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A Rigorous Stability Result for the Vlasov-Poisson System in Three Dimensions (*).

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Abstract. – *It is proven that in a neutral two-component plasma with space homogeneous positively charged background, which is governed by the Vlasov-Poisson system and for which Poisson's equation is considered on a cube in $\boldsymbol{R}^3$ with periodic boundary conditions, the space homogeneous stationary solutions g with energy gradient $\partial g/\partial\varepsilon \leqslant 0$ and compact support are (nonlinearly) stable in the L^1-norm with respect to weak solutions of the initial value problem.*

Introduction.

We consider the nonlinear Vlasov-Poisson system (VPS) in three dimensions in the form

$$\partial_t f + v\partial_x f + E(t, x)\,\partial_v f = 0\,,$$

$$E(t, x) = -\,\partial_x U(t, x)\,,$$

$$\Delta U(t, x) = \rho_0 + \gamma\rho(t, x)\,,$$

$$\rho(t, x) := \int f(t, x, v)\,dv\,, \qquad t \geqslant 0, \qquad x, v \in \boldsymbol{R}^3\,,$$

where $f = f(t, x, v)$ denotes the distribution function of an aggregation of gravitating particles or electrons, $\rho = \rho(t, x)$ their local density and $U = U(t, x)$ their Newtonian or Coulomb potential. In the description of stellar systems, we have $\rho_0 = 0$ and $\gamma = +1$. In a plàsma, $\rho_0 \geqslant 0$ stands for a constant, positively charged background density, $\gamma = -1$, and the (VPS) is often considered as a simplified model for the Vlasov-Maxwell system (VMS), in which the influence of the magnetic field is neglected. As is well known, the existence theory of these and related systems has been

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the subject of extensive mathematical research. We mention the work of ARSEN'EV [2], BARDOS and DEGOND [3], DI PERNA and LIONS [12, 13], GLASSEY and SCHAEFFER [16, 17], GLASSEY and STRAUSS [18, 19, 20], HORST [25, 26], HORST and HUNZE [27], ILLNER and NEUNZERT [28], PFAFFELMOSER [32], SCHAEFFER [35], and of the authors [5, 34] (for a survey up to the year '87 see also GANGULY and VICTORY Jr. [15]).

Comparatively little is known about the qualitative behaviour of the solutions. We know about the existence of particular, almost explicit solutions describing spherically symmetric stellar systems, which are time-periodic or contracting and then expanding [29], and of time-periodic solutions without cylindrical symmetry [7]. Moreover, there are results about the asymptotic behaviour (in space) of stationary solutions [9]. In [6] it was shown that the time averages of the potential and kinetic energy and of $I(t)/t^2$ and $\dot{I}(t)/t$ ($I(t)$ being the moment of inertia) converge for $t \to \infty$ for spherically symmetric stellar systems.

In the astronomical literature much effort has been devoted to the question of stability of certain stationary solutions g of the (VPS), particularly of the spherically symmetric models of polytropic gas spheres which have the form $g = (\varepsilon_0 - \varepsilon)_+^{\mu} F^k$, $\mu > -1$, $k > -1$, $\mu + k + 3/2 \geqslant 0$ (see also [8, p. 177]; ε is the local energy and F the angular momentum). For a good account of the development of these results, we refer the reader to the monographs of FRIDMAN and POLYACHENKO [14, Chapter III, 2] and BINNEY and TREMAINE [11, Chapter 5]. For $k = 0$ the sufficient condition $\partial g/\partial \varepsilon < 0$ appears already in the early work of ANTONOV [1] and was later investigated by BAUMANN, DOREMUS and FEIX [10], HÉNON [21], SOBOUTI [36], BARNES, GOODMAN and HUT [4], and many others. However, from a mathematical point of view, it must be pointed out that without a rigorously defined concept of stability, these investigations contribute at best to the problem of spectral stability; existence for the linearized equation and of eigenmodes [14, p. 152-153] [11, p. 291] require more detailed mathematical work. For anisotropic spherical systems, the given arguments are admittedly self-contradictory [11, p. 308].

For our purpose, we need to make precise the concept of stability of solutions of the (VPS) that we have in mind. If the distribution functions take their values in a Banach space B, a stationary solution g of the (VPS) is called (nonlinearly) stable if for every neighborhood U of g in B, there exists a neighborhood V of g such that for all initial values $f_0 \in V$ (in general satisfying an additional geometric condition)

a) a solution $t \mapsto f(t)$ of the (VPS) with $f(0) = f_0$ exists for all $t \geqslant 0$, and

b) $f(t) \in U$ for $t \geqslant 0$.

To the best of our knowledge, a (rigorous) stability result for the (VPS) does not exist. Two things however, must be added. First, in their proof of global existence for small initial data, BARDOS and DEGOND [3] come very close to what could be called asymptotic stability of the zero solution—the local density $\rho(t, x)$ corresponding to small initial data tends to zero uniformly in x (the additional geometric condition consists of a prescribed decay in $\boldsymbol{R}^6$). Secondly, in connection with investigations of the stability of plasma systems by HOLM, MARSDEN, RATIU and WEINSTEIN [22], and with an earlier application of Arnold's approach to planar Euler flows [30], MARCHIORO and PULVIRENTI proved a partial result on the way towards stability, namely a statement of type *b)* in the Banach space $L^1(Q \times \boldsymbol{R}^N)$ (Q a cube in $\boldsymbol{R}^N$) for $N = 1, 2, 3$, $\rho_0 = 1$ and $\gamma = -1$, and periodic boundary conditions for Poisson's equation on Q [31].

It is the intention of the present paper to continue the work of MARCHIORO and PULVIRENTI by establishing the existence part *a)* for $N = 3$, and thus to get a complete stability result for the (VPS). It was our first hope to be able to prove the existence of global classical solutions by applying the new ideas in the recent work on the existence of classical solutions in the full space $\boldsymbol{R}^3 \times \boldsymbol{R}^3$ [32]. Similarly, it seemed natural to study the evolution of small perturbations of the stationary solutions g which induce the characteristics of the free streaming as the zero solution does in $\boldsymbol{R}^3 \times \boldsymbol{R}^3$ [3]. But we found no substitute for the hypothesis of the decay (in space) of the initial condition, which cannot be present in the space-periodic situation on $Q \times \boldsymbol{R}^3$. Hence these two problems remain open (verifying the often encountered fact that properties of the (VPS) and (VMS) are very unstable against seemingly small perturbations).

Our main result (Theorem 5.5) says that space homogeneous stationary solutions g of mass 1 (which are decreasing functions of $|v|$ and have compact $|v|$-support) are stable with respect to weak solutions of the (VPS) with mass 1 in the norm of $L^1(Q \times \boldsymbol{R}^3)$ for $\rho_0 = 1$ and $\gamma = -1$. Although a result for weak solutions might be considered a stronger result than for classical solutions, and many known solutions on $\boldsymbol{R}^3 \times \boldsymbol{R}^3$ are not classical solutions (which makes the extension of the classical existence theory desirable), weak solutions f present particular difficulties which obstruct an easy access to stability properties: they are not known to be uniquely determined and to satisfy the conservation of energy, they may not preserve the compactness of the v-support of $f(0)$, and $f(t)$ and $f(0)$ may not be equimeasurable. We shall circumvent these problems by considering regularized solutions in Section 3, from which the weak solutions are constructed in Section 4. Because weak solutions on $\boldsymbol{R}^3 \times \boldsymbol{R}^3$ have been studied earlier by HORST and HUNZE [27], and later by DI PERNA and LIONS in a preliminary article [12], our presentation will be confined to the parts necessary to show that relevant results do hold in our situation. In Section 2 we construct and investigate the Green's function for Poisson's equation on a cube Q in $\boldsymbol{R}^3$ with periodic boundary conditions—this is an indispensable preparation for the later work, for which sufficient information does not seem to be available in the literature.

1. – Notation and preliminaries.

