbb{R}^N))} + \int_{M(X^0,\, \sigma)} \sum_{i=1}^{n} \|D_i u\|^2 dX + \int_{M(X^0,\, \sigma)} \sum_{i=1}^{n} \|f^i\|^2 dX + \int_{M(X^0,\, \sigma)} \|F^0\|^2 dX \right\}$$

(10) This because $\displaystyle\int_{I(X^0,\, \rho) \cap Q} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq \int_{I(X^0,\, \rho) \cap Q} \sum_{i=1}^{n} \|D_i w\|^2 dX.$

and, then, for every $\rho \in (0, \sigma)$

$$\int\limits_{M(X^0,\rho)} \sum_{i=1}^{n} \|D_i w\|^2 dX \leqslant c(\nu) \left\{ \left(\frac{\rho}{\sigma} \right)^{n+2} \int\limits_{M(X^0,\sigma)} \sum_{i=1}^{n} \|D_i w\|^2 dX + \right.$$
$$+ \|u\|^2_{H^{*1/2}_{t^0}(t^0,\, t^0+\sigma^2,\, L^2(B(x^0,\sigma),\, \mathbb{R}^N))} +$$
$$\left. + \int\limits_{M(X^0,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2 dX + \omega(\sigma) \int\limits_{M(X^0,\sigma)} \sum_{i=1}^{n} \|D_i w\|^2 dX + \int\limits_{M(X^0,\sigma)} \sum_{i=0}^{n} \|F^i\|^2 dX \right\}. \tag{3.10}$$

On the other hand, in view of the assumptions on u and F^i, it results:

$$\int\limits_{M(X^0,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2 dX \leqslant \sigma^\lambda \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q,\, \mathbb{R}^N)}, \tag{3.11}$$

$$\|u\|^2_{H^{*1/2}_{t^0}(t^0,\, t^0+\sigma^2,\, L^2(B(x^0,\sigma),\, \mathbb{R}^N))} \leqslant \sigma^\lambda \left\{ \|u\|^2_{L^{2,\lambda}(Q,\, \mathbb{R}^N)} + [u]^2_{H^{*0,\, 1/2,(\lambda)}_{-T}(Q,\, \mathbb{R}^N)} \right\} \tag{3.12}$$

and

$$\int\limits_{M(X^0,\sigma)} \sum_{i=0}^{n} \|F^i\|^2 dX \leqslant \sigma^\lambda \sum_{i=0}^{n} \|F^i\|^2_{L^{2,\lambda}(Q,\, \mathbb{R}^N)}. \tag{3.13}$$

From (3.10), (3.11), (3.12) and (3.13) we reach for every $0 < \rho < \sigma \leqslant \sigma_0$

$$\varphi(\rho) \leqslant c(\nu)\, \varphi(\sigma) \left\{ \left(\frac{\rho}{\sigma} \right)^{n+2} + \omega(\sigma) \right\} + c(\nu)\, \sigma^\lambda\, \mathcal{M}^2,$$

where

$$\varphi(r) = \int\limits_{M(X^0,r)} \sum_{i=1}^{n} \|D_i w\|^2 dX,$$

$$\mathcal{M}^2 = \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q,\, \mathbb{R}^N)} + \|u\|^2_{L^{2,\lambda}(Q,\, \mathbb{R}^N)} + [u]^2_{H^{*0,\, 1/2,(\lambda)}_{-T}(Q,\, \mathbb{R}^N)} + \sum_{i=0}^{n} \|F^i\|^2_{L^{2,\lambda}(Q,\, \mathbb{R}^N)}.$$

Then, applying the Lemma 2.VII of [3], in correspondence to the number $\varepsilon = n + 2 - \lambda$, there exists

$$\bar{\sigma} \in (0, \sigma_0] \quad \text{(with } \bar{\sigma} \text{ independent of } \varphi) \tag{3.14}$$

such that, if $\sigma \in (0, \bar{\sigma}]$ and $\rho \in (0, \sigma)$

$$\varphi(\rho) \leqslant c(\nu) \left(\frac{\rho}{\sigma} \right)^\lambda \varphi(\sigma) + c(\nu, n, \lambda)\, \mathcal{M}^2 \rho^\lambda,$$

from which, $\forall\rho \in (0, \bar\sigma)$

$$(3.15)\qquad \int\limits_{M(X^0,\rho)} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq c(\nu, n, \lambda)\rho^\lambda \left\{ \frac{1}{\bar\sigma^\lambda} \sum_{i=1}^{n} \|D_i w\|^2_{L^2(Q, \mathbb{R}^N)} + \mathcal{M}^2 \right\}.$$

Taking into account (3.8), from (3.15) the estimate (3.7) follows for every $\rho \in (0, \bar\sigma)$.

b) $t^0 \in (-T, 0]$. We shall prove that $\forall X^0 \in \overline{Q}_0$ con $t^0 \in (-T, 0]$ e $\forall\rho \in (0, \bar\sigma)$ (11) it results:

$$(3.16)\qquad \int\limits_{Q(X^0,\rho)\cap Q} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq c(\nu, n, \lambda)\rho^\lambda \left\{ \frac{1}{\bar\sigma^\lambda} \sum_{i=1}^{n} \|D_i w\|^2_{L^2(Q, \mathbb{R}^N)} + \right.$$
$$\left. + \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q, \mathbb{R}^N)} + \|u\|^2_{L^{2,\lambda}(Q, \mathbb{R}^N)} + [u]^2_{H^{*0,1/2,(\lambda)}_{-T}(Q, \mathbb{R}^N)} + \sum_{i=0}^{n} \|F^i\|^2_{L^{2,\lambda}(Q, \mathbb{R}^N)} \right\}.$$

Let us suppose just now we obtained this estimate.

