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Sums of Linear Operators of Parabolic Type: a priori Estimates and Strong Solutions (*).

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Summary. – *We study the equation $(A - \lambda)x + (B - \lambda)x = y$, with unknown x, in a Banach space X. $y \in X$ is the datum, $\lambda > 0$, A and B are linear closed unbounded operators in X with domains D_A, D_B. In the non commutative case, under assumptions already considered in the literature (see [7]), we show that for large values of λ any solution $x \in D_A \cap D_B$ satisfies an a priori estimate $\|x\| \leq c\lambda^{-1}\|y\|$ and we prove that for any $y \in X$ there exists a unique strong solution x, i.e. there exist $x_n \in D_A \cap D_B$ such that $x_n \to x$, $(A - \lambda)x_n + (B - \lambda)x_n \to y$ in X. We also study regularity properties of strong solutions and we show that they belong to suitable interpolation spaces between D_A (or D_B) and X.*

1. – Introduction.

Let X be a Banach space, A and B two linear operators in X with domains D_A and D_B, y a given vector in X. We want to find solutions x of the equation

$$(A - \lambda)x + (B - \lambda)x = y \tag{1.1}$$

where λ is a fixed positive number.

Such a problem provides a common framework for many applications. Results about existence and regularity of solutions of (1.1) have been applied to study linear ordinary differential equations in Banach spaces (both first order and higher order equations have been considered) and partial differential equations. Some of these applications can be found in [10], [11], [4], [5].

DEFINITION 1.1. – By *strict solution* of (1.1) we mean a vector $x \in D_A \cap D_B$ satisfying (1.1). We call x a *strong solution* of (1.1) if there exist sequences $x_n \in D_A \cap D_B$, $y_n \in X$ such that $(A - \lambda)x_n + (B - \lambda)x_n = y_n$ and $x_n \to x$, $y_n \to y$ as $n \to \infty$.

We will show that, under a certain set of hypotheses already considered in the literature, there exists a $\lambda^* > 0$ such that for any $\lambda > \lambda^*$ and for any $y \in X$ the problem

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(1.1) has a unique strong solution x. We also prove some regularity properties of x.

Existence, uniqueness and regularity of both strict and strong solutions of (1.1) have been studied in [5], under various assumptions for A and B. We are interested in the «parabolic case», of [5] i.e. we suppose that

$$\text{(H1)}\quad \begin{cases} \textit{there exist } \vartheta_A,\ \vartheta_B > 0,\ c_A,\ c_B > 0 \textit{ such that} \\ \vartheta_A + \vartheta_B < \pi, \\ (A-z)^{-1} \in \mathcal{L}(X),\ \|(A-z)^{-1}\|_{\mathcal{L}(X)} \leqslant c_A |z|^{-1} \textit{ if } z \in \boldsymbol{C},\ \ |\arg(z)| < \pi - \vartheta_A, \\ (B-z)^{-1} \in \mathcal{L}(X),\ \|(B-z)^{-1}\|_{\mathcal{L}(X)} \leqslant c_B |z|^{-1} \textit{ if } z \in \boldsymbol{C},\ \ |\arg(z)| < \pi - \vartheta_B. \end{cases}$$

Other sets of hypotheses have been proposed, in addition to (H1), in order to solve problem (1.1). The simplest case is commutativity of A and B (see [5], Sect. 3.2), but we will not deal with it. In the non commutative case two different and independent hypotheses have been proposed. The first one is considered in [5, Sect. 6], where both strict and strong solutions are studied. The second one has been proposed in [7] and, earlier but in a more particular sistuation, in [2]. It is the following:

$$\text{(H2)}\quad \begin{cases} \textit{there exist } \lambda_0 > 0,\ k \geqslant 1,\ \alpha_1, \dots, \alpha_k,\ \beta_1, \dots, \beta_k \textit{ and } c_{AB} > 0 \textit{ such that} \\ 0 \leqslant \alpha_i < \beta_i \leqslant 1, \quad i = 1, \dots, k, \\ \|(A-\lambda_0)(A-v)^{-1}[(A-\lambda_0)^{-1};(B-z)^{-1}]\|_{\mathcal{L}(X)} \leqslant c_{AB} \sum_1^k{}_i \dfrac{1}{|v|^{1-\alpha_i}|z|^{1+\beta_i}}, \\ \forall v \in \boldsymbol{C},\ \ |\arg(v)| < \pi - \vartheta_A,\ \ \forall z \in \boldsymbol{C},\ \ |\arg(z)| < \pi - \vartheta_B. \end{cases}$$

We will assume (H1) and (H2) from now on.

Under assumptions (H1) and (H2), [7] prove that for large values of λ, for any $\vartheta \in (0,1)$ and for any $y \in D_A(\vartheta, \infty)$ (resp. $y \in D_B(\vartheta, \infty)$), there exists a unique strict solution x of (1.1) and, if ϑ is sufficiently small, $Ax, Bx \in D_A(\vartheta, \infty)$, $Ax \in D_B(\vartheta, \infty)$ (resp. $Ax, Bx \in D_B(\vartheta, \infty)$, $Bx \in D_A(\vartheta, \infty)$). Here $D_A(\vartheta, \infty)$ (resp. $D_B(\vartheta, \infty)$) are real interpolation spaces between D_A and X (resp. D_B and X), which have also been characterized by [9].

However in [7] only strict solutions are considered. This is due to the lack of an a priori estimate for strict solutions of (1.1). In fact, the most natural way for proving the existence of strong solutions of (1.1) is to prove that, whenever $x \in D_A \cap D_B$ we have

$$\|x\| \leqslant c_\lambda \|(A-\lambda)x + (B-\lambda)x\| \tag{1.2}$$

where c_λ is a positive constant independent of x. Now, given $y \in X$, we may in some cases find a sequence y_n such that $y_n \to y$ and there exist x_n satisfying $(A-\lambda)x_n + (B-\lambda)x_n = y_n$. The convergence of x_n to a strong solution x is now assured by (1.2).

In order to show (1.2) we may proceed, at least formally, in the following way. We use a representation formula for strict solutions of (1.1). In [7] it is proved that if $x \in D_A \cap D_B$ satisfies (1.1) we have

$$x = (A - \lambda)^{-1}(1 + J_\lambda)^{-1}(A - \lambda)S_\lambda y \tag{1.3}$$

where J_λ and S_λ are properly chosen bounded linear operators in X (see (2.12), (2.14) below). Here we use another formula, namely

$$x = (\lambda - B)^{-1}(1 + J_\lambda)^{-1}(\lambda - B)S'_\lambda y + (\lambda - B)^{-1}(1 + J_\lambda)^{-1}M_\lambda y \tag{1.4}$$

which is obtained by the previous one by interchanging, in some sense, the roles of A and B (see definitions (2.12)-(2.15)). Neither (1.3) nor (1.4) are meaningful for all $y \in X$. However, given $\rho > 0$, (1.4) may be written

$$\begin{aligned} x = (\lambda - B)^{\rho - 1}\{1 + (\lambda - B)^{-\rho}J_\lambda(\lambda - B)^\rho\}^{-1}(\lambda - B)^{1-\rho}S'_\lambda y + \\ + (\lambda - B)^{-1}(1 + J_\lambda)^{-1}M_\lambda y =: U_\lambda y. \end{aligned} \tag{1.5}$$

It turns out that, if ρ is sufficiently small, $U_\lambda \in \mathcal{L}(X)$, so that (1.2) holds. Of course, formula (1.5) has to be given a more precise meaning; in particular, we observe that the operator $J_\lambda(\lambda - B)^\rho$ is not defined in all of X, and we have to replace it in (1.5) by a bounded operator K_λ. The crucial point is proving that K_λ is the limit in $\mathcal{L}(X)$ of a sequence of bounded operators $J_{\lambda, n}(\lambda - B_n)^\rho$ that approximate $J_\lambda(\lambda - B)^\rho$ (see Lemma 3.5)). In order to prove this and to justify (1.5) we have to enter many technical details. In particular we use Yosida approximations of both A and B, and we first prove an analogue of (1.5) with A and B replaced by their Yosida approximations. This forces us to impose some density assumptions to the domains of A and B, in order to prove (1.2) (see Theorem 4.2).

Once the existence and uniqueness of strong solutions of (1.1) is established, we study their regularity properties. We can show that they belong to $D_A(1, \infty) \cap D_B(1, \infty)$, where

$$D_A(1, \infty) := \{x \in X: \sup_{0 < t < \infty} \|tA^2(A - t)^{-2}x\| < \infty\} \tag{1.6}$$

with the norm $\|x\|_{D_A(1, \infty)} := \|x\| + \sup_{0 < t < \infty} \|tA^2(A - t)^{-1}x\|$, and $D_B(1, \infty)$ is defined similarly (see [5], [9]). The same regularity result is proved for strong solutions in [5], both in the commutative and non commutative case.

Finally, we show how the investigations about problem (1.1) can be applied to a linear parabolic evolution equation in Banach space. In particular we recover, at least partially, some results obtained in [3] concerning a priori estimates for strict solutions. Regularity properties of strong solutions in this case are studied by [1]: some of his results are connected with ours (remark that in [1] the space $D_A(1, \infty)$, defined by (1.6), is denoted by $D_{A^2}(1/2, \infty)$; if A a bounded inverse it is the real interpolation space $(X, D_{A^2})_{1/2, \infty}$ (see e.g. [12, Th. 1.14.2])).

2. – Assumptions and notations.

Throughout this paper, X is a complex Banach space, $\mathcal{L}(X)$ (resp. $\mathcal{L}(X, Y)$) is the Banach algebra (resp. the Banach space) of linear bounded operators in X (resp. from X to Y, where Y is another complex Banach space). If C is a linear operator in X, we denote by $\sigma(C)$ its spectrum and by $\rho(C)$ its resolvent set. For any $\vartheta \in [0, \pi)$ we define

$$\Sigma_\vartheta := \{z \in \boldsymbol{C}:\ z \neq 0,\ |\arg(z)| < \pi - \vartheta\} \tag{2.1}$$

($\arg(z)$ is assumed to take values in $(-\pi, \pi]$). We also use the notation

$$[P, Q] := PQ - QP, \qquad \text{for } P, Q \in \mathcal{L}(X).$$

We make the following assumptions on the operators A and B.

(H1) *A and B are linear operators in X with domains D_A and D_B and there exist $\vartheta_A, \vartheta_B \in (0, \pi)$, $c_A, c_B > 0$ such that*

$$\begin{cases} \rho(A) \supset \Sigma_{\vartheta_A}, & \|(z - A)^{-1}\|_{\mathcal{L}(X)} \leq \dfrac{c_A}{|z|}, \quad \forall z \in \Sigma_{\vartheta_A}, \\ \vartheta_A + \vartheta_B < \pi, & \\ \rho(B) \supset \Sigma_{\vartheta_B}, & \|(z - B)^{-1}\|_{\mathcal{L}(X)} \leq \dfrac{c_B}{|z|}, \quad \forall z \in \Sigma_{\vartheta_B}, \end{cases} \tag{2.2}$$

(H2) *There exist $\lambda_0 > 0$, $k \geq 1$, $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_k$ and $c_{AB} > 0$ such that $0 \leq \alpha_i < \beta_i \leq 1$, $i = 1, \ldots, k$*

$$\|(A - \lambda_0)(A - v)^{-1}[(A - \lambda_0)^{-1}; (B - z)^{-1}]\|_{\mathcal{L}(X)} \leq c_{AB} \sum_{1}^{k}{}_i \frac{1}{|v|^{1-\alpha_i} |z|^{1+\beta_i}} \tag{2.3}$$

$$\forall v \in \Sigma_{\vartheta_A}, \quad \forall z \in \Sigma_{\vartheta_B}.$$

We also assume (without loss of generality) $\delta := \min_{1 \leq i \leq k} (\beta_i - \alpha_i) \in (0, 1)$ (this causes no loss of generality, since the behavior of the right member of (2.3) turns out to be relevant only for large values of $|z|$ and $|v|$).

For all integers $m \geq 1$, $n \geq 1$ we denote by A_m, B_n the Yosida approximations of A and B: $A_m := mA(m - A)^{-1}$, $B_n := nB(n - B)^{-1}$.

LEMMA 2.1. – *Assume* (2.2). *Then for any $m \geq 1$, $n \geq 1$, A_m and B_n satisfy*

$$\begin{cases} \rho(A_m) \supset \Sigma_{\vartheta_A}, & \|(z - A_m)^{-1}\|_{\mathcal{L}(X)} \leq \dfrac{c}{|z|}, \quad \forall z \in \Sigma_{\vartheta_A}, \\ \rho(B_n) \supset \Sigma_{\vartheta_B}, & \|(z - B_n)^{-1}\|_{\mathcal{L}(X)} \leq \dfrac{c}{|z|}, \quad \forall z \in \Sigma_{\vartheta_B}, \end{cases} \tag{2.4}$$

with $c > 0$ independent of m and n.