We shall use the following notation. $Q := \{x \in \boldsymbol{R}^3;\ 0 \le x_i \le 1,\ i = 1, 2, 3\}$ is a cube in $\boldsymbol{R}^3$ and $S := Q \times \boldsymbol{R}^3$ the corresponding strip in $\boldsymbol{R}^6$. Functions in the set $\mathcal{P}(Q) := \{f\colon \boldsymbol{R}^3 \to \boldsymbol{R};\ f(x + e_i) = f(x),\ x \in \boldsymbol{R}^3,\ i = 1, 2, 3\}$ for $e_i := (\delta_{ij})_{j=1}^3 \in \boldsymbol{R}^3$ are called periodic. Similarly, we define $\mathcal{P}(S) := \{f\colon \boldsymbol{R}^6 \to \boldsymbol{R};\ f(\cdot, v) \in \mathcal{P}(Q),\ v \in \boldsymbol{R}^3\}$. Elements in $\mathcal{P}(Q)$ and $\mathcal{P}(S)$ are often identified with their restrictions to Q and S respectively. Let $\boldsymbol{R}_0^+ := [0, \infty[$, $\boldsymbol{N} := \{1, 2, \ldots\}$, $\boldsymbol{N}_0 := \{0, 1, 2, \ldots\}$. For $p \in [1, \infty]$ we consider the spaces

$$L^p_\pi(Q) := \{f \in \mathcal{P}(Q); f \in L^p(Q)\},$$

$$L^p_\pi(S) := \{f \in \mathcal{P}(S); f \in L^p(S)\},$$

$$C^n_\pi(Q) := C^n(\boldsymbol{R}^3) \cap \mathcal{P}(S), \qquad n \in \boldsymbol{N} \cup \{\infty\},$$

$$C^1_{c,\pi}(S) := \{f \in \mathcal{P}(S) \cap C^1(\boldsymbol{R}^6);\ \exists P > 0\ \forall (x, v) \in \boldsymbol{R}^6,\ |v| \geq P\colon f(x, v) = 0\}.$$

The plus sign appearing as an upper index refers to the nonnegative functions. By p' we always denote the conjugate to p. The pair (x, v) is often written z. Unless otherwise indicated, integration will always be extended over Q or $\boldsymbol{R}^3$ or S. Note that for $f \in L^1_\pi(Q)$ the integral $\int f(x+y)\,dy$ is independent of $x \in \boldsymbol{R}^3$. The Lebesgue measure is denoted by μ. Occasionally we use the weak L^p-spaces:

$$L^p_{w,\pi}(Q) := \{f \in \mathcal{P}(Q); f \cdot \chi_Q \in L^p_w(\boldsymbol{R}^3)\},$$

[33, p. 30]. For $1 \leq \lambda, r, q \leq \infty$ with $1/\lambda + 1/r = 1 + 1/q$ the convolution of $f \in L^\lambda_\pi(Q)$ and $g \in L^r_\pi(Q)$,

$$f * g(x) := \int f(x-y)\,g(y)\,dy,$$

is a well defined element in $L^q_\pi(Q)$, and $\|f * g\|_q \leq \|f\|_\lambda \|g\|_r$. Similarly, for $1 < \lambda, r, q < \infty$ with $1/\lambda + 1/r = 1 + 1/q$ the convolution of $f \in L^\lambda_{w,\pi}(Q)$ and $g \in L^r_\pi(Q)$ lies in $L^q_\pi(Q)$ and $\|f * g\|_q \leq C_{\lambda,r} \|f\|_{\lambda,w} \|g\|_r$. These are the Young's and extended Young's inequalities [33, p. 32].

2. – Construction of the Green's function for Poisson's equation on Q with periodic boundary conditions.

For this, there does not seem to exist a better reference than [37, p. 32]. We shall need the following proposition.

2.1. PROPOSITION. – *There exists a function $G \in C^\infty(\boldsymbol{R}^3 \setminus \boldsymbol{Z}^3) \cap \mathcal{P}(Q)$ with the following properties*:

(2.1.1) $\Delta G = -1$ *on* $\boldsymbol{R}^3 \setminus \boldsymbol{Z}^3$,

(2.1.2) *for a constant* $\gamma \in \boldsymbol{R}$, $\int G(x+y)\,dy = \gamma$, *and* $\int \partial_x G(x+y)\,dy = 0$ *for* $x \in \boldsymbol{R}^3$, (2.1.3) *G is a even function in x_1, x_2, x_3,*

(2.1.4) *there exists a function $G_0 \in C^\infty((\boldsymbol{R}^3 \setminus \boldsymbol{Z}^3) \cup \{(0,0,0)\})$ such that*

$$G(x) = -\frac{1}{4\pi r} + G_0(x) \quad on\ \boldsymbol{R}^3 \setminus \boldsymbol{Z}^3, \quad r := |x|,$$

(2.1.5) *for every Hölder-continuous $F \in \mathcal{P}(Q)$ with $\int F(x)\,dx = 0$*

$$U(x) := \int G(x-y)\,F(y)\,dy$$

is in $C^2_\pi(Q)$ the unique solution of the problem

$$\Delta U(x) = F(x), \qquad \int U(x)\,dx = 0.$$

PROOF. – We shall obtain G as

$$G(x) = -G_1(x) - G_2(x),$$

with two functions G_1, G_2 to be specified. Let

$$\varepsilon_j := 1 \text{ for } j = 0 \quad \text{and} \quad \varepsilon_j := 2 \text{ for } j \in \boldsymbol{N},$$

$$\varphi_j(\xi, s) := \varepsilon_j \cos(2\pi j \xi) \exp(-4\pi^2 j^2 s) \quad \text{for } j \in \boldsymbol{N}_0 \quad \text{and} \quad \xi, s \in \boldsymbol{R},$$

$$\Phi(x, s) := \sum_{(0,0,0) \neq (l,m,n) \in \boldsymbol{N}_0^3} \varphi_l(x_1, s)\,\varphi_m(x_2, s)\,\varphi_n(x_3, s), \qquad s \geq 1,$$

and define

$$G_1(x) := \int_1^\infty \Phi(x, s)\,ds, \qquad x = (x_1, x_2, x_3) \in \boldsymbol{R}^3.$$

Then $G_1 \in C^\infty(\boldsymbol{R}^3) \cap \mathcal{P}(Q)$ and

$$\Delta G_1(x) = \int_1^\infty \Delta\Phi(x, s)\,ds = \int_1^\infty \partial_s \Phi(x, s)\,ds = -\Phi(x, 1).$$

It follows from [38, p. 475] with the third theta function ϑ_3 that

$$\Psi(\xi, s) := \sum_{l=0}^\infty \varphi_l(\xi, s) = \vartheta_3(\pi\xi, 4\pi i s) = \frac{1}{2\sqrt{\pi s}} \sum_{l=-\infty}^{+\infty} \exp\left(-\frac{(\xi - l)^2}{4s}\right) =$$

$$=: \Psi_0(\xi, s) + \frac{1}{2\sqrt{\pi s}} \exp\left(-\frac{\xi^2}{4s}\right), \qquad \xi \in \boldsymbol{R} \setminus \boldsymbol{Z}, \quad 0 < s \leq 1.$$

Now we define

$$G_2(x) := \int_0^1 \Psi(x_1, s)\, \Psi(x_2, s)\, \Psi(x_3, s)\, ds.$$

Then $G_2 \in C^\infty(\boldsymbol{R}^3 \setminus \boldsymbol{Z}^3) \cap \mathscr{P}(Q)$ and on $\boldsymbol{R}^3 \setminus \boldsymbol{Z}^3$

$$\Delta G_2(x) = \int_0^1 \Delta(\Psi(x_1, s)\, \Psi(x_2, s)\, \Psi(x_3, s))\, ds =$$

$$= \int_0^1 \partial_s (\Psi(x_1, s)\, \Psi(x_2, s)\, \Psi(x_3, s))\, ds = \Psi(x_1, 1)\, \Psi(x_2, 1)\, \Psi(x_3, 1) = \Phi(x, 1) + 1.$$

We also have

$$G_2(x) = \int_0^1 \prod_{j=1}^3 \left(\Psi_0(x_j, s) + \frac{1}{2\sqrt{\pi s}} \exp\left(-\frac{x_j^2}{4s} \right) \right) ds.$$

Multiplying out the terms in the integrand, we see that there exists a function $G_3 \in C^\infty((\boldsymbol{R}^3 \setminus \boldsymbol{Z}^3) \cup \{(0, 0, 0)\})$ such that

$$G_2(x) = \frac{1}{\sqrt{4\pi}^3} \int_0^1 s^{-3/2} \exp\left(-\frac{r^2}{4s} \right) ds + G_3(x), \qquad r := |x|.$$

