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On a Large Class of Symmetric Systems of Linear PDEs, for Tensor Functions, Useful in Mathematical Physics (*) (**).

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Abstract. - *We study a class of symmetric systems of linear partial differential equations which involve tensor functions relating tensor spaces on a three dimensional vector space, on the real field, equipped with an inner product. These systems arise by coupling certain simpler symmetric systems studied in a previous paper. In order to investigate some questions, related to constitutive equations for bodies of the differential type, certain classes of physically privileged solutions are characterized for some of the aforementioned systems.*

1. - Introduction.

In M [4] the general symmetric system (1.3) below, of linear PDEs for tensor functions, has been studied. By using some results established there, here we study certain symmetric systems obtained by coupling simpler symmetric systems of the kind (1.3)—see (1.6)-(1.8).

Furthermore a class of physically privileged solutions to systems (1.6)-(1.7) is characterized. Such a class is used in M&P [5] to find the maximal indetermination of the response function for the heat flux in general differential bodies; such an indetermination turns to coincide with the one found in M [2] in connection with thermo-elastic bodies.

In the present paper another class of solutions to systems of the kind (1.6)-(1.7) is characterized. It is used in M&P [6] to prove (i) a uniqueness theorem for the response function of the stress in any differential body (of arbitrary complexity) and (ii) a uniqueness theorem for the response function of the internal energy for certain (large) classes of such bodies.

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In more details assume that

$n,\ \upsilon,\ \tau \in \{1, 2, ...\}$, $\mathbb{R}$ is the field of real numbers,

$\mathscr{V}_n$ is an inner product space of dimension n on $\mathbb{R}$,

$\mathscr{U}_\tau$ is an open connected subset of the space $\mathscr{T}_\tau(\mathscr{V}_n)$ of tensors of (covariant) order τ on $\mathscr{V}_n$,

$\mathscr{T}^\upsilon\{\mathscr{V}_n)$ is the space of tensors of (controvariant) order υ on $\mathscr{V}_n$,

$$\hat{Q}\colon \mathscr{U}_\tau \to \mathscr{T}^\upsilon(\mathscr{V}_n), \qquad X \mapsto Q = \hat{Q}(X), \tag{1.1}$$

is a (tensor) function from $\mathscr{U}_\tau$ to $\mathscr{T}^\upsilon(\mathscr{V}_n)$.

Moreover, as is customary,

$$X^{\dots(a\dots b)\dots} = \frac{1}{2}(X^{\dots a \dots b \dots} + X^{\dots b \dots a \dots}) \tag{1.2}$$

denotes the symmetric part of X with respect to the indexes a, b.

From now onward we shall refer to a fixed orthonormal basis of $\mathscr{V}$. Consider the symmetric system of linear partial differential equations

$$\frac{\partial Q^{\alpha_1 \dots \alpha_\mu - 1(\alpha_\mu \alpha_\mu + 1 \dots \alpha_\nu}}{\partial X_{\beta_1 \dots \beta_\eta - 1, \beta_\eta)\beta_\eta + 1 \dots \beta_\tau}} = 0, \tag{1.3}$$

namely

$$[\mathrm{Grad}\, Q]^{\alpha_1 \dots (\alpha_\mu \dots \alpha_\nu, \beta_1 \dots \beta_\eta) \dots \beta_\tau} = 0\ (^1), \tag{1.3'}$$

where any index ranges from 1 to n and $\mu \in \{1, ..., \upsilon\}$, $\eta \in \{1, ..., \tau\}$ are fixed. In M [4] the general smooth solution of such a system has been studied for each υ, τ and n in $\{1, 2, ...\}$.

In order to consider certain applications to mathematical physics—see M&P [5-6]—from now on we shall limit our considerations to the cases

$$n = 3, \qquad \tau = 2, \qquad \upsilon \in \{1, 2, \dots\};$$

thus we set

$$\mathscr{V} = \mathscr{V}_3, \qquad a \in \{1, 2, 3\}^{\upsilon - 1}, \qquad A \in \{1, 2, 3\}, \qquad (a, A) \in \{1, 2, 3\}^\upsilon, \tag{1.4}$$

(where a must be dropped if $\upsilon = 1$); let $[X_{bB}]$, $[Y_{bB}]$ and $[Q^{aA}]$ denote the representations of the tensors $X, Y \in \mathscr{T}_2(\mathscr{V})$ and $Q \in {}^\upsilon(\mathscr{V})$ respectively, with respect to the given vector basis of $\mathscr{V}$. The summation on repeated indexes is understood. In M [4, § 5], see (5.4), the following assertion is proved.

(1.A) *The three assertions* (1.B) *through* (1.D) *below are equivalent.*

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(1) Notice that system (1.3) is symmetric with respect to $X_{\beta_1 \dots \beta_\eta - 1, \beta_\eta, \beta_\eta + 1 \dots \beta_\tau}$ and $X_{\beta_1 \dots \beta_\eta - 1 \alpha_\mu, \beta_\eta + 1 \dots \beta_\tau}$. In words (1.3′) means that the symmetric part of tensor Grad Q with respect to α_μ and β_η vanishes, that is Grad Q is skew-symmetric in α_μ and β_η.

(1.B) [(1.C)] *The function* (1.1) *is a* C^2-[C^∞-] *solution on* $\mathcal{U}_2$ *of the first order system* (1.3).

(1.D) *For* $i = 0, 1, 2$ *there are tensors* $\overset{[i]}{\varphi}$ *such that*

$$Q^{aA} = \overset{[0]}{\varphi}{}^{aA} + \overset{[1]}{\varphi}{}^{abAB} X_{bB} + \overset{[2]}{\varphi}{}^{abcABC} X_{bB} X_{cC} \qquad ((a, A) \in \{1, 2, 3\}^{\upsilon}), \tag{1.5}$$

where $\overset{[1]}{\varphi}$ *is skew-symmetric in* $\{A, B\}$ *and* $\overset{[2]}{\varphi}$ *is totally skew-symmetric in* $\{A, B, C\}$.

Incidentally note that in the last assertion tensor $\overset{[2]}{\varphi}$ can be chosen skew-symmetric in $\{b, c\}$.

By the equivalence of assertions (1.B) and (1.C) sometimes *smooth solution* will mean indifferently solution of class C^2 or of class C^∞.