Then if $t^0 = 0$, for every $\rho \in (0, \bar\sigma)$, it results:

$$I(X^0, \rho) \cap Q = Q(X^0, \rho) \cap Q$$

and hence (3.16) coincides with (3.7); if $t^0 \in (-T, 0)$, for every $\rho \in (0, \bar\sigma)$, we have:

$$(3.17)\qquad I(X^0, \rho) \cap Q = [Q(X^0, \rho) \cap Q] \cup [B(x^0, \rho) \times \{t^0\}] \cup$$
$$\cup [M(X^0, \rho) \cap Q] = [Q(X^0, \rho) \cap Q] \cup [B(x^0, \rho) \times \{t^0\}] \cup [Q(X^*, \rho) \cap Q]$$

where $X^* = (x^0, t^*)$, $t^* = \min(0, t^0 + \rho^2)$. Taking into account that $X^* = (x^0, t^*)$ belongs to $\overline{Q}_0$ and that $t^* \in (-T, 0]$, from (3.16), written with X^* instead of X^0, it follows, $\forall\rho \in (0, \bar\sigma)$

$$(3.18)\qquad \int\limits_{Q(X^*,\rho)\cap Q} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq c(\nu, n, \lambda)\rho^\lambda \left\{ \frac{1}{\bar\sigma^\lambda} \sum_{i=1}^{n} \|D_i w\|^2_{L^2(Q, \mathbb{R}^N)} + \right.$$
$$\left. + \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q, \mathbb{R}^N)} + \|u\|^2_{L^{2,\lambda}(Q, \mathbb{R}^N)} + [u]^2_{H^{*0,1/2,(\lambda)}_{-T}(Q, \mathbb{R}^N)} + \sum_{i=0}^{n} \|F^i\|^2_{L^{2,\lambda}(Q, \mathbb{R}^N)} \right\}.$$

Then (3.7) is a consequence of (3.16), (3.17) and (3.18). Hence it is nough to show the estimate (3.16).

Setting $\sigma^* = \min(\bar\sigma, \sqrt{T + t^0})$, we have $Q(X^0, \sigma^*) \subset Q$; then, for every $\sigma \in (0, \sigma^*]$, the function $w \in L^2(-T, 0, H^1(B(x^0, \sigma), \mathbb{R}^N)) \cap H^{1/2}_{-T}(-T, 0, L^2(B(x^0, \sigma), \mathbb{R}^N))$ is

(11) $\bar\sigma$ is the constant defined by (3.14).

solution in $Q(X^0, \sigma) \subset B(x^0, \sigma) \times (-T, 0)$ to the system (3.9), with $f^i = F^i + \sum_{j=1}^{n} (A_{ij}(X^0) - A_{ij}(X)) D_j w \in L^2(B(x^0, \sigma) \times (-T, 0), \mathbb{R}^N)$, $i = 1, 2, \dots, n$.

The estimate (2.4) of the Lemma 2.1 assures that $\forall \rho \in (0, \sigma)$

$$\int_{Q(X^0,\rho)} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq c(\nu) \left\{ \left(\frac{\rho}{\sigma}\right)^{n+2} \int_{Q(X^0,\sigma)} \sum_{i=1}^{n} \|D_i w\|^2 dX + \right.$$

$$\left. + \int_{Q(X^0,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2 dX + \int_{Q(X^0,\sigma)} \sum_{i=1}^{n} \|f^i\|^2 dX + \int_{Q(X^0,\sigma)} \|F^0\|^2 dX \right\}$$

and hence, for every $0 < \rho < \sigma \leq \sigma^*$

$$\int_{Q(X^0,\rho)} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq c(\nu) \int_{Q(X^0,\sigma)} \sum_{i=1}^{n} \|D_i w\|^2 dX \left\{ \left(\frac{\rho}{\sigma}\right)^{n+2} + \omega(\sigma) \right\} + \tag{3.19}$$

$$+ c(\nu) \sigma^\lambda \left\{ \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} + \sum_{i=0}^{n} \|F^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} \right\}.$$

This last estimate coincides with (2.16) of the Lemma 2.VII of [3], if we set

$$\varphi(\sigma) = \int_{Q(X^0,\sigma)} \sum_{i=1}^{n} \|D_i w\|^2 dX,$$

and hence, being $\sigma^* \leq \bar{\sigma}$, if $\sigma \in (0, \sigma^*]$ and $\rho \in (0, \sigma)$ it results:

$$\varphi(\rho) \leq c(\nu) \left(\frac{\rho}{\sigma}\right)^\lambda \varphi(\sigma) + c(\nu, n, \lambda) \rho^\lambda \left\{ \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} + \sum_{i=0}^{n} \|F^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} \right\},$$

from which we get $\forall \rho \in (0, \sigma^*)$

$$\int_{Q(X^0,\rho)} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq c(\nu, n, \lambda) \rho^\lambda \left\{ \frac{1}{\sigma^{*\lambda}} \int_{Q(X^0,\sigma^*)} \sum_{i=1}^{n} \|D_i w\|^2 dX + \right. \tag{3.20}$$

$$\left. + \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} + \sum_{i=0}^{n} \|F^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} \right\}.$$

Now we have two possibilities.

If $\sigma^* = \bar{\sigma}$, from (3.20) it follows (3.16) $\forall \rho \in (0, \bar{\sigma})$. On the contrary $\sigma^* = \sqrt{T + t^0} < \bar{\sigma}$, it results:

$$Q(X^0, \sigma^*) = M(X^*, \sigma^*), \qquad X^* = (x^0, -T)$$

and (3.20) can be rewritten, $\forall \rho \in (0, \sigma^*)$, in the following way

$$(3.21)\qquad \int_{Q(X^0,\rho)} \sum_{i=1}^n \|D_i w\|^2 dX \le c(\nu, n, \lambda)\left(\frac{\rho}{\sigma^*}\right)^\lambda \int_{M(X^*,\sigma^*)} \sum_{i=1}^n \|D_i w\|^2 dX +$$

$$+ c(\nu, n, \lambda)\rho^\lambda \left\{ \sum_{i=1}^n \|D_i u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} + \sum_{i=0}^n \|F^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} \right\}.$$

Making use of (3.15), with $\rho = \sigma^* < \bar\sigma$ and $X^0 = X^*$, we get:

$$(3.22)\qquad \frac{1}{\sigma^{*\lambda}} \int_{M(X^*,\sigma^*)} \sum_{i=1}^n \|D_i w\|^2 dX \le c(\nu, n, \lambda)\left\{ \frac{1}{\bar\sigma^\lambda} \sum_{i=1}^n \|D_i w\|^2_{L^2(Q,\mathbb{R}^N)} + \mathfrak{M}^2 \right\}.$$