PROOF. – See [5, formula (6.11)].

LEMMA 2.2. – *Assume* (2.2), (2.3). *Then for any* $\bar{\lambda} > 0$ *there exists* $c(\bar{\lambda}) > 0$ *such that*

$$\|(A-\lambda)(A-\lambda-v)^{-1}[(A-\lambda)^{-1};(B-z)^{-1}]\|_{\mathcal{L}(X)} \leqslant c(\bar{\lambda})\sum_1^k {}_i \frac{1}{|v+\lambda|^{1-\alpha_i}|z|^{1+\beta_i}} \tag{2.5}$$
$$\forall v \in \Sigma_{\vartheta_A},\quad \forall z \in \Sigma_{\vartheta_B},\quad \forall \lambda > \bar{\lambda},$$

and the same inequality also holds if A (*resp.* B, *resp.* A *and* B) *is replaced by* A_m (*resp.* B_n, *resp.* A_m *and* B_n), *with* $c(\bar{\lambda})$ *independent of* m, n.

PROOF. – (2.5) holds for A and B (see [7, Lemma 1.2]). [7, Lemma 3.1] shows that it holds for A_m and B. The general case can be proved in a similar way.

To represent the solution of problem (1.1) we use operator-valued integrals over paths of the complex plane, which are similar to Dunford integrals (see e.g. [6]). We always assume that paths are piecewise continuously differentiable. We use the symbol $⨍$ instead of $\frac{1}{2\pi i}\int$. We have to integrate over unbounded paths. We now define some of them.

Let ϑ_0 be a number such that

$$\begin{cases} \vartheta_B < \vartheta_0 < \pi - \vartheta_A, & \vartheta_0 > \pi/2 \quad \text{if } \vartheta_A < \pi/2, \\ \vartheta_B < \vartheta_0 < \pi - \vartheta_A, & \vartheta_0 < \pi/2 \quad \text{if } \vartheta_A \geqslant \pi/2. \end{cases} \tag{2.6}$$

Let ϑ_1 be a number such that

$$\begin{cases} \vartheta_B < \vartheta_1 < \vartheta_0, & \vartheta_1 > \pi/2 \quad \text{if } \vartheta_A < \pi/2, \\ \pi - \vartheta_0 < \vartheta_1 < \pi - \vartheta_B & \quad \text{if } \vartheta_A \geqslant \pi/2. \end{cases} \tag{2.7}$$

Let ϑ_2 be a number such that

$$\vartheta_B < \vartheta_2 < \vartheta_0. \tag{2.8}$$

For $k = 0, 1, 2$ define $\gamma_k := \gamma_{k1} + \gamma_{k2}$ where

$$\begin{cases} \gamma_{k1} := \{-t\exp(-i\vartheta_k) \colon t \in (-\infty, 0]\}, \\ \gamma_{k2} := \{t\exp(i\vartheta_k) \colon t \in [0, \infty)\}. \end{cases} \tag{2.9}$$

For any $R > 0$ define $\gamma_0^R := \gamma_{01}^R + \gamma_{02}^R + \gamma_{03}^R$ where

$$\begin{cases} \gamma_{01}^R := \{-t\exp(-i\vartheta_0) \colon t \in [-R, 0]\}, \\ \gamma_{02}^R := \{t\exp(i\vartheta_0) \colon t \in [0, R]\}, \\ \gamma_{03}^R := \{R\exp(i\vartheta) \colon \vartheta \in [\vartheta_0, 2\pi - \vartheta_0]\}. \end{cases} \tag{2.10}$$

For $k = 0, 1, 2$ and for r, s with $0 < s < r$ define $\gamma_k' := \gamma_{k1}^{rs} + \gamma_{k2}^{s} + \gamma_{k3}^{rs} + \gamma_{k4}^{r}$ where

$$\text{(2.11)} \qquad \begin{cases} \gamma_{k1}^{rs} := \{-t\exp(-i\vartheta_k) \colon t \in [-r, -s]\}, \\ \gamma_{k2}^{s} := \{s\exp(i\vartheta) \colon \vartheta \in [-\vartheta_k, \vartheta_k]\}, \\ \gamma_{k3}^{rs} := \{t\exp(i\vartheta_k) \colon t \in [s, r]\}, \\ \gamma_{k4}^{r} := \{r\exp(-i\vartheta) \colon \vartheta \in [-\vartheta_k, \vartheta_k]\}. \end{cases}$$

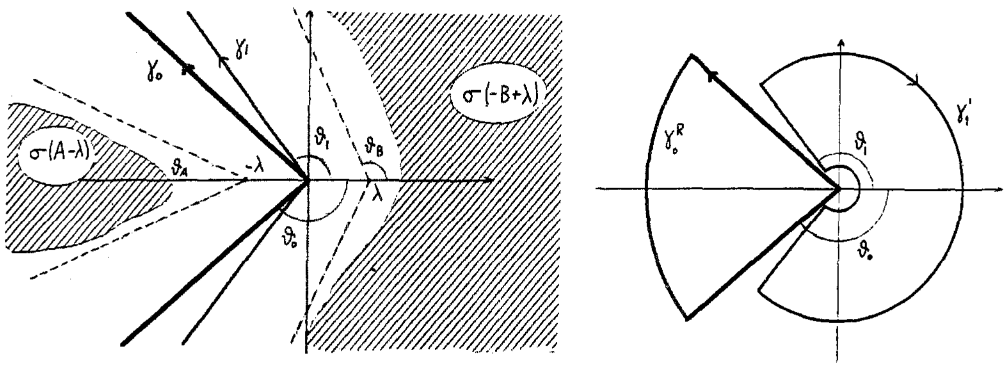

Figure 1.

Define

$$\text{(2.12)} \qquad S_\lambda := -\int_{\gamma_0} (A - \lambda - z)^{-1}(B - \lambda + z)^{-1}\, dz,$$

$$\text{(2.13)} \qquad S_\lambda' := -\int_{\gamma_0} (B - \lambda + z)^{-1}(A - \lambda - z)^{-1}\, dz,$$

$$\text{(2.14)} \qquad J_\lambda := -\int_{\gamma_0} z(A-\lambda)(A-\lambda-z)^{-1}[(A-\lambda)^{-1}; (B-\lambda+z)^{-1}]\, dz,$$

$$\text{(2.15)} \qquad M_\lambda := -\int_{\gamma_0} z(A-\lambda)(A-\lambda-z)^{-1}[(A-\lambda)^{-1}; (B-\lambda+z)^{-1}]z(A-\lambda-z)^{-1}\, dz.$$

Call $S_{\lambda,n}$, $S_{\lambda,n}'$, $J_{\lambda,n}$, $M_{\lambda,n}$ (resp. $S_{m,\lambda}$, $S_{m,\lambda}'$, $J_{m,\lambda}$, $M_{m,\lambda}$; resp. $S_{m,\lambda,n}$, $S_{m,\lambda,n}'$, $J_{m,\lambda,n}$, $M_{m,\lambda,n}$) the integrals obtained from S_λ, S_λ', J_λ, M_λ, by replacing B with B_n (resp. A with A_m; resp. A and B with A_m and B_n).

Whenever K is a bounded subset of the complex plane and γ is a closed path, we use the expression «γ surrounds K» to mean that γ is contained in $\boldsymbol{C} \setminus K$ and $\mathrm{Ind}_\gamma(z) = 1$, $\forall z \in K$ (or $\mathrm{Ind}_\gamma(z) = -1$, $\forall z \in K$).

We also use fractional powers of operators. Since they are used only as a technical tool, we do not need full generality. We only recall that if γ is a closed path surrounding the spectrum of an operator $C \in \mathcal{L}(X)$ such that $\sigma(C) \cap (-\infty, 0] = \emptyset$ and if γ does not intersect $(-\infty, 0]$ we have, for any $\rho \in \boldsymbol{R}$

$$C^{\rho} := \fint_{\gamma} z^{\rho}(z - C)^{-1}\, dz$$

where z^{ρ} is taken to be positive whenever $z > 0$. The formula

$$C^{\rho} C^{\nu} = C^{\rho + \nu}$$

holds $\forall \rho, \nu \in \boldsymbol{R}$. We occasionally use fractional powers of unbounded operators. In this case we always give an explicit formula for them.

3. - Some preliminary results.

We begin this section with some technical lemmas about boundedness and convergence properties of the operators defined in (2.12)-(2.15). The most difficult one is Lemma 3.5, which provides the key to the proof of a representation formula and an a priori estimate for strict solutions of problem (1.1) (see Definition 1.1). They are given in Theorem 4.2. Hypotheses (H1), (H2) are assumed throughout.

LEMMA 3.1. - *For any $\rho \in (0,1)$ there exists $c(\rho) > 0$ such that*

$$\|(\lambda - B_n)^{-\rho}\|_{\mathcal{L}(X)} \leq c(\rho)\, \lambda^{-\rho}, \qquad \forall n, \ \forall \lambda > 0. \tag{3.1}$$

Moreover we have, in the norm of $\mathcal{L}(X)$,

$$\lim_{n \to \infty} (\lambda - B_n)^{-\rho} = -\fint_{\gamma_0} z^{-\rho}(B - \lambda + z)^{-1}\, dz =: (\lambda - B)^{-\rho}, \qquad \forall \lambda > 0. \tag{3.2}$$

PROOF. - It is easily seen that $(\lambda - B_n)^{-\rho} = -\fint_{\gamma_0} z^{-\rho}(B_n - \lambda + z)^{-1}\, dz$, so that

$$\|(\lambda - B_n)^{-\rho}\|_{\mathcal{L}(X)} \leq \fint_{\gamma_0} \frac{c_B |z|^{-\rho}}{|z - \lambda|} |dz| = (z = \lambda v) = \fint_{\gamma_0} \frac{c_B |v|^{-\rho} \lambda^{-\rho}}{|v - 1|\lambda} \lambda |dv| = c(\rho)\, \lambda^{-\rho}$$

by (2.4). The same estimate and the dominated convergence theorem imply (3.2).

LEMMA 3.2. - *For any $\bar{\lambda} > 0$ there exists $c(\bar{\lambda}) > 0$ such that $\forall n$, $\forall m$, $\forall \lambda > \bar{\lambda}$,*

$$\|J_{m,\lambda,n}\|_{\mathcal{L}(X)} \leq c(\bar{\lambda})\, \lambda^{-\delta}, \qquad \|J_{\lambda,n}\|_{\mathcal{L}(X)} \leq c(\bar{\lambda})\, \lambda^{-\delta}, \qquad \|J_{m,\lambda}\|_{\mathcal{L}(X)} \leq c(\bar{\lambda})\, \lambda^{-\delta}, \tag{3.3}$$

$$\|M_{m,\lambda,n}\|_{\mathcal{L}(X)} \leq c(\bar{\lambda})\, \lambda^{-\delta}, \qquad \|M_{\lambda,n}\|_{\mathcal{L}(X)} \leq c(\bar{\lambda})\, \lambda^{-\delta}, \qquad \|M_{m,\lambda}\|_{\mathcal{L}(X)} \leq c(\bar{\lambda})\, \lambda^{-\delta}, \tag{3.4}$$

(*see* (2.14), (2.15)). *Moreover we have, in the norm of* $\mathcal{L}(X)$, $\forall n$, $\forall m$, $\forall \lambda > 0$,

$$\left\{\begin{array}{lll}\lim_{m\to\infty} J_{m,\lambda,n} = J_{\lambda,n}, & \lim_{n\to\infty} J_{m,\lambda,n} = J_{m,\lambda}, & \lim_{m\to\infty} J_{m,\lambda} = \lim_{n\to\infty} J_{\lambda,n} = J_\lambda, \\ \lim_{m\to\infty} M_{m,\lambda,n} = M_{\lambda,n}, & \lim_{n\to\infty} M_{m,\lambda,n} = M_{m,\lambda}, & \lim_{m\to\infty} M_{m,\lambda} = \lim_{n\to\infty} M_{\lambda,n} = M_\lambda.\end{array}\right. \tag{3.5}$$

PROOF. – By (2.4), (2.5) we have, $\forall \lambda > \bar{\lambda}$,

$$\|M_{m,\lambda,n}\|_{\mathcal{L}(X)} \leqslant \int_{\gamma_0} \sum_{1}^{k}{}_i \frac{c(\bar{\lambda})|z|}{|z+\lambda|^{1-\alpha_i}|z-\lambda|^{1+\beta_i}} \frac{c_A|z|}{|z+\lambda|}|dz| = (z = v\lambda) =$$

$$\leqslant \int_{\gamma_0} \sum_{1}^{k}{}_i \frac{c(\bar{\lambda})|v|\lambda}{|v+1|^{1-\alpha_i}|v-1|^{1+\alpha_i}\lambda^{2+\beta_i-\alpha_i}} \frac{c_A|v|\lambda}{|v+1|\lambda}\lambda|dv| \leqslant c(\bar{\lambda})\lambda^{-\varepsilon}.$$

This proves the first part of (3.3), (3.4). Similar passages hold for $J_{m,\lambda,n}$. (3.5) then follows from the dominated convergence theorem, and it yields the rest of (3.3), (3.4).