In the present paper we consider functions of the form $Q = \hat{Q}(X, Y)$ and $Q = \hat{Q}(X, Y, Z)$—see (2.1) and (4.7) below—and we first find the general smooth solution of the twofold system

$$\frac{\partial Q^{a(A}}{\partial X_{bB)}} = 0, \qquad \frac{\partial Q^{a(A}}{\partial Y_{bB)}} = 0 \qquad (a, b, A, B = 1, 2, 3) \text{ —see § 2}, \tag{1.6}$$

second, we find the one of the threefold system

$$\frac{\partial Q^{(A}}{\partial X_{bB)}} = 0, \qquad \frac{\partial Q^{(A}}{\partial Y_{bB)}} = 0 \qquad \frac{\partial Q^{(A}}{\partial Z_{B)}} = 0 \qquad (b, A, B = 1, 2, 3) \text{ —see § 4}. \tag{1.7}$$

Then, in Section 4, we characterize a class of physically privileged solutions to (1.6)-(1.7) in the case $\upsilon = 1$. In Section 5 the solutions of the system

$$\frac{\partial Q^{a(A}}{\partial \underset{p}{Y}_{bB)}} = 0, \qquad (p = 0, 1, \ldots, P), \ (a, b, A, B = 1, 2, 3), \tag{1.8}$$

which is the $(P + 1)$-fold version of the twofold system (1.6), are found; lastly, in Section 6 a class of physically privileged solutions to (1.8) is characterized.

This characterization is used in M&P [5] to prove that the maximal indetermination in the response function for the heat flux in some classes of differential bodies exactly coincides with the one found in M [2] with regard to a thermo-elastic body. In order to give more details, we consider the conditions below.

(*A*) *The body* $\mathcal{B}$ *is of the differential type and complexity* $(P, 0, 0)$, *and* $\mathcal{K}$ *is a reference configuration for it;*

(*B*) $\hat{q}_{\mathcal{K}}$ *is an admissible response function for the heat flux in* $(\mathcal{B}, \mathcal{K})$, *i.e. in* $\mathcal{B}$ *referred to* $\mathcal{K}$;

(*C*) $\tilde{q}_{\mathcal{K}}$ *is a function of class* C^2 *in the domain of* $\hat{q}_{\mathcal{K}}$, *and* $\hat{Q} = \hat{q}_{\mathcal{K}} - \tilde{q}_{\mathcal{K}}$;

(*D*) $\hat{q}_{\mathcal{K}}$ *is an admissible response function for the heat flux in* $(\mathcal{B}, \mathcal{K})$;

(*E*) *there is a function* $\hat{\varphi} \in C^3(\mathbb{R}^+ \times \mathbb{R}^3)$ *such that*

$$\hat{\boldsymbol{Q}}(\theta, \boldsymbol{G}, \boldsymbol{X}) = \boldsymbol{G} \times \mathrm{GRAD}\,\hat{\varphi}(\theta, \boldsymbol{X}).$$

In M&P [5] it is proved that, under the assumptions in (*A*) through (*C*), conditions (*D*) and (*E*) are equivalent (such a proof relies on the proof of the same fact given in M [1] with regard to thermo-elastic bodies).

Furthermore in M&P [6] the physical solutions to systems (1.6)-(1.7) for $\upsilon = 2$ are characterized and this allowed us to prove there the uniqueness theorem for (j) the response function for the stress in any thermodynamic differential body (of arbitrary complexity) and (jj) the response function for the internal energy in many classes of such bodies; incidentally let us note that the statements of these theorems do coincide with those found in the thermo-elastic case.

Therefore the present paper can be considered as the purely mathematical prerequisite to the work M&P [5] and to a part of M&P [6].

Indeed in our opinion the solutions of systems of the kind considered here can furnish the natural algorithms for dealing with general uniqueness and indeterminateness questions related with constitutive equations of differential bodies.

2. – Twofold symmetric systems.

Let us consider smooth functions of the kind

$$\hat{\boldsymbol{Q}}: \mathcal{U} \times \mathcal{U}' \to \mathcal{T}^{\upsilon}(\mathcal{V}), \qquad (\boldsymbol{X}, \boldsymbol{Y}) \mapsto \boldsymbol{Q} = \hat{\boldsymbol{Q}}(\boldsymbol{X}, \boldsymbol{Y}), \tag{2.1}$$

where (i) $\upsilon \in \{1, 2, \ldots\}$, $\mathcal{V} = \mathcal{V}_3$; (ii) both $\mathcal{U}$ and $\mathcal{U}'$ are open connected subsets of $\mathcal{T}_2(\mathcal{V})$.

Consider the twofold system (1.6); by assertion (1.A) with regard to both the systems $(1.6)_1$ and $(1.6)_2$, one can easily prove the assertion below.

(2.A) *The function* (2.1) *is a smooth solution to the twofold system* (1.6) *if and only if for* $i = 0, 1, 2$ *there are tensors* $\overset{[i]}{\varphi}$ *and* $\overset{[i]}{\psi}$, *which are skew-symmetric in their capital indexes, such that for each* $(a, A) \in \{1, 2, 3\}^{\upsilon}$ —see (1.4)— Q^{aA} *writes in a twofold way as*

$$Q^{aA} = \overset{[0]}{\varphi}{}^{aA} + \overset{[1]}{\varphi}{}^{abAB} X_{bB} + \overset{[2]}{\varphi}{}^{abcABC} X_{bB} X_{cC}, \tag{2.2}$$

$$Q^{aA} = \overset{[0]}{\psi}{}^{aA} + \overset{[1]}{\psi}{}^{abAB} Y_{bB} + \overset{[2]}{\psi}{}^{abcABC} Y_{bB} Y_{cC}. \tag{2.3}$$

Incidentally note that

(2.A′) *the tensors* $\overset{[2]}{\varphi}$ *and* $\overset{[2]}{\psi}$ *in assertion* (2.A) *can be chosen skew-symmetric in* $\{b, c\}$.

Now consider the assertions (2.B) through (2.D) below.

(2.B) *The function* (2.1) *is a* C^2*-solution in* $\mathcal{U} \times \mathcal{U}'$ *of the twofold system* (1.6).

(2.C) *The function* (2.1) *is a* C^∞*-solution in* $\mathcal{U} \times \mathcal{U}'$ *of the twofold system* (1.6).

(2.D) *There are tensors* $\overset{[00]}{\tau} \in \mathcal{T}^{\upsilon}(\mathcal{V})$, $\overset{[10]}{\tau}, \overset{[10]}{\tau} \in \mathcal{T}^{2\upsilon}(\mathcal{V})$ *and* $\overset{[11]}{\tau}, \overset{[20]}{\tau} \overset{[02]}{\tau} \in \mathcal{T}^{3\upsilon}(\mathcal{V})$ *such that, for each* $(a, A) \in \{1, 2, 3\}^{\upsilon}$ —see (1.4)—, Q^{aA} *writes as*

$$(2.4) \qquad Q^{aA} = \overset{[00]}{\tau}{}^{aA} + \overset{[10]}{\tau}{}^{abAB} X_{bB} + \overset{[01]}{\tau}{}^{abAB} Y_{bB} + \\ + \overset{[11]}{\tau}{}^{abcABC} X_{bB} Y_{cC} + \overset{[20]}{\tau}{}^{abcABC} X_{bB} X_{cC} + \overset{[02]}{\tau}{}^{abcABC} Y_{bB} Y_{cC},$$

where the tensors $\overset{[hk]}{\tau}$ *are totally skew-symmetric in their capital indexes; further* $\overset{[20]}{\tau}$ *and* $\overset{[02]}{\tau}$ *can be chosen skew-symmetric in* b, c.