Now

$$\frac{1}{\sqrt{4\pi}^3} \int_0^1 s^{-3/2} \exp\left(-\frac{r^2}{4s} \right) ds = \frac{1}{4\pi^{3/2}} \frac{1}{r} \int_{r^2/4}^\infty t^{-1/2} e^{-t}\, dt =$$

$$= \frac{1}{4\pi^{3/2}} \frac{1}{r} \left(\Gamma(1/2) - \int_0^{r^2/4} t^{-1/2} e^{-t}\, dt \right) = \frac{1}{4\pi^{3/2}} \frac{1}{r} \left(\sqrt{\pi} - \sum_{j=0}^\infty \frac{(-1)^j}{j!(j+1/2)} \left(\frac{r}{2} \right)^{2j+1} \right) =$$

$$=: \frac{1}{4\pi r} + G_4(x),$$

with $G_4 \in C^\infty(\boldsymbol{R}^3)$, and (2.1.1)-(2.1.4) are immediate. As for (2.1.5), it follows from (2.1.4) and classical arguments that $U \in C^2_\pi(Q)$. Let $W := \{x \in \boldsymbol{R}^3;\ |x|_\infty \leqslant 1/2\}$. If $\bar{x} \in \boldsymbol{R}^3$, then for all $x \in \bar{x} + (1/2)W$ and $y \in \bar{x} + W$ one has $x - y \in (3/2)W$, where G_0 is regular with $\Delta G_0 = -1$, and hence

$$U(x) = \int G(x-y)F(y)\,dy = -\frac{1}{4\pi}\int_{\bar{x}+W} \frac{1}{|x-y|}F(y)\,dy + \int_{\bar{x}+W} G_0(x-y)F(y)\,dy\,,$$

$$\Delta U(x) = F(x) + \int_{\bar{x}+W} \Delta G_0(x-y)F(y)\,dy = F(x) - \int_{\bar{x}+W} F(y)\,dy = F(x)\,,$$

and

$$\int U(x)\,dx = \int\int G(x-y)F(y)\,dy\,dx = \int\left(\int G(x)\,dx\right)F(y)\,dy = 0\,.$$

For the difference V of any two solutions with equal integrals, we have

$$0 = \int V(x)\Delta V(x)\,dx = -\int |\partial_x V(x)|^2\,dx\,, \qquad \int V(x)\,dx = 0\,,$$

and hence $V = 0$.

2.2. COROLLARY. – *a) We have $G \in L^\lambda_\pi(Q)$ for $1 \leqslant \lambda < 3$, and for $1 \leqslant r,\ q \leqslant \infty$ with $1/\lambda + 1/r = 1 + 1/q$ ($\lambda = r'/2$ for $q = r'$) the operator G: $u \mapsto G * u$ maps $L^r_\pi(Q)$ into $L^q_\pi(Q)$ and is compact, with norm $\leqslant \|G\|_\lambda$. Furthermore, $G \in L^3_{w,\pi}(Q)$ and for $1 < r$, $q < \infty$ with $1/r = 2/3 + 1/q$ the operator G maps $L^r_\pi(Q)$ into $L^q_\pi(Q)$ and has norm $\leqslant C_r\|G\|_{3,w}$.*

b) We have $\partial_x G \in L^{\lambda_1}_\pi(Q)^3$ for $1 \leqslant \lambda_1 < 3/2$, and for $1 \leqslant r,\ q \leqslant \infty$ with $1/\lambda_1 + 1/r = 1 + 1/q$ ($\lambda_1 = r'/2$ for $q = r'$) the operator $\partial_x G$: $u \mapsto \partial_x G * u$ maps $L^r_\pi(Q)$ into $L^q_\pi(Q)^3$ and is compact, with norm $\leqslant \|\partial_x G\|_{\lambda_1}$. Furthermore, $\partial_x G \in L^{3/2}_{w,\pi}(Q)^3$, and for $1 < r$, $q < \infty$ with $1/r = 2/3 + 1/q$ the operator $\partial_x G$ maps $L^r_\pi(Q)$ into $L^q_\pi(Q)^3$ and has norm $\leqslant C'_r\|\partial_x G\|_{3/2,w}$.

3. – Regularized solutions.

We need the existence of regularized solutions to obtain weak solutions in Section 4.

3.1. – For $0 < \eta < 1/2$ let $\omega_\eta \in C^\infty_\pi(Q)$ be such that $\omega_\eta \geqslant 0$, $\omega_\eta(x) = 0$ for $|x| \geqslant \eta$, ω_η is an even function in x_1, x_2, x_3 and $\int \omega_\eta\,dx = 1$. For $\sigma \in L^p_\pi(Q)$ we let $\sigma_\eta := \omega_\eta * \sigma \in L^p_\pi(Q)$ be the regularization of σ; we have $\|\sigma_\eta\|_p \leqslant \|\sigma\|_p$, $1 \leqslant p \leqslant \infty$. If $\sigma \in L^{1,+}_\pi(Q)$, then $\sigma_\eta \in L^{1,+}_\pi(Q)$ and $\|\sigma\|_1 = \|\sigma_\eta\|_1$. We note some properties of the regularization G_η of G (see Proposition 2.1):

(3.1.1) for all $x \in \boldsymbol{R}^3$, $\int G_\eta(x+y) = \gamma$, hence $\int \partial_x G_\eta \, dx = 0$,

(3.1.2) G_η is an even, $\partial_x G_\eta$ is an odd function in x_1, x_2, x_3,

(3.1.3) $\|G_\eta\|_\infty \leq \|\omega_\eta\|_\infty \|G\|_1$, $\|\partial_x G_\eta\|_\infty \leq \|\omega_\eta\|_\infty \|\partial_x G\|_1$.

3.2. – Let us assume $f_0 \in L_\pi^{1,+}(S)$ and $\|f_0\|_1 = 1$. We have to convince ourselves that for $\eta \in]0, 1/2[$, the initial value problem

$$\partial_t f + v\partial_x f + E(t, x)\partial_v f = 0, \tag{3.2.1}$$

$$E(t, x) = -\partial_x U(t, x), \tag{3.2.2}$$

$$U(t, x) = G * (1 - \rho_\eta(t))(x) = G_\eta * (1 - \rho(t))(x), \tag{3.2.3}$$

$$\rho_\eta(t, x) := \omega_\eta * \rho(t)(x), \qquad \rho(t, x) := \int f(t, x, v)\, dv, \tag{3.2.4}$$

$$f(0) = f_0, \tag{3.2.5}$$

is globally solvable by periodic functions f, ρ, U, E. A quick way to see this is to construct the phase flow induced by the corresponding system of characteristics by solving an abstract differential equation (this was done for the (VPS) on $\boldsymbol{R}^6$ in [24]). For the weight $w(z) := 1 + |v|$, we consider the Banach space

$$L := \{A \in C_\pi(S)^6;\ w^{-1}A \text{ is bounded on } S\},$$

with the norm $\|A\|_L := \|w^{-1}A\|_\infty$. It is easy to see that for $\varphi \in L_\pi^1(S)$ the mapping $Q_\varphi : L \times L \to L$,

$$Q_\varphi(A, B)(z) := \left(v + A_2(z), \int \partial_x G_\eta(x + A_1(z) - x' - B_1(z'))\, \varphi(z')\, dz'\right),$$

$A = (A_1, A_2)$, $B = (B_1, B_2) \in L$, is well defined and there exists a constant c_η such that for A, B, $\overline{A}$, $\overline{B} \in L$

$$\|Q_\varphi(A, B) - Q_\varphi(\overline{A}, B)\|_L \leq (1 + c_\eta \|\varphi\|_1)\|A - \overline{A}\|_L, \qquad \text{if } \varphi \in L_\pi^1(S),$$

$$\|Q_\varphi(A, B) - Q_\varphi(\overline{A}, \overline{B})\|_L \leq (1 + c_\eta \|\varphi\|_1)\|A - \overline{A}\|_L + c_\eta \|w\varphi\|_1 \|B - \overline{B}\|_L, \qquad \text{if } w\varphi \in L_\pi^1(S),$$

$$\|Q_\varphi(A, A) - Q_\psi(\overline{A}, \overline{A})\|_\infty \leq (1 + c_\eta \|\varphi\|_1)\|A - \overline{A}\|_\infty + c_\eta \|\varphi - \psi\|_1,$$

$$\text{if } \varphi, \psi \in L_\pi^1(S) \text{ and } A - \overline{A} \text{ is bounded}.$$

