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Comparison Results for Some Types of Relaxation of Variational Integral Functionals (*).

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Abstract. – *A comparison between some relaxation methods of an integral functional is carried out. The following relaxed functionals of the variational integral* $I(\Omega, u) = \int_\Omega f(x, Du)$*:*

$$\hat{I}(\Omega, u) = \inf\{\liminf_h I(\Omega, u_h),\ u_h \in C^1(R^n),\ u_h \to u \text{ in } L^1(\Omega)\} \qquad u \in L^1(\Omega),$$

$$\bar{I}(\Omega, u) = \inf\{\liminf_h I(\Omega, u_h),\ u_h \in C^1(\Omega),\ u_h \to u \text{ in } L^1_{\rm loc}(\Omega)\} \qquad u \in L^1_{\rm loc}(\Omega)$$

are introduced. It is proved, by means of examples, that in general such functionals are different even if Ω *is a regular bounded open set and criteria for identity on the whole* $L^1(\Omega)$ *are proved. If* f *does not depend on* x *it is proved that* $\hat{I}$ *and* $\bar{I}$ *agree if* Ω *has Lipschitz boundary and an integral representation formula for their common values on* $BV(\Omega)$ *is proved. Similar results and comparison ones with* $\hat{I}$ *and* $\bar{I}$ *are proved also for other kinds of relaxed functionals of* I.

0. – Introduction.

Let (U, τ) be a topological space satisfying the first countability axiom, let X be a τ-dense subset of U and let I be a real extended functional defined on X.

In many problems of Calculus of Variations dealing with extremal properties of the functional I one is naturally led to consider the so called relaxed functional $\mathrm{sc}^-(\tau) I$ of I, defined on the whole space U as

$$\mathrm{sc}^-(\tau) I(u) = \inf\{\liminf_h I(u_h) \,|\, u_h \in X,\ u_h \xrightarrow{\tau} u\}.$$

In fact in many cases it occurs that the functional $\mathrm{sc}^-(\tau) I$ has a minimum value on U that agrees with the infimum of I on X (see for example [Bu]).

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Let us now consider a function f verifying the following assumptions

$$\begin{cases} f:(x,z)\in \boldsymbol{R}^N\times \boldsymbol{R}^N \to f(x,z)\in [0,+\infty[\,, \\ f \text{ measurable in } x \text{ and convex in } z\,, \end{cases} \tag{0.1}$$

$$\text{for every } z \text{ in } \boldsymbol{R}^N \quad f(\cdot, z)\in L^1_{\mathrm{loc}}(\boldsymbol{R}^N)\,, \tag{0.2}$$

and let us consider the integral

$$I(\Omega, u)=\int_\Omega f(x, Du) \tag{0.3}$$

defined for every bounded open set Ω of $\boldsymbol{R}^N$ and every u in a set of functions X in general containing $C^1(\boldsymbol{R}^N)$.

We observe that $I(\Omega, u)$ exists and is finite for every u in $C^1(\boldsymbol{R}^N)$.

Several choices of the set X and of the couple (U,τ) are possible.

For example in many interesting cases it turns out to be convenient to choose X equal to $C^1(\boldsymbol{R}^N)$ and (U,τ) equal to $L^1(\Omega)$ endowed with its strong topology (see for example [DG1], [CS], [BDM1], [MS2]).

In this case the relaxed functional of I is given by

$$\hat{I}(\Omega, u)=\inf\left\{\liminf_h \int_\Omega f(x, Du_h)\,|\, u_h\in C^1(\boldsymbol{R}^N),\ u_h\to u \text{ in } L^1(\Omega)\right\} \tag{0.4}$$

and is defined for every bounded open set Ω, u in $L^1(\Omega)$.

On the other side, given a bounded open set Ω, in many papers (see example [AMT], [DM2], [GMS], [B], [DT]) it has been considered the case in which X is equal to $C^1(\Omega)$ and (U,τ) is given by $L^1_{\mathrm{loc}}(\Omega)$ endowed with its strong topology, getting therefore the following relaxed functional of I

$$\bar{I}(\Omega, u)=\inf\left\{\liminf_h \int_\Omega f(x, Du_h)\,|\, u_h\in C^1(\Omega),\ u_h\to u \text{ in } L^1_{\mathrm{loc}}(\Omega)\right\} \tag{0.5}$$

defined for every u in $L^1_{\mathrm{loc}}(\Omega)$.

Other choices natural enough consist in assuming as X the class $W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^N)$ and as (U,τ) the space $L^1(\Omega)$ with its strong topology, that is the same topological space used to construct $\hat{I}$, or as X the class $W^{1,p}_{\mathrm{loc}}(\Omega)$ and as (U,τ) the space $L^1_{\mathrm{loc}}(\Omega)$ with its strong topology, i.e. the same one considered to define $\bar{I}$.

These functionals may sometimes be different from the ones already introduced (see [DA2] and § 3).

Limiting ourselves in this introduction, for the sake of simplicity, to discuss the case of the functionals $\hat{I}$ and $\bar{I}$, we in general have that

$$\bar{I}(\Omega, u)\leq \hat{I}(\Omega, u) \quad \text{for every bounded open set } \Omega,\ u \text{ in } L^1(\Omega)\,, \tag{0.6}$$

and that strict inequality in (0.6) may occur for some bounded open set Ω and some function u, if Ω is sufficiently irregular, even if f is a smooth function independent on x (see Example 3.1).

On the other side it is possible to prove that strict inequality in (0.6) may hold, for some u, even if Ω is a bounded open set with Lipschitz boundary provided that the function f explicitly depends on x (see § 3).

In this paper we intend to examine more closely the reciprocal behaviour of the functionals $\hat{I}$ and $\bar{I}$, and representation formulas for them, as u and Ω vary.

It can be easily established that if $N = 1$ identity between $\hat{I}$ and $\bar{I}$ always holds (see Proposition 3.5).

Moreover, once recalled that a family $\mathfrak{J}$ of open subsets of $\boldsymbol{R}^N$ is said to be dense if for every couple of open sets of $\boldsymbol{R}^N A_1$ and A_2 with $\bar{A}_1 \subseteq A_2$ there exists $B \in \mathfrak{J}$ such that $\bar{A}_1 \subseteq B$ and $\bar{B} \subseteq A_2$, it can be observed that for every u in $L^1_{\text{loc}}(\boldsymbol{R}^N)$ there exists a dense family of bounded open sets such that $\hat{I}(\Omega, u) = \bar{I}(\Omega, u)$ for every open set Ω in such a family (Proposition 3.4).

This result will be deduced via techniques of increasing set functions (see [DGL] and [DM1]), by proving that for every u in $L^1_{\text{loc}}(\boldsymbol{R}^N)$ the inner regular envelope of $\hat{I}$, $\hat{I}_-(\Omega, u) = \sup_{\bar{A} \subseteq \Omega} \hat{I}(A, u)$ is the restriction of a measure to the set of all bounded open sets of $\boldsymbol{R}^N$ (Theorem 2.5).

A sufficiently significant dense family independent on u can be selected under more restrictive assumptions on the function f.

For example it will be proved (see Corollary 5.3) that identity between $\hat{I}$ and $\bar{I}$ holds for every bounded open set Ω with Lipschitz boundary and every u in $L^1(\Omega)$, provided that the function f verifies the following estimates

$$\begin{cases} \phi(z) \leq f(x, z) \leq \Lambda(a(x) + \phi(z)) & x \text{ a.e. in } \boldsymbol{R}^N, \ z \text{ in } \boldsymbol{R}^N, \\ \Lambda \geq 1, \quad a \in L^1_{\text{loc}}(\boldsymbol{R}^N), & \phi \text{ convex finite function}. \end{cases}$$

To this aim we will prove again that the functional $\hat{I}$ agrees with its inner regular envelope $\hat{I}_-$ for every bounded open set Ω with Lipschitz boundary and every u in $L^1(\Omega)$ (see § 5).

In conclusion let us explicitly observe that, by using well known results (see [GS]), it is also established an integral representation result on $BV(\Omega)$, in the case in which f does not depend on x, for the functional $\hat{I}$ and also for the functional

$$\hat{I}_0(\Omega, u) = \inf\left\{\liminf_h \int_\Omega f(x, Du_h) \,|\, u_h \in C^1_0(\Omega),\ u_h \to u \text{ in } L^1(\Omega)\right\}, \tag{0.7}$$

that is the functional that is obtained by relaxing I with a procedure similar to the one performed to get $\hat{I}$, but having in mind Dirichlet problems with null boundary data (Theorem 4.7).

Analogous results can be stated for the relaxed functionals of I constructed with the choices of X and (U, τ) already pointed out.

1. - Notations and preliminary results.

Given two bounded open sets of $\boldsymbol{R}^N$ A and B we say that $A \subset\subset B$ if $\overline{A} \subseteq B$.

A family $\mathfrak{F}$ of open sets of $\boldsymbol{R}^N$ is said to be dense if for every couple of bounded open sets A_1, A_2 of $\boldsymbol{R}^N$ there exists B in $\mathfrak{F}$ such that $A_1 \subset\subset B \subset\subset A_2$.

Let F be a real function defined on the set of all bounded open sets of $\boldsymbol{R}^N$, we say that F is increasing if

$$A_1 \subseteq A_2 \Rightarrow F(A_1) \leq F(A_2).$$

For an increasing function F we introduce the inner regular envelope F_- of F by

$$F_-(\Omega) = \sup_{A \subset\subset \Omega} F(A), \tag{1.1}$$

we refer to [DGL] and to [DM1] for the study of the properties of inner regular envelopes, here we only recall that the inner regular envelope F_- of an increasing function F is inner regular, i.e. $(F_-)_- = F_-$, and the following result (see Proposition 1.I and Theorem 1.I in [DM1]).

PROPOSITION 1.1. - Let F be an increasing function defined on the set of all bounded open sets and such that $F(\emptyset) = 0$. Then the set of all bounded open sets Ω such that $F(\Omega) = F_-(\Omega)$ is dense.

For every $p \in [1, +\infty]$ we will set $W^{1,p}_{\mathrm{loc}} = W^{1,p}_{\mathrm{loc}}(\boldsymbol{R}^N)$.

Let f be a function as in (0.1), (0.2); let us introduce the following functionals defined for every bounded open set Ω, u in $L^1(\Omega)$ and p in $[1, +\infty]$

$$\hat{J}^p(\Omega, u) = \inf\left\{\liminf_h \int_\Omega f(x, Du_h) \,|\, u_h \in W^{1,p}_{\mathrm{loc}},\ u_h \to u \text{ in } L^1(\Omega)\right\}, \tag{1.2}$$

$$\bar{J}^p(\Omega, u) = \inf\left\{\liminf_h \int_\Omega f(x, Du_h) \,|\, u_h \in W^{1,p}_{\mathrm{loc}}(\Omega),\ u_h \to u \text{ in } L^1_{\mathrm{loc}}(\Omega)\right\}, \tag{1.3}$$

$$\hat{J}^p_0(\Omega, u) = \inf\left\{\liminf_h \int_\Omega f(x, Du_h) \,|\, u_h \in W^{1,p}_0(\Omega),\ u_h \to u \text{ in } L^1(\Omega)\right\}. \tag{1.4}$$

For simplicity when $p = +\infty$ we will write $\hat{J}$ instead of $\hat{J}^\infty$ and so for the other functionals.

Let us observe that since $L^1(\Omega)$ and $L^1_{\mathrm{loc}}(\Omega)$ topologies satisfy the first countability axiom, infima in (1.2) ÷ (1.4), together with those in (0.4), (0.5) and (0.7), are attained.

We explicitly remark that for every bounded open set Ω, p in $[1, +\infty]$, the functionals $\hat{J}^p(\Omega, \cdot)$, $\bar{J}^p(\Omega, \cdot)$ and $\hat{J}^p_0(\Omega, \cdot)$ are $L^1(\Omega)$-lower semicontinuous on $L^1(\Omega)$.

Moreover, being $\hat{J}^p_-(\Omega, \cdot)$ the supremum of a family of $L^1(\Omega)$-lower semicontinuous functionals, $\hat{J}_{-}^{p}(\Omega, \cdot)$ turns out to be $L^1(\Omega)$-lower semicontinuous on $L^1(\Omega)$.

We recall that, see [DA2], in general the functionals $\hat{J}^p$ effectively depend on p, hence the whole family of functionals $\hat{J}^p$ and $\hat{J}_0^p$ must be considered.

The following inequalities are soon verified

$$(1.5) \qquad \hat{J}_{-}^{p}(\Omega, u) \leqslant \bar{J}^p(\Omega, u) \leqslant \hat{J}^p(\Omega, u)$$

for every bounded open set Ω, u in $L^1(\Omega)$, p in $[1, +\infty]$.

Moreover it is easy to prove that if Ω_1 and Ω_2 are bounded open sets with $\Omega_1 \subseteq \Omega_2$ then

$$(1.6) \qquad \hat{J}^p(\Omega_1, u) \leqslant \hat{J}^p(\Omega_2, u) \qquad \text{for every } u \text{ in } L^1(\Omega_2),\ p \text{ in } [1, +\infty],$$

$$(1.7) \qquad \bar{J}^p(\Omega_1, u) \leqslant \bar{J}^p(\Omega_2, u) \qquad \text{for every } u \text{ in } L^1(\Omega_2),\ p \text{ in } [1, +\infty]$$

and

$$(1.8) \qquad \hat{J}_0^p(\Omega_1, u) \geqslant \hat{J}_0^p(\Omega_2, u) - \int_{\Omega_2 \setminus \Omega_1} f(x, 0)$$

for every u in $L^1(\Omega_2)$ with $u = 0$ in $\Omega_2 \setminus \Omega_1$, p in $[1, +\infty]$.