THEOREM 2.1. – *The three assertions* (2.B) *though* (2.D) *are equivalent.*

PROOF. – Assume (2.B) (or equivalently (2.C)); by (1.A) and (2.A) one easily finds the following expression for the general solution to (1.6)

$$(2.5) \qquad Q^{aA} = \overset{[00]}{\tau}{}^{aA} + \overset{[10]}{\tau}{}^{abAB} X_{bB} + \overset{[01]}{\tau}{}^{adAD} Y_{dD} + \overset{[11]}{\tau}{}^{abcABD} X_{bB} Y_{cD} + \\ + \overset{[20]}{\tau}{}^{abcABC} X_{bB} X_{cC} + \overset{[02]}{\tau}{}^{abcABC} Y_{bB} Y_{cC} + \overset{[21]}{\tau}{}^{abcdABCD} X_{bB} X_{cC} Y_{dD} + \\ \overset{[12]}{\tau}{}^{abcdABCD} X_{bB} Y_{cC} Y_{dD} + \overset{[22]}{\tau}{}^{abcdeABCDE} X_{bB} X_{cC} Y_{dD} Y_{eE},$$

where $\overset{[pq]}{\tau}$ ($p, q = 0, 1, 2$) are arbitrary tensors which are totally skew-symmetric in their capital indexes—see M [4]. Thus $0 = \overset{[22]}{\tau} = \overset{[12]}{\tau} = \overset{[21]}{\tau}$, as these tensors are totally skew-symmetric in more than three indexes. The last part of the assertion (2.D) follows from (2.A′). Hence (2.D) holds. The converse implication is a trivial task to verify. q.e.d.

Next we remark that the solutions found can be written by means of the Ricci tensor ε; in more details by the elementary lemma below, assertion (1.D) [(2.D)] is equivalent to assertion (1.D′) [(2.D′)] below.

(1.D′) *For* $i = 0, 1, 2$ *there are tensors* $\overset{[i]}{\varphi}$ *such that* Q^{aA} *writes as*

$$(1.5') \qquad Q^{aA} = \overset{[0]}{\varphi}{}^{aA} + \overset{[1]}{\varphi}{}^{abC}\varepsilon^{ABC}X_{bB} + \overset{[2]}{\varphi}{}^{ah}\varepsilon^{hbc}\varepsilon^{ABC}X_{bB}X_{cC} \qquad (a, A = 1, 2, 3).$$

(2.D′) *There are tensors* $\overset{[00]}{\tau}, \overset{[20]}{\tau}, \overset{[02]}{\tau} \in \mathcal{T}^{\nu}(\mathcal{V})$ *and* $\overset{[10]}{\tau}, \overset{[01]}{\tau}, \overset{[11]}{\tau} \in \mathcal{T}^{\nu+1}(\mathcal{V})$ *such that, for each* $(a, A) \in \{1, 2, 3\}^{\nu}$ —see (1.4)—, Q^{aA} *writes as*

$$(2.4') \qquad Q^{aA} = \overset{[00]}{\tau}{}^{aA} + \overset{[10]}{\tau}{}^{abM}\varepsilon^{ABM}X_{bB} + \overset{[01]}{\tau}{}^{adN}\varepsilon^{ADN}Y_{dD} +$$
$$\overset{[11]}{\tau}{}^{abd}\varepsilon^{ABD}X_{bB}Y_{dD} + \overset{[20]}{\tau}{}^{ah}\varepsilon^{hbc}\varepsilon^{ABC}X_{bB}X_{cC} + \overset{[02]}{\tau}{}^{ah}\varepsilon^{hde}\varepsilon^{ADE}Y_{dD}Y_{eE}.$$

LEMMA 2.1. – *Let* $\boldsymbol{T} = [T^{\alpha AB}] \in \mathcal{T}^{\mu+2}(\mathcal{V})$ $[\boldsymbol{T} = [T^{\alpha ABC}] \in \mathcal{T}^{\mu+3}(\mathcal{V})]$ *be [totally] skew-symmetric in* $\{A, B\}$ $[\{A, B, C\}]$, *where* $\alpha \in \{1, 2, 3\}^{\mu}$. *There is a tensor* $\boldsymbol{U} \in \mathcal{T}^{\mu+1}(\mathcal{V}_n)$ $[\boldsymbol{U} \in \mathcal{T}^{\mu}(\mathcal{V})]$ *such that* $T^{\alpha AB} = U^{\alpha C}\varepsilon^{ABC}$ $[T^{\alpha ABC} = U^{\alpha}\varepsilon^{ABC}]$.

3. – Isotropic tensors.

Recall that we refer $\mathcal{V}$ to an orthonormal basis. Let I denote the identity tensor, and let

$$\mathrm{Orth}^{+} = \{\boldsymbol{A} \in \mathcal{T}_2(\mathcal{V}) \,|\, \boldsymbol{A}\boldsymbol{A}^T = \boldsymbol{I}, \det \boldsymbol{A} = 1\}$$

be the group of proper rotations. The Lemmas 3.1-3.2 below characterize isotropic tensors of orders 1 to 3. They are proved in many textbooks of Tensor Calculus—see e.g. F&P [1].

LEMMA 3.1 [3.2]. – *A tensor* $\boldsymbol{V} \in \mathcal{T}_2(\mathcal{V})$ $[\boldsymbol{U} \in \mathcal{V}]$ *satisfies*

$$(3.1) \qquad V_{ab}R_{ai}R_{bj} = V_{ij} \quad [U_a R_{ai} = U_i] \qquad \forall \boldsymbol{R} \in \mathrm{Orth}^{+}$$

if and only if

$$(3.2) \qquad V_{ij} = d\delta_{ij} \quad \textit{for some } d \in \mathbb{R} \;\; [\boldsymbol{U} = 0] \;\; (i, j = 1, 2, 3, \; \delta_{ij} = \text{Kronecker delta}).$$

LEMMA 3.3. – *A tensor* $\boldsymbol{V} \in \mathcal{T}_3(\mathcal{V})$ *satisfies*

$$(3.3) \qquad V_{abc}R_{ai}R_{bj}R_{ck} = V_{ijk} \qquad \forall \boldsymbol{R} \in \mathrm{Orth}^{+}$$

if and only if

$$(3.4) \qquad V_{ijk} = d\varepsilon_{ijk} \quad \textit{for some } d \in \mathbb{R} \;\; (i, j, k = 1, 2, 3).$$

4. – Physically remarkable solutions of the above twofold systems in the case $\upsilon = 1$. Threefold symmetric systems.