This implies that for $w\varphi \in L_\pi^1(Q)$ there exists a unique (strong) solution $W_\varphi = (W_{\varphi,1}, W_{\varphi,2}) : \boldsymbol{R}_0^+ \to L$ of the initial value problem

$$\dot{W} = Q_\varphi(W, W), \qquad W(0) = 0.$$

Because the set of these φ is dense in $L_\pi^1(S)$ and because the difference of two solutions is t-locally bounded over S, this statement also holds for $\varphi \in L^1_\pi(S)$. W_φ is also the unique solution of the initial value problem

$$\dot{W} = Q_\varphi(W, W_\varphi), \qquad W(0) = 0.$$

It follows that there exists a unique solution $Z(\cdot, s, z)$: $\boldsymbol{R}_0^+ \to \boldsymbol{R}^6$ of the initial value problem

$$\dot{X} = V, \qquad \dot{V} = \int \partial_x G_\eta(X - x' - W_{f_0,1}(t)(z'))\, f_0(z')\, dz', \qquad Z(s, s, z) = z,$$

(note that the right hand side is continuous on $\boldsymbol{R}_0^+ \times \boldsymbol{R}^6$ and C^∞ with respect to (X, V) with bounded derivatives). In particular,

$$Z(t, 0, z) = W_{f_0}(t)(z) + z \tag{3.2.6}$$

and

$$Z(t, s, (x + e_i, v)) = Z(t, s, (x, v)) + (e_i, 0), \qquad i = 1, 2, 3.$$

$Z(t, s)$ is C^∞ with continuous derivatives on $\boldsymbol{R}_0^+ \times \boldsymbol{R}_0^+ \times \boldsymbol{R}^6$ and $Z(t, s)$: $\boldsymbol{R}^6 \to \boldsymbol{R}^6$ is a measure preserving homeomorphism with $Z(t, s)^{-1} = Z(s, t)$. If κ denotes the canonical projection of $\boldsymbol{R}^6$ onto $[0, 1[^3 \times \boldsymbol{R}^3$, it is easy to see that

$$\varphi(\kappa \circ Z(t, s)) \sim \varphi, \tag{3.2.7}$$

that is, these functions are equimeasurable on S (see (5.1)), and hence

$$\int \varphi(\kappa \circ Z(t, s, z))\, dz = \int \varphi(z)\, dz. \tag{3.2.8}$$

In the sequel we shall use the notation Z for $\kappa \circ Z$. We define

$$f(t, z) := f_0(Z(0, t, z)), \qquad t \geq 0, \qquad z \in \boldsymbol{R}^6, \tag{3.2.9}$$

ρ and ρ_η by (3.2.4), and U and E by (3.2.3) and (3.2.2). Then with (3.2.7) and (3.2.6)

$$E(t, x) = \partial_x G_\eta * \rho(t)(x) = \int \partial_x G_\eta(x - y) f_0(Z(0, t, y, v))\, dv\, dy =$$

$$= \int \partial_x G_\eta(x - X(t, 0, z'))\, f_0(z')\, dz' = \int \partial_x G_\eta(x - x' - W_{f_0,1}(t)(z'))\, f_0(z')\, dz',$$

and Z is the phase flow induced by the solutions of

$$\dot{X} = V, \qquad \dot{V} = E(t, X). \tag{3.2.10}$$

f is constant along the characteristics, and satisfies (3.2.1) if f_0 is, in addition, continuously differentiable.

3.3. – We denote the functions constructed in 3.2 by f^η, ρ^η, ρ_η^η, U^η, and E^η. Let $f_0 \in L_\pi^{1,+}(S)$, $\|f_0\|_1 = 1$ and let $p \in [1, \infty]$ be such that $f_0 \in L_\pi^{p,+}(S)$. Then:

(3.3.1) $\quad f^\eta \in C(\boldsymbol{R}_0^+, L_\pi^{p,+}(S))$ and $\|f^\eta(t)\|_p = \|f_0\|_p$ on $\boldsymbol{R}_0^+$,

(3.3.2) $\quad \rho^\eta \in C(\boldsymbol{R}_0^+, L_\pi^{1,+}(Q))$ and $\|\rho^\eta(t)\|_1 = 1$ on $\boldsymbol{R}_0^+$,

(3.3.3) $\quad U^\eta \in C(\boldsymbol{R}_0^+, C_\pi^n(Q))$, $E^\eta \in C(\boldsymbol{R}_0^+, C_\pi^n(Q))$ for all $n \in \boldsymbol{N}$,

$\int U^\eta(t, x)\, dx = 0$ (see (2.1.5)),

$\|U^\eta(t)\|_\infty \leqslant 2\|G_\eta\|_\infty$, $\|E^\eta(t)\|_\infty \leqslant 2\|\partial_x G_\eta\|_\infty$ on $\boldsymbol{R}_0^+$.

3.4. DEFINITION. – For $\varphi \in L_\pi^{1,+}(S)$ and $\sigma(x) := \int \varphi(x, v)\, dv$ we define

$$E_{\text{kin}}\varphi := \int\int v^2 \varphi(x, v)\, dv\, dx \leqslant \infty\,,$$

$$E_{\text{pot}}^\eta \varphi := -\int G_\eta * \sigma(x)\,\sigma(x)\, dx = -\int G * \sigma_\eta(x)\,\sigma(x)\, dx\,.$$

It follows with (3.1.1) and (3.3.3) that

(3.4.1)
$$\begin{aligned} E_{\text{pot}}^\eta f^\eta(t) &= -\int G_\eta * \rho^\eta(t)(x)\,\rho^\eta(t, x)\, dx = \\ &= \int G_\eta * (1 - \rho^\eta(t))(x)\,\rho^\eta(t, x)\, dx - \int G_\eta * 1(x)\,\rho^\eta(t, x)\, dx = \\ &= \int U^\eta(t, x)\,\rho^\eta(t, x)\, dx - \gamma = -\int U^\eta(t, x)(1 - \rho^\eta(t, x))\, dx - \gamma = \\ &= -\int G_\eta * (1 - \rho^\eta(t))(x)(1 - \rho^\eta(t, x))\, dx - \gamma\,. \end{aligned}$$

3.5. – For later use we remark the following: If $\varphi \in L_\pi^{1,+}(S)$ is a function which does not depend on x and for which $\|\varphi\|_1 = 1$, then $\sigma = 1$, and hence

$$\begin{aligned} E_{\text{pot}}^\eta f^\eta(t) - E_{\text{pot}}^\eta \varphi = E_{\text{pot}}^\eta f^\eta(t) + \gamma &= -\int\int G_\eta * (1 - \rho^\eta(t))(1 - \rho^\eta(t, x))\, dx = \\ &= -\int G_\eta * (\rho^\eta(t) - 1)(x) \int (f^\eta(t, x, v) - \varphi(v))\, dv\, dx = E_{\text{pot}}^\eta (f^\eta(t) - \varphi)\,. \end{aligned}$$

3.6. PROPOSITION (Conservation of energy). – *If $f_0 \in L_\pi^{1,+}(S)$, $\|f_0\|_1 = 1$ and $E_{\text{kin}} f_0 < \infty$, then for all η and $t \geqslant 0$*

(3.6.1)
$$E_{\text{kin}} f^\eta(t) + E_{\text{pot}}^\eta f^\eta(t) = E_{\text{kin}} f_0 + E_{\text{pot}}^\eta f_0\,.$$

PROOF. – With (3.2.8), (3.2.3), and (3.2.10) we have

$$\frac{d}{dt}E_{\rm kin} f^{\eta}(t) = \frac{d}{dt}\int V^2(t, 0, z) f_0(z)\, dz =$$

$$= 2\int\int V(t, 0, z)\cdot \partial_x G_\eta(X(t, 0, z) - X(t, 0, z')) f_0(z') f_0(z)\, dz'\, dz,$$

$$\frac{d}{dt}E^{\eta}_{\rm pot} f^{\eta}(t) = -\frac{d}{dt}\int\int G_\eta(X(t, 0, z) - X(t, 0, z')) f_0(z') f_0(z)\, dz'\, dz =$$

$$= -\int\int (V(t, 0, z) - V(t, 0, z'))\, \partial_x G_\eta(X(t, 0, z) - X(t, 0, z')) f_0(z') f_0(z)\, dz'\, dz =$$

$$= -2\int\int V(t, 0, z)\, \partial_x G_\eta(X(t, 0, z) - X(t, 0, z')) f_0(z') f_0(z)\, dz'\, dz,$$

because of (3.1.2), and the sum is zero.

3.7. LEMMA [25, p. 22], [12, p. 656]. – *For all* $s \in]1, \infty]$ *there exists* $c_s > 0$ *such that for all measurable* $\varphi \geq 0$ *on* S *with* $\sigma(x) := \int \varphi(x, v)\, dv$

$$\|\sigma\|_{r_s} \leq c_s \|\varphi\|_s^{2s/(5s-3)} (E_{\rm kin}\varphi)^{(3s-3)/(5s-3)},$$

where $r_s := (5s - 3)/(3s - 1)$ *if* $s < \infty$, *and* $r_\infty := 5/3$ (*note that* $r_{9/7} = 6/5$ *and* $s \mapsto r_s$ *is strictly increasing*).

