Annali di Matematica pura ed applicata
(IV), Vol. CLXIV (1993), pp. 103-120

Gradient Estimates for a Class of Elliptic Systems (*).

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Summary. – *Gradient bounds are proved for solutions to a class of second order elliptic systems in divergence form. The main condition on this class is a generalization of the assumption that the system be the Euler-Lagrange system of equations for a functional depending only on the modulus of the gradient of the solution.*

0. – Introduction.

In this work, we consider solutions of certain second order elliptic systems of the form

$$D_\alpha(A_i^\alpha(x, u, Du)) + B_i(x, u, Du) = 0 \quad \text{in } \Omega \tag{0.1}$$

for some domain Ω in $\boldsymbol{R}^n$, $n > 1$, where we use the summation convention with Greek indices going from 1 to n and Latin indices going from 1 to N. To describe our basic additional structure, let $(G^{\alpha\beta}), (g_{ij})$ be positive define matrices on $\Omega \times \boldsymbol{R}^N$ and set

$$v = (G^{\alpha\beta}(x, z)\, g_{ij}(x, z)\, p_\alpha^i p_\beta^j)^{1/2}.$$

(Here we use $z = (z^i)$ and $p = (p_\alpha^i)$ as dummy variables for u and Du, respectively.) Our main assumption will be that there is a scalar function F on $\Omega \times \boldsymbol{R}^N \times \boldsymbol{R}^{nN}$ satisfying

$$A_i^\alpha(x, z, p) = \partial F(v)/\partial p_\alpha^i$$

and additional conditions described below. When $F(v) = v^m$ for some $m \geqslant 2$ and $B = -\partial F(v)/\partial z$, such systems were studied by FUSCO and HUTCHINSON [2] as generaliza-

(*) Entrata in Redazione il 27 aprile 1990.

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tions of the variational problem associated with the functional J defined by

$$J[z] = \int_\Omega |Du|^m \, dx,$$

which is fairly well-understood (see, e.g. [3, Sect. VI.4] and [13]).

An important distinction between the results for v of the form given here and $v = |Du|$ (with $F(v) = v^m$, $m > 1$ in both cases) is the nature of the regularity results. In the first case, only partial regularity has been proved, i.e. there is an open dense set $\Omega_0 \subset \Omega$ such that any weak solution of (0.1) is in $C^{1,\beta}(\Omega_0)$ for some $\beta > 0$ (and Ω_0 may depend on the solution). In the second case $\Omega_0 = \Omega$. Moreover, TOLKSDORF [12] has shown everywhere regularity if G and g are independent of z.

Here we prove a gradient bound for solutions of (0.1) under more general hypotheses. Because of the nature of our estimates, it will not always be clear whether we are proving directly that the solutions are Lipschitz but that point is not our present interest. We will, however, reproduce both partial and everywhere regularity results.

Our approach is a modification of the Moser iteration scheme [10] along the lines of Simon's interior gradient estimatesfor single equations [11]. We begin by proving a gradient bound for constant G and g in Section 1. This special case will demonstrate the relevant new ideas, which will then be applied to the general case in Section 2. Examples appear in Section 3.

Some results on bounds for the solution are given in [8] under related structure conditions.

1. – Gradient estimates for constant G and q.

When the matrices G and g are constant, various simplifications arise which make our calculations more transparent. We define the function f by $f(t) = F'(t)/t$ and note that our system can be written as

$$D_\alpha(f(v)\, G^{\alpha\beta} g_{ij} D_\beta u^j) + B_i(x, u, Du) = 0. \tag{1.1}$$

From now on, we suppress the arguments of f and B (which are always assumed to be v and (x, u, Du), respectively) and of their derivatives. If f, u, and B are smooth enough we can differentiate (1.1) with respect to x^δ, thus obtaining

$$0 = D_\alpha(f'\, G^{\alpha\beta} g_{ij} D_\beta u^j D_\delta v + f G^{\alpha\beta} g_{ij} D_{\delta\beta} u^j) + D_\delta B_i.$$

Next we multiply by $G^{\delta\varepsilon}(D_\varepsilon u^i/v)\zeta$ for some $\zeta \in L^\infty \cap W^{1,2}$ with compact support in