In this section we characterize the class of the solutions to the twofold system (1.6), in the case $\upsilon = 1$, which satisfy the following conditions.

(4.A) $\hat{\boldsymbol{Q}}(\boldsymbol{RX}, \boldsymbol{RY}) = \hat{\boldsymbol{Q}}(\boldsymbol{X}, \boldsymbol{Y})$ for any $\boldsymbol{R} \in \mathrm{Orth}^+$ and any $(\boldsymbol{X}, \boldsymbol{Y}) \in U \times U'$

$$\text{[in components } \hat{Q}^A(R_{bi}X_{iB}, R_{di}Y_{iD}) = \hat{Q}^A(X_{bB}, Y_{dD})] .$$

(4.B) $\hat{\boldsymbol{Q}}(\boldsymbol{RX}, \dot{\boldsymbol{R}}X + \boldsymbol{RY}) = \hat{\boldsymbol{Q}}(\boldsymbol{X}, \boldsymbol{Y})$ for any $\hat{\boldsymbol{R}} \in C^1([-\varepsilon, 0], \mathrm{Orth}^+)$ and any $(\boldsymbol{X}, \boldsymbol{Y}) \in U \times U'$, where $\varepsilon > 0$, $t \mapsto \hat{\boldsymbol{R}}(t)$, $\boldsymbol{R} = \hat{\boldsymbol{R}}(0)$, $\dot{\boldsymbol{R}} = (d\hat{\boldsymbol{R}}/dt)(0)$

$$\text{[in components } \hat{Q}^A(X_{bB}, \dot{R}_{di}X_{iD} + R_{di}Y_{iD}) = \hat{Q}^A(R_{bi}X_{iB}, R_{di}Y_{iD})] .$$

Remark that (4.B) implies (4.A); further, recall that

(4.B*) if $\hat{\boldsymbol{R}} \in C^1([-\varepsilon, 0], \mathrm{Orth}^+)$ *satisfies* $\hat{\boldsymbol{R}}(0) = \boldsymbol{I}$, *then* $\dot{\boldsymbol{R}} = (d\hat{\boldsymbol{R}}/dt)(0) \in$ $\in$ Skew.

(Indeed $\boldsymbol{RR}^T = \boldsymbol{I}$ implies $\dot{\boldsymbol{R}}\boldsymbol{R}^T + \boldsymbol{R}\dot{\boldsymbol{R}}^T = O$, and thus $\boldsymbol{R} = \boldsymbol{I}$ yields $\dot{\boldsymbol{R}} \in$ Skew.)

PHYSICAL INTERPRETATIONS OF CONDITIONS (4.A)-(4.B). – *Let both $\hat{\boldsymbol{q}}$ and $\tilde{\boldsymbol{q}}$ be admissible response functions for the heat flux of a differential body $\mathcal{B}$ of complexity one, connected with the same reference configuration, and let*

(4.1) $\quad \hat{\boldsymbol{Q}} = \hat{\boldsymbol{q}} - \tilde{\boldsymbol{q}}, \qquad \boldsymbol{X} = \boldsymbol{F}$ (position gradient), $\boldsymbol{Y} = \dot{\boldsymbol{F}}$ (material derivative of $\boldsymbol{F}$).

Condition (4.A) [(4.B)] *is a consequence of the property of Galilean* [*Euclidean*] *invariance imposed to the response functions for the stress*—see M&P [6].

Now consider assertions (4.C) through (4.F) below.

(4.C) *The function* (2.1) *is a smooth solution in* $\mathcal{U} \times \mathcal{U}'$ *to the twofold system* (1.6) *and satisfies condition* (4.A).

(4.D) *The function* (2.1) *is a smooth solution in* $\mathcal{U} \times \mathcal{U}'$ *to the twofold system* (1.6) *and satisfies condition* (4.B).

(4.E) [(4.F)] *There are a vector* τ *and a scalar* d [*there is a vector* τ] *such that*

(4.2) [(4.3)] $\qquad Q^A = \tau^A + d\varepsilon^{ABC} X_{bB} Y_{bC} \qquad [Q^A = \tau^A] \; (A = 1, 2, 3)$.

THEOREM 4.1. – *Assertions* (4.C) *and* (4.E) *are equivalent.*

THEOREM 4.2. – *Assertions* (4.D) *and* (4.F) *are equivalent.*

PROOF OF THEOREM 4.1. – Let $\hat{Q}$ as in (2.1) be a smooth solution to (1.6) for $\upsilon = 1$ ($\{a\} = \emptyset$)—see (1.4); by Theorem 2.1 we have that equality (2.3) holds (with a

dropped); thus condition (4.A) yields—see (2.4)

$$
\begin{aligned}
(4.4)\qquad & \overset{[00]}{\tau}{}^{A} + \overset{[10]}{\tau}{}^{bM}\varepsilon^{ABM}R_{bi}X_{iB} + \overset{[01]}{\tau}{}^{dN}\varepsilon^{ADN}R_{di}Y_{iD} + \overset{[11]}{\tau}{}^{bd}\varepsilon^{ABD}R_{bi}X_{iB}R_{dj}Y_{jD} + \\
& + \overset{[20]}{\tau}{}^{h}\varepsilon^{hbc}\varepsilon^{ABC}R_{bi}X_{iB}R_{cj}X_{jC} + \overset{[02]}{\tau}{}^{h}\varepsilon^{hde}\varepsilon^{ADE}R_{di}Y_{iD}R_{ej}Y_{jE} = \overset{[00]}{\tau}{}^{A} + \overset{[10]}{\tau}{}^{bM}\varepsilon^{ABM}X_{bB} + \\
& + \overset{[01]}{\tau}{}^{dN}\varepsilon^{ADN}Y_{dD} + \overset{[11]}{\tau}{}^{bd}\varepsilon^{ABD}X_{bB}Y_{dD} + \overset{[20]}{\tau}{}^{h}\varepsilon^{hbc}\varepsilon^{ABC}X_{bB}X_{cC} + \overset{[02]}{\tau}{}^{h}\varepsilon^{hde}\varepsilon^{ADE}Y_{dD}Y_{eE}.
\end{aligned}
$$