We are now in the position to derive estimates which are uniform in η.

3.8. LEMMA. – *If* $f_0 \in L_\pi^{9/7,+}(S)$, *and* $E_{\rm kin} f_0 < \infty$ *and* $\|f_0\|_1 = 1$, *then there exists* $J > 0$ *such that for all* η *and* $t \geq 0$

$$E_{\rm kin} f^{\eta}(t) \leq J.$$

PROOF. – We use Hölder's inequality and Corollary 2.2*a*) with $r = 6/5$, $r' = 6 = q$ to obtain

$$|E^{\eta}_{\rm pot} f^{\eta}(t)| \leq \int |G * \rho^{\eta}_{\eta}(t)(x)\, \rho^{\eta}(t, x)|\, dx \leq \|G * \rho^{\eta}_{\eta}(t)\|_6 \|\rho^{\eta}(t)\|_{6/5} \leq$$

$$\leq C_r \|G\|_{3,\omega} \|\rho^{\eta}_{\eta}(t)\|_{6/5} \|\rho^{\eta}(t)\|_{6/5} \leq C_r \|G\|_{3,w} \|\rho^{\eta}(t)\|^2_{6/5}.$$

By Lemma 3.7 with $s = 9/7$ and $r_s = 6/5$ and (3.2.8)

$$\|\rho^{\eta}(t)\|_{6/5} \leq c_{9/7} \|f^{\eta}(t)\|_{9/7}^{3/4} (E_{\rm kin} f^{\eta}(t))^{1/4} = c_{9/7} \|f_0\|_{9/7}^{3/4} (E_{\rm kin} f^{\eta}(t))^{1/4},$$

so that

$$|E^{\eta}_{\rm pot} f^{\eta}(t)| \leq C_r \|G\|_{3,w}\, c^2_{9/7} \|f_0\|_{9/7}^{3/2} (E_{\rm kin} f^{\eta}(t))^{1/2}.$$

The assertion now follows from (3.6.1), because we have shown that $E^{\eta}_{\mathrm{pot}} f(0)$ is bounded in η.

3.9. COROLLARY. – *Let* $f_0 \in L^{9/7,+}_{\pi}(S)$, $E_{\mathrm{kin}} f_0 < \infty$, $\|f_0\| = 1$, *and let* $p \in [1, \infty]$ *be such that* $f_0 \in L^{p,+}_{\pi}(S)$. *Then we have:*

a) For $s \in [1, p]$, *all* η *and* $t \geq 0$

$$\|\rho^{\eta}(t)\|_{r_s} \leq c_s \|f_0\|_s^{2s/(5s-3)} J^{(3s-3)/(5s-3)}.$$

b) If $\psi \in C_{c,\pi}(S)$ *and* $\rho^{\eta}(\psi, t, x) := \int \psi(x, v) f^{\eta}(t, x, v)\, dv$, *then for all* $s \in [1, p]$, *all* η *and* $t \geq 0$

$$\|\rho^{\eta}(\psi, t)\|_s \leq c_s(\psi)\|f_0\|_s, \qquad \textit{with } c_s(\psi) := \sup_{x \in Q} \|\psi(x, \cdot)\|_{s'}.$$

PROOF. – *a)* follows from Lemma 3.7, (3.3.1), and Lemma 3.8; *b)* follows from Hölder's inequality and (3.3.1).

Next we show that $E^{\eta}_{\mathrm{pot}} f^{\eta}(t) + \gamma$ is uniformly «almost» nonnegative.

3.10. LEMMA. – *Let* $f_0 \in L^{p,+}_{\pi}(S)$ *for some* $p > 9/7$, $E_{\mathrm{kin}} f_0 < \infty$ *and* $\|f_0\|_1 = 1$. *Then there is a positive continuous function* Δ *on* $]0, 1/2[$ *with* $\Delta(\eta) \to 0$ $(\eta \to 0)$ *such that for all* η *and* $t \geq 0$

$$-E^{\eta}_{\mathrm{pot}} f^{\eta}(t) \leq \Delta(\eta) + \gamma. \tag{3.10.1}$$

PROOF. – From (3.4.1)

$$-E^{\eta}_{\mathrm{pot}} f^{\eta}(t) = \int (G_{\eta} - G) * (1 - \rho^{\eta}(t))(x)(1 - \rho^{\eta}(t, x))\, dx + \gamma + \\ + \int G * (1 - \rho^{\eta}(t))(x)(1 - \rho^{\eta}(t, x))\, dx.$$

To estimate the first term, we use Hölder's inequality and Corollary 2.2*a)* with $r := r_p > 6/5$, $q := r' < 6$, $\lambda := r'/2 < 3$ to obtain with Corollary 3.9*a)*

$$\left| \int (G_{\eta} - G) * (1 - \rho^{\eta}(t))(x)(1 - \rho^{\eta}(t, x))\, dx \right| \leq \\ \leq \|(G_{\eta} - G) * (1 - \rho^{\eta}(t))\|_{r'} \|1 - \rho^{\eta}(t)\|_r \leq \|G_{\eta} - G\|_{\lambda} \|1 - \rho^{\eta}(t)\|_r^2 \leq \\ \leq \|G_{\eta} - G\|_{\lambda} (1 + c_p \|f_0\|_p^{2p/(5p-3)} J^{(3p-3)/(5p-3)})^2.$$

For the second term we have with a regularization by ω_δ

$$\int G * (1-\rho^\eta(t))(x)(1-\rho^\eta(t,x))\,dx = \lim_{\delta\to 0}\int G * (1-\rho_\delta^\eta(t))(x)(1-\rho_\delta^\eta(t,x))\,dx =$$

$$= \lim_{\delta\to 0}\int G * (1-\rho_\delta^\eta(t))(x)\,\Delta_x(G * (1-\rho_\delta^\eta(t))(x))\,dx =$$

$$= -\lim_{\delta\to 0}\int (\partial_x G * (1-\rho_\delta^\eta(t))(x))^2\,dx \leq 0,$$

after integrating by parts and using Proposition 2.1.

4. - Weak solutions.

In this section we prove the existence of weak solutions and study their properties. For any space L_π^s over Q or S, $s \in [1,\infty]$, we denote by σ_s the topology $\sigma(L_\pi^s, L_\pi^{s'})$.

4.1. DEFINITION. - Let $p \in [1,\infty]$ and $f_0 \in L_\pi^{1,+}(S) \cap L_\pi^p(S)$, $\|f_0\|_1 = 1$. Let $f\colon \boldsymbol{R}_0^+ \to L_\pi^1(\boldsymbol{R}^6)$ be a function and define

$$\rho(t,x) := \int f(t,x,v)\,dv, \qquad U(t) := G * (1-\rho(t)), \tag{4.1.1}$$

$$E(t,x) := -\partial_x U(t) = \partial_x G * \rho(t). \tag{4.1.2}$$

We call f a *weak p-solution* of the (VPS) with initial condition f_0, if

a) $f \in C(\boldsymbol{R}_0^+, (L_\pi^s(S), \sigma_s))$ for $s \in [1,p]$ and $\|f(t)\|_1 = \|f_0\|_1$, $\|f(t)\|_s \leq \|f_0\|_s$ for $s \in]1,p]$,

b) $\rho \in C(\boldsymbol{R}_0^+, (L_\pi^s(S), \sigma_s))$ for $s \in [1, r_p]$,

c) $E \in C(\boldsymbol{R}_0^+, L_\pi^{p'}(Q))$,

d) for all $\psi \in C_{c,\pi}^1(S)$ and $t \geq 0$

$$\int \psi(z)(f(t,z) - f_0(z))\,dz = \int_0^t \int (v\partial_x\psi(z) + E(\tau,x)\,\partial_v\psi(z))\,f(\tau,z)\,dz\,d\tau. \tag{4.1.3}$$

4.2. REMARK. – For $\psi \in C^1_{c,\pi}(S)$ we have

$$\frac{d}{dt}\int \psi(z) f^{\eta}(t,z)\,dz = \frac{d}{dt}\int \psi(Z(t,0,z))\,f_0(z)\,dz =$$

$$= \int (V(t,0,z)\,\partial_x \psi(Z(t,0,z)) + E^{\eta}(t, X(t,0,z))\,\partial_v \psi(Z(t,0,z)))\,f_0(z)\,dz =$$

$$= \int v\partial_x \psi(z) f^{\eta}(t,z)\,dz + \int E^{\eta}(t,x)\,\partial_v \psi(z) f^{\eta}(t,z)\,dz\,,$$

hence (after integrating over t) f^{η} and E^{η} satisfy (4.1.3).