LEMMA 3.3. – *For any* $\rho \in (0,2)$ *there exists* $c(\rho) > 0$ *such that*

$$\|(\lambda - B_n)^{1-\rho} S'_{\lambda,n}\|_{\mathcal{L}(X)} \leqslant c(\rho)\lambda^{-\rho}, \qquad \forall n, \ \forall \lambda > 0, \tag{3.6}$$

(*see* (2.13)) *with* $c(\rho)$ *independent of* λ, n, *and*

$$\lim_{n\to\infty} (\lambda - B_n)^{1-\rho} S'_{\lambda,n} = \int_{\gamma_2} v^{1-\rho}(B-\lambda+v)^{-1}(A-\lambda-v)^{-1}\,dv =: (\lambda - B)^{1-\rho} S'_\lambda \tag{3.7}$$

exists in the norm of $\mathcal{L}(X)$, $\forall \lambda > 0$.

PROOF. – Fix m, n, $\lambda > 0$. By (2.4) we can take $R > 0$ so large that

$$S'_{m,\lambda,n} = -\int_{\gamma_0^R} (B_n - \lambda + z)^{-1}(A_m - \lambda - z)^{-1}\,dz.$$

We can take $s > 0$ so small and r so large than the closed path γ'_2 (see (2.11)) surrounds $\sigma(\lambda - B_n)$. So we have

$$(\lambda - B_n)^{1-\rho} = -\int_{\gamma'_2} v^{1-\rho}(B_n - \lambda + v)^{-1}\,dv.$$

Therefore

$$(\lambda - B_n)^{1-\rho} S'_{m,\lambda,n} = \int_{\gamma_0^R}\int_{\gamma_2'} v^{1-\rho}(B_n - \lambda + v)^{-1}(B_n - \lambda + z)^{-1}(A_m - \lambda - z)^{-1}\, dv\, dz =$$

$$= \int_{\gamma_0^R}\int_{\gamma_2'} \frac{v^{1-\rho}}{z-v}\{(B_n - \lambda + v)^{-1} - (B_n - \lambda + z)^{-1}\}(A_m - \lambda - z)^{-1}\, dv\, dz.$$

Since

$$\int_{\gamma_2'} \frac{v^{1-\rho}}{z-v}\, dv = 0, \quad \forall z \in \gamma_0^R, \quad \text{and} \quad \int_{\gamma_0^R} \frac{1}{z-v}(A_m - \lambda - z)^{-1}\, dz = (A_m - \lambda - v)^{-1}, \quad \forall v \in \gamma_2'$$

we obtain

$$(\lambda - B_n)^{1-\rho} S'_{m,\lambda,n} = \int_{\gamma_2'} v^{1-\rho}(B_n - \lambda + v)^{-1}(A_m - \lambda - v)^{-1}\, dv.$$

Letting $s \to 0$ and $r \to \infty$ we have, since $0 \in \rho(A_m - \lambda) \cap \rho(B_n - \lambda)$,

$$(3.8) \qquad (\lambda - B_n)^{1-\rho} S'_{m,\lambda,n} = \int_{\gamma_2} v^{1-\rho}(B_n - \lambda + v)^{-1}(A_m - \lambda - v)^{-1}\, dv, \qquad \forall m,\ \forall n,\ \forall \lambda > 0.$$

As $m \to \infty$ we have

$$(\lambda - B_n)^{1-\rho} S'_{\lambda,n} = \int_{\gamma_2} v^{1-\rho}(B_n - \lambda + v)^{-1}(A - \lambda - v)^{-1}\, dv, \qquad \forall n,\ \forall \lambda > 0,$$

by (2.4) and the dominated convergence theorem. By (2.4) again we have

$$\|(\lambda - B_n)^{1-\rho} S'_{\lambda,n}\|_{\mathcal{L}(X)} \leq \int_{\gamma_2} \frac{c_A c_B |v|^{1-\rho}}{|v-\lambda|\,|v+\lambda|}\, dv = (\text{put } v = w\lambda) =$$

$$= \int_{\gamma_2} \frac{c_A c_B |w|^{1-\rho}\lambda^{1-\rho}}{|w-1|\,|w+1|\lambda^2}\lambda\, dw = c(\rho)\,\lambda^{-\rho}.$$

It also follows that

$$\lim_{n\to\infty}(\lambda - B_n)^{1-\rho}S'_{\lambda,n} = \int_{\gamma_2} v^{1-\rho}(B-\lambda+v)^{-1}(A-\lambda-v)^{-1}\,dv =: (\lambda - B)^{1-\rho}S'_{\lambda}$$

exists in $\mathcal{L}(X)$, $\forall\lambda > 0$, by the dominated convergence theorem.

COROLLARY 3.4. – *For any* $\rho \in (0,2)$ *there exists* $c(\rho) > 0$ *such that*

$$\|(\lambda - B_n)^{1-\rho}S'_{m,\lambda,n}\|_{\mathcal{L}(X)} \leqslant c(\rho)\,\lambda^{-\rho}, \qquad \forall m,\ \forall n,\ \forall\lambda > 0, \tag{3.9}$$

with $c(\rho)$ *independent of* λ, n, m, *and the following limits exist in the norm of* $\mathcal{L}(X)$, $\forall\lambda > 0$:

$$\begin{cases} \lim_{m\to\infty}(\lambda - B_n)^{1-\rho}S'_{m,\lambda,n} = (\lambda - B_n)^{1-\rho}S'_{\lambda,n}, \\ \lim_{n\to\infty}(\lambda - B_n)^{1-\rho}S'_{m,\lambda,n} = \int_{\gamma_2} v^{1-\rho}(B-\lambda+v)^{-1}(A_m-\lambda-v)^{-1}\,dv =: (\lambda - B)^{1-\rho}S'_{m,\lambda}, \\ \lim_{m\to\infty}\{\lim_{n\to\infty}(\lambda - B_n)^{1-\rho}S'_{m,\lambda,n}\} = \lim_{n\to\infty}(\lambda - B_n)^{1-\rho}S'_{\lambda,n} =: (\lambda - B)^{1-\rho}S'_{\lambda}. \end{cases} \tag{3.10}$$

PROOF. – This follows from (3.8), from the estimate

$$\|v^{1-\rho}(B_n - \lambda + v)^{-1}(A_m - \lambda - z)^{-1}\|_{\mathcal{L}(X)} \leqslant \frac{c_A c_B |v|^{1-\rho}}{|v-\lambda|\,|v+\lambda|}, \qquad \forall v \in \gamma_2,\ \forall m,\ \forall n,\ \forall\lambda > 0,$$

(see (2.4)) and from the dominated convergence theorem.

LEMMA 3.5. – *For any* $\rho \in (0,\delta)$ *and for any* $\bar\lambda > 0$, *there exists* $c = c(\rho,\bar\lambda) > 0$ *such that*

$$\begin{cases} \|J_{\lambda,n}(\lambda - B_n)^{\rho}\|_{\mathcal{L}(X)} \leqslant c\lambda^{\rho-\delta}, & \forall n,\ \forall\lambda > \bar\lambda, \\ \|J_{m,\lambda,n}(\lambda - B_n)^{\rho}\|_{\mathcal{L}(X)} \leqslant c\lambda^{\rho-\delta}, & \forall m,\ \forall n,\ \forall\lambda > \bar\lambda. \end{cases} \tag{3.11}$$

Moreover, the following limits exist in the norm of $\mathcal{L}(X)$, $\forall\lambda > \bar\lambda$:

$$\begin{cases} \lim_{n\to\infty} J_{m,\lambda,n}(\lambda - B_n)^{\rho} =: K_{m,\lambda}, \quad \lim_{m\to\infty} J_{m,\lambda,n}(\lambda - B_n)^{\rho} = J_{\lambda,n}(\lambda - B_n)^{\rho}, \\ \lim_{m\to\infty}\{\lim_{n\to\infty} J_{m,\lambda,n}(\lambda - B_n)^{\rho}\} = \lim_{n\to\infty}\{\lim_{m\to\infty} J_{m,\lambda,n}(\lambda - B_n)^{\rho}\} = \lim_{n\to\infty} J_{\lambda,n}(\lambda - B_n)^{\rho} =: K_{\lambda} \end{cases} \tag{3.12}$$

PROOF. – The proof of this lemma is rather long and it is given in Section 6.

In the next lemma we give a representation formula for strict solutions of

$$(A - \lambda)x + (B_n - \lambda)x = y,$$

which can be considered as a smoothed version of problem (1.1).

LEMMA 3.6. – *For any* $\rho \in (0, \delta)$ *there exists a* $\lambda^* > 0$ *such that* $\forall x \in D_A$ *we have*

$$x = U_{\lambda,n} y \qquad \forall \lambda > \lambda^*, \ \forall n \tag{3.13}$$

where $y := (A - \lambda)x + (B_n - \lambda)x$,

$$U_{\lambda,n} = (\lambda - B_n)^{\rho - 1}\{1 + (\lambda - B_n)^{-\rho} J_{\lambda,n}(\lambda - B_n)^{\rho}\}^{-1}(\lambda - B_n)^{1-\rho} S'_{\lambda,n} + + (\lambda - B_n)^{-1}\{1 + J_{\lambda,n}\}^{-1} M_{\lambda,n}. \tag{3.14}$$

Furthermore there exists $c = c(\rho, \lambda^*) > 0$ *such that*

$$\|U_{\lambda,n}\|_{\mathcal{L}(X)} \leq c\lambda^{-1}, \qquad \forall \lambda > \lambda^*, \ \forall n \tag{3.15}$$

and

$$U_\lambda := \lim_{n \to \infty} U_{\lambda,n} \tag{3.16}$$

exists in the norm of $\mathcal{L}(X)$, $\forall \lambda > \lambda^*$; *hence*

$$\|U_\lambda\|_{\mathcal{L}(X)} \leq c\lambda^{-1}, \qquad \forall \lambda > \lambda^*, \tag{3.17}$$

and we have (see (3.2), (3.7), (3.12))

$$U_\lambda = (\lambda - B)^{\rho-1}\{1 + (\lambda - B)^{-\rho} K_\lambda\}^{-1}(\lambda - B)^{1-\rho} S'_\lambda + (\lambda - B)^{-1}\{1 + J_\lambda\}^{-1} M_\lambda. \tag{3.18}$$

PROOF. – Fix $\bar{\lambda} > 0$ and consider $\lambda > \bar{\lambda}$. Suppose $x \in D_A$ and $(A - \lambda)x + (B_n - - \lambda)x =: y$. A slight modification of the passages of [7, Prop. 2.1-2.3] shows that

$$S_{\lambda,n} y \in D_A \quad \text{and} \quad (A - \lambda)x + J_{\lambda,n}(A - \lambda)x = (A - \lambda)S_{\lambda,n} y$$

(see (2.12)), so that

$$y - (B_n - \lambda)x + J_{\lambda,n} y - J_{\lambda,n}(B_n - \lambda)x = (A - \lambda)S_{\lambda,n} y,$$

$$(\lambda - B_n)x + J_{\lambda,n}(\lambda - B_n)x = (A - \lambda)S_{\lambda,n} y - y - J_{\lambda,n} y. \tag{3.19}$$

We now prove that

$$(A - \lambda)S_{\lambda,n} y - y - J_{\lambda,n} = (\lambda - B_n)S'_{\lambda,n} + M_{\lambda,n}. \tag{3.20}$$

Using (2.4), (2.5) it is easily seen that we can take $s > 0$ so small and r so large that we have

$$S_{\lambda,n} = -\oint_{\gamma'_0} (A - \lambda - z)^{-1}(B_n - \lambda + z)^{-1}\, dz,$$

$$J_{\lambda,n} = -\oint_{\gamma'_0} z(A - \lambda)(A - \lambda - z)^{-1}[(A - \lambda)^{-1}; (B_n - \lambda + z)^{-1}]\, dz.$$