Let α be a mollifier, that is α belongs to $C^\infty(\boldsymbol{R}^N)$ and is a nonnegative function with support contained in the unit ball of $\boldsymbol{R}^N$ such that $\int_{\boldsymbol{R}^N} \alpha = 1$, and let us define for every $\eta > 0$

$$(1.9) \qquad \alpha^{(\eta)}(x) = \frac{1}{\eta^N} \alpha\left(\frac{x}{\eta}\right).$$

For every u in $L^1_{\text{loc}}(\boldsymbol{R}^N)$ we define the regularization of u by

$$(1.10) \qquad u_\eta(x) = (\alpha^{(\eta)} * u)(x) = \int_{\boldsymbol{R}^N} \alpha^{(\eta)}(x - y)\, u(y)\, dy.$$

Moreover for every bounded open set Ω, $\varepsilon > 0$ let us set

$$\Omega_\varepsilon^- = \{x \in \Omega \,|\, \operatorname{dist}(x, \partial\Omega) > \varepsilon\}, \qquad \Omega_\varepsilon^+ = \{x \in \boldsymbol{R}^N \,|\, \operatorname{dist}(x, \Omega) < \varepsilon\}.$$

The relationship between functionals in (1.2) and (1.4) with $p = +\infty$ and those in (0.4) and (0.7) is given by the following result.

PROPOSITION 1.2. – Let f be as in (0.1) and (0.2) and let $\hat{I}, \hat{I}_0, \hat{J}, \hat{J}_0$ be the functionals defined in (0.4), (0.7), (1.2) and (1.4) with $p = +\infty$. Then

$$(1.11) \qquad \hat{I}(\Omega, u) = \hat{J}(\Omega, u),$$

$$(1.12) \qquad \hat{I}_0(\Omega, u) = \hat{J}_0(\Omega, u)$$

for every bounded open set Ω, u in $L^1(\Omega)$.

PROOF. – Obviously

$$\begin{cases} \hat{I}(\Omega, u) \geq \hat{J}(\Omega, u), \\ \hat{I}_0(\Omega, u) \geq \hat{J}_0(\Omega, u) \end{cases} \quad \text{for every bounded open set } \Omega,\ u \text{ in } L^1(\Omega). \tag{1.13}$$

In order to prove the reverse inequalities let us consider first the case of $\hat{I}$ and $\hat{J}$.

Let Ω be a bounded open set, u in $L^1(\Omega)$.

Let $\{u_h\}_h \subseteq W^{1,\infty}_{\text{loc}}$ be such that $u_h \to u$ in $L^1(\Omega)$ and

$$\hat{J}(\Omega, u) \geq \liminf_h \int_\Omega f(x, Du_h). \tag{1.14}$$

For every fixed $h \in N$ let $u_{h,\eta}$ be a regularization of u_h.

For every $\eta > 0$ we have $\| |Du_{h,\eta}| \|_{L^\infty(\Omega)} \leq \| |Du_h| \|_{L^\infty(\Omega_\eta^+)}$ and $Du_{h,\eta} \to Du_h$ almost everywhere on Ω as $\eta \to 0^+$.

Then the dominated convergence theorem yields

$$\int_\Omega f(x, Du_h) = \lim_{\eta \to 0^+} \int_\Omega f(x, Du_{h,\eta}). \tag{1.15}$$

By (1.14), (1.15) and a diagonal process we can select $\{\eta_h\}_h$, with $\eta_h \to 0^+$ as $h \to \infty$, such that $u_{h,\eta_h} \to u$ in $L^1(\Omega)$ and

$$\hat{J}(\Omega, u) \geq \liminf_h \int_\Omega f(x, Du_{h,\eta_h}) \geq \hat{I}(\Omega, u). \tag{1.16}$$

By (1.13) and (1.16), (1.11) follows.

We now consider the case of (1.12).

As before let $\{v_h\}_h \subseteq W_0^{1,\infty}(\Omega)$ be such that $v_h \to u$ in $L^1(\Omega)$ and

$$\hat{J}_0(\Omega, u) \geq \liminf_h \int_\Omega f(x, Dv_h). \tag{1.17}$$

Let $\{\sigma_h\}_h$ be a sequence of positive numbers, with $\sigma_h \to 0^+$ as $h \to \infty$, that will be specified later.

Let us define the functions β_h as

$$\beta_h(t) = \begin{cases} t - \sigma_h & \text{if } t > \sigma_h, \\ 0 & \text{if } -\sigma_h \leq t \leq \sigma_h, \\ t + \sigma_h & \text{if } t < -\sigma_h, \end{cases} \tag{1.18}$$

and set

$$\tilde{v}_h(x) = \beta_h(v_h(x)). \tag{1.19}$$

Then $\tilde{v}_h \in W_0^{1,\infty}(\Omega)$, $\operatorname{spt}(\tilde{v}_h) \subset\subset \Omega$ and $\tilde{v}_h \to u$ in $L^1(\Omega)$.

For every fixed $h \in N$ let $\tilde{v}_{h,\eta}$ be a regularization of $\tilde{v}_h$.

For every h fixed and for every η sufficiently small and depending on h we have

$\tilde{v}_{h,\eta} \in C_0^1(\Omega)$ and, as in (1.15), it results

$$\int_\Omega f(x, D\tilde{v}_h) = \lim_{\eta \to 0^+} \int_\Omega f(x, D\tilde{v}_{h,\eta}). \tag{1.20}$$

Moreover we have

$$\int_\Omega f(x, D\tilde{v}_h) \leq \int_\Omega f(x, Dv_h) + \int_{\{x \in \Omega \mid 0 < |v_h(x)| < \sigma_h\}} f(x, 0). \tag{1.21}$$

By (1.17), (1.20) and (1.21) we can prove that

$$\hat{J}_0(\Omega, u) \geq \hat{I}_0(\Omega, u) \tag{1.22}$$

as in (1.16) and by choosing a suitable sequence $\{\sigma_h\}_h$.

By (1.13) and (1.22) (1.12) follows. ■

For every measurable set A we will denote by $|A|$ the Lebesgue measure of A.

In the following we will need to select a particular class of star-shaped open sets.

DEFINITION 1.3. – We say that an open set Ω is strongly star-shaped if it is star-shaped with respect to some point x_0 in Ω and if for every x in $\overline{\Omega}$ the half open line segment joining x_0 to x and not containing x is contained in Ω.

Let Ω be a strongly star-shaped bounded open set, for simplicity let Ω be star-shaped with respect to 0, then it is obvious that for every $t > 0$ the open set $t\Omega$ is still strongly star-shaped and that, if $t > 1$, $\overline{\Omega} \subset\subset t\Omega$.

This implies that for $0 \leq s < 1 < t$ it results $s\Omega \subset\subset \Omega \subset\subset t\Omega$.

Let Ω be an open set, we say that Ω has Lipschitz boundary if $\partial\Omega$ is locally the graph of a Lipschitz continuous function.

By using a proof already performed in [ET], page 309-310, we can prove the following result.

LEMMA 1.4. – Let Ω be a bounded open set with Lipschitz boundary, then there exists a finite open covering of $\overline{\Omega}$ $\{\tilde{\Omega}_j\}_{j=1,\dots,s}$ such that for every $j = 1, \dots, s$ $\tilde{\Omega}_j \cap \Omega$ is strongly star-shaped with Lipschitz boundary.

PROOF. – Let x be in $\partial\Omega$, then in a cylindrical neighbourhood I_x of x we have

$$I_x \cap \Omega = \{y \in \mathbf{R}^N \mid y_N \leq \vartheta(\tilde{y}),\ \tilde{y} \in B\} \tag{1.23}$$

where $\tilde{y} \in \mathbf{R}^{N-1}$, $\vartheta : \mathbf{R}^{N-1} \to \mathbf{R}$ is a Lipschitz continuous function and B is the $(N-1)$-dimensional basis of I_x.

Obviously we can assume that in the coordinate system of B $x = (\tilde{0}, \vartheta(\tilde{0}))$ with $\vartheta(\tilde{0}) > 0$.

In [ET], page 309-310, using the same notations, it is proved that if k is the Lipschitz constant of ϑ and if $|\tilde{y}| < \vartheta(\tilde{0})/2k$ then

$$(1.24) \qquad 0 < \vartheta(\tilde{0}) - 2k\lambda|\tilde{y}| \leq \vartheta(\lambda\tilde{y}) - k\lambda|\tilde{y}| \quad \text{for every } \lambda \in [0, 1[\,.$$

By (1.24) we deduce

$$(1.25) \qquad \lambda\vartheta(\tilde{y}) \leq \lambda(\vartheta(\tilde{y}) - \vartheta(\lambda\tilde{y})) + \lambda\vartheta(\lambda\tilde{y}) \leq k\lambda(1-\lambda)|\tilde{y}| + \lambda\vartheta(\lambda\tilde{y}) <$$
$$< (1-\lambda)\,\vartheta(\lambda\tilde{y}) + \lambda\vartheta(\lambda\tilde{y}) = \vartheta(\lambda\tilde{y}) \quad \text{for every } \lambda \in [0, 1[\,.$$

By (1.23) and (1.25) we deduce that, in the coordinate system of B, for every $\tilde{y}$ with $|\tilde{y}| < \vartheta(\tilde{0})/2k$ the half open line segment joining $(\tilde{0}, 0)$ to $(\tilde{y}, \vartheta(\tilde{y}))$ but not containing this last point is contained in $I_x \cap \Omega$, that is x possesses a cylindrical neighbourhood J_x such that $J_x \cap \Omega$ is strongly star-shaped with Lipschitz boundary.

Since for every x in Ω there exists a ball B_x centered at x with $B_x \subsetneq \Omega$, taking into account the compactness of $\overline{\Omega}$, we deduce the thesis. ■

For every bounded open set Ω we denote by $BV(\Omega)$ the set of the functions in $L^1(\Omega)$ having distributional partial derivatives that are Radon measures with bounded total variations on Ω.

We recall that for every u in $BV(\Omega)$ the total variation of Du on Ω is given by

$$(1.26) \qquad \int_\Omega |Du| = \sup\left\{ \int_\Omega u \operatorname{div} g \,|\, g \in C_0^1(\Omega; \boldsymbol{R}^N),\ |g(x)| \leq 1 \text{ for every } x \in \Omega \right\}.$$

Moreover let us recall that for every BV-function u, according to Lebesgue's decomposition theorem, we have

$$(1.27) \qquad (Du)(E) = \int_E RDu\, dx + (SDu)(E) \quad \text{for every Borel set } E$$

where we have denoted with RDu the Radon-Nikodym derivative of Du and with SDu the singular part of Du both taken with respect to Lebesgue measure, and that (see [G]) if Ω has Lipschitz boundary and if $\boldsymbol{n}$ is the unit inward normal to $\partial\Omega$ then

$$(1.28) \qquad Du|_{\partial\Omega} = \boldsymbol{n} u H^{N-1}|_{\partial\Omega} \quad \text{for every } u \text{ in } BV(\boldsymbol{R}^N) \text{ with } u \equiv 0 \text{ in } \boldsymbol{R}^N \setminus \Omega\,,$$

H^{N-1} being the $(N-1)$-dimensional-Hausdorff measure on $\boldsymbol{R}^N$.

For a survey on BV-functions we refer to [G].

DEFINITION 1.5. – Let f be a nonnegative convex finite function on $\boldsymbol{R}^N$ and let ν be a positive Borel measure on $\boldsymbol{R}^N$.

Then for every $\boldsymbol{R}^N$-valued Borel measure μ and every Borel set E we define

$$(1.29)\qquad f\mu(E) = \sup\left\{\sum_i \nu(E_i) f\left(\frac{\mu(E_i)}{\nu(E_i)}\right) \,\middle|\, \{E_i\} \text{ is a finite partition of } E \text{ into Borel sets}\right\}.$$

In [GS] (Theorem 2′) it is proved that $f\mu$ is a Borel measure and that if

$$(1.30)\qquad \mu(E) = \int_E a\,d\nu + \beta(E) \quad \text{for every Borel set } E$$

with $a \in L^1(d\nu)$, β singular with respect to ν, then

$$(1.31)\qquad f\mu(E) = \int_E f(a)\,d\nu + f^*\beta(E) \quad \text{for every Borel set } E$$

where f^* is the recession function of f given by

$$(1.32)\qquad f^*(z) = \lim_{t \to 0^+} tf\left(\frac{z}{t}\right) \qquad z \in \boldsymbol{R}^N.$$

In particular, taking as ν the Lebesgue measure, by (1.27), (1.30) and (1.31) it follows that

$$(1.33)\qquad (fDu)(\Omega) = \int_\Omega f(RDu) + f^*(SDu)(\Omega) = \int_\Omega f(RDu) + \int_\Omega f^*\left(\frac{dSDu}{d|Du|}\right) d|Du|$$

for every bounded open set Ω, u in $BV(\Omega)$.

In (1.33) $dSDu/d|Du|$ denotes the Radon-Nikodym derivative of SDu with respect to $|Du|$.

Let Ω_h, $h \in \boldsymbol{N}$, Ω be bounded open sets, we say that $\Omega_h \to \Omega$ if for every compact subset K of Ω it definitively results $\Omega_h \supseteq K$.

Let f be a function as in (0.1) and (0.2), in [Se1] J. SERRIN introduced the following functional defined for every bounded open set Ω, u in $L^1(\Omega)$

$$(1.34)\qquad \mathfrak{I}(\Omega, u) = \inf\left\{\liminf_h \int_{\Omega_h} f(x, Du_h) \,|\, u_h \in C^1(\Omega_h),\ \Omega_h \to \Omega,\ \ u_h \to u \text{ in } L^1_{\text{loc}}(\Omega)\right\}.$$

The following representation result holds for the functional $\mathfrak{I}$ (see [Se2], page 144 and [GS], page 174).