Ω, sum over δ, and integrate the expression involving D_α by parts. It then follows that

$$0 = \int_\Omega v^{-1} f' G^{\alpha\beta} G^{\delta\varepsilon} g_{ij} D_\beta u^j D_{\alpha\varepsilon} u^i D_\delta v \zeta \, dx +$$

$$+ \int_\Omega v^{-1} f G^{\alpha\beta} G^{\delta\varepsilon} g_{ij} D_{\delta\beta} u^j D_{\alpha\varepsilon} u^j D_{\alpha\varepsilon} u^i \zeta \, dx +$$

$$+ \int_\Omega v^{-1} f' G^{\alpha\beta} G^{\delta\varepsilon} g_{ij} D_\beta u^j D_\varepsilon u^i D_\delta v D_\alpha \zeta \, dx +$$

$$+ \int_\Omega v^{-1} f G^{\alpha\beta} G^{\delta\varepsilon} g_{ij} D_{\delta\beta} u^j D_\varepsilon u^i D_\alpha \zeta \, dx -$$

$$- \int_\Omega v^{-2} f' G^{\alpha\beta} G^{\delta\varepsilon} g_{ij} D_\beta u^j D_\varepsilon u^i D_\delta v D_\alpha v \zeta \, dx -$$

$$- \int_\Omega v^{-2} f G^{\alpha\beta} G^{\delta\varepsilon} g_{ij} D_{\delta\beta} u^j D_\varepsilon u^i D_\alpha v \zeta \, dx -$$

$$- \int_\Omega v^{-1} G^{\delta\varepsilon} D_\varepsilon u^i (\partial B_i / \partial p^j_\alpha) D_{\alpha\delta} u^j \zeta \, dx -$$

$$- \int_\Omega v^{-1} G^{\delta\varepsilon} D_\varepsilon u^i [(\partial B_i / \partial z^i) D_\delta u^j + (\partial B_i / x^\delta)] \zeta \, dx .$$

To proceed, we introduce some notation:

$$\mathcal{C}_0^2 = v^{-2} G^{\alpha\beta} G^{\delta\varepsilon} g_{ij} D_{\delta\beta} u^j D_{\alpha\varepsilon} u^i , \qquad B_j^{\alpha\delta} = - G^{\delta\varepsilon} D_\varepsilon u^j (\partial B_i / \partial p^j_\alpha) ,$$

$$\tilde{G}^{\alpha\delta} = v^{-2} G^{\alpha\beta} G^{\delta\varepsilon} g_{ij} D_\beta u^j D_\varepsilon u^i , \qquad \Gamma^{\alpha\beta} = v f' \tilde{G}^{\alpha\beta} + f G^{\alpha\beta} ,$$

$$\overline{B} = - G^{\delta\varepsilon} D_\varepsilon u^i [(\partial B_i / \partial z^j) D_\delta u^j + (\partial B_i / \partial x^\delta)] ,$$

and we observe that

$$D_\varepsilon v = v^{-1} G^{\alpha\beta} g_{ij} D_{\alpha\varepsilon} u^i D_\beta u^j .$$

Our integral identity then becomes

$$(1.2) \qquad 0 = \int_\Omega v f \mathcal{C}_0^2 \zeta \, dx + \int_\Omega (f' G^{\alpha\beta} - v^{-1} \Gamma^{\alpha\beta}) D_\alpha v D_\beta v \zeta \, dx +$$

$$+ \int_\Omega \Gamma^{\alpha\beta} D_\alpha \zeta D_\beta v \, dx + \int_\Omega (v^{-1} B_i^{\alpha\delta} D_{\alpha\delta} u^i + v^{-1} \overline{B}) \zeta \, dx .$$

Let us note here an important distinction between the case of a single equation (as in [11]) and the case of a system of equations. Writing (g^{ij}) for the inverse matrix to

(g_{ij}), we see that there are tensors E and F, satisfying

$$E_{ij}^{\alpha\beta}\zeta_\alpha^i\zeta_\beta^j, \qquad F_{ij}^{\alpha\beta}\zeta_\alpha^i\zeta_\beta^j \geq 0$$

for all matrices ζ, such that

$$vf\mathcal{C}_0^2 + (f' G^{\alpha\beta} - v^{-1}\Gamma^{\alpha\beta}) D_\alpha v D_\beta v = E_{ij}^{\alpha\beta} F_{km}^{\delta\varepsilon} g^{ik} D_{\alpha\delta} u^m D_{\beta\varepsilon} u^j.$$

This quantity is clearly nonnegative if $N = 1$, but in the present case its nonnegativity is not clear even if G and g are identity matrices.