From (3.21) and (3.22) the estimate (3.16) follows for every $\rho \in (0, \sqrt{T + t^0})$. While if $\sqrt{T+t^0} \le \rho < \bar\sigma$ we have $Q(X^0,\rho) \cap Q \subset M(X^*,\rho) \subset Q$, $X^* = (x^0, -T)$, and hence from (3.15), with $X^0 = X^*$, we obtain

$$(3.23)\qquad \int_{Q(X^0,\rho)\cap Q} \sum_{i=1}^n \|D_i w\|^2 dX \le$$

$$\le \int_{M(X^*,\rho)} \sum_{i=1}^n \|D_i w\|^2 dX \le c(\nu, n, \lambda)\rho^\lambda \left\{ \frac{1}{\bar\sigma^\lambda} \sum_{i=1}^n \|D_i w\|^2_{L^2(Q,\mathbb{R}^N)} + \mathfrak{M}^2 \right\}.$$

Now let us show the following

THEOREM 3.2. - *If* $w \in L^2(-T, 0, \tilde H_0^1(B^+(1), \mathbb{R}^N)) \cap H_{-T}^{1/2}(-T, 0, L^2(B^+(1), \mathbb{R}^N))$ *is solution in* $Q = Q^+(1) = B^+(1) \times (-T, 0)$ *to the system* (3.1), *if the conditions* (1.3), (1.5), (1.6), (3.2) *are satisfied with* $\Omega = B^+(1)$, $Q = Q^+(1)$ *and if* $u \in L^{2,\lambda}(Q^+(1), \mathbb{R}^N) \cap H_{-T}^{*0,1/2,(\lambda)}(Q^+(1), \mathbb{R}^N)$, $D_i u \in L^{2,\lambda}(Q^+(1), \mathbb{R}^N)$, $0 < \lambda < n+2$, $i = 1, 2, \ldots, n$, *then,* $\forall R \in (0,1)$, *it results*:

$$(3.24)\qquad D_i w \in L^{2,\lambda}(Q^+(R), \mathbb{R}^N), \qquad i = 1, 2, \ldots, n,$$

and moreover

$$(3.25)\qquad \sum_{i=1}^n \|D_i w\|^2_{L^{2,\lambda}(Q^+(R),\mathbb{R}^N)} \le c(\nu, n, \lambda)\left\{ \frac{1}{\bar\sigma^\lambda} \sum_{i=1}^n \|D_i w\|^2_{L^2(Q,\mathbb{R}^N)} + \right.$$

$$\left. + \sum_{i=1}^n \|D_i u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} + \|u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} + [u]^2_{H_{-T}^{*0,1/2,(\lambda)}(Q,\mathbb{R}^N)} + \sum_{i=0}^n \|F^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} \right\},$$

where $Q^+(R) = B^+(R) \times (-T, 0)$, $Q = Q^+(1)$, $\bar\sigma$ *is the positive number defined by* (3.32).

PROOF. – Let us fix $R \in (0,1)$ and let us set

$$\sigma_0 = \frac{1}{2}\min(1-R, R, \sqrt{T}), \tag{3.26}$$

$$\omega(\sigma) = \sup_{\substack{X,Y\in\overline{Q}\\ d(X,Y)\le\sigma}} \sum_{ij=1}^{n} \|A_{ij}(X) - A_{ij}(Y)\|^2, \qquad \sigma > 0. \tag{3.27}$$

We shall prove the theorem if we shall be able to determine a constant $\bar\sigma \in (0, \sigma_0]$ such that $\forall X^0 \in \overline{Q^+(R)}$ and $\forall \rho \in (0, \bar\sigma/2)$ we have an estimate of the type

$$\int_{I(X^0,\rho)\cap Q} \sum_{i=1}^{n} \|D_i w\|^2 dX \le c(\nu, n, \lambda)\rho^\lambda \left\{ \frac{1}{\bar\sigma^\lambda}\sum_{i=1}^{n} \|D_i w\|^2_{L^2(Q,\mathbb{R}^N)} + \right. \tag{3.28}$$

$$\left. + \sum_{i=1}^{n}\|D_i u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} + \|u\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} + [u]^2_{H^{*0,1/2,(\lambda)}_{-T}(Q,\mathbb{R}^N)} + \sum_{i=0}^{n}\|F^i\|^2_{L^{2,\lambda}(Q,\mathbb{R}^N)} \right\}.$$

Let $X^0 = (x^0, t^0) = (x_1^0, x_2^0, \ldots, x_n^0, t^0)$ be a point of $\overline{Q^+(R)}$; we must consider several possibilities, first let us show (3.28) when

a) $x_n^0 = 0$ and $t^0 = -T$. For every $\sigma \in (0, \sigma_0]$ we have:

$$I(X^0,\sigma)\cap Q = B^+(x^0,\sigma)\times(-T, -T+\sigma^2) = M^+(X^0,\sigma) \subset Q; \tag{3.29}$$

the function $w \in L^2(-T, 0, \tilde H_0^1(B^+(x^0,\sigma), \mathbb{R}^N)) \cap H^{1/2}_{-T}(-T, 0, L^2(B^+(x^0,\sigma), \mathbb{R}^N))$ and is solution in $M^+(X^0,\sigma) \subset B^+(x^0,\sigma)\times(-T,0)$ to the system (3.9), with $f^i = F^i + \sum_{j=1}^{n}(A_{ij}(X^0) - A_{ij}(X))D_j w \in L^2(B^+(x^0,\sigma)\times(-T,0), \mathbb{R}^N)$, $i = 1, 2, \ldots, n$.