THEOREM 1.6. – Let f be a function as in (0.1) and let $\mathfrak{I}$ be given by (1.34). Assume that f does not depend on x.

Then

$$(1.35)\qquad \mathfrak{I}(\Omega, u) = f(Du)(\Omega) \quad \text{for every bounded open set } \Omega,\ u \text{ in } BV(\Omega).$$

Functionals $\hat{I}$ in (0.4) and $\Im$ in (1.34) are linked by the following result.

PROPOSITION 1.7. – Let f be a function as in (0.1) and (0.2) and let $\hat{I}$, $\Im$ be defined by (0.4) and (1.34). Then

$$\Im(\Omega, u) = \hat{I}_-(\Omega, u) \quad \text{for every bounded open set } \Omega,\ u \text{ in } L^1(\Omega). \tag{1.36}$$

PROOF. – Let Ω, u be as above.
Let us first prove that

$$\Im(\Omega, u) \geq \hat{I}_-(\Omega, u). \tag{1.37}$$

To this aim we can assume that $\Im(\Omega, u) < +\infty$.
Let $\varepsilon > 0$, $A \subset\subset \Omega$ and let $\Omega_h \to \Omega$ $u_h \to u$ in $L^1_{\text{loc}}(\Omega)$ with $u_h \in C^1(\Omega_h)$ such that

$$\Im(\Omega, u) + \varepsilon \geq \liminf_h \int_{\Omega_h} f(x, Du_h). \tag{1.38}$$

Since $\Omega_h \to \Omega$ then definitively $\Omega_h \supset\supset A$, hence by (1.38) we have

$$\Im(\Omega, u) + \varepsilon \geq \liminf_h \int_A f(x, Du_h) \geq \hat{I}(A, u). \tag{1.39}$$

As ε and A are arbitrarily chosen we deduce (1.37) by (1.39).
Let us now prove the reverse inequality to (1.37).
Let us observe that

$$\hat{I}_-(\Omega, u) = \lim_k \hat{I}(\Omega^-_{1/k}, u), \tag{1.40}$$

then for every $k \in \boldsymbol{N}$ there exists $\{u_h^k\}_h \subseteq C^1(\boldsymbol{R}^N)$ such that $u_h^k \to u$ in $L^1(\Omega^-_{1/k})$ as $h \to +\infty$ and

$$\hat{I}(\Omega^-_{1/k}, u) \geq \liminf_h \int_{\Omega^-_{1/k}} f(x, Du_h^k). \tag{1.41}$$

Therefore by (1.40) and (1.41), and by virtue of a diagonal process, we can select a sequence $\{h_k\}_h$ such that $u_{h_k}^k \to u$ in $L^1_{\text{loc}}(\Omega)$ and

$$\hat{I}_-(\Omega, u) \geq \liminf_h \int_{\Omega^-_{1/k}} f(x, Du_{h_k}^k) \geq \Im(\Omega, u). \tag{1.42}$$

By (1.37) and (1.42) equality (1.36) follows. ■

2. – Some technical results.

In this section we will prove some measure theoretic properties of the functionals $\hat{J}^p_-$ relative to a function f as in (0.1) and (0.2).

Let us first prove the following result.

LEMMA 2.1. – Let f be a function as in (0.1) and (0.2).

Let Ω be a bounded open set, u in $L^\infty(\Omega)$, p in $[1, +\infty]$ and $\{u_h\}_h \subseteq W^{1,p}(\Omega)$ with $u_h \to u$ in $L^1(\Omega)$.

Then there exists a sequence $\{\tilde{u}_h\}_h \subseteq W^{1,p}(\Omega)$ such that $\tilde{u}_h \to u$ in $L^1(\Omega)$,

$$\|\tilde{u}_h\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)} \tag{2.1}$$

and

$$\liminf_h \int_\Omega f(x, D\tilde{u}_h) \leq \liminf_h \int_\Omega f(x, Du_h). \tag{2.2}$$

PROOF. – Let us define

$$\tilde{u}_h = -\|u\|_{L^\infty(\Omega)} \vee (u_h \wedge \|u\|_{L^\infty(\Omega)}), \tag{2.3}$$

then obviously $\tilde{u}_h \to u$ in $L^1(\Omega)$ and (2.1) holds.

Moreover we have

$$\int_\Omega f(x, D\tilde{u}_h) \leq \int_\Omega f(x, Du_h) + \int_{\{x \in \Omega:\ |\|u\|_h(x)| > \|u\|_{L^\infty(\Omega)}\}} f(x, 0). \tag{2.4}$$

Since $|\{x \in \Omega:\ |u_h(x)| > \|u\|_{L^\infty(\Omega)}\}| \to 0$ as $h \to \infty$, (2.2) follows by (2.4) and (0.2). ■

For every measurable function u and every $k \in N$ let us denote by $\tau_k u$ the function

$$(\tau_k u)(x) = -k \vee (u(x) \wedge k). \tag{2.5}$$

Then we have

LEMMA 2.2. – Let f be a function as in (0.1) and (0.2). Then

$$\lim_k \hat{J}^p(\Omega, \tau_k u) = \hat{J}^p(\Omega, u), \tag{2.6}$$

$$\lim_k \hat{J}^p_-(\Omega, \tau_k u) = \hat{J}^p_-(\Omega, u) \tag{2.7}$$

for every bounded open set Ω, u in $L^1(\Omega)$, p in $[1, +\infty]$.

PROOF. – Let Ω, u, p be as above.

Since $\hat{J}^{p}$ and $\hat{J}^{\underline{p}}$ are $L^1(\Omega)$-lower semicontinuous on $L^1(\Omega)$ we have

$$\hat{J}^{p}(\Omega, u) \leqslant \liminf_k \hat{J}^{p}(\Omega, \tau_k u), \tag{2.8}$$

$$\hat{J}^{\underline{p}}(\Omega, u) \leqslant \liminf_k \hat{J}^{\underline{p}}(\Omega, \tau_k u). \tag{2.9}$$

In order to prove the reverse inequality to (2.8) let $\{u_h\}_h \subseteq W^{1,p}_{\mathrm{loc}}$ be such that $u_h \to u$ in $L^1(\Omega)$, $u_h(x) \to u(x)$ a.e. in Ω and

$$\hat{J}^{p}(\Omega, u) \geqslant \liminf_h \int_\Omega f(x, Du_h), \tag{2.10}$$

then for every $k \in N$ $\tau_k u_h \to \tau_k u$ in $L^1(\Omega)$ as $h \to \infty$ and

$$\int_\Omega f(x, D(\tau_k u_h)) \leqslant \int_\Omega f(x, Du_h) + \int_{\{x \in \Omega:\ |u_h(x)| > k\}} f(x, 0). \tag{2.11}$$

By (2.10) and (2.11) we deduce

$$\hat{J}^{p}(\Omega, \tau_k u) \leqslant \liminf_h \int_\Omega f(x, D(\tau_k u_h)) \leqslant \hat{J}^{p}(\Omega, u) + \int_{\{x \in \Omega:\ |u(x)| \geqslant k\}} f(x, 0), \tag{2.12}$$

hence taking the limit as $k \to \infty$ in (2.12) we get

$$\limsup_k \hat{J}^{p}(\Omega, \tau_k u) \leqslant \hat{J}^{p}(\Omega, u). \tag{2.13}$$

Therefore by (2.8) and (2.13) (2.6) follows.

In order to prove the opposite inequality to (2.9) let $A \subset\subset \Omega$, then by (2.12) written with $\Omega = A$ we deduce

$$\hat{J}^{p}(A, \tau_k u) \leqslant \hat{J}^{p}(A, u) + \int_{\{x \in A:\ |u(x)| \geqslant k\}} f(x, 0) \leqslant \hat{J}^{\underline{p}}(\Omega, u) + \int_{\{x \in \Omega:\ |u(x)| \geqslant k\}} f(x, 0). \tag{2.14}$$

By (2.14), being A arbitrarily chosen, we infer

$$\hat{J}^{\underline{p}}(\Omega, \tau_k u) \leqslant \hat{J}^{\underline{p}}(\Omega, u) + \int_{\{x \in \Omega:\ |u(x)| \geqslant k\}} f(x, 0), \tag{2.15}$$

hence taking the limit as $k \to \infty$ in (2.15) we get

$$\limsup_k \hat{J}^{\underline{p}}(\Omega, \tau_k u) \leqslant \hat{J}^{\underline{p}}(\Omega, u). \tag{2.16}$$

By (2.9) and (2.16), (2.7) follows. ■

We now prove some additivity properties for $\hat{J}^{p}$ and $\hat{J}^{\underline{p}}$.

Lemma 2.3. - Let f be a function as in (0.1) and (0.2). Then

$$\hat{J}^{p}(\Omega, u) \leqslant \hat{J}^{p}(\Omega_1, u) + \hat{J}^{p}(\Omega_2, u) \tag{2.17}$$

for every triplet of bounded open sets Ω, Ω_1, Ω_2 with $\Omega \subset\subset \Omega_1 \cup \Omega_2$, u in $L^1(\Omega_1 \cup \Omega_2)$, p in $[1, +\infty]$,

$$\hat{J}^p_-(\Omega, u) \leq \hat{J}^p_-(\Omega_1, u) + \hat{J}^p_-(\Omega_2, u) \tag{2.18}$$

for every triplet of bounded open sets Ω, Ω_1, Ω_2 with $\Omega \subseteq \Omega_1 \cup \Omega_2$, u in $L^1(\Omega_1 \cup \Omega_2)$, p in $[1, +\infty]$.

PROOF. – Let Ω, Ω_1, Ω_2, p be as above.

Let us assume first that $u \in L^\infty(\Omega_1 \cup \Omega_2)$.

Let $\{u_h^1\}_h$, $\{u_h^2\}_h \subseteq W^{1,p}_{\text{loc}}$ be such that $u_h^i \to u$ in $L^1(\Omega_i)$ and a.e. on Ω_i as $h \to \infty$, $i = 1, 2$ and

$$\hat{J}^p(\Omega_i, u) \geq \limsup_h \int_{\Omega_i} f(x, Du_h^i) \qquad i = 1, 2. \tag{2.19}$$

Obviously by virtue of Lemma 2.1, we can assume that

$$\|u_h^i\|_{L^\infty(\Omega_i)} \leq \|u\|_{L^\infty(\Omega_1 \cup \Omega_2)} \qquad i = 1, 2. \tag{2.20}$$

Now let B such that $B \subset\subset \Omega_1$, $\Omega \subset\subset B \cup \Omega_2$ and let $\varphi \in C_0^1(\Omega_1)$ be such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on B and set

$$w_h = \varphi u_h^1 + (1 - \varphi) u_h^2, \tag{2.21}$$

then obviously $w_h \to u$ in $L^1(\Omega)$.

For every $t \in]0, 1[$ we have by convexity

$$\begin{aligned} \int_\Omega f(x, tDw_h) &\leq t \int_\Omega \varphi f(x, Du_h^1) + t \int_\Omega (1 - \varphi) f(x, Du_h^2) + \\ &\quad + (1 - t) \int_\Omega f\left(x, \frac{t}{1-t}(u_h^1 - u_h^2) D\varphi\right) \leq \\ &\leq t \int_{\Omega_1} f(x, Du_h^1) + t \int_{\Omega_2} f(x, Du_h^2) + (1 - t) \int_\Omega f\left(x, \frac{t}{1-t}(u_h^1 - u_h^2) D\varphi\right). \end{aligned} \tag{2.22}$$

In order to compute the limit as $h \to \infty$ of the last term in (2.22) let us observe that by (2.20) for almost every $x \in \Omega$, $h \in N$ and every $t \in]0, 1[$ the vector $t/(1-t)(u_h^1(x) - u_h^2(x)) D\varphi(x)$ lies in the cube $2t/(1-t)\|u\|_{L^\infty(\Omega_1 \cup \Omega_2)} \|D\varphi\|_{L^\infty(\Omega_1)}] - 1, 1[^N$, hence if we denote by $\bar{z}_1, \ldots, \bar{z}_{2^N}$ the vertices of $] - 1, 1[^N$, we have by convexity that

$$f\left(x, \frac{t}{1-t}(u_h^1 - u_h^2) D\varphi\right) \leq \sum_{j=1}^{2^N} f\left(x, \frac{t}{1-t} 2\|u\|_{L^\infty(\Omega_1 \cup \Omega_2)} \|D\varphi\|_{L^\infty(\Omega_1)} \bar{z}_j\right) \tag{2.23}$$

for a.e. x in Ω.

Therefore by (2.23) and Lebesgue's dominated convergence theorem we deduce

$$\limsup_h \int_\Omega f\left(x, \frac{t}{1-t}(u_h^1 - u_h^2) D\varphi\right) = \int_\Omega f(x, 0). \tag{2.24}$$

Taking the limit as $h \to \infty$ in (2.22) we obtain by (2.19) and (2.24)

$$\hat{J}^p(\Omega, tu) \leq t\hat{J}^p(\Omega_1, u) + t\hat{J}^p(\Omega_2, u) + (1-t)\int_\Omega f(x, 0), \tag{2.25}$$

therefore as $t \to 1^-$ we deduce by (2.25)

$$\hat{J}^p(\Omega, u) \leq \liminf_{t\to 1^-} \hat{J}^p(\Omega, tu) \leq \hat{J}^p(\Omega_1, u) + \hat{J}^p(\Omega_2, u). \tag{2.26}$$

By (2.26), (2.17) follows if $u \in L^\infty(\Omega_1 \cup \Omega_2)$.

Let us prove now (2.18) again if $u \in L^\infty(\Omega_1 \cup \Omega_2)$.