Next we choose $\zeta = (v-\tau)_+\chi(v)\eta^2$ for some $\tau > 0$, nonnegative C^1 increasing function χ, and $\eta \in C^{0,1}(\Omega)$ with compact support to be further specified. In this way we infer that

$$\begin{aligned} 0 = &\int_\Omega fv\mathcal{C}_0^2(v-\tau)_+\chi\eta^2\,dx + \\ &+ \int_\Omega \{f' G^{\alpha\beta} - v^{-1}\Gamma^{\alpha\beta}\} D_\alpha v D_\beta v (v-\tau)_+\chi\eta^2\,dx + \int_\Omega \Gamma^{\alpha\beta} D_\alpha v D_\beta v \{\chi + (v-\tau)_+\chi'\}\eta^2\,dx + \\ &+ 2\int_\Omega \Gamma^{\alpha\beta} D_\beta v D_\alpha \eta \chi (v-\tau)_+\eta\,dx + \int_\Omega v^{-1}[B_j^{\alpha\delta} D_{\alpha\delta} u^j + \overline{B}](v-\tau)\chi\eta^2\,dx. \end{aligned}$$

Now the matrix Γ will be nonnegative definite if

$$vf' + f \geq 0, \tag{1.3}$$

in which case we can estimate the fourth integral in this equation via Cauchy's inequality.

Hence for any $\theta \in (0,1)$ we have

$$\begin{aligned} 0 \geq &\int_\Omega fv\mathcal{C}_0^2(v-\tau)_+\chi\eta^2\,dx + \int_\Omega \{f' G^{\alpha\beta} - v^{-1}\Gamma^{\alpha\beta}\} D_\alpha v D_\beta v (v-\tau)_+\chi\eta^2\,dx + \\ &+ \int_\Omega \Gamma^{\alpha\beta}\{\chi + (v-\tau)_+\chi' - \theta\chi\} D_\alpha v D_\beta v v\eta^2\,dx - \theta^{-1}\int_\Omega \Gamma^{\alpha\beta} D_\alpha\eta D_\beta\eta (v-\tau)_+^2\chi\,dx - \\ &+ \int_\Omega v^{-1} B_i^{\alpha\delta} D_{\alpha\delta} u^i (v-\tau)_+\chi\eta^2\,dx + \int_\Omega v^{-1}\overline{B}(v-\tau)_+\chi\eta^2\,dx. \end{aligned}$$

Next we note that $v^2\mathcal{C}_0^2 \geq G^{\alpha\beta} D_\alpha v D_\beta v$, $v^2\mathcal{C}_0^2 \geq c_1|D^2 u|^2$ for some c_1 depending only on the minimum eigenvalues of the G and g matrices. As it stands, we can't control the terms involving Dv, so we strengthen (1.3) to

$$vf' + \mu f \geq 0 \tag{1.4}$$

for some $\mu \in [0,1)$. Then

$$\{f' G^{\alpha\beta} - v^{-1}\Gamma^{\alpha\beta}\} D_\alpha v D_\beta v(v-\tau)_+ \geqslant -\frac{(1+\mu)f}{2v} G^{\alpha\beta} D_\alpha v D_\beta v(v-\tau)_+ -$$

$$-\frac{1+\mu}{2} v^{-1}\Gamma^{\alpha\beta} D_\alpha v D_\beta v(v-\tau)_+ \geqslant -\frac{1+\mu}{2} fv C^2 (v-\tau)_+ - \frac{1+\mu}{2}\Gamma^{\alpha\beta} D_\alpha v D_\beta v,$$

and so, recalling that $\chi' \geqslant 0$,

$$0 \geqslant \frac{(1-\mu)}{2} \int_\Omega fv\mathcal{C}^2 (v-\tau)_+ \chi\eta^2\, dx + \left(\frac{1}{2}(1-\mu) - \theta\right) \int_\Omega \Gamma^{\alpha\beta} D_\alpha v D_\beta v \chi\eta^2\, dx -$$

$$-\theta^{-1} \int_\Omega \Gamma^{\alpha\beta} D_\alpha \eta D_\beta \eta v^2 \chi\, dx + \int_\Omega v^{-1} B_i^{\alpha\delta} D_{\alpha\delta} u^i (v-\tau)_+ \chi\eta^2\, dx + \int_\Omega v^{-1}\overline{B}(v-\tau)_+ \chi\eta^2\, dx.$$