The estimate (2.21) of the Lemma 2.4 assures that $\forall \rho \in (0,\sigma)$ we have

$$\int_{M^+(X^0,\rho)} \sum_{i=1}^{n}\|D_i w\|^2 dX \le c(\nu)\left\{ \left(\frac{\rho}{\sigma}\right)^{n+2} \int_{M^+(X^0,\sigma)} \sum_{i=1}^{n}\|D_i w\|^2 dX + \right. \tag{3.30}$$

$$+ \|u\|^2_{H^{*1/2}_{t^0}(t^0, t^0+\sigma^2, L^2(B^+(x^0,\sigma),\mathbb{R}^N))} + \int_{M^+(X^0,\sigma)} \sum_{i=1}^{n}\|D_i u\|^2 dX +$$

$$\left. + \int_{M^+(X^0,\sigma)} \sum_{i=1}^{n}\|f^i\|^2 dX + \int_{M^+(X^0,\sigma)} \|F^0\|^2 dX \right\}$$

and hence $\forall 0 < \rho < \sigma \le \sigma_0$

$$\varphi(\rho) \le c(\nu)\,\varphi(\sigma)\left\{\left(\frac{\rho}{\sigma}\right)^{n+2} + \omega(\sigma)\right\} + c(\nu)\,\mathcal{M}^2\sigma^\lambda, \tag{3.31}$$

where

$$\varphi(r) = \int\limits_{M^+(X^0, r)} \sum_{i=1}^{n} \|D_i w\|^2 \, dX,$$

$$\mathcal{M}^2 = \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q, \mathbb{R}^N)} + \|u\|^2_{L^{2,\lambda}(Q, \mathbb{R}^N)} + [u]^2_{H_{-T}^{*0,\, 1/2, (\lambda)}(Q, \mathbb{R}^N)} + \sum_{i=0}^{n} \|F^i\|^2_{L^{2,\lambda}(Q, \mathbb{R}^N)}.$$

In virtue of Lemma 2.VII of [3], there exists

$$\bar{\sigma} \in (0, \sigma_0] \quad (\bar{\sigma} \text{ independent of } \varphi) \tag{3.32}$$

such that, if $\sigma \in (0, \bar{\sigma}]$ and $\rho \in (0, \sigma)$

$$\varphi(\rho) \leq c(\nu) \left(\frac{\rho}{\sigma} \right)^{\lambda} \varphi(\sigma) + c(\nu, n, \lambda) \, \mathcal{M}^2 \rho^{\lambda},$$

from which, $\forall \rho \in (0, \bar{\sigma})$,

$$\int\limits_{M^+(X^0, \rho)} \sum_{i=1}^{n} \|D_i w\|^2 \, dX \leq c(\nu, n, \lambda) \rho^{\lambda} \left\{ \frac{1}{\bar{\sigma}^{\lambda}} \sum_{i=1}^{n} \|D_i w\|^2_{L^2(Q, \mathbb{R}^N)} + \mathcal{M}^2 \right\}.$$

From this estimate, taking into account (3.29), (3.28) follows, for every $\rho \in (0, \bar{\sigma})$.

b) $x_n^0 = 0$ and $-T < t^0 \leq 0$. Taking into account the technique applied in the case *b)* of the proof of the Theorem 3.1, it is enough to show that $\forall \rho \in (0, \bar{\sigma}/2)$ it results

$$\int\limits_{Q^+(X^0, \rho) \cap Q} \sum_{i=1}^{n} \|D_i w\|^2 \, dX \leq c(\nu, n, \lambda) \rho^{\lambda} \left\{ \frac{1}{\bar{\sigma}^{\lambda}} \sum_{i=1}^{n} \|D_i w\|^2_{L^2(Q, \mathbb{R}^N)} + \mathcal{M}^2 \right\}, \tag{3.33}$$

because from (3.33) the estimate (3.28) easily follows for every $\rho \in (0, \bar{\sigma}/2)$. The estimate (3.33) can be achieved using the same procedure followed in the case *b)* of the proof of Theorem 3.1, making use of the Lemma 2.3, instead of the Lemma 2.1.

c) $0 < x_n^0 \leq \bar{\sigma}/2$ and $t^0 = -T$. For every $\sigma \in (0, x_n^0]$ we have:

$$M(X^0, \sigma) = B(x^0, \sigma) \times (-T, -T + \sigma^2) \subset Q;$$

the function w belongs to $L^2(-T, 0, H^1(B(x^0, \sigma), \mathbb{R}^N)) \cap H_{-T}^{1/2}(-T, 0, L^2(B(x^0, \sigma), \mathbb{R}^N))$ and is solution in $M(X^0, \sigma) \subset B(x^0, \sigma) \times (-T, 0)$ to the system (3.9), with

$$f^i = F^i + \sum_{j=1}^{n} (A_{ij}(X^0) - A_{ij}(X)) D_j w \in L^2(B(x^0, \sigma) \times (-T, 0), \mathbb{R}^N), \qquad i = 1, 2, \ldots, n.$$

The estimate (2.7) of the Lemma 2.2 assures that $\forall\rho\in(0,\sigma)$,

$$\int\limits_{M(X^0,\rho)}\sum_{i=1}^n\|D_iw\|^2dX\leqslant c(\nu)\left\{\left(\frac{\rho}{\sigma}\right)^{n+2}\int\limits_{M(X^0,\sigma)}\sum_{i=1}^n\|D_iw\|^2dX+\right.$$

$$+\|u\|^2_{H^{*1/2}_{t^0}(t^0,t^0+\sigma^2,L^2(B(x^0,\sigma),\mathbb{R}^N))}+\int\limits_{M(X^0,\sigma)}\sum_{i=1}^n\|D_iu\|^2dX+$$

$$\left.+\int\limits_{M(X^0,\sigma)}\sum_{i=1}^n\|f^i\|^2dX+\int\limits_{M(X^0,\sigma)}\|F^0\|^2dX\right\}$$

and then $\forall\sigma\in(0,x_n^0]$, $\forall\rho\in(0,\sigma)$

$$\int\limits_{M(X^0,\rho)}\sum_{i=1}^n\|D_iw\|^2dX\leqslant c(\nu)\int\limits_{M(X^0,\sigma)}\sum_{i=1}^n\|D_iw\|^2dX\left\{\left(\frac{\rho}{\sigma}\right)^{n+2}+\omega(\sigma)\right\}+c(\nu)\sigma^\lambda\mathfrak{M}^2. \tag{3.34}$$