Then

$$(A-\lambda) S_{\lambda,n} y - y - J_{\lambda,n} y = -\oint_{\gamma_0'} (A-\lambda)(A-\lambda-z)^{-1}(B_n-\lambda+z)^{-1}\,dz + \oint_{\gamma_0'} (B_n-\lambda+z)^{-1}\,dz +$$

$$+\oint_{\gamma_0'} z(A-\lambda)(A-\lambda-z)^{-1}[(A-\lambda)^{-1}; (B_n-\lambda+z)^{-1}]\,dz =$$

$$=\oint_{\gamma_0'} \{-(A-\lambda)(A-\lambda-z)^{-1}(B_n-\lambda+z)^{-1}+(B_n-\lambda+z)^{-1}+$$

$$+z(A-\lambda-z)^{-1}(B_n-\lambda+z)^{-1}-z(A-\lambda)(A-\lambda-z)^{-1}(B_n-\lambda+z)^{-1}(A-\lambda)^{-1}\}\,dz =$$

$$=-\oint_{\gamma_0'} z(A-\lambda)(A-\lambda-z)^{-1}(B_n-\lambda+z)^{-1}(A-\lambda)^{-1}\,dz = -\oint_{\gamma_0'} z(B_n-\lambda+z)^{-1}(A-\lambda-z)^{-1}\,dz -$$

$$-\oint_{\gamma_0'} z[(A-\lambda)(A-\lambda-z)^{-1}; (B_n-\lambda+z)^{-1}](A-\lambda)^{-1}\,dz =$$

$$=\oint_{\gamma_0'} (B_n-\lambda)(B_n-\lambda+z)^{-1}(A-\lambda-z)^{-1}\,dz - \oint_{\gamma_0'} z^2[(A-\lambda-z)^{-1}; (B_n-\lambda+z)^{-1}](A-\lambda)^{-1}\,dz =$$

$$=(\lambda-B_n)\oint_{\gamma_0'} -(B_n-\lambda+z)^{-1}(A-\lambda-z)^{-1}\,dz -$$

$$-\oint_{\gamma_0'} z^2(A-\lambda)(A-\lambda-z)^{-1}[(A-\lambda)^{-1}; (B_n-\lambda+z)^{-1}](A-\lambda-z)^{-1}\,dz .$$

As before we can see that these integrals can be calculated on γ_0 instead of γ_0', so that (3.20) is proved.

Then applying $(\lambda - B_n)^{-\rho}$ to both sides of (3.19) we obtain

$$(\lambda-B_n)^{1-\rho}x+(\lambda-B_n)^{-\rho}J_{\lambda,n}(\lambda-B_n)^{\rho}(\lambda-B_n)^{1-\rho}x=(\lambda-B_n)^{1-\rho}S'_{\lambda,n}y+(\lambda-B_n)^{-\rho}M_{\lambda,n}y\,,$$

$$\{1+(\lambda-B_n)^{-\rho}J_{\lambda,n}(\lambda-B_n)^{\rho}\}(\lambda-B_n)^{1-\rho}x=(\lambda-B_n)^{1-\rho}S'_{\lambda,n}y+(\lambda-B_n)^{-\rho}M_{\lambda,n}y\,.$$

By (3.1), (3.11), (3.3) we can now choose $\lambda^* > \bar{\lambda}$ so large that $\forall \lambda > \lambda^*$ we have

$$\|(\lambda-B_n)^{-\rho}J_{\lambda,n}(\lambda-B_n)^{\rho}\|_{\mathcal{L}(X)} \le 1/2, \quad \forall n,$$
$$\|J_{\lambda,n}\|_{\mathcal{L}(X)} \le 1/2, \quad \forall n.$$

For these λ the operators $\{1+(\lambda-B_n)^{-\rho}J_{\lambda,n}(\lambda-B_n)^{\rho}\}$ and $\{1+J_{\lambda,n}\}$ are invertible and we obtain

$$x=(\lambda-B_n)^{\rho-1}\{1+(\lambda-B_n)^{-\rho}J_{\lambda,n}(\lambda-B_n)^{\rho}\}^{-1}(\lambda-B_n)^{1-\rho}S'_{\lambda,n}\,y+$$

$$+(\lambda-B_n)^{\rho-1}\{1+(\lambda-B_n)^{-\rho}J_{\lambda,n}(\lambda-B_n)^{\rho}\}^{-1}(\lambda-B_n)^{-\rho}M_{\lambda,n}\,y=$$

$$=(\lambda-B_n)^{\rho-1}\{1+(\lambda-B_n)^{-\rho}J_{\lambda,n}(\lambda-B_n)^{\rho}\}^{-1}(\lambda-B_n)^{1-\rho}S'_{\lambda,n}\,y+$$

$$+(\lambda-B_n)^{-1}\{1+J_{\lambda,n}\}^{-1}M_{\lambda,n}\,y=U_\lambda\,y\,.$$

Furthermore we have

$$\|\{1+(\lambda-B_n)^{-\rho}J_{\lambda,n}(\lambda-B_n)^{\rho}\}^{-1}\|_{\mathcal{L}(X)}\leq 2 \quad \text{and} \quad \|\{1+J_{\lambda,n}\}^{-1}\|_{\mathcal{L}(X)}\leq 2,\ \forall n,\ \forall\lambda\geq\lambda^*.$$

By (3.1), (3.6), (2.4), (3.4) we have $\|x\|\leq c(\rho,\bar{\lambda})\,\lambda^{-1}\|y\|$. So (3.13), (3.14), (3.15) are proved. (3.16), (3.18) follow from (3.2), (3.12), (3.10), (3.5), and (3.17) follows from (3.15), (3.16).

COROLLARY 3.7. – *For any* $\rho\in(0,\delta)$ *there exists a* $\lambda^*>0$ *such that* $\forall x\in X$ *we have*

(3.21) $$x=U_{m,\lambda,n}\,y \qquad \forall\lambda>\lambda^*,\ \forall n,\ \forall m,$$

where: $y:=(A_m-\lambda)x+(B_n-\lambda)x$,

(3.22) $$U_{m,\lambda,n}=(\lambda-B_n)^{\rho-1}\{1+(\lambda-B_n)^{-\rho}J_{m,\lambda,n}(\lambda-B_n)^{\rho}\}^{-1}(\lambda-B_n)^{1-\rho}S'_{m,\lambda,n}+$$
$$+(\lambda-B_n)^{-1}\{1+J_{m,\lambda,n}\}^{-1}M_{m,\lambda,n}\,.$$

Furthermore there exists a $c=c(\rho,\lambda^*)>0$ *such that*

(3.23) $$\|U_{m,\lambda,n}\|_{\mathcal{L}(X)}\leq c\lambda^{-1},\qquad \|U_{m,\lambda}\|_{\mathcal{L}(X)}\leq c\lambda^{-1},\qquad \forall m,\ \forall n,\ \forall\lambda>\lambda^*,$$

where (see (3.2), (3.10), (3.12))

(3.24) $$U_{m,\lambda}:=(\lambda-B)^{\rho-1}\{1+(\lambda-B)^{-\rho}K_{m,\lambda}\}^{-1}(\lambda-B)^{1-\rho}S'_{m,\lambda}+$$
$$+(\lambda-B)^{-1}\{1+J_{m,\lambda}\}^{-1}M_{m,\lambda}$$

and the following limits exist in the norm of $\mathcal{L}(X)$:

(3.25) $$\lim_{n\to\infty}U_{m,\lambda,n}=U_{m,\lambda}\,,\qquad \lim_{m\to\infty}U_{m,\lambda,n}=U_{\lambda,n}\,,\qquad \lim_{m\to\infty}U_{m,\lambda}=U_\lambda\,,$$
$$\forall\lambda>\lambda^*,\ \forall n,\ \forall m.$$

PROOF. – We can adapt the proof of Lemma 3.6, remarking that all the estimates we obtain are uniform with respect to m.

4. – A priori estimates for strict solutions. Existence, uniqueness and regularity of strong solutions.

The first theorem of this section shows that hypotheses (2.2) and (2.3) imply some regularity property for strict solutions of problem (1.1) (see Definition 1.1), under some density assumptions on the domains of A and B.

THEOREM 4.1. – *If D_B is dense in X and $x \in D_A \cap D_B$ then*

$$Bx \in \overline{D_A}. \tag{4.1}$$

If D_A is dense in X and $x \in D_A \cap D_B$ then

$$Ax \in \overline{D_B}. \tag{4.2}$$

PROOF. – First we prove (4.1). By (2.3) we have

$$\|(A-\lambda_0)(A-v)^{-1}[(A-\lambda_0)^{-1};\ B(B-z)^{-1}]\|_{\mathcal{L}(X)} \leq c_{AB}\sum_{1}^{k}{}_i\,|v|^{\alpha_i-1}|z|^{-\beta_i}.$$

Taking $z = n$, $v = \lambda_0 + n$, and recalling that $B = nB(n-B)^{-1}$ we obtain

$$\|(A-\lambda_0)(A-\lambda_0-n)^{-1}[(A-\lambda_0)^{-1};B_n\|_{\mathcal{L}(X)} \leq c_{AB}\sum_{1}^{k}{}_i\, n\,|\lambda_0+n|^{\alpha_i-1}|n|^{-\beta_i} = O(n^{-\delta}). \tag{4.3}$$

Then

$$B_n x = B_n(A-\lambda_0)^{-1}(A-\lambda_0)x = (A-\lambda_0)^{-1}B_n(A-\lambda_0)x - [(A-\lambda_0)^{-1};B_n](A-\lambda_0)x$$

and

$$(A-\lambda_0)(A-\lambda_0-n)^{-1}B_n x = (A-\lambda_0-n)^{-1}B_n(A-\lambda_0)x -$$
$$-(A-\lambda_0)(A-\lambda_0-n)^{-1}[(A-\lambda_0)^{-1};\ B_n](A-\lambda_0)x.$$

So we finally have

$$B_n x = -n(A-\lambda_0-n)^{-1}B_n x + (A-\lambda_0-n)^{-1}B_n(A-\lambda_0)x -$$
$$-(A-\lambda_0)(A-\lambda_0-n)^{-1}[(A-\lambda_0)^{-1};\ B_n](A-\lambda_0)x =: I_{1n} + I_{2n} + I_{3n}.$$

Since $B_n x \to Bx$, $I_{1n} + I_{2n} \in D_A$, $I_{3n} \to 0$ by (4.3), we have $Bx \in \overline{D_A}$. (4.2) can be proved in a similar way.

Under some assumptions on the domains of A and B, the next theorem gives an a priori estimate for strict solutions of problem (1.1), and an explicit representation formula for them.

THEOREM 4.2. – *Let D_A or D_B be dense in X. There exists $\lambda^* > 0$ such that whenever $\lambda > \lambda^*$, $x \in D_A \cap D_B$ and $y := (A - \lambda)x + (B - \lambda)x$ we have*

$$x = U_\lambda y \tag{4.4}$$

(where U_λ is given by (3.18)) *so that*

$$\|x\| \leq c(\lambda^*)\lambda^{-1}\|y\| \quad \forall \lambda > \lambda^*, \tag{4.5}$$

with $c(\lambda^*) > 0$ independent of x and λ.

PROOF. – Take any $\rho \in (0, \delta)$ and choose λ^* as in Lemma 3.6. First suppose D_B dense in X. x satisfies $(A - \lambda)x + (B_n - \lambda)x = y - Bx + B_n x$, $\forall \lambda > \lambda^*$. By (3.13) we have $x = U_{\lambda, n}(y - Bx + B_n x)$ and by (3.16) we have (4.4), as $n \to \infty$.

Now suppose that D_A is dense in X. We first show that

$$\rho(A_m + B_n) \supset (2\lambda^*, \infty) \quad \forall m, \ \forall n. \tag{4.6}$$

In fact, if we have $2\mu \in \rho(A_m + B_n)$ for a $\mu > \lambda^*$, then $\forall y \in X$ the vector $z := (A_m + B_n - 2\mu)^{-1} y$ is a strict solution of $(A_m - \mu)z + (B_n - \mu)z = y$, so that by [7, Prop. 2.4] we have $z = (A_m - \mu)^{-1}(1 + J_{m, \mu, n})^{-1}(A_m - \mu)S_{m, \mu, n} y$ and, as in [7, Lemma 3.3], $\|z\| \leq c_{m, n}\mu^{-1}\|y\|$. So we have proved that $\mu > \lambda^*$, $2\mu \in \rho(A_m + B_n)$ implies $\|(A_m + B_n - 2\mu)^{-1}\|_{\mathcal{L}(X)} \leq c_{m, n}\mu^{-1}$. By a standard argument (see [5, Th. 2.1] or [7, Prop. 3.1]) this implies (4.6).