Now by equating the corresponding terms of the same degree in the two sides of equality (4.4), and by the arbitrariness of the independent variables, one finds the equalities below

$$
(4.5)\qquad \begin{cases}
\overset{[10]}{\tau}{}^{bM}\varepsilon^{ABM} = \overset{[10]}{\tau}{}^{bM}\varepsilon^{ABM}R_{ib}, \\
\qquad\qquad \text{i.e. } \overset{[10]}{\tau}{}^{bM} = \overset{[10]}{\tau}{}^{iM}R_{ib} \text{ (a similar equality holds for } \overset{[01]}{\tau}), \\
\overset{[11]}{\tau}{}^{bd}\varepsilon^{ABD} = \overset{[11]}{\tau}{}^{ij}\varepsilon^{ABD}R_{ib}R_{jd}, \\
\qquad\qquad \text{i.e. } \overset{[11]}{\tau}{}^{bd} = \overset{[11]}{\tau}{}^{ij}R_{ib}R_{jd}.
\end{cases}
$$

Lastly the equality $\overset{[20]}{\tau}{}^{h}\varepsilon^{hij}\varepsilon^{ABC}R_{ib}R_{jc} = \overset{[20]}{\tau}{}^{h}\varepsilon^{hbc}\varepsilon^{ABC}$, i.e. $\overset{[20]}{\tau}{}^{h}\varepsilon^{hij}R_{ib}R_{jc} = \overset{[20]}{\tau}{}^{h}\varepsilon^{hbc}$, holds; thus $\overset{[20]}{\tau}{}^{h}\varepsilon^{kbc}\varepsilon^{hij}R_{ib}R_{jc} = 2\overset{[20]}{\tau}{}^{k}$, and by $\varepsilon^{kbc}\varepsilon^{hij}R_{ib}R_{jc} = 2\det(\boldsymbol{R}^{-1})_{hk}$ the last equality yields

$$
(4.6)\qquad \overset{[20]}{\tau}{}^{h}R_{kh} = \overset{[20]}{\tau}{}^{k} \quad \text{(a similar equality holds for } \overset{[02]}{\tau}).
$$

Hence by (4.5)-(4.6), by the arbitrariness of $\boldsymbol{R}$ and Lemma 3.1, it follows that in equality (4.2) the vector $\overset{[00]}{\tau}$ can be taken ad arbitrium, that $\overset{[10]}{\tau} = \boldsymbol{O} = \overset{[01]}{\tau}$, $\overset{[20]}{\tau} = \boldsymbol{O} = \overset{[02]}{\tau}$ and $\overset{[11]}{\tau}{}^{ij} = d\delta^{ij}$ for some scalar d; by these equalities and (2.4) the theorem is proved. q.e.d.

PROOF OF THEOREM 4.2. – Assume (4.D); as (4.B) implies (4.A), it follows that (4.C) holds; thus by Theorem 4.1 assertion (4.E) holds too; now equality (4.2) and (4.B) together yield

$$
\tau^{A} + d\varepsilon^{ABC}X_{bB}Y_{bC} = \tau^{A} + d\varepsilon^{ABC}R_{bs}X_{sB}(\dot{R}_{bi}X_{iB} + R_{bj}Y_{jC});
$$

hence

$$
\begin{aligned}
& d\varepsilon^{ABC}X_{bB}Y_{bC} = d\varepsilon^{ABC}R_{bs}X_{sB}(\dot{R}_{bi}X_{iC} + R_{bj}Y_{jC}) = \\
& \qquad = d\varepsilon^{ABC}R_{bs}\dot{R}_{bi}X_{sB}X_{iC} + d\varepsilon^{ABC}X_{sB}Y_{sC} \quad \text{(the last equality holds as } R_{bs}R_{bj} = \delta_{sj});
\end{aligned}
$$

thus the arbitrariness of X_{aA} and Y_{aA} yields $d = 0$. q.e.d.

Now let us consider more complex functions, that is functions of the kind

$$\hat{Q}: \mathcal{U} \times \mathcal{U}' \times \mathcal{W} \to \mathcal{V} \ (\upsilon = 1), \qquad (X, Y, Z) \mapsto Q = \hat{Q}(X, Y, Z), \tag{4.7}$$

where all the conditions below (2.1) still hold, and moreover $\mathcal{W}$ is an open connected subset of $\mathcal{V}$.

Consider the system of equations below and the assertions (4.C*) through (4.F*) below

$$\frac{\partial Q^A}{\partial Z_B} + \frac{\partial Q^B}{\partial Z_A} = 0, \qquad (A, B = 1, 2, 3). \tag{4.8}$$

(4.C*) *The function* (4.7) *is a smooth solution in* $\mathcal{U} \times \mathcal{U}'$ *to both systems* (1.6), (4.8) *and satisfies condition* (4.A) *at any* $Z \in \mathcal{W}$.

(4.D*) *The function* (4.7) *is a smooth solution in* $\mathcal{U} \times \mathcal{U}'$ *to both systems* (1.6), (4.8) *and satisfies condition* (4.B) *at any* $Z \in \mathcal{W}$.

(4.E*) [(4.F*)] *There are* $\sigma, \tau \in \mathcal{V}$ *and* $d \in R$ [*There are* $\sigma, \tau \in \mathcal{V}$] *such that the equalities* (4.9) [(4.10)] *below hold*

$$Q^A = \sigma^A + \tau^C \varepsilon^{ABC} Z_B + d\varepsilon^{ABC} X_{bB} Y_{bC} \qquad (A = 1, 2, 3), \tag{4.9}$$

$$Q^A = \sigma^A + \tau^C \varepsilon^{ABC} Z_B \qquad (A = 1, 2, 3). \tag{4.10}$$

The following well known lemma is used in the proof of the theorems below.

LEMMA 4.1. – *A function*

$$\hat{\boldsymbol{Q}} = [\hat{Q}^A]: \mathcal{W} \to \mathcal{V}, \qquad \boldsymbol{Q} = \hat{\boldsymbol{Q}}(\boldsymbol{Z}), \quad \mathcal{W} \subset \mathcal{V} \text{ open connected,} \tag{4.11}$$

is a C^1*-solution of the first order system* (4.8) *if and only if*

$$\hat{\boldsymbol{Q}}(\boldsymbol{Z}) = \boldsymbol{WZ} + \boldsymbol{V} \quad \text{for some } \boldsymbol{W} \in \text{Skew and } \boldsymbol{V} \in \mathcal{V} \tag{4.12}$$

[in components $Q^A = W^{AB} Z_B + V^A$].

An elegant proof of this lemma, due to Gurtin and Williams, is expounded in T [7, pp. 98, 258]. Under the assumption $\hat{\boldsymbol{Q}} \in C^2$, a different proof is given in M [2] and in M&P [6, Lemma 4.2].