Being $x_n^0<\bar\sigma$, in virtue of the Lemma 2.VII of [3], from (3.34) we deduce, $\forall\sigma\in(0,x_n^0]$, $\forall\rho\in(0,\sigma)$

$$\int\limits_{M(X^0,\rho)}\sum_{i=1}^n\|D_iw\|^2dX\leqslant c(\nu)\left(\frac{\rho}{\sigma}\right)^\lambda\int\limits_{M(X^0,\sigma)}\sum_{i=1}^n\|D_iw\|^2dX+c(\nu,n,\lambda)\mathfrak{M}^2\rho^\lambda, \tag{3.35}$$

from which, $\forall\rho\in(0,x_n^0)$,

$$\int\limits_{M(X^0,\rho)}\sum_{i=1}^n\|D_iw\|^2dX\leqslant c(\nu,n,\lambda)\rho^\lambda\left\{\frac{1}{(x_n^0)^\lambda}\int\limits_{M(X^0,x_n^0)}\sum_{i=1}^n\|D_iw\|^2dX+\mathfrak{M}^2\right\}. \tag{3.36}$$

If $x_n^0=\bar\sigma/2$, from (3.36) the estimate (3.28) easily follows for every $\rho\in(0,\bar\sigma/2)$.

If $x_n^0<\bar\sigma/2$, setting $X^*=(x_1^0,x_2^0,\dots,x_{n-1}^0,0,-T)=(x^*,-T)$, for every $\sigma\in(0,\bar\sigma]$ we have:

$$M^+(X^*,\sigma)=B^+(x^*,\sigma)\times(-T,-T+\sigma^2)\subset Q;$$

the function w belongs to $L^2(-T,0,\tilde H_0^1(B^+(x^*,\sigma),\mathbb{R}^N))\cap H^{1/2}_{-T}(-T,0,L^2(B^+(x^*,\sigma),\mathbb{R}^N))$ and is solution in $M^+(X^*,\sigma)\subset B^+(x^*,\sigma)\times(-T,0)$ to the system (3.9), with

$$f^i=F^i+\sum_{j=1}^n(A_{ij}(X^0)-A_{ij}(X))D_jw\in L^2(B^+(x^*,\sigma)\times(-T,0),\mathbb{R}^N),\qquad i=1,2,\dots,n.$$

The estimate (2.21) of the Lemma 2.4 assures that $\forall \rho \in (0, \sigma)$ one has

$$(3.37)\qquad \int_{M^+(X^*,\rho)} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq c(\nu) \left\{ \left(\frac{\rho}{\sigma}\right)^{n+2} \int_{M^+(X^*,\sigma)} \sum_{i=1}^{n} \|D_i w\|^2 dX + \right.$$

$$+ \|u\|^2_{H_t^{*1/2}(t^0, t^0+\sigma^2, L^2(B^+(x^*,\sigma), \mathbb{R}^N))} + \int_{M^+(X^*,\sigma)} \sum_{i=1}^{n} \|D_i u\|^2 dX +$$

$$\left. + \int_{M^+(X^*,\sigma)} \sum_{i=1}^{n} \|f^i\|^2 dX + \int_{M^+(X^*,\sigma)} \|F^0\|^2 dX \right\}$$

and hence $\forall \sigma \in (0, \bar\sigma]$, $\forall \rho \in (0, \sigma)$

$$(3.38)\qquad \int_{M^+(X^*,\rho)} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq c(\nu) \int_{M^+(X^*,\sigma)} \sum_{i=1}^{n} \|D_i w\|^2 dX \left\{ \left(\frac{\rho}{\sigma}\right)^{n+2} + \omega(\sigma) \right\} + c(\nu)\, \mathfrak{M}^2 \sigma^\lambda .$$

Taking into account the Lemma 2.VII of [3], from (3.38) we get, $\forall \sigma \in (0, \bar\sigma]$, $\forall \rho \in (0, \sigma)$

$$(3.39)\qquad \int_{M^+(X^*,\rho)} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq c(\nu) \left(\frac{\rho}{\sigma}\right)^{\lambda} \int_{M^+(X^*,\sigma)} \sum_{i=1}^{n} \|D_i w\|^2 dX + c(\nu, n, \lambda)\, \mathfrak{M}^2 \rho^\lambda ,$$

from which, setting $\sigma = \bar\sigma$ and $\rho = 2x_n^0$ we obtain

$$\frac{1}{(x_n^0)^\lambda} \int_{M^-(X^*, 2x_n^0)} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq c(\nu, n, \lambda) \left\{ \frac{1}{\bar\sigma^\lambda} \int_{M^+(X^*,\bar\sigma)} \sum_{i=1}^{n} \|D_i w\|^2 dX + \mathfrak{M}^2 \right\}$$

and hence, being

$$M(X^0, x_n^0) \subset M^+(X^*, 2x_n^0) \subset M^+(X^*, \bar\sigma) \subset Q ,$$

we deduce

$$(3.40)\qquad \frac{1}{(x_n^0)^\lambda} \int_{M(X^0, x_n^0)} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq c(\nu, n, \lambda) \left\{ \frac{1}{\bar\sigma^\lambda} \sum_{i=1}^{n} \|D_i w\|^2_{L^2(Q, \mathbb{R}^N)} + \mathfrak{M}^2 \right\}.$$

Then, from (3.36) and (3.40), we get $\forall \rho \in (0, x_n^0)$:

$$\int_{M(X^0,\rho)} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq c(\nu, n, \lambda)\, \rho^\lambda \left\{ \frac{1}{\bar\sigma^\lambda} \sum_{i=1}^{n} \|D_i w\|^2_{L^2(Q, \mathbb{R}^N)} + \mathfrak{M}^2 \right\},$$

from which the estimate (3.28) follows for every $\rho \in (0, x_n^0)$.