We now choose $\theta \in (0,(1/2)(1-\mu))$ and estimate the integral involving $B_i^{\alpha\delta}$ via Cauchy's inequality. Then we set

$$\Lambda_0 = v^2 (f')_+ + vf, \qquad \mathcal{C}^2 = fv\mathcal{C}_0^2, \qquad \mathcal{E} = v^{-1} f G^{\alpha\beta} D_\alpha v D_\beta v, \qquad d\mu = v\, dx,$$

and we write $\bar\mu$ for the maximum eigenvalue of the matrix $v\Gamma$. If also

$$|B_p|^2/f + \overline{B}/v \leqslant \beta_1^2 \Lambda_0, \tag{1.5}$$

we obtain from routine calculations, and the observation that (1.4) implies $\Gamma \geqslant$ $\geqslant c(\mu) fG$, that

$$\int_\Omega \left[\mathcal{C}^2\left(1 - \frac{\tau}{v}\right)_+ + \mathcal{E}\right] \chi\eta^2\, d\mu \leqslant c_2 \int_\Omega [\beta_1^2 \Lambda_0 \eta^2 + \bar\mu |D\eta|^2] \chi\, d\mu$$

for some constant c_2 determined by c_1 and μ. This inequality is an energy inequality analogous to [11, (2.11)] or [6, Lemma 4.1], which is a key to our gradient bound. Note that the estimate does not depend on an upper bound for $(v-\tau)\chi'/\chi$ because our calculations correspond to the case $a_i \equiv 0$ in [11].

Another important tool is a suitable Sobolev inequality, which we give in terms of the matrix γ defined by

$$\gamma^{\alpha\beta} = \begin{cases} \delta^{\alpha\beta} - \delta_{ij} |Du|^{-2} D_\alpha u^i D_\beta u^j & \alpha, \beta \leqslant n, \\ \delta^{\alpha\beta} & \alpha = n+1 \text{ or } \beta = n+1 \end{cases}$$

and the vector $\mathcal{H}$ given by

$$\mathcal{H}_\alpha = \sum_{\delta, \beta, i, j} [|Du|^{-3} D_\delta u^i D_{\delta_\beta} u^j - |Du|^{-1} Du^i] D_\alpha u^i / |Du|.$$

Writing δ for the operator with components $\delta_\alpha = \gamma^{\alpha\beta} D_\beta$ and setting $d\tilde\mu = |Du|\, dx$, we see that $\operatorname{tr}\gamma = n$ and

$$\int_\Omega \delta h \, d\tilde{\mu} = \int_\Omega h \mathcal{H} \, d\tilde{\mu}$$

for any $h \in C^1(\Omega)$ with compact support. If h also vanishes wherever $|Du| < 1$ (or if we modify γ, $\mathcal{H}$, δ, $d\tilde{\mu}$, appropriately there), we combine Michael and Simon's Sobolev inequality [9, (1.3)] with the argument in [6, Lemma 1.3] to see that

$$\int_\Omega h^{2(n+2)/n} d\tilde{\mu} \leq C(n) \left(\int_\Omega h^2 \, d\tilde{\mu} \right)^{2/n} \left(\int_\Omega [|\delta h|^2 + h^2 |\mathcal{H}|^2] \, d\tilde{\mu} \right).$$

A straightforward calculation shows that, for $\lambda_0 = vf$, we have

$$\lambda_0 |\mathcal{H}|^2 \leq \mathcal{C}^2, \qquad v^{-2} \lambda_0 |\delta v|^2 \leq \mathcal{E}.$$

Then the proof of [11, Lemma 1] allows us to estimate the supremum of v over a ball in Ω in terms of integral norm of v over a larger ball provided the structure condition

$$\lambda_0 (1 + (v \lambda_0' / \lambda_0)^2) \gamma \leq \lambda_0 G,$$

holds and $(\Lambda_0/\lambda_0)^{(n+2)/2}/\Lambda$ behaves like a rational function near infinity. (A more precise statement of this property will be given below.) In fact we can replace λ_0 by any smaller function λ, Λ_0 by any larger function Λ and v by an increasing function w of v provided w, λ, Λ are suitably connected. (The case of general w, λ, Λ for single equations is the one given in [11].) Setting $\Omega_\tau = \{x \in \Omega : v(x) > \tau\}$, we have as our first part of the estimate the following result.