4.3. THEOREM. – *Let* $p_0 := (12 + 3\sqrt{5})/11$ $(> 9/7)$ *and assume that* $f_0 \in L^{1,+}_{\pi}(S) \cap L^p_{\pi}(S)$ *for some* $p \in]p_0, \infty]$, $E_{\text{kin}} f_0 < \infty$ *and* $\|f_0\|_1 = 1$. *Then for every sequence* $\eta_n \downarrow 0$ *there exists a subsequence* (η_{n_i}) *and a function* $f \in C(\boldsymbol{R}_0^+, (L^{s,+}_{\pi}(S), \sigma_s))$ *such that (with* (4.1.1), (4.1.2) *and writing* i *for the index* η_{n_i}*) we have t-locally uniformly on* $\boldsymbol{R}_0^+$

(4.3.1) $f^i(t) \to f(t)$ *in* $(L^s_{\pi}(S), \sigma_s)$ *for* $s \in [1, p]$,

(4.3.2) $\rho^i(t) \to \rho(t)$ *in* $(L^s_{\pi}(Q), \sigma_s)$ *for* $s \in [1, r_p]$,

(4.3.3) $\rho^i(\psi, t) \to \rho(\psi, t)$ *in* $(L^s_{\pi}(Q), \sigma_s)$ *for* $s \in [1, p]$ *and all* $\psi \in C^1_{c,\pi}(\boldsymbol{R}^6)$,

(4.3.4) $E^i(t) \to E(t)$ *in the norm of* $L^{q'}_{\pi}(Q)$ *for every* $q \in]p_0, p]$.

The limit f *is a weak p-solution of the* (VPS) *with initial condition* f_0, *and satisfies*

$$E_{\text{kin}} f(t) \leq \liminf_{i\to\infty} E_{\text{kin}} f^i(t)\,, \qquad E_{\text{pot}} f(t) \leq \lim_{i\to\infty} E^i_{\text{pot}} f^i(t)\,. \tag{4.3.5}$$

PROOF. – We indicate the main ideas; for a comparable situation see [27].

STEP 1. – We note: For all $t \geq 0$ the set $\{f^{\eta}(t);\ \eta \in]0, 1/2[\}$ is relatively sequentially compact in $(L^s_{\pi}(S), \sigma_s)$ for $s \in [1, p]$. In view of (3.3.1) this fact relies on the reflexivity of $L^s_{\pi}(S)$ for $1 < s < \infty$, on the separability of $L^1_{\pi}(S)$ for $s = \infty$, and the criterion of B. J. PETTIS for $s = 1$ (the uniform integrability of the set is a consequence of the measure preservation of the flow (see (3.2)) and the uniform smallness of the integrals over complements of big sets follows from

$$\int_Q \int_{|v| \geq R} f^{\eta}(t,x,v)\,dv\,dx \leq \frac{1}{R^2} E_{\text{kin}} f^{\eta}(t) \leq \frac{1}{R^2} J \to 0 \qquad (R \to \infty) \tag{4.3.6}$$

with Lemma 3.8).

STEP 2. – We have: For all $F \in L^{s'}_{\pi}(S)$ the set $\{\langle F, f^{\eta}(\cdot)\rangle;\ \eta \in]0, 1/2[\}$ is equicontinuous. By a density argument for $1 < s \leq \infty$ and with the help of (4.3.6) for $s = 1$, it is sufficient to show this for $F := \psi \in C^1_{c,\pi}(S)$ with Remark 4.2. The first term on the right hand side is bounded by Corollary 3.9*b*). We prove the boundedness of the second term. We observe that p_0 is determined by the equation $1/p_0 + 1/r_{p_0} = 4/3 = 2 - 1/(3/2)$ and that $q \mapsto 1/q + 1/r_q$ is strictly decreasing. Hence for all finite $q \in]p_0, p]$ there exists $\lambda_1 \in [1, 3/2[$ such that $1/q + 1/r_q = 2 - 1/\lambda_1$ or $1/\lambda_1 + 1/r_q = 1 + 1/q'$. Applying Corollaries 2.2*b*) and 3.9 yields

$$\left| \int \partial_x G_\eta * \rho^\eta(t)(x)\rho(\partial_v\psi, t, x)\,dx \right| \leq \|\partial_x G_\eta * \rho^\eta(t)\|_{q'} \|\rho(\partial_v\psi, t)\|_q \leq$$

$$\leq \|\partial_x G_\eta\|_{\lambda_1} \|\rho^\eta(t)\|_{r_q} \|\rho(\partial_v\psi, t)\|_q \leq \|\partial_x G\|_{\lambda_1} c_q c_q(\partial_v\psi) \|f_0\|_q^{(7q-3)/(5q-3)} J^{(3q-3)/(5q-3)}.$$

STEP 3. – Steps 1 and 2 imply the existence of f with (4.3.1), (4.3.2), and (4.3.3) first for $s = 1$ and then for the other s (see [27, p. 269]). To prove (4.3.4) we consider $q \in]p_0, p]$ and λ_1 as in Step 2. The operator $\partial_x G: L^{r_q}_\pi(Q) \to L^{q'}_\pi(Q)$ is compact by Corollary 2.2 and maps weakly convergent sequences into strongly convergent sequences. Hence with Corollary 3.9*a*) and (4.3.3)

$$\|E^i(t) - E(t)\|_{q'} = \|\partial_x G_i * \rho^i(t) - \partial_x G * \rho(t)\|_{q'} \leq$$

$$\leq \|(\partial_x G_i - \partial_x G) * \rho^i(t)\|_{q'} + \|\partial_x G * (\rho^i(t) - \rho(t))\|_{q'} \leq$$

$$\leq \|\partial_x G_i - \partial_x G\|_{\lambda_1} \sup_{t,i} \|\rho^i(t)\|_{r_q} + \|\partial_x G * (\rho^i(t) - \rho(t))\|_{q'} \to 0,$$

locally t-uniformly.

STEP 4. – We show that f is a weak p-solution. Only *d*) remains to be proven. By Remark 4.2,

$$\int \psi(z)(f^i(t,z) - f_0(z))\,dz = \int_0^t \int v\partial_x\psi(z) f^i(\tau, z)\,dz\,d\tau + \int_0^t \int E^i(\tau, x)\,\partial_v\psi(z) f^i(\tau, z)\,dz\,d\tau$$

for $i \in \boldsymbol{N}$. If we let $i \to \infty$, by (4.3.1) the first three terms tend to the corresponding terms in (4.1.3). As for the last term, we have t-locally uniformly with Corollary 3.9*b*)

$$\left| \int (E^i(\tau, x)\,\partial_v\psi(z) f^i(\tau, z) - E(\tau, x)\,\partial_v\psi(z) f(\tau, z))\,dz \right| \leq$$

$$\leq \left| \int (E^i(\tau, x) - E(\tau, x)) \int \partial_v\psi(x, v) f^i(\tau, x, v)\,dv\,dx \right| +$$

$$+ \left| \int E(\tau, x)(\rho^i(\partial_v\psi, \tau, x) - \rho(\partial_v\psi, \tau, x))\,dx \right| =$$

$$= \left| \int (E^i(\tau, x) - E(\tau, x))\rho^i(\partial_v\psi, \tau, x)\,dx \right| +$$

$$+ \left| \int\int \partial_x G(x-y)\rho(\tau, y)(\rho^i(\partial_v\psi, \tau, x) - \rho(\partial_v\psi, \tau, x))\,dy\,dx \right| \leq$$

$$\leq \|E^i(\tau) - E(\tau)\|_{q'} \sup_{t,i} \|\rho^i(\partial_v\psi, \tau)\|_q + \sup_\tau \|\rho(\tau)\|_{r_q} \|\partial_x G * (\rho^i(\partial_v\psi, \tau) - \rho(\partial_v\psi, \tau))\|_{r_q'} \to 0\,.$$

Here we have used (3.1.2) and the results of Step 3, and applied a similar argument to the operator $\partial_x G$: $L^q_\pi(Q) \to L^{r_q'}_\pi(Q)^3$ (note that also $1/\lambda_1 + 1/q = 1 + 1/r_q'$).