THEOREM 4.1*. – *The assertions* (4.C*) *and* (4.E*) *are equivalent.*

THEOREM 4.2*. – *The assertions* (4.D*) *and* (4.F*) *are equivalent.*

PROOF. – Assume (4.C*) [(4.D*)]; by Theorem 4.1 [4.2] and Lemma 4.1, equalities (4.2) [(4.3)] and (4.12) must hold together. That is, we must have

$$(4.13)_2 \qquad Q^A = W^{AB} Z_B + V^A \quad \text{and} \quad Q^A = \tau^A + d\varepsilon^{ABC} X_{bB} Y_{bC}$$

$$[\text{and } Q^A = \tau^A] \ (A = 1, 2, 3),$$

($W \in$ Skew, $V, \tau \in \mathcal{V}$, $d \in \mathbb{R}$), where τ, d are functions at most of Z, and W, V are functions of X and Y. Therefore one easily finds (4.9) [(4.10)] and (4.E*) [(4.D*)] is proved. The converse implication is a trivial task to verify. q.e.d.

5. – Multiple $(P+1)$-fold symmetric systems.

From now on let us consider smooth functions of the kind

$$(5.1) \quad \hat{Q}\colon \mathcal{U} \times \mathcal{U}^1 \times \ldots \times \mathcal{U}^P \to \mathcal{T}^{\nu}(\mathcal{V}), \qquad (Y, \overset{1}{Y}, \ldots, \overset{P}{Y}) \mapsto Q = \hat{Q}(Y, \overset{1}{Y}, \ldots, \overset{P}{Y}),$$

where $P \geq 1$, conditions (i) through (iii) at the beginning of § 2 hold, and $\mathcal{U}, \mathcal{U}^1, \ldots, \mathcal{U}^P$ are open connected subsets of $\mathcal{T}_2(\mathcal{V})$.

Next we consider the $(P+1)$-fold symmetric system of linear partial differential equations

$$(5.2) \qquad \frac{\partial Q^{a(A}}{\partial \overset{p}{Y}_{bB)}} = 0, \quad (p = 0, 1, \ldots, P), \ (a, b, A, B = 1, 2, 3), \ (\overset{0}{Y} = Y) \text{ —see (1.2);}$$

by assertion (1.A) with regard to any system $(5.2)_p$, for $p = 0, \ldots, P$, we deduce the assertion below as in § 2.

The next theorem, which generalizes Theorem 2.1, will state the equivalence of assertions (5.A) through (5.C) below.

(5.A) *The function* (5.1) *is a C^2-solution in $\mathcal{U} \times \mathcal{U}^1 \times \ldots \times \mathcal{U}^P$ to system* (5.2).

(5.B) *The function* (5.1) *is a C^∞-solution in $\mathcal{U} \times \mathcal{U}^1 \times \ldots \times \mathcal{U}^P$ to system* (5.2).

(5.C) *There are tensors $\overset{[0]}{\tau}, \overset{[ii]}{\tau} \in \mathcal{T}^{\nu}(\mathcal{V})$ and $\overset{[i]}{\tau}, \overset{[ij]}{\tau} \in \mathcal{T}^{\nu+1}(\mathcal{V})$ $(i = 0, \ldots, P)$ such that, for each $(a, A) \in \{1, 2, 3\}^{\nu}$, Q^{aA} writes as*

$$(5.3) \quad Q^{aA} = \overset{[0]}{\tau}{}^{aA} + \sum_{i=0}^{P} \overset{[i]}{\tau}{}^{abC} \varepsilon^{ABC} \overset{i}{Y}_{bB} + \sum_{i=0}^{P} \overset{[ii]}{\tau}{}^{ah} \varepsilon^{hbc} \varepsilon^{ABC} \overset{i}{Y}_{bB} \overset{i}{Y}_{cC} + \sum_{\substack{i=0=j \\ i \neq j}}^{P} \overset{[ij]}{\tau}{}^{abc} \varepsilon^{ABC} \overset{i}{Y}_{bB} \overset{j}{Y}_{cC}.$$

THEOREM 5.1. – *The three assertions* (5.A) *through* (5.C) *are equivalent.*

To prove the theorem one has to proceed as in the proof of Theorem 2.1.

6. – Physically remarkable solutions of the above multiple systems in the case $\upsilon = 1$.

This section is the analogue of the first part of § 4, in that here we characterize the class of the solutions to the multiple system (5.2) in the case $\upsilon = 1$, which satisfy the conditions (6.A)-(6.B) below. The former express the property of Galilean invariance, the latter the property of Euclidean invariance—see below (4.B*).

(6.A) $\hat{\boldsymbol{Q}}(\boldsymbol{RY}, \boldsymbol{R}\overset{1}{\boldsymbol{Y}}, \dots, \boldsymbol{R}\overset{P}{\boldsymbol{Y}}) = \hat{\boldsymbol{Q}}(\boldsymbol{Y}, \overset{1}{\boldsymbol{Y}}, \dots, \overset{P}{\boldsymbol{Y}})$ *for any* $\boldsymbol{R} \in \mathrm{Orth}^+$ *and* $(\boldsymbol{Y}, \overset{1}{\boldsymbol{Y}}, \dots, \overset{P}{\boldsymbol{Y}}) \in \in \mathcal{U} \times \mathcal{U}^1 \times \dots \times \mathcal{U}^P$ $(\boldsymbol{Y} = \overset{0}{\boldsymbol{Y}})$—see (5.1)

[*in components* $\hat{Q}^A(R_{bi} Y_{iB}, R_{di} \overset{1}{Y}_{iD}, \dots, R_{di} \overset{P}{Y}_{iD}) = \hat{Q}^A(Y_{bB}, \overset{1}{Y}_{iD}, \dots, \overset{P}{Y}_{iD})$].