LEMMA 1.1. – *Let $\tau_0 > 1$ be a constant and let w, λ, Λ be C^1 increasing function on $[\tau_0, \infty)$ with*

$$\Lambda(t) \geq t^2 (f'(t))_+ + tf(t), \tag{1.6a}$$

$$\lambda(t)(1 + (t\lambda'(t)/\lambda(t))^2) \leq tf(t), \tag{1.6b}$$

Suppose there are nonnegative constants $\mu < 1$, β, β_1, β_2 such that

$$w^\beta (\Lambda/\lambda)^{(n+2)/2}/\Lambda \ \textit{is increasing}, \tag{1.7}$$

$$tf'(t) + \mu f(t) \geq 0 \quad \textit{for } t \geq \tau_0, \tag{1.8}$$

$$|B_p| f(v)^{-1} + v^{-1} \overline{B} \leq \beta_1^2 \Lambda(v) \quad \textit{wherever } v \geq \tau_0, \tag{1.9}$$

$$G \geq I, \ g \geq I; \qquad |G| \leq \beta_2, \ |g| \leq \beta_2. \tag{1.10}$$

Then for any $\rho > 0$, and $x_0 \in \Omega$ such that the ball $B(x_0, \rho) \subset \Omega$, there is a constant

$C_1(n, \beta, \beta_1\rho, \beta_2, \mu)$ *such that*

$$\sup_{B(x_0,\rho/2)} \left(1 - \frac{\tau}{v}\right)_+^{(n+2)/2} w(v)^2 \leq C_1 \rho^{-n} \int_{B(x_0,\rho)\cap\Omega_\tau} w(v)^2 (\Lambda/\lambda)^{(n+2)/2} v\, dx \tag{1.11}$$

for any $\tau \geq \tau_0$ *and any* $W^{2,2} \cap W^{1,\infty}$ *solution* u *of* (0,1). ∎

We note that the normalization of the minimum eigenvalues of G and g in (1.9) guarantees that $|Du| > 1$ wherever $v > \tau_0$, as required by our Sobolev inequality. Moreover, in this case, (1.7) is the only monotonicity condition we need because of the special form of our energy inequality.

For our next estimates, we introduce the shorthand notation $Du \cdot A$ to denote $D_\alpha u^i A_i^\alpha$. If we assume now that

$$(\Lambda/\lambda)^{(n+2)/2} v \leq \beta_3 w^{\beta_4} Du\cdot A \tag{1.12}$$

for nonnegative constants β_3, β_4, we reduce the estimate of $\sup w$ to one on

$$\int_{\Omega_\tau} w^q Du\cdot A\, dx,$$

which we estimate by adjusting the proof of [11, Lemma 2] along the lines of [6, Lemma 4.3] (see also [7, p. 41]).

LEMMA 1.2. – *Suppose that in addition to the hypotheses on* w, λ, Λ, f, B, G, g *in Lemma* 1.1, *that there are a positive constant* β_5 *and a positive decreasing function* ε *such that*

$$|A|(w')^2 \leq \beta_5 v f(v), \tag{1.13a}$$

$$w|A| \leq \beta_5^{1/2} Du\cdot A, \tag{1.13b}$$

$$v^3|f'| + v^2 f \leq \beta_5 Du\cdot A, \tag{1.13c}$$

$$\Lambda v \leq \varepsilon w^2 Du\cdot A. \tag{1.13d}$$

Fix x_0, ρ *in Lemma* 1 *and set* $\sigma = \operatorname{osc}_{B(x_0,\rho)} u$. *Then for any* $q > 0$, *there are constants* $C_2(n, \beta, m, q)$ *and* $C_3(n, \beta, \mu, \beta_1\rho, \beta_2, \beta_5)$ *such that if* τ_1 *is so large that*

$$C_2\beta_5(\beta_1\sigma)^2\varepsilon(\tau_1) \leq 1, \tag{1.14}$$

and if $\tau \geq \tau_0$, *then*

$$\int_{B(x_0,\rho/2)\cap\Omega_\tau} w^q Du\cdot A\, dx \leq C_3[w(\tau_1) + \sigma/\rho]^q \int_{B(x_0,\rho)\cap\Omega_\tau} Du\cdot A\, dx. \quad ∎ \tag{1.15}$$