On the other hand from (3.39), written with $\bar\sigma$ and 2ρ instead of σ and ρ, respectively, we get $\forall \rho \in (0, \bar{\sigma}/2)$

$$(3.41) \qquad \int_{M^+(X^*, 2\rho)} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq c(\nu, \lambda) \left(\frac{\rho}{\bar{\sigma}} \right)^{\lambda} \sum_{i=1}^{n} \|D_i w\|^2_{L^2(Q, \mathbb{R}^N)} + c(\nu, n, \lambda) \mathcal{M}^2 \rho^{\lambda}.$$

From this, which is true for every $\rho \in (0, \bar{\sigma}/2)$, and from the inclusion $I(X^0, \rho) \cap Q \subset M^+(X^*, 2\rho)$, which holds for every $\rho \in [x_n^0, \bar{\sigma}/2)$, the estimate (3.28) follows for every $\rho \in [x_n^0, \bar{\sigma}/2)$. Having already shown the estimate (3.28) for every $\rho \in (0, x_n^0)$, it will hold for every $\rho \in (0, \bar{\sigma}/2)$.

d) $\bar{\sigma}/2 < x_n^0 \leq R$ and $t^0 = -T$. For every $\sigma \in (0, \bar{\sigma}/2]$, we have:

$$M(X^0, \sigma) \subset Q;$$

arguing as in the case *c)*, we obtain (3.35) for every $\sigma \in (0, \bar{\sigma}/2]$ and for every $\rho \in (0, \sigma)$; from (3.35) it follows for $\sigma = \bar{\sigma}/2$:

$$\int_{M(X^0, \rho)} \sum_{i=1}^{n} \|D_i w\|^2 dX \leq c(\nu, n, \lambda) \rho^{\lambda} \left\{ \frac{1}{\bar{\sigma}^{\lambda}} \int_{M(X^0, \bar{\sigma}/2)} \sum_{i=1}^{n} \|D_i w\|^2 dX + \mathcal{M}^2 \right\}, \qquad \forall \rho \in \left(0, \frac{\bar{\sigma}}{2}\right),$$

from which (3.28) for every $\rho \in (0, \bar{\sigma}/2)$.

e) $\bar{\sigma}/2 < x_n^0 \leq R$ and $-T < t^0 \leq 0$. In this case we achieve the estimate (3.28) using the same technique applied in the case *b)* of the proof of Theorem 3.1.

f) $0 < x_n^0 \leq \bar{\sigma}/2$ and $-T < t^0 \leq 0$. The estimate (3.28) will follow taking into account the techniques used in connexion with the case *b)* of the Theorem 3.1 and the above case *c)*.

REMARK 3.1. – The Theorem 3.2 holds also if w is solution to a system of the type:

$$-\sum_{ij=1}^{n} D_i(A_{ij}(X) D_j w) + \sum_{i=1}^{n} a_i(X) D_i w + \frac{\partial w}{\partial t} = \frac{\partial u}{\partial t} - \sum_{i=1}^{n} D_i F^i + F^0,$$

with coefficients $a_i(X) \in C^0(\overline{Q}, \mathbb{R}^{N^2})$, other assumptions being unchanged.

4. – Regularity of the solution to the problem (1.1), (1.2).

Now we are able to show the following result concerning the $L^{2,\lambda}$ regularity of the spatial gradient of the solution v to the problem (1.1), (1.2).

THEOREM 4.1. – *If* $v \in L^2(-T, 0, H^1(\Omega, \mathbb{R}^N)) \cap H^{*1/2}_{-T}(-T, 0, L^2(\Omega, \mathbb{R}^N))$ *is solution in* Q *to the problem* (1.1), (1.2) (8), *if the conditions* (1.3), (1.5), (1.6) *are fulfilled, if* $f^i \in L^{2,\lambda}(Q, \mathbb{R}^N)$, $0 < \lambda < n+2$, $i = 0, 1, \ldots, n$, *and if* $u \in L^{2,\lambda}(Q, \mathbb{R}^N) \cap$

$\cap H^{*0,\,1/2,(\lambda)}_{-T}(Q, \mathbb{R}^N)$, $D_i u \in L^{2,\lambda}(Q, \mathbb{R}^N)$, $i = 1, 2, \ldots, n$, *then it results:*

$$D_i v \in L^{2,\lambda}(Q, \mathbb{R}^N), \qquad i = 1, 2, \ldots, n, \tag{4.1}$$

and the following estimate holds:

$$\sum_{i=1}^{n} \|D_i v\|^2_{L^{2,\lambda}(Q,\,\mathbb{R}^N)} \leq \tag{4.2}$$

$$\leq c \left\{ \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q,\,\mathbb{R}^N)} + \|u\|^2_{L^{2,\lambda}(Q,\,\mathbb{R}^N)} + [u]^2_{H^{*0,\,1/2,(\lambda)}_{-T}(Q,\,\mathbb{R}^N)} + \sum_{i=0}^{n} \|f^i\|^2_{L^{2,\lambda}(Q,\,\mathbb{R}^N)} \right\}.$$

PROOF. – In virtue of the assumptions it's possible to find a finite number of open set of $\mathbb{R}^n$, $\Omega_0, \Omega_1, \ldots, \Omega_m$ such that

$$\Omega_0 \subset\subset \Omega, \qquad \Omega \subset \bigcup_{r=0}^{m} \Omega_r$$

and for each Ω_r $(r = 1, 2, \ldots, m)$ there exists an homeomorphism

$$x \to \mathfrak{z}_{(r)}(x)$$

of class C^2 (12) that maps $\overline{\Omega_r \cap \Omega}$ on $\overline{B^+(1)}$ and, consequently, an homeomorphism τ_r

$$\tau_r : (x, t) \to (\mathfrak{z}_{(r)}(x), t) = (y, t)$$

of class C^2 that maps the cylinder $\overline{(\Omega_r \cap \Omega) \times (-T, 0)}$ in the cylinder $\overline{Q^+(1)} = \overline{B^+(1) \times (-T, 0)}$.

Setting $Q_r = (\Omega_r \cap \Omega) \times (-T, 0)$ $(r = 0, 1, \ldots, m)$, let w_r and u_r the restrictions of $w = u - v$ and u to Q_r.