(6.B) $\hat{\boldsymbol{Q}}(\boldsymbol{RY}, D^1(\boldsymbol{RY}), \dots, D^P(\boldsymbol{RY})) = \hat{\boldsymbol{Q}}(\boldsymbol{Y}, \overset{1}{\boldsymbol{Y}}, \dots, \overset{P}{\boldsymbol{Y}})$ *for any* $\hat{\boldsymbol{R}} \in C^p([-\varepsilon, 0], \mathrm{Orth}^+)$ *and* $\hat{\boldsymbol{Y}} \in C^p([-\varepsilon, 0], \mathcal{U})$ *such that*

$$\left(\frac{d}{dt}\boldsymbol{Y}(t), \dots, \frac{d^P}{(dt)^P}\boldsymbol{Y}(t)\right) \in \mathcal{U}^1 \times \dots \times \mathcal{U}^P \qquad \forall t \in [-\varepsilon, 0],$$

where $\varepsilon > 0$ *and*

$$\text{(6.1)} \qquad \begin{cases} D^P(\boldsymbol{RY}) := \sum_{h=0}^{P} \binom{p}{h} \overset{h}{Y}_{sD} \overset{p-h}{R}_{bs}, & \boldsymbol{R} = \hat{\boldsymbol{R}}(0), \qquad \boldsymbol{Y} = \hat{\boldsymbol{Y}}(0), \\ \overset{i}{\boldsymbol{R}} = \dfrac{d^i \hat{\boldsymbol{R}}}{(dt)^i}(0), \quad \overset{i}{\boldsymbol{Y}} = \dfrac{d^i \hat{\boldsymbol{Y}}}{(dt)^i}(0) & (p, i = 1, \dots, P), \end{cases}$$

[*in components* $\hat{Q}^A(R_{bi} Y_{iB}, \dot{R}_{di} Y_{iD} + R_{di} \overset{1}{Y}_{iD}, \dots) = \hat{Q}^A(Y_{bB}, \overset{1}{Y}_{dD}, \dots, \overset{P}{Y}_{dD})$].

Note that (6.B) implies (6.A).

Consider the assertions (6.C) through (6.F) below. The next Theorems 6.1-6.2 generalize Theorems 4.1-4.2 respectively.

(6.C) *The function* (5.1) *is a smooth solution in* $\mathcal{U} \times \mathcal{U}^1 \times \dots \times \mathcal{Y}^P$ *to system* (5.2) *and satisfies condition* (6.A).

(6.D) *The function* (5.1) *is a smooth solution in* $\mathcal{U} \times \mathcal{U}^1 \times \dots \times \mathcal{Y}^P$ *to system* (5.2) *and satisfies condition* (6.B).

(6.E) *There are a vector* τ *and scalars* $\overset{[ij]}{d}$, *which are skew-symmetric in* i, j, *such that*

$$Q^A = \tau^A + \sum_{i,j=0}^{P} \overset{[ij]}{d}\, \varepsilon^{ABC} \overset{i}{Y}_{bB} \overset{j}{Y}_{bC} \qquad (A = 1, 2, 3). \tag{6.2}$$

(6.F) *There is a vector* τ *such that*

$$Q^A = \tau^A \qquad (A = 1, 2, 3). \tag{6.3}$$

THEOREM 6.1. – *The assertions* (6.C) *and* (6.E) *are equivalent.*

THEOREM 6.2. – *The assertions* (6.D) *and* (6.F) *are equivalent.*

PROOF OF THEOREM 6.1. – Assume (6.C) with $\upsilon = 1$; by Theorem 5.1 equality (5.3) holds (with a dropped)—see (1.4); now (6.A) and (5.3) yield

$$\overset{[0]}{\tau}{}^A + \sum_{i=0}^{P} \overset{[i]}{\tau}{}^{bc} \varepsilon^{ABC} \overset{i}{Y}_{bB} + \sum_{i=0}^{P} \overset{[ii]}{\tau}{}^{h} \varepsilon^{hbc} \varepsilon^{ABC} \overset{i}{Y}_{bB} \overset{i}{Y}_{cC} + \sum_{\substack{i,j=0\\ i\neq j}}^{P} \overset{[ij]}{\tau}{}^{bc} \varepsilon^{ABC} \overset{i}{Y}_{bB} \overset{j}{Y}_{cC} = \tag{6.4}$$

$$= \overset{[0]}{\tau}{}^A + \sum_{i=0}^{P} \overset{[i]}{\tau}{}^{bC} \varepsilon^{ABC} R_{bl} \overset{i}{Y}_{lB} + \sum_{i=0}^{P} \overset{[ii]}{\tau}{}^{h} \varepsilon^{hbc} \varepsilon^{ABC} R_{bl} \overset{i}{Y}_{lB} R_{cs} \overset{i}{Y}_{sC} + \sum_{\substack{i,j=0\\ i\neq j}}^{P} \overset{[ij]}{\tau}{}^{bc} \varepsilon^{ABC} R_{bl} \overset{i}{Y}_{lB} R_{cs} \overset{j}{Y}_{sC}.$$

By equating the terms of equal degree in the two sides of equality (6.4), and by the arbitrariness of the independent variables and of R in Orth$^+$, we have

$$\overset{[i]}{\tau}{}^{bC} = \overset{[i]}{\tau}{}^{lC} R_{lb}, \qquad \overset{[ii]}{\tau}{}^{h} \varepsilon^{hls} = \overset{[ii]}{\tau}{}^{h} \varepsilon^{hbc} R_{bl} R_{cs}, \qquad \overset{[ij]}{\tau}{}^{ls} = \overset{[ij]}{\tau}{}^{bc} R_{bl} R_{cs}.$$

The arbitrariness of $\boldsymbol{R}$, Lemmas 3.1-3.2 and the first [third] equality above yield $(6.5)_1$ [$(6.5)_3$] below

$$\overset{[i]}{\tau} = O = \overset{[ii]}{\tau}, \qquad \overset{[ij]}{\tau}{}^{ls} = \overset{[ij]}{d}\, \delta^{ls} \qquad \text{if } i \neq j \ (i, j = 1, \dots, P). \tag{6.5}$$

Lastly, the second equality above (6.5) yields

$$2\, \overset{[ii]}{\tau}{}^{k} = \overset{[ii]}{\tau}{}^{h} \varepsilon^{hbc} \varepsilon^{kls} R_{bl} R_{cs};$$

and as $\varepsilon^{hbc} \varepsilon^{kls} R_{bl} R_{cs} = 2 \det(\boldsymbol{R})(\boldsymbol{R}^{-1})_{kh}$, it follows that $\overset{[ii]}{\tau}{}^{k} = \overset{[ii]}{\tau}{}^{h} R_{hk}$. Thus by Lemma 3.1 equality $(6.5)_2$ holds. Note that in equality (6.4) vector $\overset{[0]}{\tau}$ can be taken ad arbitrium. Thus (6.5) yields

$$Q^A = \tau^A + \sum_{\substack{i,j=0\\ i\neq j}}^{P} \overset{[ij]}{d}\, \varepsilon^{ABC} \overset{i}{Y}_{bB} \overset{j}{Y}_{bC} \qquad (A = 1, 2, 3). \tag{6.2$'$}$$

By the skew-symmetry of $\varepsilon^{ABC} \overset{i}{Y}_{bB} \overset{j}{Y}_{bC}$ with respect to i, j one deduces that scalars $\overset{[ij]}{d}$ can be taken skew-symmetric and (6.E) is proved. To prove the converse implication, one only has to verify that if $\hat{\boldsymbol{Q}}$ is given by (6.2), then it solves (5.2) and satisfies (6.A). q.e.d.