The final estimate, on the integral of $Du\cdot A$, is slightly delicate. If we assume that $u = \varphi$ on $\partial\Omega$ for a known Lipschitz function φ, we have a simple estimate in terms of $\Omega'_\tau = \{x \in \Omega: |Du| > \tau\}$.

LEMMA 1.3 A. – *If* $u = \varphi$ *on* $\partial\Omega$, *and if there are constants* $\beta_6 \in [0, 1)$, $\beta_7 > 0$, $\Phi > 0$ *and an increasing function* ψ *such that*

$$(u^i - \varphi^i) B_i \leq \beta_6 Du \cdot A \tag{1.16a}$$

$$Du \cdot A \geq |Du| \psi(|Du|), \ |A| \leq \beta_7 \psi(|Du|) \tag{1.16b}$$

on Ω'_τ *and if* $|D\varphi| \leq \Phi$, *then*

$$\int_{\Omega'_\tau} Du \cdot A \, dx \leq \frac{2}{(1-\beta_6)} |\Omega| \left(\sup_{|p| \leq \tau} \{(u^i - \varphi^i) B_i - Du \cdot A + D\varphi \cdot A\}_+ + \beta_7 \Phi \psi\left(\frac{2\beta_7 \Phi}{1-\beta_6}\right)\right). \tag{1.17}$$

PROOF. – Multiply (0.1) by $u^i - \varphi^i$, sum on i, and integrate by parts to see that

$$\int_{\Omega'_\tau} \{Du \cdot A - (u^i - \varphi^i) B_i\} \, dx = \int_{\Omega - \Omega'_\tau} \{(u^i - \varphi^i) B_i - Du \cdot A\} \, dx + \int_{\Omega'_\tau} D\varphi \cdot A \, dx + \int_{\Omega - \Omega'_\tau} D\varphi \cdot A \, dx .$$

If we write Σ for the supremum on the right hand side of (1.17), then (1.16a) and (1.16b) give, for any $\varepsilon > 0$,

$$(1-\beta_6) \int_{\Omega'_\tau} Du \cdot A \, dx \leq |\Omega| \Sigma + \beta_7 \int_{\Omega'_\tau} |D\varphi| \psi(|Du|) \, dx \leq |\Omega| \Sigma + \int_{\Omega'_\tau} \varepsilon |Du| \psi(|Du|) \, dx +$$

$$+ \int_{\Omega'_\tau} \beta_7 |D\varphi| \psi\left(\frac{\beta_7 |D\varphi|}{\varepsilon}\right) dx \leq |\Omega| \Sigma + \varepsilon \int_{\Omega'_\tau} Du \cdot A \, dx + |\Omega| \beta_7 \Phi \psi(\beta_7 \Phi / \varepsilon) .$$

The desired conclusion follows from this inequality by taking $\varepsilon = (1 - \beta_6)/2$ and rearranging the resulting inequality. ■

In particular, if L^∞ bounds for u and φ are known and if A the form assumed in this paper, then a bound on the integral of $Du \cdot A$ is guaranteed if $|B| = o(|Du| F'(|Du|))$.

So far our estimates have been purely local in nature. To derive a local estimate for the integral of $Du \cdot A$ requires some additional hypotheses, which do not appear in the scalar case. (In fact SIMON [11] showed that the hypotheses $|B| = O(Du \cdot A)$, $|Du| \, |A| = O(Du \cdot A)$ as $|Du| \to \infty$ suffice in the scalar case.)