The functions $w = u - v$ and u verify the assumptions of the Theorem 3.1, with $F^i = \sum_{j=1}^{n} A_{ij}(X) D_j u - f^i \in L^{2,\lambda}(Q, \mathbb{R}^N)$, $i = 1, 2, \ldots, n$, $F^0 = -f^0 \in L^{2,\lambda}(Q, \mathbb{R}^N)$; then the derivatives $D_i w_0$ belong to $L^{2,\lambda}(Q_0, \mathbb{R}^N)$ and the following estimate holds

$$\sum_{i=1}^{n} \|D_i w_0\|^2_{L^{2,\lambda}(Q_0,\,\mathbb{R}^N)} \leq \tag{4.3}$$

$$\leq c_0 \left\{ \sum_{i=1}^{n} \|D_i w\|^2_{L^2(Q,\,\mathbb{R}^N)} + \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q,\,\mathbb{R}^N)} + \|u\|^2_{L^{2,\lambda}(Q,\,\mathbb{R}^N)} + [u]^2_{H^{*0,\,1/2,(\lambda)}_{-T}(Q,\,\mathbb{R}^N)} + \sum_{i=0}^{n} \|f^i\|^2_{L^{2,\lambda}(Q,\,\mathbb{R}^N)} \right\},$$

where c_0 depends on ν, n, λ, Q_0 and $\omega(\sigma)$.

---

(12) Let A and B be two bounded open sets of $\mathbb{R}^n$ and $x \to \mathfrak{z}(x)$ an application that maps $\overline{A}$ in $\overline{B}$, having components $\mathfrak{z}_h(x)$, $h = 1, 2, \ldots, n$. We shall say that $x \to \mathfrak{z}(x)$ is of class C^2 if the functions $\mathfrak{z}_h \in C^2(\overline{A})$.

Let us fix r, $1 \leq r \leq m$, and let us set

$$\tilde{w}(y, t) = w_r(\mathfrak{Z}_{(r)}^{-1}(y), t).$$

The function $\tilde{w}$ is defined in $Q^+(1)$, belongs to $L^2(-T, 0, \tilde{H}_0^1(B^+(1), \mathbb{R}^N)) \cap H_{-T}^{1/2}(-T, 0, L^2(B^+(1), \mathbb{R}^N))$ and is solution in $Q^+(1)$ to the system

$$(4.4) \qquad -\sum_{hk=1}^{n} D_h(B_{hk}(y, t) D_k \tilde{w}) + \sum_{k=1}^{n} b_k(y, t) D_k \tilde{w} + \frac{\partial \tilde{w}}{\partial t} = \frac{\partial \tilde{u}}{\partial t} - \sum_{h=1}^{n} D_h F^h(y, t) + F^0(y, t),$$

where

$$(4.5) \qquad B_{hk}(y, t) = \sum_{ij=1}^{n} A_{ij}(\mathfrak{Z}_{(r)}^{-1}(y), t) D_j \mathfrak{Z}_{(r)k}(\mathfrak{Z}_{(r)}^{-1}(y)) D_i \mathfrak{Z}_{(r)h}(\mathfrak{Z}_{(r)}^{-1}(y)),$$

$$(4.6) \qquad b_k(y, t) = -\sum_{ijh=1}^{n} A_{ij}(\mathfrak{Z}_{(r)}^{-1}(y), t) D_j \mathfrak{Z}_{(r)k}(\mathfrak{Z}_{(r)}^{-1}(y)) D_i \mathfrak{Z}_{(r)h}(\mathfrak{Z}_{(r)}^{-1}(y)) \frac{D_h J_{(r)}^{-1}(y)}{J_{(r)}^{-1}(y)},$$

$$(4.7) \qquad \tilde{u}(y, t) = u_r(\mathfrak{Z}_{(r)}^{-1}(y), t),$$

$$(4.8) \qquad F^h(y, t) = \sum_{i=1}^{n} \left\{ -f^i(\mathfrak{Z}_{(r)}^{-1}(y), t) + \sum_{jk=1}^{n} A_{ij}(\mathfrak{Z}_{(r)}^{-1}(y), t) D_k \tilde{u}(y, t) D_j \mathfrak{Z}_{(r)k}(\mathfrak{Z}_{(r)}^{-1}(y)) \right\} D_i \mathfrak{Z}_{(r)h}(\mathfrak{Z}_{(r)}^{-1}(y)),$$

$$(4.9) \qquad F^0(y, t) = \sum_{ih=1}^{n} \left\{ f^i(\mathfrak{Z}_{(r)}^{-1}(y), t) - \sum_{jk=1}^{n} A_{ij}(\mathfrak{Z}_{(r)}^{-1}(y), t) D_k \tilde{u}(y, t) D_j \mathfrak{Z}_{(r)k}(\mathfrak{Z}_{(r)}^{-1}(y)) \right\} D_i \mathfrak{Z}_{(r)h}(\mathfrak{Z}_{(r)}^{-1}(y)) \cdot \frac{D_h J_{(r)}^{-1}(y)}{J_{(r)}^{-1}(y)} - f^0(\mathfrak{Z}_{(r)}^{-1}(y), t)$$

being $J_{(r)}^{-1}(y)$ the Jacobian of the transformation $\mathfrak{Z}_{(r)}^{-1}(y)$ (13). Since $\tau_r : (x, t) \to (\mathfrak{Z}_{(r)}(x), t)$

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(13) It is enough to assume in the (1.7)

$$\varphi(x, t) = \frac{\psi(\mathfrak{Z}_{(r)}(x), t)}{J_{(r)}^{-1}(\mathfrak{Z}_{(r)}(x))},$$

with

$$\psi(y, t) \in L^2(-T, 0, H_0^1(B^+(1), \mathbb{R}^N)) \cap [H_{-T}^{1/2} \cap H_0^{1/2}](-T, 0, L^2(B^+(1), \mathbb{R}^N)).$$

is a C^2-homeomorphism it is easy to verify that

$$\tilde{u} \in L^2(-T, 0, H^1(B^+(1), \mathbb{R}^N)) \cap L^{2,\lambda}(Q^+(1), \mathbb{R}^N) \cap H_{-T}^{*0, 1/2, (\lambda)}(Q^+(1), \mathbb{R}^N),$$

$$D_i \tilde{u} \in L^{2,\lambda}(Q^+(1), \mathbb{R}^N), \qquad i = 1, 2, \ldots, n,$$

$$F^h \in L^{2,\lambda}(Q^+(1), \mathbb{R}^N), \qquad h = 0, 1, \ldots, n,$$

$$b_k \in C^0(\overline{Q^+(1)}, \mathbb{R}^{N^2}), \qquad k = 1, 2, \ldots, n,$$

and that the $N \times N$ matrices B_{hk}, $h, k = 1, 2, ..., n$, belong to the class $C^0(\overline{Q^+(1)}, \mathbb{R}^{N^2})$ and satisfy in $Q^+(1)$ the strong ellipticity condition with a constant $K\nu$.