PROOF OF THEOREM 6.2. – Assume (6.C) with $\upsilon = 1$—see (1.4); as (6.B) implies (6.A), by Theorem 6.1 also assertion (6.E) holds. Thus by (6.A) equality (6.2) yields

$$\tau^A + \sum_{i,j=0}^{p} \overset{[ij]}{d}\, \varepsilon^{ABC} \sum_{h=0}^{i} \binom{i}{h} \overset{h}{Y}_{lB} \overset{i-h}{R}_{bl} \sum_{k=0}^{j} \binom{j}{k} \overset{k}{Y}_{sC} \overset{j-k}{R}_{bs} = \tag{6.6}$$

$$= \tau^A + \sum_{i,j=0}^{p} \overset{[ij]}{d}\, \varepsilon^{ABD} \overset{i}{Y}_{bB} \overset{j}{Y}_{bD} \qquad \left(\binom{0}{0} = 1\right).$$

Choose $m, n \in \{0, 1, \dots, P\}$ and $c, C, f, F \in \{1, 2, 3\}$; by taking the derivatives of both the sides of equality (6.6) first with respect to $\overset{m}{Y}_{cC}$ and then with respect to $\overset{n}{Y}_{fF}$ we find

$$\sum_{\substack{j=0\\ i=m}}^{p} \overset{[ij]}{d}\, \varepsilon^{ACE} \binom{i}{m} \overset{i-m}{R}_{bc} \sum_{k=0}^{j} \binom{j}{k} \overset{k}{Y}_{sE} \overset{j-k}{R}_{bs} + \sum_{\substack{j=m\\ i=0}}^{p} \overset{[ij]}{d}\, \varepsilon^{ADC} \sum_{h=0}^{i} \binom{i}{h} \overset{h}{Y}_{lD} \overset{i-h}{R}_{bl} \binom{j}{m} \overset{j-m}{R}_{bc} = \tag{6.7}$$

$$= \sum_{i=0}^{p} \overset{[im]}{d}\, \varepsilon^{ADC} \overset{i}{Y}_{cD} + \sum_{j=0}^{p} \overset{[mj]}{d}\, \varepsilon^{ACE} \overset{j}{Y}_{cE}$$

and

$$\sum_{\substack{j=n\\ i=m}}^{p} \overset{[ij]}{d}\, \varepsilon^{ACF} \binom{i}{m} \overset{i-m}{R}_{bc} \binom{j}{n} \overset{j-n}{R}_{bf} + \sum_{\substack{j=m\\ i=n}}^{p} \overset{[ij]}{d}\, \varepsilon^{AFC} \binom{i}{n} \overset{i-n}{R}_{bf} \binom{j}{m} \overset{j-m}{R}_{bc} = \tag{6.8}$$

$$= \overset{[nm]}{d}\, \varepsilon^{AFC} \delta_{cf} + \overset{[mn]}{d}\, \varepsilon^{ACF} \delta_{cf},$$

respectively. Multiply each side of equality (6.8) by ε^{ACF}; the skew-symmetry of $\overset{[ij]}{d}$ yields

$$\sum_{\substack{j=n\\ i=m}}^{p} \overset{[ij]}{d} \binom{i}{m} \overset{i-m}{R}_{bc} \binom{j}{n} \overset{j-n}{R}_{bf} = \overset{[nm]}{d}\, \delta_{cf}. \tag{6.9}$$

Now let us consider uniform rotations $\boldsymbol{R}$ about the e-th reference axis; by (2.1) in [4] we have

$$R_{ts} = \delta_t^e \delta_s^e + (\delta_t^{e+1} \delta_s^{e+1} + \delta_t^{e+2} \delta_s^{e+2}) \cos(\omega t) + (\delta_t^{e+2} \delta_s^{e+1} - \delta_t^{e+1} \delta_s^{e+2})(\delta_e^1 + \delta_e^2 - \delta_e^3) \sin(\omega t),$$

with $e+3=e$ and $\omega\in\mathbb{R}$; thus the expressions for the derivatives of $\boldsymbol{R}$ at time $t=0$ are

$$\begin{cases} \overset{2r}{R}_{ts} = (\delta^t_{e+1}\delta^s_{e+1} + \delta^t_{e+2}\delta^s_{e+2})\,\omega^{2r}(-1)^r, \\ \overset{2r-1}{R}_{ts} = -(\delta^t_{e+2}\delta^s_{e+1} + \delta^t_{e+1}\delta^s_{e+2})(\delta^1_e + \delta^2_e - \delta^3_e)\,\omega^{2r-1}(-1)^r \qquad (r=1,2,\ldots). \end{cases} \tag{6.10}$$

By replacing (6.10) into (6.9) the left-hand side of the resulting equality becomes a polynomial in ω. As ω is arbitrary, the coefficients of any monomial of degree ≥ 1 of such polynomial vanish. The monomials of degree one are found by the choices $(i,j)=(m,n+1)$ or $(i,j)=(m+1,n)$; hence

$$\overset{[m\,n+1]}{d}\binom{m}{m}R_{bc}\binom{n+1}{n}\overset{1}{R}_{bf} + \overset{[m+1\,n]}{d}\binom{m+1}{m}\overset{1}{R}_{bc}\binom{n}{n}R_{bf} = 0. \tag{6.11}$$

Now at $t=0$ we have $\boldsymbol{R}=\boldsymbol{I}$ and $\overset{1}{\boldsymbol{R}}\in\text{Skew}$; thus (6.11) becomes

$$[(n+1)\overset{[m\,n+1]}{d} - (m+1)\overset{[m+1\,n]}{d}](\delta^c_{e+2}\delta^f_{e+1} - \delta^c_{e+1}\delta^f_{e+2})(\delta^1_e + \delta^2_e - \delta^3_e)\,\omega = 0, \tag{6.12}$$

and the arbitrariness of ω yields

$$(n+1)\overset{[m\,n+1]}{d} = (m+1)\overset{[m+1\,n]}{d}. \tag{6.13}$$

By setting $n=m+1$ and $n=m$ into (6.13) we find

$$\overset{[m\,m+2]}{d} = 0 \tag{6.14}$$

and

$$\overset{[m\,m+1]}{d} = \overset{[m+1\,m]}{d}, \tag{6.15}$$

respectively. Now the skew-symmetry of $\overset{[ij]}{d}$ and (6.15) yield

$$\overset{[m\,m+1]}{d} = 0. \tag{6.16}$$

Lastly, from (6.13) and (6.14) [(6.15)] it follows that any term $\overset{[ij]}{d}$ with $i+j$ even [odd] vanishes. Hence $\overset{[ij]}{d}=0$ for all $i,j=0,1,\ldots,P$ and (6.3) holds. q.e.d.

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