Let $A \subset\subset \Omega$ and let B_1, B_2 be such that $B_i \subset\subset \Omega_i$, $i = 1, 2$, $A \subset\subset B_1 \cup B_2$.

Then by (2.17) written with $\Omega = A$, $\Omega_1 = B_1$ and $\Omega_2 = B_2$ we deduce

$$\hat{J}^p(A, u) \leq \hat{J}^p(B_1, u) + \hat{J}^p(B_2, u) \leq \hat{J}^p_-(\Omega_1, u) + \hat{J}^p_-(\Omega_2, u). \tag{2.27}$$

By (2.27), being A arbitrary, (2.18) follows.

Finally let u be in $L^1(\Omega_1 \cup \Omega_2)$ and let us prove (2.17), the proof for (2.18) being analogous.

For every $k \in N$ let $\tau_k u$ be defined by (2.5), then by lower semicontinuity, (2.17) for bounded functions and Lemma 2.2 it follows

$$\hat{J}^p(\Omega, u) \leq \liminf_k \hat{J}^p(\Omega, \tau_k u) \leq \limsup_k \hat{J}^p(\Omega_1, \tau_k u) + \tag{2.28}$$

$$+ \limsup_k \hat{J}^p(\Omega_2, \tau_k u) = \hat{J}^p(\Omega_1, u) + \hat{J}^p(\Omega_2, u),$$

that is (2.17). ∎

LEMMA 2.4. – Let f be a function as in (0.1) and (0.2). Then

$$\hat{J}^p(\Omega, u) \geq \hat{J}^p(\Omega_1, u) + \hat{J}^p(\Omega_2, u) \tag{2.29}$$

$$\hat{J}^p_-(\Omega, u) \geq \hat{J}^p_-(\Omega_1, u) + \hat{J}^p_-(\Omega_2, u) \tag{2.30}$$

for every triplet of bounded open sets Ω, Ω_1, Ω_2 with $\Omega \supseteq \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$, u in $L^1(\Omega)$, p in $[1, +\infty]$.

PROOF. – Let Ω, Ω_1, Ω_2, u, p be as above.

Let us prove (2.29).

Let $\{u_h\}_h \subseteq W^{1,p}_{\mathrm{loc}}$ be such that $u_h \to u$ in $L^1(\Omega)$ and

$$\hat{J}^p(\Omega, u) \geq \liminf_h \int_\Omega f(x, Du_h). \tag{2.31}$$

Then obviously $u_h \to u$ in $L^1(\Omega_1)$ and in $L^1(\Omega_2)$, hence by (2.31) we obtain

$$\hat{J}^p(\Omega, u) \geq \liminf_h \int_{\Omega_1} f(x, Du_h) + \liminf_h \int_{\Omega_2} f(x, Du_h) \geq \hat{J}^p(\Omega_1, u) + \hat{J}^p(\Omega_2, u), \tag{2.32}$$

that is (2.29).

Let us prove now (2.30).

Let $B_1 \subset\subset \Omega_1$, $B_2 \subset\subset \Omega_2$, then by (2.29) written with $\Omega_i = B_i$, $i = 1, 2$, we deduce

$$\hat{J}^p_-(\Omega, u) \geq \hat{J}^p(B_1 \cup B_2, u) \geq \hat{J}^p(B_1, u) + \hat{J}^p(B_2, u). \tag{2.33}$$

By (2.33), being B_1 and B_2 arbitrarily chosen, we deduce (2.30). ∎

We can now prove the following result.

THEOREM 2.5. – Let f be a function as in (0.1) and (0.2) and let $\hat{J}^p$ be defined by (1.2).

Then for every u in $L^1_{\mathrm{loc}}(\boldsymbol{R}^N)$, p in $[1, +\infty]$, the set function $\hat{J}^p_-(\cdot, u)$ is the restriction to the set of all bounded open sets of a Borel measure.

PROOF. – Let u be in $L^1_{\mathrm{loc}}(\boldsymbol{R}^N)$, p in $[1, +\infty]$, then by Lemma 2.3 and Lemma 2.4 the set function $\hat{J}^p_-(\cdot, u)$ is an additive and sub-additive inner regular increasing function such that $\hat{J}^p_-(\emptyset, u) = 0$.

The thesis now follows by Proposition 5.5 and Theorem 5.6 in [DGL]. ∎

Let $u \in L^1_{\mathrm{loc}}(\boldsymbol{R}^N)$, p in $[1, +\infty]$, let us denote by $\hat{J}^p_*(u)$ the Borel measure extending $\hat{J}^p_-(\cdot, u)$ given by Theorem 2.5.

Then since obviously for every bounded Borel set E

$$\hat{J}^p_*(u)(E) = \inf\{\hat{J}^p_-(A, u) \mid A \text{ open set, } A \supseteq E\} \tag{2.34}$$

and since $\hat{J}^p_-(u)$ verifies the following locality property

$$u, v \in L^1_{\mathrm{loc}}(\boldsymbol{R}^N), \quad u = v \text{ a.e. in a bounded open set } \Omega \Rightarrow \Rightarrow \hat{J}^p_-(\Omega, u) = \hat{J}^p_-(\Omega, v), \tag{2.35}$$

by (2.34) and (2.35) it follows that

$$(2.36) \qquad u, v \in L^1_{\mathrm{loc}}(\mathbf{R}^N), \qquad u = v \text{ a.e. in a bounded open set } \Omega \Rightarrow$$
$$\Rightarrow \hat{J}^p_*(u)(E) = \hat{J}^p_*(v)(E) \quad \text{for every Borel set } E \subseteq \Omega .$$

3. – Some examples and general identity results.

Let $\hat{I}$ and $\bar{I}$ be the functionals defined by (0.4) and (0.5) relative to a function f as in (0.1) and (0.2).

In this section we will first discuss two examples showing that the functionals $\hat{I}$ and $\bar{I}$ can be different and then we will prove some identity results.

We first report, for the sake of completeness, an example showing that, if Ω is not sufficiently regular, then $\hat{I}(\Omega, u)$ may be different from $\bar{I}(\Omega, u)$ for some u, even if f is a smooth function independent on x.

EXAMPLE 3.1. – Let $N = 1, f(x, z) = |z|^2$, $z \in \mathbf{R}$; let $\Omega =]-1, 0[\cup]0, 1[$ and let u be the function defined by

$$u(x) = \begin{cases} 1 & \text{if } x > 0 , \\ 0 & \text{if } x < 0 . \end{cases}$$

Then obviously $u \in C^1(\Omega)$, hence taking $u_h = u$ for every $h \in \mathbf{N}$, it results

$$(3.1) \qquad \bar{I}(\Omega, u) \leqslant \liminf_h \int_\Omega |Du_h|^2 = 0 .$$

On the other side it is easy to see that

$$(3.2) \qquad \hat{I}(\Omega, u) = +\infty . \qquad \blacksquare$$

We observe that if $N = 2$ examples similar to Example 3.1 can be given in which the open set Ω is also connected, in fact it suffices to take for example

$$f(x, z) = |z|^2, \qquad \Omega = \{(x, y) \mid 1 < \sqrt{x^2 + y^2} < 2\} \setminus \{(x, y) \mid -2 < x < -1, y = 0\}$$

and

$$u(x, y) = \operatorname{tg}\left(\frac{1}{2} \operatorname{arctg}\left(\frac{y}{x}\right)\right).$$

We now present an example shaped on the one in [DA2] proving that, if f depends also on x, identity between $\hat{I}$ and $\bar{I}$ may fail even for very regular bounded open sets.

Let $N = 2$, $q \in [1, 2[$ and let

$$(3.3)\qquad f(x, z) = \frac{|x_2|}{|x|^3} |\langle z, x\rangle| + |z|^q \qquad x = (x_1, x_2) \in \boldsymbol{R}^2, \qquad z = (z_1, z_2) \in \boldsymbol{R}^2,$$

$$B_\rho = \{x \in \boldsymbol{R}^2 \mid |x| < \rho\}, \qquad B = B_1, \qquad B^+ = \{x \in B \mid x_1 > 0\}, \qquad u^*(x_1, x_2) = \frac{x_2}{|x|}.$$

Obviously $u^* \in W^{1,q}_{\mathrm{loc}}(\boldsymbol{R}^2)$.

The function f verifies the following growth conditions

$$(3.4)\qquad |z|^q \leq f(x, z) \leq \frac{1}{|x|} |z| + |z|^q \leq \frac{r-1}{r} |x|^{-r/(r-1)} + \frac{1}{r} |z|^r + |z|^q$$

for every $x \in \boldsymbol{R}^2$, $z \in \boldsymbol{R}^2$ and $r > 1$,

hence (0.2) follows if $r > 2$.

Let us recall that by Proposition 1.3 in [DA2] we get for $\hat{J}$ given by (1.2) with $p = +\infty$ and relative to f in (3.3)

$$(3.5)\qquad \hat{J}(B, u) = \pi + \int_B |Du^*|^q.$$

LEMMA 3.2. – Let f be given by (3.3). Then

$$(3.6)\qquad \hat{J}(B^+, u) = \frac{\pi}{2} + \int_{B^+} |Du^*|^q.$$

PROOF. – Let $\{u_h\}_h \subseteq W^{1,\infty}_{\mathrm{loc}}$ be such that $u_h \to u^*$ in $L^1(B^+)$, let us define the functions $\tilde{u}_h$ by

$$(3.7)\qquad \tilde{u}_h(x_1, x_2) = \begin{cases} u_h(x_1, x_2) & \text{if } x_1 \geq 0, \\ u_h(-x_1, x_2) & \text{if } x_1 < 0. \end{cases}$$

Then $\tilde{u}_h \in W^{1,\infty}_{\mathrm{loc}}$ for every $h \in \boldsymbol{N}$, $\tilde{u}_h \to u^*$ in $L^1(B)$ and by (3.5) we get

$$(3.8)\qquad \liminf_h \int_{B^+} f(x, Du_h) = \frac{1}{2} \liminf_h \int_B f(x, D\tilde{u}_h) \geq$$

$$\geq \frac{\pi}{2} + \frac{1}{2} \int_B |Du^*|^q = \frac{\pi}{2} + \int_{B^+} |Du^*|^q,$$

hence

$$(3.9)\qquad \hat{J}(B^+, u^*) \geq \frac{\pi}{2} + \int_{B^+} |Du^*|^q.$$

In order to prove the equality in (3.9) let us observe that

$$\langle x, Du^*(x)\rangle = 0 \quad \text{a.e. in } \boldsymbol{R}^2 \tag{3.10}$$

and for every $h \in \boldsymbol{N}$ let us define the functions v_h by

$$v_h(x_1, x_2) = \begin{cases} u^*(x_1, x_2) & \text{if } |x| > \dfrac{1}{h}, \\ hx_2 & \text{if } |x| < \dfrac{1}{h}, \end{cases} \tag{3.11}$$

then $v_h \in W^{1,\infty}_{\text{loc}}$ for every $h \in \boldsymbol{N}$ and $v_h \to u^*$ in $W^{1,q}_{\text{loc}}(\boldsymbol{R}^2)$; this yields that

$$\int_{B^+} |Dv_h|^q \to \int_{B^+} |Du^*|^q. \tag{3.12}$$

On the other side by (3.10) we get

$$\int_{B^+} \frac{|x_2|}{|x|^3} |\langle x, Dv_h(x)\rangle| \, dx = \int_{B^+ \cap (1/h)B} \frac{|x_2|}{|x|^3} |hx_2| \, dx = h \int_0^{1/h} \int_0^{\pi} |\sin^2 \vartheta| \, d\vartheta \, d\rho = \frac{\pi}{2}. \tag{3.13}$$

By (3.12) and (3.13) we deduce the opposite inequality to (3.9) and therefore (3.6) follows. ■

By virtue of Lemma 3.2, of (3.10), recalling that $u^* \in C^1(B^+)$ and taking $u_h = u^*$ for every $h \in \boldsymbol{N}$ we get

$$\hat{J}(B^+, u^*) = \frac{\pi}{2} + \int_{B^+} |Du^*|^q > \int_{B^+} |Du^*|^q = \liminf_h \int_{B^+} f(x, Du_h) \geq \bar{I}(B^+, u). \tag{3.14}$$

Therefore by (3.14) and Proposition 1.2 it follows that

$$\hat{I}(B^+, u^*) = \hat{J}(B^+, u^*) > \bar{I}(B^+, u^*). \tag{3.15}$$

REMARK 3.3. – Let us observe that by virtue of a computation similar to the one of Proposition 1.3 in [DA2] we can prove that

$$\hat{J}(B_\rho, u^*) = \pi + \int_{B_\rho} |Du^*|^q \quad \text{for every } \rho > 0, \tag{3.16}$$

hence by (3.5) and (3.16) we soon deduce that

$$\hat{J}(B, u^*) = \hat{J}_-(B, u^*). \tag{3.17}$$

Therefore by (1.5), (3.17), (3.5) and recalling that $u^* \in W^{1,q}(B)$ we get

$$(3.18) \quad \bar{J}(B, u^*) \geq \hat{J}_-(B, u^*) = \hat{J}(B, u^*) = \pi + \int_B |Du^*|^q >$$

$$> \int_B |Du^*|^q = \int_B f(x, Du^*) \geq \hat{J}^q(B, u^*) = \bar{J}^q(B, u^*),$$

that is $\bar{J}(B, u^*)$ is different from $\bar{J}^q(B, u^*)$ as $q < 2$.

We now prove a first identity result.