Taking into account Theorem 3.2, and more precisely the generalization made in the Remark 3.1, we obtain that

$$D_i \tilde{w} \in L^{2,\lambda}(Q^+(R), \mathbb{R}^N), \qquad i = 1, 2, \ldots, n, \tag{4.10}$$

where $0 < R < 1$ and $Q^+(R) = B^+(R) \times (-T, 0)$, and the following estimate holds

$$\sum_{i=1}^{n} \|D_i \tilde{w}\|^2_{L^{2,\lambda}(Q^+(R), \mathbb{R}^N)} \leq C_r^* \left\{ \sum_{i=1}^{n} \|D_i \tilde{w}\|^2_{L^2(Q^+(1), \mathbb{R}^N)} + \right.$$
$$\left. + \sum_{i=1}^{n} \|D_i \tilde{u}\|^2_{L^{2,\lambda}(Q^+(1), \mathbb{R}^N)} + \|\tilde{u}\|^2_{L^{2,\lambda}(Q^+(1), \mathbb{R}^N)} + [\tilde{u}]^2_{H_{-T}^{*0, 1/2, (\lambda)}(Q^+(1), \mathbb{R}^N)} + \sum_{i=0}^{n} \|F^i\|^2_{L^{2,\lambda}(Q^+(1), \mathbb{R}^N)} \right\}. \tag{4.11}$$

Setting

$$Q_r(R) = \tau_r^{-1}(Q^+(R)) = \Omega_r(R) \times (-T, 0),$$

from (4.11), by means of the homeomorphism τ_r, it follows

$$\sum_{i=1}^{n} \|D_i w_r\|^2_{L^{2,\lambda}(Q_r(R), \mathbb{R}^N)} \leq C_r \left\{ \sum_{i=1}^{n} \|D_i w\|^2_{L^2(Q, \mathbb{R}^N)} + \right.$$
$$\left. + \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q, \mathbb{R}^N)} + \|u\|^2_{L^{2,\lambda}(Q, \mathbb{R}^N)} + [u]^2_{H_{-T}^{*0, 1/2, (\lambda)}(Q, \mathbb{R}^N)} + \sum_{i=0}^{n} \|f^i\|^2_{L^{2,\lambda}(Q, \mathbb{R}^N)} \right\}. \tag{4.12}$$

Now we can choose R so close to 1 that the sets $\Omega_0, \Omega_1(R), \ldots, \Omega_m(R)$ cover Ω; hence from (4.3) and (4.12) we get

$$\sum_{i=1}^{n} \|D_i w\|^2_{L^{2,\lambda}(Q, \mathbb{R}^N)} \leq C \left\{ \sum_{i=1}^{n} \|D_i w\|^2_{L^2(Q, \mathbb{R}^N)} + \right.$$
$$\left. + \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q, \mathbb{R}^N)} + \|u\|^2_{L^{2,\lambda}(Q, \mathbb{R}^N)} + [u]^2_{H_{-T}^{*0, 1/2, (\lambda)}(Q, \mathbb{R}^N)} + \sum_{i=0}^{n} \|f^i\|^2_{L^{2,\lambda}(Q, \mathbb{R}^N)} \right\}, \tag{4.13}$$

from which, being $v = u - w$, it follows

$$(4.14)\qquad \sum_{i=1}^{n} \|D_i v\|^2_{L^{2,\lambda}(Q,\, \mathbb{R}^N)} \leq C \Bigg\{ \sum_{i=1}^{n} \|D_i v\|^2_{L^{2}(Q,\, \mathbb{R}^N)} + \sum_{i=1}^{n} \|D_i u\|^2_{L^{2,\lambda}(Q,\, \mathbb{R}^N)} + \|u\|^2_{L^{2,\lambda}(Q,\, \mathbb{R}^N)} + [u]^2_{H^{*0,\,1/2,(\lambda)}_{-T}(Q,\, \mathbb{R}^N)} + \sum_{i=0}^{n} \|f^i\|^2_{L^{2,\lambda}(Q,\, \mathbb{R}^N)} \Bigg\}.$$

On the other hand, since v is solution in Q to the problem (1.1), (1.2), in virtue of Theorem 1.1, it results:

$$(4.15)\qquad \sum_{i=1}^{n} \|D_i v\|^2_{L^{2}(Q,\, \mathbb{R}^N)} \leq C \Bigg\{ \|u\|^2_{H^{*1/2}_{-T}(-T,\,0,\,L^2(\Omega,\, \mathbb{R}^N))} + \sum_{i=1}^{n} \|D_i u\|^2_{L^{2}(Q,\, \mathbb{R}^N)} + \sum_{i=0}^{n} \|f^i\|^2_{L^{2}(Q,\, \mathbb{R}^N)} \Bigg\}.$$

Then the assert follows from (4.14) and (4.15).

REFERENCES

[1] S. CAMPANATO, *Equazioni paraboliche del secondo ordine e spazi* $\mathcal{L}^{2,\theta}(\Omega, \delta)$, Ann. Mat. Pura Appl., **73** (1966), pp. 55-102.

[2] S. CAMPANATO, *Partial Hölder continuity of solutions of quasilinear parabolic systems of second order with linear growth*, Rend. Sem. Mat. Padova, **64** (1981), pp. 59-75.

[3] S. CAMPANATO, *Hölder continuity of the solutions of some non-linear elliptic systems*, Advances Math., **48** (1983), pp. 16-43.

[4] S. KAPLAN, *Abstract boundary value problems for linear parabolic equations*, Ann. Scuola Norm. Sup. Pisa (3), **20** (1966), pp. 395-419.

[5] M. MARINO - A. MAUGERI, *Partial Hölder continuity of solutions of nonlinear parabolic systems of second order with quadratic growth*, Boll. Un. Mat. Ital. (7), **3**-B (1989), pp. 397-435.

[6] M. MARINO - A. MAUGERI, *Maximum principle for parabolic systems*, Algebra and Analysis, **3** (1991), pp. 155-163.