STEP 5. – We show the second relation of (4.3.5): Using the same r, q, λ as in the proof of Lemma 3.10 and exploiting the compactness of G: $L^r_\pi(Q) \to L^{r'}_\pi(Q)$ as in Step 3, we get with a suitable constant $M_p > 0$,

$$\left| \int G_i * \rho^i(t)(x)\rho^i(t, x)\,dx - \int G * \rho(t)(x)\rho(t, x)\,dx \right| \leq$$

$$\leq \left| \int (G_i - G) * \rho^i(t)(x)\rho^i(t, x)\,dx \right| + \left| \int G * (\rho^i(t) - \rho(t))(x)\rho^i(t, x)\,dx \right| +$$

$$+ \left| \int \rho(t, y)\, G * (\rho^i(t) - \rho(t))(y)\,dy \right| \leq$$

$$\leq \|G_i - G\|_\lambda M_p^2 + \|G * (\rho^i(t) - \rho(t))\|_{r'} M_p + M_p \|G * (\rho^i(t) - \rho(t))\|_{r'} \to 0\,.$$

4.4. REMARK. – Any classical solution (as long as it exists) is the limit of the f^η for $\eta \to 0$ even in a stronger sense if the initial condition $f_0 \in C^{1,+}_\pi(S)$ satisfies the decay condition

$$|f_0(x, v)| \leq K(1 + |v|)^{-\alpha},$$

for some $K > 0$, $\alpha > 0$, together with its derivatives. The argument given in [27, p. 265] can be extended to the present situation.

5. – Stability.

5.1. – In this section, we consider initial values f_0 in the neighborhood of certain stationary solutions g and estimate the difference $f(t) - g$ in the norm of $L^1_\pi(S)$. We choose g in the class

$$\mathcal{M} := \{\varphi \in L^{\infty,+}_\pi(S); \varphi \text{ depends only on } |v|, (|v| \mapsto \varphi(|v|)) \text{ is decreasing in } |v| \geq 0,\ P_\varphi < \infty,\ \|\varphi\|_1 = 1\},$$

where for a function $\varphi \in \mathcal{P}(S)$ we define

$$P_\varphi := \inf\{P > 0;\ \varphi(x, v) = 0 \ \text{for} \ |v| > P\} \in [0, \infty]$$

($\inf \emptyset := +\infty$). Obviously, $g \in \mathfrak{M}$ satisfies the conditions for the initial values in Theorem 4.3 for $p = \infty$, the constant function $g(t) := g$ is a weak ∞-solution of the initial value problem for g (and, as will follow from the results of this section, is the only one). Because then $U = 0$, g is a function of the energy $\varepsilon = v^2/2$, and $\partial g/\partial\varepsilon \leqslant 0$ a.e. The initial conditions f_0 are allowed in the class

$$\mathfrak{J} := \{\varphi \in L_\pi^{\infty,+}(S);\ P_\varphi < \infty,\ \|\varphi\|_1 = 1\}.$$

f_0 also induces a (weak) ∞-solution f of the initial value problem. We call two measurable functions $\varphi, \psi \in \mathcal{P}^+(S)$ equimeasurable and write $\varphi \sim \psi$, if for all $\lambda \geqslant 0$

$$\mu(\{z \in S;\ \varphi(z) > \lambda\}) = \mu(\{z \in S;\ \psi(z) > \lambda\}),$$

which implies $\|\varphi\|_p = \|\psi\|_p$ for all $p \in [1, \infty]$.

5.2. LEMMA. – *For $\varphi \in \mathfrak{J}$ there exists $\varphi^* \in \mathfrak{M}$ such that $\varphi \sim \varphi^*$, $P_{\varphi^*} \leqslant P_\varphi$ and $\|\varphi^* - g\|_1 \leqslant \|\varphi - g\|_1$ for all $g \in \mathfrak{M}$.*

PROOF. – For $\varphi \in \mathfrak{J}$ and $\lambda \geqslant 0$ let

$$A_\lambda(\varphi) := \{z \in S;\ \varphi(z) > \lambda\}, \qquad y(\lambda) := \left(\frac{3}{4\pi}\mu(A_\lambda(\varphi))\right)^{1/3}.$$

The function y is decreasing and right continuous. The rearrangement φ^* of φ is given by

$$\varphi^*(|v|) := \sup\{\lambda;\ 0 < \lambda < \|\varphi\|_\infty,\ y(\lambda) \geqslant |v|\}.$$

It is standard that $\varphi^* \in \mathfrak{M}$, $\varphi^* \sim \varphi$ and $P_{\varphi^*} \leqslant P_\varphi$. Let $g \in \mathfrak{M}$. The function $\chi_{A_\lambda(\varphi^*)} - \chi_{A_\lambda(g)}$ does not change sign on S. Hence

$$\begin{aligned}
\mu(A_\lambda(|\varphi^* - g|)) &= \mu(A_\lambda(\varphi^*)) + \mu(A_\lambda(g)) - 2\mu(A_\lambda(\min(\varphi^*, g))) = \\
&= \mu(A_\lambda(\varphi^*)\Delta A_\lambda(g)) = \int |\chi_{A_\lambda(\varphi^*)} - \chi_{A_\lambda(g)}|\,dz = \\
&= \left|\int (\chi_{A_\lambda(\varphi^*)} - \chi_{A_\lambda(g)})\,dz\right| = \left|\int (\chi_{A_\lambda(\varphi)} - \chi_{A_\lambda(g)})\,dz\right| \leqslant \\
&\leqslant \int |\chi_{A_\lambda(\varphi)} - \chi_{A_\lambda(g)}|\,dz = \mu(A_\lambda(\varphi)\Delta A_\lambda(g)) \leqslant \mu(A_\lambda(|\varphi - g|)),
\end{aligned}$$

and integration over λ yields the last assertion of the lemma.

5.3. LEMMA. – *For h, $\varphi \in \mathfrak{J}$ with $h \sim \varphi$ we have*

$$\|h - \varphi^*\|_1^2 \leq 8\pi P_\varphi \|\varphi\|_\infty (E_{\text{kin}} h - E_{\text{kin}} \varphi^*) .$$

PROOF. – We let $\alpha := \|h\|_\infty = \|\varphi^*\|_\infty$ and for $n \geq 2$,

$$A_k := \left\{ z \in S;\ h(z) > \frac{k\alpha}{n} \right\},$$

$$A_k^* := \left\{ z \in S;\ \varphi^*(z) > \frac{k\alpha}{n} \right\}, \qquad 1 \leq k \leq n-1,$$

$$h_n := \sum_{k=1}^{n-1} \frac{\alpha}{n} \chi_{A_k}, \qquad \varphi_n^* := \sum_{k=1}^{n-1} \frac{\alpha}{n} \chi_{A_k^*},$$

$$\beta_k := \mu(A_k \setminus A_k^*) = \mu(A_k^* \setminus \Delta A_k) = \frac{1}{2} \mu(A_k \Delta A_k^*) = \frac{1}{2} \|\chi_{A_k} - \chi_{A_k^*}\|_1$$

(note that $h \sim \varphi^*$). Since $\varphi^* \in \mathfrak{M}$ and $\beta_k \leq \mu(A_k^*)$, there exist constants r_k, δ_k, $\delta_k' \geq 0$ such that

$$A_k^* = Q \times B_{r_k}, \qquad \mu(Q \times B_{r_k, r_k + \delta_k}) = \beta_k = \mu(Q \times B_{r_k - \delta_k', r_k})$$

for $1 \leq k \leq n-1$, where $B_r := \{v \in \boldsymbol{R}^3;\ |v| < r\}$ and $B_{r,s} := B_s \setminus B_r$. As an immediate consequence of these definitions, we obtain

$$\int\limits_{(A_k \setminus A_k^*) \setminus (Q \times B_{r_k, r_k + \delta_k})} \int v^2 \, dx \, dv \geq \int\limits_{(Q \times B_{r_k, r_k + \delta_k}) \setminus (A_k \setminus A_k^*)} \int v^2 \, dx \, dv$$

and thus

$$\int\limits_{A_k \setminus A_k^*} \int v^2 \, dx \, dv \geq \int\limits_{Q \times B_{r_k, r_k + \delta_k}} \int v^2 \, dx \, dv = \int\limits_{B_{r_k r_k + \delta_k}} v^2 \, dv = \frac{4\pi}{5} ((r_k + \delta_k)^5 - r_k^5) .$$

Similarly,

$$\int\limits_{A_k^* \setminus A_k} \int v^2 \, dx \, dv \leq \int\limits_{B_{r_k - \delta_k', r_k}} v^2 \, dv = \frac{4\pi}{5} ((r_k - \delta_k')^5 - r_k^5) .$$