Now take any $\lambda > \lambda^*$ and $\forall m$, $\forall n$, let $z_{m, n} := (A_m + B_n - 2\mu)^{-1}(y - Ax + A_m x)$ be the strict solution of $(A_m - \lambda)z_{m, n} + (B_n - \lambda)z_{m, n} = y - Ax + A_m x$. Then by [7, Prop. 2.4] $z_{m, n}$ is given by

$$z_{m, n} = (A_m - \lambda)^{-1}(1 + J_{m, \lambda, n})^{-1}(A_m - \lambda)S_{m, \lambda, n}(y - Ax + A_m x), \qquad \forall m, \ \forall n, \ \forall \lambda > \lambda^*.$$

On the other hand x satisfies $(A_m - \lambda)x + (B - \lambda)x = y - Ax + A_m x$ so that by [7, Prop. 2.4]

$$x = (A_m - \lambda)^{-1}(1 + J_{m, \lambda})^{-1}(A_m - \lambda)S_{m, \lambda}(y - Ax + A_m x), \qquad \forall m, \ \forall n, \ \forall \lambda > \lambda^*.$$

It follows that $z_{m, n} \to x$ as $n \to \infty$, $\forall m$, $\forall \lambda > \lambda^*$.

By (3.21) we have $z_{m, n} = U_{m, \lambda, n}(y - Ax + A_m x)$, $\forall m$, $\forall n$, $\forall \lambda > \lambda^*$. By (3.25), letting $n \to \infty$, we see that $x = U_{m, \lambda}(y - Ax + A_m x)$, $\forall m$, $\forall \lambda > \lambda^*$. Letting now $m \to \infty$ we have (4.4), by (3.25). Finally (4.5) follows from (3.17).

REMARK 4.3. – For any linear operator C in X, denote by $\overline{C}$ the closure of its graph $\mathcal{G}(C)$ in $X \times X$ (C need not to be closable in the usual sense). If, for a $\lambda \in \mathbf{C}$, $(\overline{C} - \lambda)^{-1} := \{(x, y) \in X \times X : (y, x + \lambda y) \in \overline{C}\}$ is the graph of a bounded linear operator in X, we write $\lambda \in \rho(\overline{C})$ and identify $(\overline{C} - \lambda)^{-1}$ with that operator (see [5, Sect. 2.2]). Now suppose, as in the previous theorem, that D_A (or D_B) is dense in X. By [7, Th. 4.1] and by (4.5) we can apply [5, Th. 2.7] and we obtain $\rho(\overline{A + B}) \supset (2\lambda^*, \infty)$ and $\|(\overline{A + B} - 2\lambda)^{-1}\|_{\mathcal{L}(X)} \leq c\lambda^{-1}$, $\forall \lambda > \lambda^*$. It is also easy to prove that $(\overline{A + B} - 2\lambda)^{-1} = U_\lambda$, $\forall \lambda > \lambda^*$.

REMARK 4.4. – If $D_A \cap D_B$ is dense in X, then $A+B$ is closable, $\rho(\overline{A+B}) \supset (2\lambda^*, \infty)$ and $(\overline{A+B} - 2\lambda)^{-1} = U_\lambda$, $\forall \lambda > \lambda^*$. This follows from [5, Th. 2.1] and from the previous remark.

Now recall definition (1.1). Under some conditions on the domains of the operators A and B we can prove the existence and uniqueness of strong solutions for large values of λ and any datum $y \in X$.

THEOREM 4.5. – *Let D_A or D_B be dense in X. Then there exists $\lambda^* > 0$ such that for any $y \in X$ there exists a unique strong solution x of*

$$(A-\lambda)x + (B-\lambda)x = y, \qquad \lambda > \lambda^*$$

given by $x = U_\lambda y$ (see (3.18)).

PROOF. – Take λ^* as in Lemma 3.6. Take $y \in X$. Since D_A (resp. D_B) is dense in X, we can choose a sequence $y_n \in D_A$ (resp. D_B) such that $y_n \to y$ in X. By [7, Th. 4.1] there exists a unique strict solution x_n of

$$(A-\lambda)x_n + (B-\lambda)x_n = y_n, \qquad \forall \lambda > \lambda^*.$$

By (4.4) we have $x_n = U_\lambda y_n$, so that, $\forall \lambda > \lambda^*$, x_n converges to $x := U_\lambda y$. Since the strong solution is given by $U_\lambda y$, it is unique.

Now we study regularity properties of strong solutions. As in [5] we can prove that strong solutions belong to $D_A(1, \infty) \cap D_B(1, \infty)$ (see (1.6)).

THEOREM 4.6. – *Suppose D_A or D_B is dense in X. Then there exists $\lambda^* > 0$ such that*

$$U_\lambda \in \mathcal{L}(X, D_A(1, \infty)), \qquad \|U_\lambda\|_{\mathcal{L}(X, D_A(1,\infty))} \le c(\lambda^*), \qquad \forall \lambda > \lambda^* \tag{4.7}$$

$$U_\lambda \in \mathcal{L}(X, D_B(1, \infty)), \qquad \|U_\lambda\|_{\mathcal{L}(X, D_B(1,\infty))} \le c(\lambda^*), \qquad \forall \lambda > \lambda^* \tag{4.8}$$

(*see* (3.18)) *with $c(\lambda^*)$ independent of λ. So $\forall y \in X$ the strong solution x of*

$$(A-\lambda)x + (B-\lambda)x = y, \qquad \lambda > \lambda^*$$

belongs to $D_A(1, \infty) \cap D_B(1, \infty)$ and satisfies

$$\|x\|_{D_A(1,\infty)} + \|x\|_{D_B(1,\infty)} \le c(\lambda^*)\|y\|. \tag{4.9}$$

PROOF. – Take any $\rho \in (0, \delta)$, choose λ^* as in Lemma 3.6 and fix $\lambda > \lambda^*$. We have (see (3.14))

$$U_{\lambda,n} = (\lambda - B_n)^{\rho-1}\{1 + (\lambda - B_n)^{-\rho} J_{\lambda,n}(\lambda - B_n)^\rho\}^{-1} \cdot$$

$$\cdot (\lambda - B_n)^{1-\rho} S'_{\lambda,n} + (\lambda - B_n)^{-1}\{1 + J_{\lambda,n}\}^{-1} M_{\lambda,n} = S'_{\lambda,n} - (\lambda - B_n)^{-1} J_{\lambda,n}(\lambda - B_n)^\rho \cdot$$

$$\cdot \{1 + (\lambda - B_n)^{-\rho} J_{\lambda,n}(\lambda - B_n)^\rho\}^{-1} \cdot (\lambda - B_n)^{1-\rho} S'_{\lambda,n} + (\lambda - B_n)^{-1}\{1 + J_{\lambda,n}\}^{-1} M_{\lambda,n}$$

so that

$$U_\lambda := \lim_{n \to \infty} U_{\lambda, n} = S'_\lambda + (\lambda - B)^{-1} Q_\lambda \tag{4.10}$$

where

$$Q_\lambda := -K_\lambda \{1 + (\lambda - B)^{-\rho} K_\lambda\}^{-1} (\lambda - B)^{1-\rho} S'_\lambda + \{1 + J_\lambda\}^{-1} M_\lambda$$

and $\|Q_\lambda\|_{\mathcal{L}(X)} \leq c(\lambda^*) \lambda^{-\delta} \leq c(\lambda^*)$ independent of λ, as in Lemma 3.6.

In order to prove (4.8) it is therefore enough to prove $S'_\lambda \in \mathcal{L}(X, D_B(1, \infty))$, $\|S'_\lambda\|_{\mathcal{L}(X, D_B(1, \infty))} \leq c(\lambda^*)$. This can be done as in [5, Lemma 6.4].

We now prove (4.7). Clearly, it is sufficient to consider $t \geq 1$ in (1.6). Since D_A (resp. D_B) is dense in X, it is enough to prove $\|A^2 (A - t)^{-2} U_\lambda y\| \leq c(\lambda^*) t^{-1} \|y\|$, $\forall y \in D_A$ (resp. D_B). By [7, Th. 4.1] there exists a unique solution $x \in D_A \cap D_B$ of the problem $(A - \lambda) x + (B - \lambda) x = y$. By (4.4) we have $x = U_\lambda y$. We then obtain

$$A^2 (A - t)^{-2} x = (A - \lambda) A (A - t)^{-2} x + \lambda A (A - t)^{-2} x =$$

$$= A (A - t)^{-2} (y - (B - \lambda) x) + \lambda A (A - t)^{-2} x .$$

Since $\|A(A - t)^{-2} y\| \leq ct^{-1} \|y\|$ and $\|\lambda A(A - t)^{-2} x\| \leq \lambda c t^{-1} \|x\| \leq ct^{-1} \|y\|$, by (4.5), it is enough to prove $\|A(A - t)^{-2} (\lambda - B) x\| \leq ct^{-1} \|y\|$.

By (4.10) this reduces to proving

$$\|A(A - t)^{-2} (\lambda - B) S'_\lambda y\| \leq ct^{-1} \|y\| . \tag{4.11}$$

With passages similar to [7, Prop. 2.2, 3.2] we can show that $S'_\lambda y \in D_B$ and $(\lambda - B_n) S'_{\lambda, n} y \to (\lambda - B) S'_\lambda y$, so that

$$A(A - t)^{-2} (\lambda - B_n) S'_{\lambda, n} \to A(A - t)^{-2} (\lambda - B) S'_\lambda y . \tag{4.12}$$

Observe that

$$-A(A - t)^{-2} (\lambda - B_n) S'_{\lambda, n} y = \fint_{\gamma_0} A(A - t)^{-2} (\lambda - B_n)(B_n - \lambda + z)^{-1} (A - \lambda - z)^{-1} y \, dz =$$

$$= \fint_{\gamma_0} (\lambda - B_n)(B_n - \lambda + z)^{-1} A(A - t)^{-2} (A - \lambda - z)^{-1} \, dz +$$

$$+ \fint_{\gamma_0} [A(A - t)^{-2}; (\lambda - B_n)(B_n - \lambda + z)^{-1}] (A - \lambda - z)^{-1} \, dz =: \fint_{\gamma_0} P_{1n} y \, dz + \fint_{\gamma_0} P_{2n} y \, dz .$$

Define $P_2 := [A(A-t)^{-2}; (\lambda - B)(B-\lambda+z)^{-1}](A-\lambda-z)^{-1}$. We will show

$$\left\| \oint_{\gamma_0} P_{1n} y\, dz \right\|_{\mathcal{L}(X)} \leq c(\lambda^*) t^{-1} \|y\|, \tag{4.13}$$

$$\left\| \oint_{\gamma_0} P_2 y\, dz \right\|_{\mathcal{L}(X)} \leq c(\lambda^*) t^{-1} \|y\|, \tag{4.14}$$

$$\oint_{\gamma_0} P_{2n} y\, dz \to \oint_{\gamma_0} P_2 y\, dz. \tag{4.15}$$

Together with (4.12) this proves (4.11).

(4.13) can be proved by passages similar to [5, Lemma 6.4].

Observe that

$$\begin{aligned}(4.16)\quad & [A(A-t)^{-2}; (B-\lambda+z)^{-1}] = \\ &= A(A-t)^{-1}[(A-t)^{-1}; (B-\lambda+z)^{-1}] + [(A-t)^{-1}; (B-\lambda+z)^{-1}]\, t(A-t)^{-1}.\end{aligned}$$