PROPOSITION 3.4. – Let f be a function as in (0.1) and (0.2).
Let $\hat{I}, \bar{I}, \hat{J}^p, \bar{J}^p$ be the functionals defined by (0.4), (0.5), (1.2) and (1.3) and let u be in $L^1_{\text{loc}}(\mathbf{R}^N)$.
Then there exists a dense family $\mathfrak{F}$ of bounded open sets such that

$$(3.19) \quad \hat{I}(\Omega, u) = \bar{I}(\Omega, u) = \hat{I}_-(\Omega, u) = \hat{J}_-(\Omega, u) = \bar{J}(\Omega, u) = \hat{J}(\Omega, u)$$

for every Ω in $\mathfrak{F}$;

moreover for every p in $[1, +\infty]$ there exists a dense family $\mathfrak{F}_p$ of bounded open sets such that

$$(3.20) \quad \hat{J}^p(\Omega, u) = \bar{J}^p(\Omega, u) = \hat{J}^p_-(\Omega, u) \quad \text{for every } \Omega \text{ in } \mathfrak{F}_p.$$

PROOF. – The proof easily follows by Proposition 1.2, Proposition 1.1 and (1.5). ■

In the one dimensional case it is possible to prove the following result.

PROPOSITION 3.5. – Let $N = 1$, f be a function as in (0.1) and (0.2) and let $\hat{I}, \bar{I}, \hat{J}^p, \bar{J}^p$ be defined by (0.4), (0.5), (1.2) and (1.3). Then

$$(3.21) \quad \hat{I}(\Omega, u) = \bar{I}(\Omega, u) = \hat{I}_-(\Omega, u) = \hat{J}^p_-(\Omega, u) = \bar{J}^p(\Omega, u) = \hat{J}^p(\Omega, u)$$

for every bounded open interval Ω, u in $L^1(\Omega)$, p in $[1, +\infty]$.

PROOF. – Let $\Omega =]a, b[$, u, p be as above.
Let us first assume in addition that $u \in L^\infty(\Omega)$ and prove that

$$(3.22) \quad \hat{J}^p_-(\Omega, u) \geq \hat{J}^p(\Omega, u).$$

Obviously it results

$$(3.23) \quad \hat{J}^p_-(\Omega, u) = \lim_h \hat{J}^p(\Omega^-_{1/h}, u)$$

and for every $h \in \mathbf{N}$ there exists $\{u^h_k\}_k \subseteq W^{1,p}_{\text{loc}}$, with $u^h_k \to u$ in $L^1(\Omega^-_{1/h})$ as $k \to \infty$, such that

$$\hat{J}^p(\Omega^-_{1/h}, u) \geq \liminf_k \int_{\Omega^-_{1/h}} f(x, (u^h_k)'). \tag{3.24}$$

Moreover by Lemma 2.1 we can assume that

$$\|u^h_k\|_{L^\infty(\Omega^-_{1/h})} \leq \|u\|_{L^\infty(\Omega)} \quad \text{for every } h, k \in \boldsymbol{N}. \tag{3.25}$$

By virtue of (3.23), (3.24) and of a diagonal process we can construct a sequence of integer numbers $\{k_h\}_h$ such that, setting $u_h = u^h_{k_h}$, we have $u_h \to u$ in $L^1_{\text{loc}}(\Omega)$ and

$$\hat{J}^p_-(\Omega, u) \geq \liminf_h \int_{\Omega^-_{1/h}} f(x, u'_h). \tag{3.26}$$

Moreover by (3.25) it follows

$$\|u_h\|_{L^\infty(\Omega^-_{1/h})} \leq \|u\|_{L^\infty(\Omega)} \quad \text{for every } h \in \boldsymbol{N}. \tag{3.27}$$

For every $h \in \boldsymbol{N}$ let us define the functions $\tilde{u}_h$ as

$$\tilde{u}_h(x) = \begin{cases} u_h(a + 1/h) & \text{if } x < a + 1/h, \\ u_h(x) & \text{if } a + 1/h \leq x \leq b - 1/h, \\ u_h(b - 1/h) & \text{if } x > b - 1/h. \end{cases} \tag{3.28}$$

Then obviously $\tilde{u}_h \in W^{1,p}_{\text{loc}}$ and, by virtue of (3.27), $\tilde{u}_h \to u$ in $L^1(\Omega)$.
By (3.26) and (0.2) we get

$$\hat{J}^p(\Omega, u) \leq \liminf_h \int_\Omega f(x, \tilde{u}'_h) \leq \liminf_h \int_{\Omega^-_{1/h}} f(x, u'_h) + \\ + \limsup_h \int_a^{a+1/h} f(x, 0) + \limsup_h \int_{b-1/h}^b f(x, 0) \leq \hat{J}^p_-(\Omega, u), \tag{3.29}$$

that is (3.22) when $u \in L^\infty(\Omega)$.
If u only is in $L^1(\Omega)$ let, for $k \in \boldsymbol{N}$, $\tau_k u$ be defined by (2.5).
Then by lower semicontinuity, (3.22) for bounded functions and Lemma 2.2 it results

$$\hat{J}^p(\Omega, u) \leq \liminf_k \hat{J}^p(\Omega, \tau_k u) \leq \liminf_k \hat{J}^p_-(\Omega, \tau_k u) = \hat{J}^p_-(\Omega, u) \tag{3.30}$$

that is (3.22).
In order to complete the proof let us prove that

$$\hat{J}^p(\Omega, u) \geq \hat{J}(\Omega, u). \tag{3.31}$$

To this aim let $\{u_h\}_h \subseteq W^{1,p}_{\mathrm{loc}}$ be such that $u_h \to u$ in $L^1(\Omega)$ and

$$\hat{J}^p(\Omega, u) \geq \liminf_h \int_\Omega f(x, u_h'). \tag{3.32}$$

Let $\{\sigma_h\}_h$ be a sequence of positive numbers converging to 0 and let, for every h, $k \in \boldsymbol{N}$, u_h^k be the function defined by

$$u_h^k(x) = u_h(a) + \int_a^x [-k \vee (u_h'(t) \wedge k)]\, dt.$$

Obviously it results

$$|u_h(x) - u_h^k(x)| \leq \int_a^k |u_h'(t) - [-k \vee (u_h'(t) \wedge k)]|\, dt \leq \tag{3.33}$$

$$\leq \int_{\{t \in \Omega:\ |u_h'(t)| > k\}} ||u_h'(t)| - k|\, dt \quad \text{for every } x \text{ in } \Omega,$$

hence if we choose k_h so that

$$\int_{\{t \in \Omega:\ |u_h'(t)| > k_h\}} ||u_h'(t)| - k_h|\, dt < \sigma_h, \qquad |\{t \in \Omega:\ |u_h'(t)| > k_h\}| < \sigma_h \tag{3.34}$$

by (3.33) and (3.34) it follows that $u_h^{k_h} \in W^{1,\infty}_{\mathrm{loc}}$, $u_h^{k_h} \to u$ in $L^1(\Omega)$ as $h \to \infty$ and that

$$\hat{J}(\Omega, u) \leq \liminf_h \int_\Omega f(x, (u_h^{k_h})') \leq \tag{3.35}$$

$$\leq \liminf_h \left[\int_{\{x \in \Omega:\ |u_h'(x)| \leq k_h\}} f(x, u_h') + \int_{\{x \in \Omega:\ u_h'(x) > k_h\}} f(x, k_h) + \int_{\{x \in \Omega:\ u_h'(x) < -k_h\}} f(x, -k_h) \right].$$

Moreover by (3.35), convexity, (3.34) and (3.32) it results

$$\hat{J}(\Omega, u) \leq \liminf_h \left[\int_{\{x \in \Omega:\ |u_h'(x)| \leq k_h\}} f(x, u_h') + \right. \tag{3.36}$$

$$+ \int_{\{x \in \Omega:\ u_h'(x) > k_h\}} [(k_h / u_h'(x)) f(x, u_h') + (1 - k_h / u_h'(x)) f(x, 0)] +$$

$$+ \int_{\{x \in \Omega:\, u_h'(x) < -k_h\}} [(-k_h/u_h'(x))\, f(x, u_h') + (1 + k_h/u_h'(x))\, f(x, 0)] \Bigg] \leq$$

$$\leq \liminf_h \int_\Omega f(x, u_h') + \limsup_h \int_{\{x \in \Omega:\, |u_h'(x)| > k_h\}} f(x, 0) \leq \hat{J}^p(\Omega, u).$$

Therefore (3.31) follows by (3.36).

Now by (1.5), (3.22) and (3.31) it follows

$$\hat{I}(\Omega, u) \geq \bar{I}(\Omega, u) \geq \hat{I}_-(\Omega, u) \geq \hat{J}^p_-(\Omega, u) \geq \hat{J}^p(\Omega, u) \geq \hat{J}(\Omega, u), \tag{3.37}$$

hence by (3.37) and Proposition 1.2 (3.21) follows. ■

4. - The case of integrand not depending on x.

Let $\hat{I}$, $\hat{I}_0$ be the functionals defined in (0.4) and (0.7).

Throughout this section we will assume that the function f in (0.1) does not depend on x.

By adopting a proof already used in [DA1] (Theorem 2.5 in [DA1]) we first prove the following result.

LEMMA 4.1. - Let f be a nonnegative convex finite funtion on $\boldsymbol{R}^N$ and let $\hat{J}$ be defined by (1.2) with $p = +\infty$. Then

$$\hat{J}(\Omega, u) = \hat{J}_-(\Omega, u) \tag{4.1}$$

for every strongly star-shaped bounded open set Ω, u in $L^1(\Omega)$.

PROOF. - Let Ω, u be as above.

For simplicity let us assume that Ω is star-shaped with respect to 0.

Obviously by (1.5) we only have to prove that

$$\hat{J}_-(\Omega, u) \geq \hat{J}(\Omega, u). \tag{4.2}$$

Let $s \in]0, 1[$ and let $\{u_h\}_h \subseteq W^{1,\infty}_{\text{loc}}$ be such that $u_h \to u$ in $L^1(s\Omega)$ and

$$\hat{J}(s\Omega, u) \geq \liminf_h \int_{s\Omega} f(Du_h). \tag{4.3}$$

By virtue of our assumptions on Ω let $t \in]0, s[$, then $(1/t)s\Omega \supset\supset \Omega$.

Define the functions u_h^t, u^t by $u_h^t(y) = (1/t)\, u_h(ty)$, $u^t(y) = (1/t)\, u(ty)$, then

$$u_h^t \to u^t \quad \text{in } L^1(\Omega) \text{ as } h \to \infty. \tag{4.4}$$

By (4.3), (4.4) we deduce

$$\tilde{J}(s\Omega, u) \geq \liminf_h t^N \int_{(1/t)s\Omega} f(D_x u_h(ty))\, dy \geq \tag{4.5}$$

$$\geq \liminf_h t^N \int_\Omega f(D_y u_h^t(y))\, dy \geq t^N \tilde{J}(\Omega, u^t).$$

Letting first $s \to 1^-$ and then $t \to 1^-$ we deduce (4.2) by (4.5), therefore (4.1) follows. ■

We need the following result of measure theory.

LEMMA 4.2. – Let Ω be a bounded open set and let μ_h, $h \in \boldsymbol{N}$, μ be nonnegative Borel measures on Ω such that

$$\limsup_h \mu_h(\Omega) \leq \mu(\Omega) < +\infty, \tag{4.6}$$

$$\liminf_h \mu_h(A) \geq \mu(A) \quad \text{for every open set } A \subseteq \Omega. \tag{4.7}$$

Then the following limit exists and it results

$$\lim_h \int_\Omega \varphi\, d\mu_h = \int_\Omega \varphi\, d\mu \quad \text{for every } \varphi \text{ in } C^0(\overline{\Omega}). \tag{4.8}$$

PROOF. – For every $h \in \boldsymbol{N}$ let us define the measure $\bar{\mu}_h$ on $\overline{\Omega}$ as

$$\bar{\mu}_h(E) = \mu_h(E \cap \Omega) \quad \text{for every Borel set } E \subseteq \overline{\Omega}. \tag{4.9}$$

By virtue of (4.6) the sequence $\{\bar{\mu}_h(\overline{\Omega})\}_h$ is bounded, therefore there exists a nonnegative Borel measure ν, finite on $\overline{\Omega}$, such that

$$\lim_h \int_{\overline{\Omega}} \varphi\, d\bar{\mu}_h = \int_{\overline{\Omega}} \varphi\, d\nu \quad \text{for every } \varphi \text{ in } C^0(\overline{\Omega}). \tag{4.10}$$

Actually (4.10) would only hold for a subsequence $\{\bar{\mu}_{h_k}\}_k$ but, since we will describe the limit measure ν, we can assume that (4.10) holds.

Let us prove the inequality

$$\nu(B) \geq \mu(B) \quad \text{for every open set } B \subseteq \Omega. \tag{4.11}$$

Take $B \subseteq \Omega$, $A \subset\subset B$ and $\varphi \in C_0^0(B)$ with $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on A, then by (4.10) and (4.7) we deduce

$$\nu(B) \geq \int_B \varphi\, d\nu = \lim_h \int_B \varphi\, d\bar{\mu}_h = \lim_h \int_B \varphi\, d\mu_h \geq \liminf_h \mu_h(A) \geq \mu(A). \tag{4.12}$$

As A increasingly converges to B we deduce (4.11) from (4.12).