It follows with Schwarz' inequality that

$$E_{\mathrm{kin}} h_n - E_{\mathrm{kin}} \varphi_n^* = \frac{\alpha}{n} \sum_{k=1}^{n-1} \left(\int\limits_{A_k \setminus A_k^*} \int v^2 \, dx \, dv - \int\limits_{A_k^* \setminus A_k} \int v^2 \, dx \, dv \right) \geqslant$$

$$\geqslant \frac{\alpha}{n} \frac{4\pi}{5} \sum_{k=1}^{n-1} ((r_k + \delta_k)^5 - 2r_k^5 + (r_k - \delta_k')^5) =$$

$$= \frac{\alpha}{n} \frac{4\pi}{5} \sum_{k=1}^{n-1} r_k^5 \left(\left(1 + \frac{3}{4\pi} \frac{\beta_k}{r_k^3}\right)^{5/3} + \left(1 - \frac{3}{4\pi} \frac{\beta_k}{r_k^3}\right)^{5/3} - 2 \right) \geqslant \frac{1}{2\pi} \frac{\alpha}{n} \sum_{k=1}^{n-1} r_k^5 \left(\frac{\beta_k}{r_k^3} \right)^2 =$$

$$= \frac{1}{2\pi} \frac{\alpha}{n} \sum_{k=1}^{n-1} r_k^{-1} \beta_k^2 \geqslant \frac{1}{2\pi} \frac{\alpha}{n} P_{\varphi^*}^{-1} \left(\sum_{k=1}^{n-1} \frac{\alpha}{n} \beta_k \right)^2 \left((n-1) \frac{\alpha^2}{n^2} \right)^{-1} =$$

$$= \frac{1}{2\pi} \frac{n}{n-1} \alpha^{-1} P_{\varphi^*}^{-1} \left(\sum_{k=1}^{n-1} \frac{\alpha}{n} \frac{1}{2} \|\chi_{A_k} - \chi_{A_k^*}\|_1 \right)^2 \geqslant (8\pi\alpha P_{\varphi^*})^{-1} \|h_n - \varphi_n^*\|_1^2,$$

and for $n \to \infty$ we obtain the assertion.

5.4. REMARK. – The preceding lemmas may be used for $\varphi := f_0 \in \mathfrak{Z}$ and $h := f^\eta(t)$. In fact, by (3.2.7) and (3.2.9), $f^\eta(t) \sim f_0$; also, because the field E^η is uniformly bounded by (3.3.3) and because $P_{f_0} < \infty$ we have $P_{f^\eta(t)} < \infty$. These properties and the conservation of energy are not guaranteed for the weak solutions. Therefore we shall argue for f^η first and obtain the stability result for f by approximation.

5.5. THEOREM. – *Let $g \in \mathfrak{M}$ be a stationary solution. Then we have for any weak solution f constructed in Theorem 4.3 with initial condition $f_0 \in \mathfrak{Z}$*

$$\|f(t) - g\|_1 \leqslant C(f_0) \|f_0 - g\|_1^{1/2} + \|f_0 - g\|_1, \qquad t \geqslant 0,$$

where

$$C(f_0) := \left(16\pi P_{f_0} \|f_0\|_\infty (P_{f_0}^2 + \|G\|_1 (\|\rho(0)\|_\infty + 1))\right)^{1/2}.$$

PROOF. – By conservation of energy (3.6.1) and the estimate (3.10.1),

$$E_{\mathrm{kin}} f^\eta(t) = E_{\mathrm{kin}} f_0 + E_{\mathrm{pot}}^\eta f_0 - E_{\mathrm{pot}}^\eta f^\eta(t) \leqslant E_{\mathrm{kin}} f_0 + E_{\mathrm{pot}}^\eta f_0 + \gamma + \Delta(\eta).$$

If we subtract $E_{\mathrm{kin}} f_0^*$ and use (3.5) for $t = 0$ and $\varphi := f_0^*$, we get

$$E_{\mathrm{kin}} f^{\eta}(t) - E_{\mathrm{kin}} f_0^* \leqslant E_{\mathrm{kin}} f_0 - E_{\mathrm{kin}} f_0^* + E^{\eta}_{\mathrm{pot}} f_0 - E^{\eta}_{\mathrm{pot}} f_0^* + \Delta(\eta) =$$

$$= E_{\mathrm{kin}}(f_0 - f_0^*) - \int G_\eta * (\rho(0) - 1)(x) \int (f_0(x, v) - f_0^*(|v|))\, dv\, dx + \Delta(\eta) \leqslant$$

$$\leqslant P_{f_0}^2 \|f_0 - f_0^*\|_1 + \|G_\eta * (\rho(0) - 1)\|_\infty \|f_0 - f_0^*\|_1 + \Delta(\eta) \leqslant$$

$$\leqslant (P_{f_0}^2 + \|G_\eta\|_1 (\|\rho(0)\|_\infty + 1))\|f_0 - f_0^*\|_1 + \Delta(\eta) \leqslant$$

$$\leqslant (P_{f_0}^2 + \|G\|_1 (\|\rho(0)\|_\infty + 1))\|f_0 - f_0^*\|_1 + \Delta(\eta).$$

Hence by the preceding remark (Lemmas 5.3 and 5.2)

$$\|f^{\eta}(t) - f_0^*\|_1 \leqslant (8\pi P_{f_0} \|f_0\|_\infty)^{1/2} (E_{\mathrm{kin}} f^{\eta}(t) - E_{\mathrm{kin}} f_0^*)^{1/2} \leqslant$$

$$\leqslant (8\pi P_{f_0} \|f_0\|_\infty)^{1/2} ((P_{f_0}^2 + \|G\|_1 (\|\rho(0)\|_\infty + 1))^{1/2} \|f_0 - f_0^*\|_1^{1/2} + \Delta(\eta)^{1/2}) \leqslant$$

$$\leqslant \frac{C(f_0)}{\sqrt{2}} (\|f_0 - g\|_1 + \|g - f_0^*\|_1)^{1/2} + (8\pi)^{1/2} P_{f_0}^{1/2} \|f\|_\infty^{1/2} \Delta(\eta)^{1/2} \leqslant$$

$$\leqslant \frac{C(f_0)}{\sqrt{2}} (\|f_0 - g\|_1 + \|g - f_0\|_1)^{1/2} + (8\pi)^{1/2} P_{f_0}^{1/2} \|f_0\|_\infty^{1/2} \Delta(\eta)^{1/2} =$$

$$= C(f_0) \|f_0 - g\|_1^{1/2} + (8\pi)^{1/2} P_{f_0}^{1/2} \|f_0\|_\infty^{1/2} \Delta(\eta)^{1/2},$$

so that

$$\|f^{\eta}(t) - g\|_1 \leqslant \|f^{\eta}(t) - f_0^*\|_1 + \|f_0^* - g\|_1 \leqslant \tag{5.5.1}$$

$$\leqslant C(f_0) \|f_0 - g\|_1^{1/2} + \|f_0 - g\|_1 + (8\pi)^{1/2} P_{f_0}^{1/2} \|f_0\|_\infty^{1/2} \Delta(\eta)^{1/2}.$$

For a suitable sequence $\eta^i \downarrow 0$ we have from Theorem 4.3,

$$f^{\eta_i}(t) - g \to f(t) - g \quad \text{in } (L_\pi^1(S), \sigma_1),$$

which implies

$$\|f(t) - g\|_1 \leqslant \liminf_{i \to \infty} \|f^{\eta_i} - g\|_1,$$

and the assertion follows from (5.5.1) together with $\Delta(\eta) \to 0$ $(\eta \to 0)$.

5.6. COROLLARY. – *Given* $g \in \mathfrak{M}$, $\varepsilon > 0$ *and a constant* $C > 0$, *there exists* $\delta > 0$ *such that for all* $f_0 \in \mathfrak{J}$ *with* $\|f_0 - g\|_1 < \delta$ *and satisfying the geometric condition* $C(f_0) \leqslant C$ *we have* $\|f(t) - g\|_1 < \varepsilon$ *for* $t \geqslant 0$.

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Note added in proof.

The above mentioned problem of the existence of global classical solutions on $Q \times \boldsymbol{R}^3$ has been solved meanwhile; see J. BATT - G. REIN, *Global classical solutions of the periodic Vlasov-Poisson system in three dimensions*, C.R. Acad. Sci. Paris, **313**, Série I (1991), pp. 411-416.

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