It follows that

$$\begin{aligned}(4.17)\quad & P_2 = z[A(A-t)^{-2}; (B-\lambda+z)^{-1}](A-\lambda-z)^{-1} = A(A-t)^{-1} z[(A-t)^{-1}; (B-\lambda+z)^{-1}]\cdot \\
&\cdot (A-\lambda-z)^{-1} + z[(A-t)^{-1}; (B-\lambda+z)^{-1}](A-\lambda-z)^{-1} t(A-t)^{-1} = \\
&= A(A-t)^{-1} z(A-\lambda)(A-t)^{-1}[(A-\lambda)^{-1}; (B-\lambda+z)^{-1}](A-\lambda)(A-t)^{-1}(A-\lambda+z)^{-1} + \\
&+ z(A-\lambda)(A-t)^{-1}[(A-\lambda)^{-1}; (B-\lambda+z)^{-1}](A-\lambda)(A-t)^{-1}(A-\lambda-z)^{-1} t(A-t)^{-1} = \\
&= A(A-t)^{-1} z(A-\lambda)(A-\lambda-z)^{-1}[(A-\lambda)^{-1}; (B-\lambda+z)^{-1}](A-\lambda)(A-t)^{-1}(A-\lambda-z)^{-1} + \\
&+ A(A-t)^{-1} z(A-\lambda)(A-t)^{-1}(A-\lambda-z)^{-1}[(A-\lambda)^{-1}; (B-\lambda+z)^{-1}](A-\lambda)(A-t)^{-1}\cdot \\
&\cdot (A-\lambda-z)^{-1}(t-\lambda-z) + z(A-\lambda)(A-\lambda-z)^{-1}[(A-\lambda)^{-1}; (B-\lambda+z)^{-1}]\cdot \\
&\cdot (A-\lambda)(A-t)^{-1}(A-\lambda-z)^{-1} t(A-t)^{-1} + z(A-\lambda)(A-t)^{-1}(A-\lambda-z)^{-1}\cdot \\
&\cdot [(A-\lambda)^{-1}; (B-\lambda+z)^{-1}](A-\lambda)(A-t)^{-1}(A-\lambda-z)^{-1} t(A-t)^{-1}(t-\lambda-z) = \\
&= A(A-t)^{-1} z(A-\lambda)(A-\lambda-z)^{-1}[(A-\lambda)^{-1}; (B-\lambda+z)^{-1}](A-\lambda)(A-t)^{-1}(A-\lambda-z)^{-1} + \\
&+ A(A-t)^{-2} z(A-\lambda)(A-\lambda-z)^{-1}[(A-\lambda)^{-1}; (B-\lambda+z)^{-1}](A-\lambda)(A-t)^{-1} - \\
&- A(A-t)^{-2} z(A-\lambda)(A-\lambda-z)^{-1}[(A-\lambda)^{-1}; (B-\lambda+z)^{-1}](A-\lambda)(A-\lambda-z)^{-1} + \\
&+ z(A-\lambda)(A-\lambda-z)^{-1}[(A-\lambda)^{-1}; (B-\lambda+z)^{-1}](A-\lambda)(A-\lambda-z)^{-1} t(A-t)^{-2} + \\
&+ (A-t)^{-1} z(A-\lambda)(A-\lambda-z)^{-1}[(A-\lambda)^{-1}; (B-\lambda+z)^{-1}](A-\lambda)(A-t)^{-1} t(A-t)^{-1} -\end{aligned}$$

$$-(A-t)^{-1}z(A-\lambda)(A-\lambda-z)^{-1}[(A-\lambda)^{-1};(B-\lambda+z)^{-1}]\cdot$$

$$\cdot(A-\lambda)(A-\lambda-z)^{-1}t(A-t)^{-1}=:\sum_1^6{}_i P^{(i)}(z,\lambda,t).$$

By (2.5) we have

$$\|P^{(1)}(z,\lambda,t)\|_{\mathcal{L}(X)}+\|P^{(3)}(z,\lambda,t)\|_{\mathcal{L}(X)}\leqslant$$

$$\leqslant (1+c_A)\sum_1^k{}_i \frac{|z|c(\lambda^*)}{|z+\lambda|^{1-\alpha_i}|z-\lambda|^{1+\beta_i}}\left(1+\frac{c_A|z|}{|z+\lambda|}\right)\frac{c_A}{t}=:\psi_1(z,\lambda,t)$$

and

$$\|P^{(4)}(z,\lambda,t)\|_{\mathcal{L}(X)}+\|P^{(6)}(z,\lambda,t)\|_{\mathcal{L}(X)}\leqslant$$

$$\leqslant \frac{c_A^2}{t}\sum_1^k{}_i \frac{|z|c(\lambda^*)}{|z+\lambda|^{1-\alpha_i}|z-\lambda|^{1+\beta_i}}\left(1+\frac{c_A|z|}{|z+\lambda|}\right)=:\psi_2(z,\lambda,t).$$

Since

$$P^{(2)}(z,\lambda,t)=A(A-t)^{-2}z(A-t)(A-\lambda-z)^{-1}[(A-t)^{-1};(B-\lambda+z)^{-1}]$$

and

$$P^{(5)}(z,\lambda,t)=(A-t)^{-1}z(A-t)(A-\lambda-z)^{-1}[(A-t)^{-1};(B-\lambda+z)^{-1}]\,t(A-t)^{-1}$$

we have, by [7, Lemma 1.2], (recall that $t\geqslant 1$),

$$\|P^{(2)}(z,\lambda,t)\|_{\mathcal{L}(X)}\leqslant(1+c_A)\sum_1^k{}_i \frac{c_{AB}|z|}{|z+\lambda|^{1-\alpha_i}|z-\lambda|^{1+\beta_i}}\left(1+\frac{c_A|t-\lambda_0|}{t}\right)\frac{c_4}{t}=:\psi_3(z,\lambda,t),$$

$$\|P^{(5)}(z,\lambda,t)\|_{\mathcal{L}(X)}\leqslant\frac{c_A^2}{t}\sum_1^k{}_i \frac{c_{AB}|z|}{|z+\lambda|^{1-\alpha_i}|z-\lambda|^{1+\beta_i}}\left(1+\frac{c_A|t-\lambda_0|}{t}\right)=:\psi_4(z,\lambda,t).$$

So we have

$$\left\|\oint_{\gamma_0}P_2\,dz\right\|_{\mathcal{L}(X)}\leqslant\oint_{\gamma_0}(\psi_1+\psi_2+\psi_3+\psi_4)(z,\lambda,t)\,dz=(\text{putting } z=v\lambda)\leqslant$$

$$\leqslant c(\lambda^*)\,\lambda^{-\delta}\oint_{\gamma_0}(\psi_1+\psi_2+\psi_3+\psi_4)(v,1,t)\,dv<c\lambda^{-\delta}t^{-1}\leqslant ct^{-1}$$

and (4.14) is proved.

To prove (4.15) we remark that $P_{2n}=\sum_1^6{}_i P_n^{(i)}(z,\lambda,t)$, where $P_n^{(i)}(z,\lambda,t)$ is the same as $P^{(i)}(z,\lambda,t)$ with B replaced by B_n (in fact, the passages leading from (4.16) to (4.17) still hold in this case). By (4.17) we have

$$\left\| \sum_1^6 {}_i P_n^{(i)}(z, \lambda, t) \right\|_{\mathcal{L}(X)} \leq c(\lambda, t) \sum_1^k {}_i \frac{|z|}{|z+\lambda|^{1-\alpha_i} |z-\lambda|^{1+\beta_i}}$$

with $c(\lambda, t)$ independent of n. Then (4.15) follows from the dominated convergence theorem.

Finally (4.9) follows from (4.7), (4.8) and Theorem 4.5.

5. - Applications.

Suppose E is a complex Banach space, T.013, λ.0130. We want to solve the problem

$$u'(t) = \Lambda(t) u(t) - \lambda u(t) + f(t), \qquad t \in [0, T], \tag{5.1}$$

$$u(0) = 0, \tag{5.2}$$

where $u: [0, T] \to E$ is the unknown function, $f: [0, T] \to E$ is the datum and, for any $t \in [0, T]$, $\Lambda(t)$ is a linear operator in E, with domain $D_{\Lambda(t)}$, possibly varying with t and not necessarily dense in E.

In order to apply the abstract results of the preceding sections to problem (5.1) we choose X to be one of the following spaces, with their usual norms:

$$X = L^p([0, T], E), \quad p \in [1, \infty); \quad \text{or } X = C([0, T], E); \quad \text{or } X = C_0([0, T], E), \tag{5.3}$$

$C_0([0, T], E)$ being the subspace of $C([0, T], E)$ consisting of functions that vanish in $t = 0$. We assume $f \in X$.

By *strict solution* of (5.1), (5.2) we mean a function $u \in X$ such that $u' \in X$, $u(t) \in D_{\Lambda(t)}$, $\forall t \in [0, T]$, $t \to \Lambda(t) u(t)$ belongs to X, and (5.1), (5.2) are satisfied (in case $X = L^p([0, T], E)$ we only require that $u(t) \in D_{\Lambda(t)}$ a.e. in $[0, T]$ and that (5.1) is satisfied a.e. in $[0, T]$).

We call u a *strong solution* of (5.1), (5.2) if there exist sequences $u_n, f_n \in X$ such that $u_n \to u$ in X, $f_n \to f$ in X and u_n is a strict solution of

$$\begin{cases} u_n'(t) = \Lambda(t) u_n(t) - \lambda u_n(t) + f_n(t), & t \in [0, T], \\ u_n(0) = 0. \end{cases}$$

We now define operators A and B in X

$$Bu := -u', \qquad D_B := \{u \in X \colon u' \in X, \ u(0) = 0\}; \tag{5.4}$$

$$\begin{cases} (Au)(t) := \Lambda(t) u(t), \\ D_A := \{u \in X \colon u(t) \in D_{\Lambda(t)}, \ \forall t \in [0, T], \ t \to \Lambda(t) u(t) \text{ belongs to } X\}, \end{cases} \tag{5.5}$$

(in case $X = L^p([0, T], E)$ we only require $u(t) \in D_{\Lambda(t)}$ a.e. in $[0, T]$).

With these definitions strict (resp. strong) solutions of problem (5.1), (5.2) are precisely the strict (resp. strong) solutions of problem (1.1), according to Definition 1.1. In order to verify assumptions (2.2), (2.3) for A and B we assume (cf. [2])

(H1)′ *$\Lambda(t)$ is a linear operator in E for any $t \in [0, T]$ and there exist $M > 0$, $\vartheta_\Lambda \in (0, \pi/2)$ such that*

$$\rho(\Lambda(t)) \supset \Sigma_{\vartheta_\Lambda}, \quad \forall t \in [0,T] \quad \text{and} \quad \|(\Lambda(t)-z)^{-1}\|_{\mathcal{L}(X)} \leq \frac{M}{|z|}, \quad \forall z \in \Sigma_{\vartheta_\Lambda}, \forall t \in [0,T]. \tag{5.6}$$

(H2)′ *There exist $\lambda_0 > 0$, $k \geq 1$, $\alpha_1, \dots, \alpha_k$, $\beta_1, \dots, \beta_k$ and $C > 0$ such that $0 \leq \alpha_i < \beta_i \leq 1$, $i = 1, \dots, k$*

$$\|(\Lambda(t) - \lambda_0)(\Lambda(t) - v)^{-1}((\Lambda(t) - \lambda_0)^{-1} - (\Lambda(s) - \lambda_0)^{-1})\|_{\mathcal{L}(X)} \leq C \sum_1^k {}_i \frac{(t-s)^{\beta_i}}{|v|^{1-\alpha_i}}, \quad \forall v \in \Sigma_{\vartheta_\Lambda}, \quad 0 \leq s \leq t \leq T. \tag{5.7}$$

We also assume (without loss of generality) $\delta := \min_{1 \leq i \leq k} (\beta_i - \alpha_i) \in (0, 1)$.

LEMMA 5.1. – *Assume* (5.3), (5.6). *Then A and B, defined by* (5.4), (5.5), *are closed linear operators in X, and they satisfy* (2.2) *with $\vartheta_A = \vartheta_\Lambda$ and $\vartheta_B = \pi/2 + \varepsilon$, for any $\varepsilon > 0$ sufficiently small.*

PROOF. – Part of the proof is contained in [5, Prop. 7.1, 7.2]. Recalling that

$$((B - z)^{-1} u)(t) = - \int_0^t e^{-z(t-s)} u(s)\, ds, \tag{5.8}$$

the rest is easy to prove.

LEMMA 5.2. – *Assume* (5.3), (5.6) *and* (5.7). *Then A and B, defined by* (5.4), (5.5), *satisfy* (2.3).

PROOF. – Choose ϑ_A, ϑ_B as in Lemma 5.1. By (5.8) we have, $\forall v \in \Sigma_{\vartheta_A}$, $\forall z \in \Sigma_{\vartheta_B}$, $0 \leq s \leq t \leq T$, $\forall u \in X$,

$$((A-\lambda_0)(A-v)^{-1}[(A-\lambda_0)^{-1}; (B-z)^{-1}]\, u)(t) =$$

$$= ((A-\lambda_0)(A-v)^{-1}) \left(-(\Lambda(t)-\lambda_0)^{-1} \int_0^t e^{-z(t-s)} u(s)\, ds + \int_0^t e^{-z(t-s)} (\Lambda(s)-\lambda_0)^{-1} u(s)\, ds \right) =$$

$$= (\Lambda(t)-\lambda_0)(\Lambda(t)-v)^{-1} \int_0^t e^{-z(t-s)} ((\Lambda(s)-\lambda_0)^{-1} - (\Lambda(t)-\lambda_0)^{-1})\, u(s)\, ds$$

so that, by (5.7), recalling that $\vartheta_B > \pi/2$,

$$\|(A-\lambda_0)(A-v)^{-1}[(A-\lambda_0)^{-1};(B-z)^{-1}]\|_{\mathcal{L}(X)} \leq C\sum_1^k{}_i \int_0^T e^{-\mathrm{Re}(z)s}\frac{s^{\beta_i}}{|v|^{1-\alpha_i}}ds =$$

$$= (r:=\mathrm{Re}(z)s) = C\sum_1^k{}_i \int_0^{T\,\mathrm{Re}(z)} e^{-r}\frac{r^{\beta_i}}{(\mathrm{Re}(z))^{\beta_i}|v|^{1-\alpha_i}}\frac{dr}{\mathrm{Re}(z)} \leq$$

$$\leq C\sum_1^k{}_i \frac{1}{(\mathrm{Re}(z))^{\beta_i+1}|v|^{1-\alpha_i}}\int_0^\infty e^{-r}r^{\beta_i}dr \leq c\sum_1^k{}_i \frac{1}{|z|^{\beta_i+1}|v|^{1-\alpha_i}}.$$

THEOREM 5.3. – *Assume* (5.3), (5.6), (5.7). *In case* $X = C([0,T],E)$ *also assume that* $D_{A(t)}$ *is dense in* E, $\forall t\in[0,T]$ *and that* $-n(A(t)-n)^{-1}y\to y$, $\forall y\in E$, *uniformly in* $[0,T]$. *Then there exists* $\lambda^*>0$ *such that* $\forall\lambda>\lambda^*$ *any strict solution of* (5.1), (5.2) *satisfies*

$$\|u\|_X \leq c\lambda^{-1}\|f\|_X \tag{5.9}$$

and $\forall\lambda>\lambda^*$, $\forall f\in X$ *there exists a unique strong solution* u *of* (5.1), (5.2).