Let us observe now that, choosing $\varphi \equiv 1$ in (4.10), we obtain by (4.6)

$$\mu(\Omega) \geq \limsup_h \mu_h(\Omega) = \limsup_h \bar{\mu}_h(\Omega) = \nu(\bar{\Omega}) \geq \nu(\Omega). \tag{4.13}$$

By (4.11) and (4.13) we deduce that

$$\nu(\partial\Omega) = 0 \tag{4.14}$$

and also, by standard arguments, that

$$\nu = \mu \quad \text{on } \Omega. \tag{4.15}$$

By (4.15), (4.14) and (4.10) it follows

$$\int_\Omega \varphi \, d\mu = \int_\Omega \varphi \, d\nu = \int_{\bar{\Omega}} \varphi \, d\nu = \lim_h \int_{\bar{\Omega}} \varphi \, d\bar{\mu}_h = \lim_h \int_\Omega \varphi \, d\mu_h \quad \text{for every } \varphi \in C^0(\bar{\Omega}), \tag{4.16}$$

hence the thesis follows. ∎

LEMMA 4.3. – Let f be a nonnegative convex finite funtion on $\boldsymbol{R}^N$ and let $\hat{J}$ be defined by (1.2) with $p = +\infty$. Then

$$\hat{J}_-(\Omega, u) = \hat{J}(\Omega, u) \tag{4.17}$$

for every bounded open set Ω with Lipschitz boundary Ω, u in $L^1(\Omega)$.

PROOF. – Let Ω be as above. Let us assume first that $u \in L^\infty(\Omega)$.
Let us prove that

$$\hat{J}_-(\Omega, u) \geq \hat{J}(\Omega, u). \tag{4.18}$$

To this aim we can assume that $\hat{J}_-(\Omega, u) < +\infty$.
By virtue of Lemma 1.4 let $\{\tilde{\Omega}_j\}_{j=1,\dots,s}$ be a finite open covering of $\bar{\Omega}$ such that each $\Omega_j = \tilde{\Omega}_j \cap \Omega$ is strongly star-shaped with Lipschitz boundary.
Let $\{\alpha_j\}_{j=1,\dots,s}$ be functions in $C_0^\infty(\boldsymbol{R}^N)$ such that

$$0 \leq \alpha_j \leq 1, \quad \sum_{j=1}^s \alpha_j = 1 \quad \text{on } V \supset\supset \bar{\Omega}, \ \operatorname{spt}(\alpha_j) \subset\subset \tilde{\Omega}_j. \tag{4.19}$$

By virtue of Lemma 4.1 for every $j = 1, \dots, s$ let $\{u_h^j\}_h \subseteq W^{1,\infty}_{\text{loc}}$ be such that

$$\begin{cases} \text{i)}\ u_h^j \to u \quad \text{in } L^1(\Omega_j) \text{ and a.e. in } \Omega_j \text{ as } h \to \infty, \\ \text{ii)}\ +\infty > \hat{J}_-(\Omega_j, u) \geq \limsup\limits_h \int_{\Omega_j} f(Du_h^j). \end{cases} \tag{4.20}$$

By setting, for every $j = 1, \dots, s$, $\mu_s = \int_{(\cdot)} f(Du_h^j)$, $\mu = \hat{J}_*(u)$, $\hat{J}_*(u)$ being the measure given by Theorem 2.5 and verifying (2.34), it turns out that the assumptions of Lemma 4.2 are fulfilled on Ω_j.

In fact (4.6) follows by (4.20) and (4.7) by the definition of $\hat{J}$ and i) of (4.20).

Therefore Lemma 4.2 applies and we get

$$\lim_h \int_{\Omega_j} \varphi f(Du_h^j) = \int_{\Omega_j} \varphi \, d\hat{J}_*(u) \quad \text{for every } \varphi \text{ in } C^0(\overline{\Omega}_j). \tag{4.21}$$

By Lemma 2.1 we can assume that

$$\|u_h^j\|_{L^\infty(\Omega_j)} \le \|u\|_{L^\infty(\Omega)} \quad \text{for every } j = 1, \ldots, s. \tag{4.22}$$

Let $t \in]0,1[$ and define

$$u_h^t = t \sum_{j=1}^{s} \alpha_j u_h^j \in W^{1,\infty}_{\mathrm{loc}}; \tag{4.23}$$

then by (4.20) i) it results

$$u_h^t \to tu \quad \text{in } L^1(\Omega) \text{ as } h \to \infty. \tag{4.24}$$

We have by convexity

$$\begin{aligned}\int_\Omega f(Du_h^t) &\le t \int_\Omega f\left(\sum_{j=1}^{s} \alpha_j Du_h^j\right) + (1-t)\int_\Omega f\left(\frac{t}{1-t}\sum_{j=1}^{s} u_h^j D\alpha_j\right) \le \\ &\le t \sum_{j=1}^{s} \int_{\Omega_j} \alpha_j f(Du_h^j) + (1-t)\int_\Omega f\left(\frac{t}{1-t}\sum_{j=1}^{s} u_h^j D\alpha_j\right).\end{aligned} \tag{4.25}$$

As $h \to \infty$ by (4.20) i), (4.21), (4.22) and Lebesgue dominated convergence theorem we get from (4.25)

$$\begin{aligned}\hat{J}(\Omega, tu) &\le \limsup_h \int_\Omega f(Du_h^t) \le t \sum_{j=1}^{s} \int_{\Omega_j} \alpha_j \, d\hat{J}_*(u) + (1-t)\int_\Omega f\left(\frac{t}{1-t} u \sum_{j=1}^{s} D\alpha_j\right) = \\ &= t \int_\Omega d\hat{J}_*(u) + (1-t) f(0)|\Omega| = t\hat{J}_-(\Omega, u) + (1-t) f(0)|\Omega|,\end{aligned} \tag{4.26}$$

hence by (4.26) we deduce as $t \to 1^-$

$$\hat{J}(\Omega, u) \le \liminf_{t \to 1^-} \hat{J}(\Omega, tu) \le \hat{J}_-(\Omega, u). \tag{4.27}$$

In order to deduce (4.27) when $u \in L^1(\Omega)$ we only have to observe that by (4.27) and Lemma 2.2 it follows

$$\hat{J}(\Omega, u) \le \liminf_k \hat{J}(\Omega, \tau_k u) \le \liminf_k \hat{J}_-(\Omega, \tau_k u) \le \hat{J}_-(\Omega, u) \tag{4.28}$$

$\tau_k u$ being defined by (2.5).

By (4.28) and (1.5), (4.17) follows. ■

We can now prove an identity result.

THEOREM 4.4. – Let f be a nonnegative convex finite function on $\boldsymbol{R}^N$ and let $\hat{I}$, $\bar{I}$, $\hat{J}$, $\bar{J}$ be the functionals defined in (0.4), (0.5) and in (1.2), (1.3) with $p = +\infty$. Then

$$\hat{I}(\Omega, u) = \bar{I}(\Omega, u) = \hat{I}_-(\Omega, u) = \hat{J}_-(\Omega, u) = \bar{J}(\Omega, u) = \hat{J}(\Omega, u) \tag{4.29}$$

for every bounded open set Ω with Lipschitz boundary, u in $L^1(\Omega)$.

PROOF. – Let Ω be a bounded open set with Lipschitz boundary, u in $L^1(\Omega)$.
By Proposition 1.2 and Lemma 4.3 it follows

$$\hat{I}(\Omega, u) = \hat{J}(\Omega, u) = \hat{J}_-(\Omega, u) = \hat{I}_-(\Omega, u), \tag{4.30}$$

hence (4.29) follows by (4.30) and (1.5). ■

We now consider the case of null boundary datum.
As a first step let us prove the following result by adopting a proof performed in [DA1] (Lemma 3.4 and Lemma 3.6 of [DA1]).

LEMMA 4.5. – Let f be a nonnegative convex finite function on $\boldsymbol{R}^N$ and let $\hat{J}_0$ be defined by (1.4) with $p = +\infty$. Then

$$\hat{J}_0(\Omega, u) = \hat{J}_-(\Omega', u) - f(0)|\Omega' \setminus \Omega| \tag{4.31}$$

for every couple of bounded open sets Ω, Ω' with Ω strongly star-shaped, $\Omega \subset\subset \Omega'$, u in $L^1(\boldsymbol{R}^N)$ with $u = 0$ in $\boldsymbol{R}^N \setminus \Omega$.

PROOF. – Let Ω, Ω' be two bounded open sets with $\Omega \subset\subset \Omega'$ and let u be as above.
Obviously by (1.8) and (1.5) it results

$$\hat{J}_0(\Omega, u) \geq \hat{J}_0(\Omega', u) - f(0)|\Omega' \setminus \Omega| \geq \hat{J}(\Omega', u) - f(0)|\Omega' \setminus \Omega| \geq$$
$$\geq \hat{J}_-(\Omega', u) - f(0)|\Omega' \setminus \Omega|. \tag{4.32}$$

In order to prove the reverse inequality to (4.32) let us assume that Ω is also strongly star-shaped and that $u \in L^\infty(\boldsymbol{R}^N)$, $u = 0$ in $\boldsymbol{R}^N \setminus \Omega$.
For simplicity let us assume that Ω is star-shaped with respect to 0.
Let A, B be open sets with $\Omega \subset\subset A \subset\subset B \subset\subset \Omega'$ and let $\{u_h\}_h \subseteq W^{1,\infty}_{\mathrm{loc}}$ with $u_h \to u$ in $L^1(A)$ and a.e. in A be such that

$$\hat{J}(A, u) \geq \limsup_h \int_A f(Du_h). \tag{4.33}$$

Obviously by Lemma 2.1 we can assume that

$$\|u_h\|_{L^\infty(A)} \leq \|u\|_{L^\infty(\Omega)}. \tag{4.34}$$

Let $\varphi \in C_0^1(A)$ with $0 \leq \varphi \leq 1$ be such that $\varphi \equiv 1$ on Ω and set $w_h = \varphi u_h$, then $w_h \in W_0^{1,\infty}(A)$ and, since $u = 0$ in $\boldsymbol{R}^N \setminus \Omega$, $w_h \to u$ in $L^1(A)$.

For every $t \in]0,1[$ we have by convexity

$$\int_A f(tDw_h) \leq t\int_A \varphi f(Du_h) + t\int_A (1-\varphi) f(0) + (1-t)\int_A f\left(\frac{t}{1-t} u_h D\varphi\right). \tag{4.35}$$

By (4.35), (4.33), (4.34) and Lebesgue dominated convergence theorem we get as $h \to \infty$

$$\hat{J}_0(A, tu) \leq t\hat{J}(A,u) + tf(0)|A\setminus\Omega| + (1-t)f(0)|A|, \tag{4.36}$$

hence as $t \to 1^-$ we obtain by (4.36)

$$\hat{J}_0(A,u) \leq \hat{J}(A,u) + f(0)|A\setminus\Omega|. \tag{4.37}$$

Let us now observe that by Lemma 2.4 it follows

$$\hat{J}(A,u) + \hat{J}(B\setminus\overline{A}, u) \leq \hat{J}(B,u). \tag{4.38}$$

Moreover, being the functional $u \in W^{1,\infty}_{\mathrm{loc}} \mapsto \int_{B\setminus\overline{A}} f(Du)$ $L^1(B\setminus\overline{A})$-lower semicontinuous on $W^{1,\infty}_{\mathrm{loc}}$ (see for example [Se2], [Mo]) and recalling that $u = 0$ in $B\setminus\overline{A}$, it turns out that

$$\hat{J}(B\setminus\overline{A}, u) \geq f(0)|B\setminus\overline{A}|. \tag{4.39}$$

By (4.37), (4.38) and (4.39) we deduce

$$\hat{J}_0(A,u) \leq \hat{J}(B,u) - f(0)[|B\setminus\overline{A}| - |A\setminus\Omega|], \tag{4.40}$$

hence as B increase to Ω' we obtain by (4.40)

$$\hat{J}_0(A,u) \leq \hat{J}_-(\Omega', u) - f(0)[|\Omega'\setminus\overline{A}| - |A\setminus\Omega|]. \tag{4.41}$$

Let now $t > 1$ and let $\{u_h\}_h \subseteq W_0^{1,\infty}(t\Omega)$ be such that $u_h \to u$ in $L^1(t\Omega)$ and

$$\hat{J}_0(t\Omega, u) \geq \liminf_h \int_{t\Omega} f(Du_h). \tag{4.42}$$

Setting $u_h^t(x) = 1/t\ u_h(tx)$, $u^t(x) = 1/t\ u(tx)$ we have that $u_h^t \to u^t$ in $L^1(\Omega)$ as $h \to \infty$ and that, by (4.42)

$$\hat{J}_0(t\Omega, u) \geq \liminf_h t^N \int_\Omega f(Du_h^t) \geq t^N \hat{J}_0(\Omega, u^t). \tag{4.43}$$

By (4.43) and (4.41), written with $A = t\Omega$, we deduce as $t \to 1^+$

$$\hat{J}_-(\Omega', u) - f(0)|\Omega' \setminus \Omega| \geq \liminf_{t \to 1^+} t^N \hat{J}_0(\Omega, u^t) \geq \hat{J}_0(\Omega, u). \tag{4.44}$$

In order to prove (4.44) when $u \in L^1(\boldsymbol{R}^N)$ with $u = 0$ in $\boldsymbol{R}^N \setminus \Omega$ let us define for every $k \in \boldsymbol{N}$ $\tau_k u$ as in (2.5).

Then by Lemma 2.2 and (4.44) we get

$$\hat{J}_-(\Omega', u) - f(0)|\Omega' \setminus \Omega| = \lim_k \hat{J}_-(\Omega', \tau_k u) - f(0)|\Omega' \setminus \Omega| \geq \tag{4.45}$$

$$\geq \liminf_k \hat{J}_0(\Omega, \tau_k u) \geq \hat{J}_0(\Omega, u),$$

hence by (4.45) we obtain

$$\hat{J}_-(\Omega', u) - f(0)|\Omega' \setminus \Omega| \geq \hat{J}_0(\Omega, u). \tag{4.46}$$

Therefore by (4.46) and (4.32) (4.31) follows. ■

We are now in a position to prove the following representation result analogous to Theorem 4.4.

PROPOSITION 4.6. – Let f be a nonnegative convex function on $\boldsymbol{R}^N$ and let $\hat{I}$, $\hat{I}_0$, $\hat{J}$, $\hat{J}_0$ be defined by (0.4), (0.7), (1.2) and (1.4) with $p = +\infty$. Then

$$\hat{I}_0(\Omega, u) = \hat{J}_0(\Omega, u) = \hat{J}_-(\Omega', u) - f(0)|\Omega' \setminus \Omega| = \hat{I}_-(\Omega', u) - f(0)|\Omega' \setminus \Omega| \tag{4.47}$$

for every couple of bounded open sets Ω, Ω' with Ω having Lipschitz boundary, $\Omega \subset\subset \Omega'$, u in $L^1(\boldsymbol{R}^N)$ with $u = 0$ in $\boldsymbol{R}^N \setminus \Omega$.

PROOF. – Let Ω, Ω' be as above.

The proof of the following inequality

$$\hat{J}_0(\Omega, u) \geq \hat{J}_-(\Omega', u) - f(0)|\Omega' \setminus \Omega| \quad \text{for every } u \text{ in } L^1(\boldsymbol{R}^N) \text{ with } u = 0 \text{ in } \boldsymbol{R}^N \setminus \Omega \tag{4.48}$$

comes as in (4.32).

In order to prove the reverse inequality let us first assume that $u \in L^\infty(\boldsymbol{R}^N)$, $u = 0$ in $\boldsymbol{R}^N \setminus \Omega$, moreover it is not restrictive to assume that $\hat{J}_-(\Omega', u) < +\infty$.

By Lemma 1.4 let $\{\tilde{\Omega}_j\}_{j=1,\dots,s}$ be a finite open covering of $\overline{\Omega}$ such that $\Omega_j = \tilde{\Omega}_j \cap \Omega$ is strongly star-shaped with Lipschitz boundary and let $\{\alpha_j\}_{j=1,\dots,s}$ be functions in $C_0^\infty(\boldsymbol{R}^N)$ verifying (4.19).

Obviously we can assume that $V \supset\supset \Omega'$.

For every $j = 1, \dots, s$ let us define the functions u^j as

$$u^j(x) = \begin{cases} u(x) & \text{if } x \in \Omega_j \\ 0 & \text{if } x \in \boldsymbol{R}^N \setminus \Omega_j \end{cases} \tag{4.49}$$

and let, by Lemma 4.5, $\{u_h^j\}_h \subseteq W_0^{1,\infty}(\Omega_j)$ be such that $u_h^j \to u^j$ in $L^1(\Omega_j)$ and a.e. in Ω_j as $h \to \infty$ and

$$\text{(4.50)} \qquad +\infty > \hat{J}_-(\Omega', u^j) - f(0)|\Omega' \setminus \Omega_j| \geq \limsup_h \int_{\Omega_j} f(Du_h^j).$$

Looking on the functions u_h^j as defined on the whole $\boldsymbol{R}^N$ by $u_h^j = 0$ for every $x \in \boldsymbol{R}^N \setminus \Omega_j$ we deduce by (4.50)

$$\text{(4.51)} \qquad +\infty > \hat{J}_-(\Omega', u^j) \geq \limsup_h \int_{\Omega'} f(Du_h^j).$$

As usual, by Lemma 2.1, we can assume that

$$\text{(4.52)} \qquad \|u_h^j\|_{L^\infty(\Omega_j)} \leq \|u\|_{L^\infty(\Omega)} \quad \text{for every } j = 1, \dots, s.$$

For every $t \in]0, 1[$ let us define

$$\text{(4.53)} \qquad u_h^t = t \sum_{j=1}^{s} \alpha_j u_h^j \in W_0^{1,\infty}(\Omega),$$

then, since $\alpha_j u^j = \alpha_j u$ on $\boldsymbol{R}^N$ for every j, it results

$$\text{(4.54)} \qquad u_h^t \to tu \quad \text{in } L^1(\Omega) \text{ as } h \to \infty.$$

As in (4.25) we have by convexity

$$\text{(4.55)} \qquad \int_{\Omega} f(Du_h^t) \leq t \sum_{j=1}^{s} \int_{\Omega_j} \alpha_j f(Du_h^j) + (1-t) \int_{\Omega} f\left(\frac{t}{1-t} \sum_{j=1}^{s} u_h^j D\alpha_j\right).$$

By Lemma 4.2, as in the proof of (4.21), we deduce that

$$\text{(4.56)} \qquad \lim_h \int_{\Omega'} \varphi f(Du_h^j) = \int_{\Omega'} \varphi \, d\hat{J}_*(u^j) \quad \text{for every } \varphi \in C^0(\bar{\Omega}'),$$

hence by (4.56) it follows that

$$\text{(4.57)} \qquad \lim_h \int_{\Omega} \alpha_j f(Du_h^j) = \lim_h \left\{ \int_{\Omega'} \alpha_j f(Du_h^j) - f(0) \int_{\Omega' \setminus \Omega} \alpha_j \right\} = \int_{\Omega'} \alpha_j \, d\hat{J}_*(u^j) - f(0) \int_{\Omega' \setminus \Omega} \alpha_j.$$

Since $\alpha_j \in C_0^\infty(\tilde{\Omega}_j)$ for every j, we have

$$\text{(4.58)} \qquad \int_{\Omega'} \alpha_j \, d\hat{J}_*(u^j) = \int_{\tilde{\Omega}_j \cap \Omega'} \alpha_j \, d\hat{J}_*(u^j),$$

hence, recalling that $u^j = u$ on $\tilde{\Omega}_j$, we deduce by (2.36) and (4.58) that

$$\text{(4.59)} \qquad \int_{\Omega'} \alpha_j \, d\hat{J}_*(u^j) = \int_{\tilde{\Omega}_j \cap \Omega'} \alpha_j \, d\hat{J}_*(u) = \int_{\Omega'} \alpha_j \, d\hat{J}_*(u).$$

By (4.55), (4.57), (4.59) and by (4.52) and Lebesgue dominated convergence theorem we deduce as $h \to \infty$

$$(4.60) \qquad \hat{J}_0(\Omega, tu) \le t \sum_{j=1}^{s} \left\{ \int_{\Omega'} \alpha_j \, d\hat{J}_*(u) - f(0) \int_{\Omega' \setminus \Omega} \alpha_j \right\} + (1-t) \int_{\Omega} f\left(\frac{t}{1-t} \sum_{j=1}^{s} u^j D\alpha_j \right) =$$

$$= t\hat{J}_-(\Omega', u) - tf(0)|\Omega' \setminus \Omega| + (1-t) \int_{\Omega} f\left(\frac{t}{1-t} \sum_{j=1}^{s} u^j D\alpha_j \right).$$

Let us now observe that for every $j = 1, \ldots, s$ $u^j D\alpha_j = uD\alpha_j$ on Ω and that, since $\sum_{j=1}^{s} \alpha_j = 1$, $\sum_{j=1}^{s} D\alpha_j = 0$ on Ω; hence $\sum_{j=1}^{s} u^j D\alpha_j = 0$ on Ω and by (4.60) it results

$$(4.61) \qquad \hat{J}_0(\Omega, tu) \le t\hat{J}_-(\Omega', u) + tf(0)|\Omega' \setminus \Omega| + (1-t) f(0)|\Omega|.$$

As $t \to 1^-$ we get by (4.61)

$$(4.62) \qquad \hat{J}_0(\Omega, u) \le \hat{J}_-(\Omega', u) - f(0)|\Omega' \setminus \Omega|.$$

Finally the proof of (4.62) when $u \in L^1(\boldsymbol{R}^N)$ with $u = 0$ in $\boldsymbol{R}^N \setminus \Omega$ comes as in Lemma 4.3.

By (4.48), (4.62) and Proposition 1.2 (4.47) follows. ■

By virtue of Theorem 4.4, Proposition 4.6 and Theorem 1.6 we are able to deduce the following representation result.

THEOREM 4.7. – Let f be a nonnegative convex finite function on $\boldsymbol{R}^N$ and let $\hat{I}_0$ and $\hat{I}$ be defined by (0.7) and (0.4). Then

$$(4.63) \qquad \hat{I}_0(\Omega, u) = \int_{\Omega} f(RDu) + \int_{\Omega} f^*\left(\frac{dS\,Du}{d|Du|} \right) d|Du| + \int_{\partial\Omega} f^*(\boldsymbol{n}u) \, dH^{N-1},$$

$$(4.64) \qquad \hat{I}(\Omega, u) = \int_{\Omega} f(RDu) + \int_{\Omega} f^*\left(\frac{dS\,Du}{d|Du|} \right) d|Du|$$

for every bounded open set Ω with Lipschitz boundary, u in $BV(\Omega)$.

If in addition we assume that

$$(4.65) \qquad \lim_{|z| \to +\infty} f(z) = +\infty$$

then

$$(4.66) \qquad \hat{I}(\Omega, u) = \hat{I}_0(\Omega, u) = +\infty$$

for every bounded open set Ω, u in $L^1(\Omega) \setminus BV(\Omega)$.

PROOF. – Let Ω be a bounded open set with Lipschitz boundary and let $\mathfrak{I}$ be defined by (1.34).

Let u be in $BV(\Omega)$, then by Theorem 4.4, Proposition 1.7 and Theorem 1.6 it follows

$$\hat{I}(\Omega, u) = \hat{I}_-(\Omega, u) = \mathfrak{I}(\Omega, u) = f(Du)(\Omega), \tag{4.67}$$

that is (4.64).

In order to deduce (4.63) we only have to observe that by Proposition 4.6, Theorem 4.4 and (4.64) we have for every bounded open set Ω' with Lipschitz boundary, $\Omega' \supset\supset \Omega$

$$\hat{I}_0(\Omega, u) = \hat{J}_-(\Omega', u) - f(0)|\Omega' \setminus \Omega| = \hat{I}(\Omega', u) - f(0)|\Omega' \setminus \Omega| = \\ = f(Du)(\Omega') - f(0)|\Omega' \setminus \Omega|. \tag{4.68}$$

As Ω' shrinks to Ω we deduce by (4.68) that

$$\hat{I}_0(\Omega, u) = f(Du)(\overline{\Omega}) = f(Du)(\Omega) + f(Du)(\partial\Omega). \tag{4.69}$$

At this point by (4.69), (1.28), (1.30) and (1.31) we get

$$\hat{I}_0(\Omega, u) = f(Du)(\Omega) + \int_{\partial\Omega} f^*(\boldsymbol{n}u)\, dH^{N-1}, \tag{4.70}$$

that is (4.63).

We assume now (4.65) and check that $\hat{I}(\Omega, u) = +\infty$ for every u in $L^1(\Omega)\setminus BV(\Omega)$, the proof for $\hat{I}_0(\Omega, u)$ being similar.

By (4.65) we have

$$\exists\ k > 0: k|z| - 1 \leq f(z) \quad \text{for every } z \text{ in } \boldsymbol{R}^N. \tag{4.71}$$

Let $u \in L^1(\Omega)\setminus BV(\Omega)$ and let $\{u_h\}_h \subseteq C^1(\boldsymbol{R}^N)$ be such that $u_h \to u$ in $L^1(\Omega)$, we have to prove that

$$\liminf_h \int_\Omega f(Du_h) = +\infty. \tag{4.72}$$

If (4.72) would not occur, by (4.71) a subsequence of $\{u_h\}_h$ would be bounded in $BV(\Omega)$ and then, by weak compactness, u would be in $BV(\Omega)$.

Therefore (4.72) holds and the thesis follows. ■

From Theorem 4.7 we deduce the following corollary.

COROLLARY 4.8. – Let Ω be a bounded open set with Lipschitz boundary and let u be in $BV(\Omega)$.

Then there exist $\{u_h\}_h \subseteq C^1(\boldsymbol{R}^N)$ and $\{v_h\}_h \subseteq C_0^1(\Omega)$ such that $u_h \to u$, $v_h \to u$ in

$L^1(\Omega)$ and

$$\int_\Omega |Du| = \lim_h \int_\Omega |Du_h|, \tag{4.73}$$

$$\int_\Omega |Du| + \int_{\partial\Omega} |u|\, dH^{N-1} = \lim_h \int_\Omega |Dv_h|. \tag{4.74}$$

PROOF. – Take $f(z) = |z|$ and apply Theorem 4.7. ■

A result similar to Corollary 4.8 is proved in [AG] where the above mentioned sequence $\{u_h\}_h$ is in $C^1(\Omega)$.

5. – The general case.

Given a function f as in (0.1), (0.2) in the first part of this section we will denote by $\hat{I}_f$ the functional defined by (0.4) relatively to f, we will analogously behave for the functionals $\bar{I}_f$, $\hat{J}_f$, $\bar{J}_f$.

We will compare the functionals $\hat{I}_f$, $\bar{I}_f$, $\hat{J}_f$, $\bar{J}_f$ when f depends also on x.

LEMMA 5.1. – Let f and g be functions as in (0.1), (0.2).

Assume that

$$f(x, z) \leq g(x, z) \qquad x \text{ a.e. in } \boldsymbol{R}^N,\ z \text{ in } \boldsymbol{R}^N; \tag{5.1}$$

then

$$\hat{J}_f(\Omega, u) = (\hat{J}_f)_-(\Omega, u) \tag{5.2}$$

for every bounded open set Ω, u in $L^1(\Omega)$ such that $\hat{J}_g(\Omega, u) = (\hat{J}_g)_-(\Omega, u) < +\infty$.

PROOF. – Let Ω be as above and assume at first that u is in $L^\infty(\Omega)$.