PROOF. – Notice that in case $X = L^p([0,T],E)$ or $X = C_0([0,T],E)$ we have D_B dense in X, and in case $X = C([0,T],E)$ we have D_A dense in X (see [5, Prop. 7.2]). We can then apply Theorems 4.2 and 4.5.

REMARK 5.4. – Estimate (5.9) has been obtained by [3] in case $X = C([0,T],E)$ without the additional assumption that $D_{A(t)}$ is dense in E, $\forall t\in[0,T]$ and that $-n(A(t)-n)^{-1}y\to y$, $\forall y\in E$, uniformly in $[0,T]$ (see [3, Th. 1.1]). Their estimate is a consequence of (5.9) if we choose $X = C_0([0,T],E)$ in (5.3) and if we make the slight additional assumption $f(0)=0$. If, however, $f(0)\neq 0$, then [3, Th. 1.1] is more general than (5.9). We also recall that [3] also consider cases where $u(0)\neq 0$ in (5.2).

REMARK 5.5. – Suppose $X = C_0([0,T],E)$ in (5.3). Then by Theorem 4.6, the strong solution of (5.1), (5.2) given by Theorem 5.3 belongs to $D_A(1,\infty)\cap D_B(1,\infty)$. As it can be easily seen, this implies the regularity result [1, Th. 4.3(i)], under the assumption $f\in C_0([0,T],E)$.

REMARK 5.6. – The results of Section 4 can be applied to the problem studied in [8], where only strict solutions are considered. We can therefore obtain existence and regularity results for strong solutions.

6. – Appendix.

In this section we prove Lemma 3.5. In order to do this we need some more preliminary lemmas. Lemmas 6.1 and 6.2 can be proved by elementary considerations recalling definitions (2.6), (2.7), (2.9) and observing that, by (2.1),

$$-(\boldsymbol{C}\setminus\Sigma_{\vartheta_B}) = \{0\} \cup \{w \in \boldsymbol{C}\colon |\arg(w)| \leqslant \vartheta_B\}.$$

LEMMA 6.1. – *For any $\lambda > 0$ the set*

$$\bigcup\{-(\boldsymbol{C}\setminus\Sigma_{\vartheta_B}) + \lambda - v\colon v \in \gamma_0\}$$

lies in the open region to the right of γ_1.

LEMMA 6.2. – *There exists $c = c(\vartheta_0, \vartheta_1) > 0$ such that*

$$\text{(i)}\quad |v + z - \lambda| \geqslant c|z - \lambda|, \tag{6.1}$$

$$\text{(ii)}\quad |v + z - \lambda| \geqslant c|v - \lambda|, \tag{6.2}$$

$\forall\lambda > 0,\ \ \forall v \in \vartheta_0, \forall z \in \gamma_1$.

LEMMA 6.3. – *Define*

$$T_\lambda(z, v) := v(A - \lambda)(A - \lambda - v)^{-1}[(A - \lambda)^{-1}; (B - \lambda + v)^{-1}]z^\rho(B - \lambda + z)^{-1}.$$

Then

$$T_\lambda(z, v) := \sum_1^5{}_i\, T_\lambda^{(i)}(z, v) \tag{6.3}$$

where

$$\text{(6.4)}\quad \begin{cases} T_\lambda^{(1)}(z, v) := v(A - \lambda)(A - \lambda - v)^{-1}[(A - \lambda)^{-1}; (B - \lambda + v)^{-1}]\cdot \\ \qquad\qquad\cdot(B - \lambda + z + v)^{-1}(B - \lambda + z)^{-1}\, vz^\rho, \\ T_\lambda^{(2)}(z, v) := v(A - \lambda)(A - \lambda - v)^{-1}[(A - \lambda)^{-1}; (B - \lambda + v)^{-1}]z^{\rho-1}, \\ T_\lambda^{(3)}(z, v) := -v(A - \lambda)(A - \lambda - v)^{-1}[(A - \lambda)^{-1}; (B - \lambda + v + z)^{-1}]z^{\rho-1}, \\ T_\lambda^{(4)}(z, v) := -v(B - \lambda + v)^{-1}(A - \lambda)(A - \lambda - v)^{-1}[(A - \lambda)^{-1}; (B - \lambda + v + z)^{-1}]z^\rho, \\ T_\lambda^{(5)}(z, v) := -v^2(A - \lambda)(A - \lambda - v)^{-1}[(A - \lambda)^{-1}; (B - \lambda + v)^{-1}]\cdot \\ \qquad\qquad\cdot(A - \lambda)(A - \lambda - v)^{-1}[(A - \lambda)^{-1}; (B - \lambda + v + z)^{-1}]z^\rho, \end{cases}$$

provided all the inverse operators exist and belong to $\mathcal{L}(X)$.

The same result holds with A (resp. B; resp. A and B) replaced by A_m (resp. B_n;

resp. A_m *and* B_n). *In this case we write* $T_{m,\lambda}$, $T^{(i)}_{m,\lambda}$ (*resp.* $T_{\lambda,n}$, $T^{(i)}_{\lambda,n}$; *resp.* $T_{m,\lambda,n}$, $T^{(i)}_{m,\lambda,n}$) *instead of* T_λ, $T^{(i)}_\lambda$.

PROOF.

$$T_\lambda(z,v) = v(A-\lambda)(A-\lambda-v)^{-1}[(A-\lambda)^{-1}; (B-\lambda+v)^{-1}](B-\lambda+z+v)^{-1}z^\rho +$$

$$+ v(A-\lambda)(A-\lambda-v)^{-1}[(A-\lambda)^{-1}; (B-\lambda+v)^{-1}](B-\lambda+z+v)^{-1}(B-\lambda+z)^{-1}vz^\rho =$$

$$= v(A-\lambda)(A-\lambda-v)^{-1}(B-\lambda)(B-\lambda+v)^{-1}[(A-\lambda)^{-1}; (B-\lambda)^{-1}](B-\lambda)(B-\lambda+v)^{-1}\cdot$$

$$\cdot (B-\lambda+z+v)^{-1}z^\rho + T^{(1)}_\lambda(z,v) = v(A-\lambda)(A-\lambda-v)^{-1}(B-\lambda)(B-\lambda+v)^{-1}\cdot$$

$$\cdot [(A-\lambda)^{-1}; (B-\lambda)^{-1}](B-\lambda)(B-\lambda+v)^{-1}z^{\rho-1} - v(A-\lambda)(A-\lambda-v)^{-1}(B-\lambda)(B-\lambda+v)^{-1}\cdot$$

$$\cdot [(A-\lambda)^{-1}; (B-\lambda)^{-1}](B-\lambda)(B-\lambda+z+v)^{-1}z^{\rho-1} + T^{(1)}_\lambda(z,v) = T^{(2)}_\lambda(z,v) - v(A-\lambda)\cdot$$

$$\cdot (A-\lambda-v)^{-1}(B-\lambda)(B-\lambda+z+v)^{-1}[(A-\lambda)^{-1}; (B-\lambda)^{-1}](B-\lambda)(B-\lambda+z+v)^{-1}z^{\rho-1} -$$

$$- v(A-\lambda)(A-\lambda-v)^{-1}(B-\lambda)\,z(B-\lambda+z+v)^{-1}(B-\lambda+v)^{-1}[(A-\lambda)^{-1}; (B-\lambda)^{-1}]\cdot$$

$$\cdot (B-\lambda)(B-\lambda+z+v)^{-1}z^{\rho-1} + T^{(1)}_\lambda(z,v) = T^{(2)}_\lambda(z,v) + T^{(3)}_\lambda(z,v) -$$

$$- v(A-\lambda)(A-\lambda-v)^{-1}(B-\lambda+v)^{-1}[(A-\lambda)^{-1}; (B-\lambda+z+v)^{-1}]z^\rho + T^{(1)}_\lambda(z,v) =$$

$$= T^{(2)}_\lambda(z,v) + T^{(3)}_\lambda(z,v) - v(B-\lambda+v)^{-1}(A-\lambda)(A-\lambda-v)^{-1}[(A-\lambda)^{-1}; (B-\lambda+z+v)^{-1}]z^\rho -$$

$$- v[(A-\lambda)(A-\lambda-v)^{-1}; (B-\lambda+v)^{-1}][(A-\lambda)^{-1}; (B-\lambda+z+v)^{-1}]z^\rho + T^{(1)}_\lambda(z,v) =$$

$$= T^{(2)}_\lambda(z,v) + T^{(3)}_\lambda(z,v) + T^{(4)}_\lambda(z,v) - v^2[(A-\lambda-v)^{-1}; (B-\lambda+v)^{-1}]\cdot$$

$$\cdot [(A-\lambda)^{-1}; (B-\lambda+z+v)^{-1}]z^\rho + T^{(1)}_\lambda(z,v) = T^{(2)}_\lambda(z,v) + T^{(3)}_\lambda(z,v) +$$

$$+ T^{(4)}_\lambda(z,v) - v^2(A-\lambda)(A-\lambda-v)^{-1}[(A-\lambda)^{-1}; (B-\lambda+v)^{-1}](A-\lambda)(A-\lambda-v)^{-1}\cdot$$

$$\cdot [(A-\lambda)^{-1}; (B-\lambda+z+v)^{-1}]z^\rho + T^{(1)}_\lambda(z,v) = \sum_1^5{}_i\, T^{(i)}_\lambda(z,v).$$

PROOF OF LEMMA 3.5. – For notational simplicity, and without loss of generality, we assume that (2.3) holds with $k = 1$, and $\alpha := \alpha_1$, $\beta := \beta_1$. Take any $\rho \in (0, \delta)$ and any $\bar{\lambda} > 0$. By (3.5), in order to prove (3.11), it is enough to prove $\|J_{m,\lambda,n}(\lambda - B_n)^\rho\|_{\mathcal{L}(X)} \leqslant c(\rho, \bar{\lambda})\,\lambda^{\rho-\delta}$.

Fix $\lambda > \bar{\lambda}$, m, n, $R > 0$ and define (see (2.10))

$$(6.5) \qquad J^R_{m,\lambda,n} := - \oint_{\gamma^R_{01}+\gamma^R_{02}} v(A_m - \lambda)(A_m - \lambda - v)^{-1}[(A_m - \lambda)^{-1}; (B_n - \lambda + v)^{-1}]\, dv,$$

$$V_{R,\lambda} := \bigcup \{ -(C \setminus \Sigma_{\vartheta_B}) + \lambda - v \colon v \in \gamma^R_{01} + \gamma^R_{02} \}.$$

By Lemma 5.1 this set lies in the open region to the right of γ_1. Since, by (2.4), $\sigma(-B_n) \subset -(C \setminus \Sigma_{\vartheta_B})$, the set

$$V_{R,\lambda,n} := \bigcup \{ \sigma(-B_n) + \lambda - v \colon v \in \gamma^R_{01} + \gamma^R_{02} \}$$

also lies in the open region to the right of γ_1.

By the compactness of $\sigma(-B_n)$ and of γ^R_0 we have

$$M_{R,\lambda,n} := \max \{ |w + \lambda - v| \colon w \in \sigma(-B_n),\ v \in \gamma^R_{01} + \gamma^R_{02} \} < \infty,$$

$$m_{R,\lambda,n} := \min \{ |w + \lambda - v| \colon w \in \sigma(-B_n),\ v \in \gamma^R_{01} + \gamma^R_{02} \} > 0,$$

so that $\forall s < m_{R,\lambda,n}$ and $\forall r > M_{R,\lambda,n}$ the closed path γ_1' surrounds $V_{R,\lambda,n}$ (see (2.11)). Therefore:

$$(6.6) \qquad 1) \quad \lambda - z - v \in \Sigma_{\vartheta_B}, \qquad \forall v \in \gamma_0,\ \forall z \in \gamma_1.$$

Otherwise we would have $\lambda - z - v \in C \setminus \Sigma_{\vartheta_B}$, i.e. $z \in -(C \setminus \Sigma_{\vartheta_B}) + \lambda - v$ for a $v \in \gamma_0$, which contradicts Lemma 5.1, since $z \in \gamma_1$.