Let us prove that

$$(\hat{J}_f)_-(\Omega, u) \geq \hat{J}_f(\Omega, u). \tag{5.3}$$

Let Ω', Ω'' be open sets with $\Omega'' \subset\subset \Omega' \subset\subset \Omega$ and let $\{u_h\}_h \subseteq W^{1,\infty}_{\mathrm{loc}}$ be such that $u_h \to u$ in $L^1(\Omega')$ and a.e. in Ω' and

$$\hat{J}_f(\Omega', u) \geq \limsup_h \int_{\Omega'} f(x, Du_h). \tag{5.4}$$

Moreover, being $\hat{J}_g(\Omega, u) = (\hat{J}_g)_-(\Omega, u)$, let $\{v_h\}_h \subseteq W^{1,\infty}_{\mathrm{loc}}$ be such that $v_h \to u$ in

$L^1(\Omega)$ and a.e. in Ω and

$$(\hat{J}_g)_-(\Omega, u) \geq \limsup_h \int_\Omega g(x, Dv_h). \tag{5.5}$$

By Lemma 2.1 we can assume that

$$\|u_h\|_{L^\infty(\Omega')} \leq \|u\|_{L^\infty(\Omega)}, \tag{5.6}$$

$$\|v_h\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}. \tag{5.7}$$

Let $\varphi \in C_0^1(\Omega')$ be such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on Ω'' and set

$$w_h = \varphi u_h + (1 - \varphi) v_h. \tag{5.8}$$

Then $w_h \in W^{1,\infty}_{\text{loc}}$ and $w_h \to u$ in $L^1(\Omega)$.
For every $t \in]0, 1[$ we have by convexity and (5.1)

$$\begin{aligned}\int_\Omega f(x, tDw_h) &\leq t \int_\Omega \varphi f(x, Du_h) + t \int_\Omega (1 - \varphi) f(x, Dv_h) + \\ &\quad + (1-t) \int_\Omega f\left(x, \frac{t}{1-t}(u_h - v_h) D\varphi\right) \leq \\ &\leq t \int_{\Omega'} f(x, Du_h) + t \int_\Omega (1-\varphi) g(x, Dv_h) + (1-t) \int_\Omega f\left(x, \frac{t}{1-t}(u_h - v_h) D\varphi\right).\end{aligned} \tag{5.9}$$

Since $(\hat{J}_g)_-(\Omega, u) < +\infty$ by Lemma 4.2 applied to the measures $\mu_h = \int_{(\cdot)} g(x, Dv_h)$, $\mu = (\hat{J}_g)_*(u)$, we deduce

$$\lim_h \int_\Omega (1 - \varphi) g(x, Dv_h) = \int_\Omega (1 - \varphi)\, d(\hat{J}_g)_*(u). \tag{5.10}$$

Moreover by (5.6), (5.7) and Lebesgue dominated convergence theorem we have as in (2.24)

$$\lim_h \int_\Omega f\left(x, \frac{t}{1-t}(u_h - v_h) D\varphi\right) = \int_\Omega f(x, 0). \tag{5.11}$$

By (5.9), (5.10), (5.11) and (5.4) we deduce as $h \to \infty$

$$\hat{J}_f(\Omega, tu) \leq t\hat{J}_f(\Omega', u) + t \int_{\Omega \setminus \Omega''} d(\hat{J}_g)_*(u) + (1 - t) \int_\Omega f(x, 0). \tag{5.12}$$

At this point if $t \to 1^-$ and Ω'' increases to Ω we obtain

$$\hat{J}_f(\Omega, u) \leq \liminf_{t \to 1^-} \hat{J}_f(\Omega, tu) \leq (\hat{J}_f)_-(\Omega, u), \tag{5.13}$$

that is (5.3) when $u \in L^\infty(\Omega)$.

When $u \in L^1(\Omega)$ setting, for every $k \in \boldsymbol{N}$, $\tau_k u$ as in (2.5) we deduce by (5.13) and Lemma 2.2

$$\hat{J}_f(\Omega, u) \leq \liminf_k \hat{J}_f(\Omega, \tau_k u) \leq \limsup_k (\hat{J}_f)_-(\Omega, \tau_k u) = (\hat{J}_f)_-(\Omega, u), \tag{5.14}$$

that is (5.3).

By (5.3) and (1.5) equality (5.2) follows. ∎

We can prove now the main result of this section.

THEOREM 5.2. – Let f, g be functions as in (0.1), (0.2).
Assume that

$$g(x, z) \leq f(x, z) \leq \Lambda(a(x) + g(x, z)) \quad x \text{ a.e. in } \boldsymbol{R}^N,\ z \text{ in } \boldsymbol{R}^N \tag{5.15}$$

$$\text{for some } \Lambda \geq 1,\ a \in L^1_{\text{loc}}(\boldsymbol{R}^N).$$

Then

$$\hat{J}_f(\Omega, u) = \bar{J}_f(\Omega, u) = (\hat{J}_f)_-(\Omega, u) = (\hat{I}_f)_-(\Omega, u) = \bar{I}_f(\Omega, u) = \hat{I}_f(\Omega, u) \tag{5.16}$$

for every bounded open set Ω, u in $L^1(\Omega)$ such that $\hat{I}_g(\Omega, u) = (\hat{I}_g)_-(\Omega, u)$.

PROOF. – Let Ω, u be as above.

If $\hat{I}_g(\Omega, u) < +\infty$ by Lemma 5.1, Proposition 1.2 and (1.5) we deduce

$$(\hat{J}_f)_-(\Omega, u) = \hat{J}_f(\Omega, u) = \hat{I}_f(\Omega, u) \geq \bar{I}_f(\Omega, u) \geq (\hat{I}_f)_-(\Omega, u) \geq (\hat{J}_f)_-(\Omega, u). \tag{5.17}$$

Hence by (5.17) and (1.5), (5.16) follows in this case.

If $\hat{I}_g(\Omega, u) = +\infty$ then by (5.15) and Proposition 1.2 it follows

$$+\infty = (\hat{I}_g)_-(\Omega, u) \leq (\hat{I}_f)_-(\Omega, u) \leq (\hat{J}_f)_-(\Omega, u), \tag{5.18}$$

hence (5.16) follows by (5.18) and (1.5). ∎

By Theorem 5.2 and Theorem 4.4 we obtain the following corollary.

COROLLARY 5.3. – Let f be as in (0.1) and let Φ be a nonnegative convex finite function on $\boldsymbol{R}^N$. Assume that

$$\Phi(z) \leq f(x, z) \leq \Lambda(a(x) + \Phi(z)) \quad x \text{ a.e. in } \boldsymbol{R}^N,\ z \text{ in } \boldsymbol{R}^N \tag{5.19}$$

$$\text{for some } \Lambda \geq 1,\ a \in L^1_{\text{loc}}(\boldsymbol{R}^N).$$

Then if $\hat{I}$, $\bar{I}$, $\hat{J}$, $\bar{J}$ are the functionals defined in (0.4), (0.5), (1.2) and (1.3) with $p = +\infty$ and relative to f it results

$$\hat{J}(\Omega, u) = \bar{J}(\Omega, u) = \hat{J}_-(\Omega, u) = \hat{I}_-(\Omega, u) = \bar{I}(\Omega, u) = \hat{I}(\Omega, u) \tag{5.20}$$

for every bounded open set Ω with Lipschitz boundary, u in $L^1(\Omega)$.

PROOF. – The proof follows by Theorem 5.2 and Theorem 4.4 once observed that for every bounded open set Ω with Lipschitz boundary, u in $L^1(\Omega)$ $\hat{I}_\Phi(\Omega, u) = (\hat{I}_\Phi)_-(\Omega, u)$. ■

We now compare the functionals $\hat{I}$, $\bar{I}$, $\hat{I}_0$, $\hat{J}^p$, $\bar{J}^p$, $\hat{J}^p_0$.

THEOREM 5.4. – Let f be a function as in (0.1).
Assume that f satisfies (5.19). Then

$$\hat{J}^p(\Omega, u) = \bar{J}^p(\Omega, u) = \hat{J}^p_-(\Omega, u) = \hat{I}_-(\Omega, u) = \bar{I}(\Omega, u) = \hat{I}(\Omega, u), \tag{5.21}$$

$$\hat{J}^p_0(\Omega, u) = \hat{I}_0(\Omega, u) \tag{5.22}$$

for every bounded open set Ω with Lipschitz boundary, u in $L^1(\Omega)$, p in $[1, +\infty]$.

PROOF. – Let Ω, u, p be as above.
Let us prove that

$$\hat{J}^p_-(\Omega, u) \geq \hat{I}_-(\Omega, u). \tag{5.23}$$

We can assume that $\hat{J}^p_-(\Omega, u) < +\infty$.
Let $A \subset\subset \Omega$ and let $\{u_h\}_h \subseteq W^{1,p}_{\rm loc}$ be such that $u_h \to u$ in $L^1(A)$ and

$$+\infty > \hat{J}^p(A, u) \geq \liminf_h \int_A f(x, Du_h). \tag{5.24}$$

For every $\eta > 0$ let $\alpha^{(\eta)}$ be given by (1.9) and let, for every $h \in \mathbf{N}$, $u_{h,\eta} = \alpha^{(\eta)} * u_h$ be a regularization of u_h, then for fixed $h \in \mathbf{N}$, $u_{h,\eta} \to u_h$ in $L^p(A)$ as $\eta \to 0^+$.
By Jensen inequality we deduce for almost every x in A

$$f(x, Du_{h,\eta}(x)) \leq \int_{\mathbf{R}^N} \alpha^{(\eta)}(x-y) f(x, Du_h(y))\, dy. \tag{5.25}$$

Let us observe that by (5.19) it follows

$$f(x, z) \leq \Lambda(a(x) + \Phi(z)) \leq \Lambda a(x) + \Lambda f(y, z) \tag{5.26}$$

for almost every $x, y \in \mathbf{R}^N$, z in $\mathbf{R}^N$.

By (5.25) and (5.26) we get

$$\begin{aligned} f(x, Du_{h,\eta}(x)) \leq \Lambda \int_{\mathbf{R}^N} \alpha^{(\eta)}(x-y)\, a(x)\, dy + \\ + \Lambda \int_{\mathbf{R}^N} \alpha^{(\eta)}(x-y) f(y, Du_h(y))\, dy = \Lambda a(x) + \Lambda[\alpha^{(\eta)} * f(\cdot, Du_h(\cdot))](x). \end{aligned} \tag{5.27}$$

By (5.24) $\alpha^{(\eta)} * f(\cdot, Du_h(\cdot)) \to f(\cdot, Du_h(\cdot))$ in $L^1(\Omega)$ as $\eta \to 0^+$, hence by (5.27) and Vitali convergence theorem we deduce

$$\lim_{\eta \to 0^+} \int_A f(x, Du_{h,\eta}(x))\, dx = \int_A f(x, Du_h(x))\, dx . \tag{5.28}$$

For every $h \in \boldsymbol{N}$, $\eta > 0$, $u_{h,\eta} \in C^1(\boldsymbol{R}^N)$, hence by virtue of (5.24) and (5.28) we can select $\eta_h \to 0^+$ such that $u_{h,\eta_h} \to u$ in $L^1(A)$ and

$$\hat{J}^p_-(\Omega, u) \geq \hat{J}^p(A, u) \geq \liminf_h \int_A f(x, Du_{h,\eta_h}) \geq \hat{I}(A, u) . \tag{5.29}$$

By (5.29), (5.23) follows as A increases to Ω.

Let us now observe that by (1.5) and (5.23) we deduce

$$\hat{J}^p(\Omega, u) \geq \bar{J}^p(\Omega, u) \geq \hat{J}^p_-(\Omega, u) \geq \hat{I}_-(\Omega, u) , \tag{5.30}$$

hence by Corollary 5.3 and (5.30) (5.21) follows.

We now prove (5.22).

Let us first prove that

$$\hat{J}^p_0(\Omega, u) \geq \hat{I}_0(\Omega, u) ; \tag{5.31}$$

to this aim we can assume that $\hat{J}^p_0(\Omega, u) < +\infty$.

Let $\{u_h\}_h \subseteq W^{1,p}_0(\Omega)$ be such that $u_h \to u$ in $L^1(\Omega)$ and

$$+\infty > \hat{J}^p_0(\Omega, u) \geq \limsup_h \int_\Omega f(x, Du_h) . \tag{5.32}$$

By (5.19) and (5.32) it results that $\int_\Omega \Phi(Du_h) < +\infty$ for every $h \in \boldsymbol{N}$, hence, for fixed h in $\boldsymbol{N}$, by Proposition 2.6 at page 312 in [ET], there exists a sequence $\{u_h^k\}_k \subseteq C^1_0(\Omega)$ such that $u_h^k \to u_h$ in $L^1(\Omega)$, $Du_h^k \to Du_h$ in $(L^1(\Omega))^N$ and a.e. in Ω as $k \to \infty$ and

$$\lim_k \int_\Omega \Phi(Du_h^k) = \int_\Omega \Phi(Du_h) . \tag{5.33}$$

By (5.33) and Lebesgue dominated convergence theorem we deduce for every h

$$\lim_k \int_\Omega f(x, Du_h^k) = \int_\Omega f(x, Du_h) . \tag{5.34}$$

By (5.32) and (5.34), e can select a sequence $\{k_h\}_h \subseteq \boldsymbol{N}$ such that

$$\hat{J}^p_0(\Omega, u) \geq \liminf_h \int_\Omega f(x, Du_h^{k_h}) \geq \hat{I}_0(\Omega, u) , \tag{5.35}$$

that is (5.31).

Since obviously

$$\hat{J}_0^p(\Omega, u) \leq \hat{I}_0(\Omega, u) \tag{5.36}$$

equality (5.22) follows by (5.31) and (5.36). ∎

REMARK 5.5. – We remark that in general if (5.19) does not hold, identity between $\hat{I}$ and $\hat{J}^p$ can be no more true, to this aim see [DA2].

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