2) $B_n - \lambda + z + v$ has a bounded inverse $\forall z \in \gamma_1'$, $\forall v \in \gamma^R_0$. Otherwise we would have $z \in \sigma(-B_n + \lambda - v)$ for a $v \in \gamma^R_{01} + \gamma^R_{02}$, which is impossible, since $z \in \gamma_1'$ and $\gamma_1' \cap V_{R,\lambda,n} = \emptyset$;

3) γ_1' surrounds $\sigma(-B_n + \lambda)$ and does not intersect $(-\infty, 0]$, so that

$$(-B_n + \lambda)^\rho = - \oint_{\gamma_1'} (B_n - \lambda + z)^{-1} z^\rho\, dz.$$

So we obtain (see (6.3), (6.4))

$$J^R_{m,\lambda,n}(-B_n + \lambda)^\rho = \oint_{\gamma^R_{01}+\gamma^R_{02}} \oint_{\gamma_1'} T_{m,\lambda,n}(z, v)\, dz\, dv.$$

Now observe that

$$\oint_{\gamma_1'} T^{(2)}_{m,\lambda,n}(z, v)\, dz = 0, \qquad \forall v \in \gamma^R_{01} + \gamma^R_{02},$$

by analyticity. So we have

$$J^R_{m,\lambda,n}(-B_n+\lambda)^\rho = \oint_{\gamma^R_{01}+\gamma^R_{02}} \oint_{\gamma_1'} T'_{m,\lambda,n}(z,v)\,dz\,dv,$$

where (see (6.4))

$$T'_{m,\lambda,n}(z,v) := T^{(1)}_{m,\lambda,n}(z,v) + T^{(3)}_{m,\lambda,n}(z,v) + T^{(4)}_{m,\lambda,n}(z,v) + T^{(5)}_{m,\lambda,n}(z,v).$$

For any $v\in\gamma^R_{01}+\gamma^R_{02}$ fixed, let us show

$$\text{(6.7)}\qquad \lim_{s\to 0}\oint_{\gamma^s_{12}} T'_{m,\lambda,n}(z,v)\,dz = 0, \qquad \text{and}$$

$$\text{(6.8)}\qquad \lim_{r\to 0}\oint_{\gamma^r_{14}} T'_{m,\lambda,n}(z,v)\,dz = 0.$$

(6.7) is obvious, since we have

$$T^{(3)}_{m,\lambda,n}(z,v) = O(|z|^{\rho-1}), \qquad T^{(1)}_{m,\lambda,n}(z,v) + T^{(4)}_{m,\lambda,n}(z,v) + T^{(5)}_{m,\lambda,n}(z,v) = O(|z|^\rho)$$

as $z\to 0$, z to the right of γ_1.

To prove (6.8) observe that:

1) $T^{(1)}_{m,\lambda,n}(z,v) + T^{(3)}_{m,\lambda,n}(z,v) = O(|z|^{-2+\rho})$ as $|z|\to\infty$;

2) as for $T^{(4)}_{m,\lambda,n}(z,v) + T^{(5)}_{m,\lambda,n}(z,v)$ it is enough to prove that the function

$$\varphi\colon z\mapsto (A_m-\lambda)(A_m-\lambda-v)^{-1}[(A_m-\lambda)^{-1};(B_n-\lambda+v+z)^{-1}],$$

which is holomorphic at infinity, has a zero of order at least 2 at infinity. By (6.6) we can apply (2.5) and we have

$$\|\varphi(z)\|_{\mathcal{L}(X)} \leqslant \frac{c}{|v+\lambda|^{1-\alpha}|v+z-\lambda|^{1+\beta}} = O(|z|^{-1-\beta}) \quad \text{as } |z|\to\infty,\ z\in\gamma_1,\ \forall v\in\gamma_0.$$

So (6.7) and (6.8) are proved.

By the analyticity of $z\mapsto T'_{m,\lambda,n}(z,v)$ we have, choosing $s=r^{-1}$,

$$J^R_{m,\lambda,n}(-B_n+\lambda)^\rho = \oint_{\gamma^R_{01}+\gamma^R_{02}} \left\{\lim_{r\to\infty} \oint_{\gamma^{r,r^{-1}}_{11}+\gamma^{r,r^{-1}}_{13}} T'_{m,\lambda,n}(z,v)\,dz\right\} dv.$$

Now take $\varepsilon>0$ such that $\rho+\varepsilon<\delta$. By (2.4), (2.5), (6.1), (6.2) we have the following estimates for $T^{(1)}_{m,\lambda,n}(z,v)$, $T^{(3)}_{m,\lambda,n}(z,v)$, $T^{(4)}_{m,\lambda,n}(z,v)$, $T^{(5)}_{m,\lambda,n}(z,v)$ which are valid for

$v \in \gamma_0$ and for $z \in \gamma_1$, by (6.6)

$$(6.9) \qquad \|T^{(1)}_{m,\lambda,n}(z, v)\|_{\mathcal{L}(X)} \leq \frac{|v|c(\bar{\lambda})}{|v+\lambda|^{1-\alpha}|v-\lambda|^{1+\beta}} \frac{c_B}{|z+v-\lambda|} \frac{c_B}{|z-\lambda|} |v|\,|z|^\rho \leq$$

$$\leq \frac{c|v|^2}{|v+\lambda|^{1-\alpha}|v-\lambda|^{1+\beta}|v-\lambda|^{1-\rho-\varepsilon}} \frac{|z|^\rho}{|z-\lambda|^{\rho+\varepsilon}|z-\lambda|} =:$$

$$=: \varphi_1(z, v, \lambda) = O(|v|^{-1-\delta+\varepsilon+\rho}|z|^{-1-\varepsilon}),$$

$$(6.10) \qquad \|T^{(3)}_{m,\lambda,n}(z, v)\|_{\mathcal{L}(X)} \leq \frac{|v|c(\bar{\lambda})|z|^{\rho-1}}{|v+\lambda|^{1-\alpha}|v+z-\lambda|^{1+\beta}} \leq$$

$$\leq \frac{c|v|}{|v+\lambda|^{1-\alpha}|v-\lambda|^{1+\alpha+\varepsilon}} \frac{|z|^{\rho-1}}{|z-\lambda|^{-\alpha-\varepsilon+\beta}} =: \varphi_3(z, v, \lambda) = O(|v|'^{-1-\varepsilon}|z|^{\rho-1-\delta+\varepsilon}),$$

$$(6.11) \qquad \|T^{(4)}_{m,\lambda,n}(z, v)\|_{\mathcal{L}(X)} \leq \frac{c_B|v|}{|v-\lambda|} \frac{c(\bar{\lambda})|z|^\rho}{|v+\lambda|^{1-\alpha}|v+z-\lambda|^{1+\beta}} \leq$$

$$\leq \frac{c|v|}{|v-\lambda|\,|v+\lambda|^{1-\alpha}|v-\lambda|^{\alpha+\varepsilon}} \frac{|z|^\rho}{|z-\lambda|^{1+\beta-\alpha-\varepsilon}} =: \varphi_4(z, v, \lambda) = O(|v|^{-1-\varepsilon}|z|^{\rho-1-\delta+\varepsilon}),$$

$$(6.12) \qquad \|T^{(5)}_{m,\lambda,n}(z, v)\|_{\mathcal{L}(X)} \leq \frac{c(\bar{\lambda})|v|^2}{|v+\lambda|^{1-\alpha}|v-\lambda|^{1+\beta}} \frac{c(\bar{\lambda})|z|^\rho}{|v+\lambda|^{1-\alpha}|v+z-\lambda|^{1+\beta}} \leq$$

$$\leq \frac{c|v|^2}{|v+\lambda|^{1-\alpha}|v-\lambda|^{1+\beta}|v+\lambda|^{1-\alpha}|v-\lambda|^{\alpha+\varepsilon}} \frac{|z|^\rho}{|z-\lambda|^{1+\beta-\alpha-\varepsilon}} =:$$

$$=: \varphi_5(z, v, \lambda) = O(|v|^{-\delta-1-\varepsilon}|z|^{\rho-1-\delta+\varepsilon}).$$

It follows that $T'_{m,\lambda,n}(z, v)$ is absolutely integrable on $\gamma_0 \times \gamma_1$, $\int_{\gamma_1} T'_{m,\lambda,n}(z, v)\,dz$ exists as a usual Bochner integral, $\forall v \in \gamma_0$, and

$$\int_{\gamma_1} T'_{m,\lambda,n}(z, v)\,dz = \lim_{r \to \infty} \int_{\gamma_{11}^{r^{-1}} + \gamma_{13}^{r^{-1}}} T'_{m,\lambda,n}(z, v)\,dz, \qquad \forall v \in \gamma_0$$

so that

$$J^R_{m,\lambda,n}(-B_n + \lambda)^\rho = \int_{\gamma^R_{01} + \gamma^R_{02}} \int_{\gamma_1} T'_{m,\lambda,n}(z, v)\,dz\,dv.$$

Letting $R \to \infty$ we have, by (6.5),

$$J_{m,\lambda,n}(\lambda - B_n)^\rho = \int_{\gamma_0}\int_{\gamma_1} T'_{m,\lambda,n}(z, v)\, dz\, dv\,. \tag{6.13}$$

From (6.9)-(6.12) it follows that

$$\|J_{m,\lambda,n}(\lambda - B_n)^\rho\|_{\mathcal{L}(X)} \leq \int_{\gamma_0}\int_{\gamma_1} (\varphi_1 + \varphi_3 + \varphi_4 + \varphi_5)(z, v, \lambda)|dv|\,|dz|\,.$$

By performing the change of variables $z = \lambda w$, $v = \lambda q$ we have

$$\int_{\gamma_0}\int_{\gamma_1} \varphi_i(z, v, \lambda)|dv|\,|dz| = \int_{\gamma_0}\int_{\gamma_1} \varphi_i(w, q, 1)|dw|\,|dq|\lambda^{\eta_i}$$

where $\eta_i = -\delta + \rho$ if $i = 1, 3, 4$; $\eta_i = -2\delta + \rho$ if $i = 5$. So we finally have $\|J_{m,\lambda,n}(\lambda - B_n)^\rho\|_{\mathcal{L}(X)} \leq c\lambda^{\rho-\delta}$. (3.11) is proved.

Since the estimates (6.9)-(6.12) are uniform with respect to m and n, by the dominated convergence theorem we see from (6.13) that the following limits exist in the norm of $\mathcal{L}(X)$:

$$\lim_{n\to\infty} J_{m,\lambda,n}(\lambda - B_n)^\rho = \int_{\gamma_0}\int_{\gamma_1} T'_{m,\lambda}(z, v)\, dz\, dv =: K_{m,\lambda}\,,$$

$$J_{\lambda,n}(\lambda - B_n)^\rho = \lim_{m\to\infty} J_{m,\lambda,n}(\lambda - B_n)^\rho = \int_{\gamma_0}\int_{\gamma_1} T'_{\lambda,n}(z, v)\, dz\, dv\,,$$

$$\lim_{m\to\infty}\left\{\lim_{n\to\infty} J_{m,\lambda,n}(\lambda - B_n)^\rho\right\} = \lim_{n\to\infty}\left\{\lim_{m\to\infty} J_{m,\lambda,n}(\lambda - B_n)^\rho\right\} =$$

$$= \lim_{n\to\infty} J_{\lambda,n}(\lambda - B_n)^\rho = \int_{\gamma_0}\int_{\gamma_1} T'_\lambda(z, v)\, dz\, dv =: K_\lambda$$

where (see (6.4))

$$T'_{m,\lambda}(z, v) := T^{(1)}_{m,\lambda}(z, v) + T^{(3)}_{m,\lambda}(z, v) + T^{(4)}_{m,\lambda}(z, v) + T^{(5)}_{m,\lambda}(z, v)\,,$$

$$T'_{\lambda,n}(z, v) := T^{(1)}_{\lambda,n}(z, v) + T^{(3)}_{\lambda,n}(z, v) + T^{(4)}_{\lambda,n}(z, v) + T^{(5)}_{\lambda,n}(z, v)\,,$$

$$T'_\lambda(z, v) := T^{(1)}_\lambda(z, v) + T^{(3)}_\lambda(z, v) + T^{(4)}_\lambda(z, v) + T^{(5)}_\lambda(z, v)\,.$$

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