LEMMA 1.3 B. – *Suppose there are constants* $\beta_6 \in [0, 1)$, $\beta_7 > 0$ *and an increasing function* ψ *such that* (1.16b) *and*

$$u^i B_i \leq \beta_6 Du \cdot A \tag{1.18}$$

hold on Ω'_τ. *Define* h *and* $\tilde{h}$ *by*

$$h(t) \psi \circ h(t) = t , \qquad \tilde{h}(t) = 1/\{t h(1/t)\} \tag{1.19}$$

and suppose $\tilde{h}$ is integrable near zero. If $|u| \leqslant M$ in $B(x_0, 2\rho) \subset \Omega$, then there are constants C_5 determined by n, β_6, β_7, ψ, and M/ρ and C_6 determined by n and β_6 such that

$$(1.20)\qquad \int\limits_{\Omega_\tau' \cap B(x_0, \rho)} Du\cdot A\, dx \leqslant C_5\rho^n + C_6\rho^n \sup_{\substack{|Du|<\tau\\ x\in B_{2\rho}}} \left[\{u^iB_i - Du\cdot A\}_+ + \frac{M}{\rho}|A|\right].$$

PROOF. – Now we multiply (0.1) by $\eta(1 - |x|^2/R^2)u^i$, with $R = 2\rho$ and η a smooth nonnegative function satisfying $\eta(t) = 0$ for $t \leqslant 0$ to be further specified. We then have

$$\int\limits_{\Omega_\tau'} \eta[Du\cdot A - u^iB_i]\,dx = \int\limits_{\Omega-\Omega_\tau'} \eta\{u^iB_i - Du\cdot A\}\,dx - \int\limits_{\Omega_\tau'} \eta'\,\frac{x^\alpha}{R^2}u^iA_i^\alpha\,dx - \int\limits_{\Omega-\Omega_\tau'} \eta'\,\frac{x^\alpha}{R^2}u^iA_i^\alpha\,dx.$$

Now we use our structure conditions to see that

$$-\eta'\,\frac{x^\alpha}{R^2}u^iA_i^\alpha \leqslant \beta_7|\eta'|\frac{M}{R}\psi(|Du|) \leqslant \beta_7\varepsilon\eta|Du|\psi(|Du|) + \beta_7|\eta'|\frac{M}{R}\psi\left(\frac{|\eta'|M}{\eta R\varepsilon}\right)$$

for any $\varepsilon > 0$, so, assuming $\eta \leqslant 1$, we have

$$(1.21)\qquad (1 - \beta_6 - \varepsilon\beta_7)\int\limits_{\Omega_\tau'} \eta\, Du\cdot A\,dx \leqslant$$

$$\leqslant |B_\rho| \sup_{|Du|<\tau, x\in B_R}(\{u^iB_i - Du\cdot A\}_+) + \sup_{\substack{|Du|<\tau\\ x\in B_R}} M|A|\int\limits_{B_R}|D\eta|\,dx +$$

$$+\beta_7\int\limits_{\Omega}|\eta'|\frac{M}{R}\psi\left(\frac{|\eta'|}{\eta}\,\frac{M}{R\varepsilon}\right)dx.$$

Now we observe that

$$\int\limits_{B_R}|D\eta|\,dx = c(n)\int\limits_0^R\left|\eta'\left(1-\frac{r^2}{R^2}\right)\right|\frac{r^n}{R^2}\,dr \leqslant c(n)R^{n-1}\int\limits_0^R\left|\eta'\left(1-\frac{r^2}{R^2}\right)\right|\frac{r}{R^2}\,dr.$$

If η is monotone (and hence increasing), then

$$\int\limits_0^R\left|\eta'\left(1-\frac{r^2}{R^2}\right)\right|\frac{r}{R^2}\,dr = \int\limits_0^R\eta'\left(1-\frac{r^2}{R^2}\right)\frac{r}{R^2}\,dr = \frac{1}{2}\int\limits_0^1\eta'(s)\,ds = \frac{\eta(1)}{2} \leqslant 1.$$

Therefore

$$\int_{B_R} |D\eta|\,dx \leqslant c(n)\,R^{n-1}.$$

It remains only to estimate the last integral in (1.21). To this end, we introduce a constant H (to be further specified) and consider the initial value problem

$$|\eta'(t)|\,\frac{M}{R}\,\psi\left(\frac{|\eta'(t)}{\eta(t)}\,\frac{M}{R\varepsilon}\right) = H, \qquad \eta(0) = 0.$$

Because the differential equation here can be written as

$$h(H/\eta\varepsilon) = \frac{|n'|}{\eta}\,\frac{M}{R\varepsilon}, \tag{1.22}$$

it follows that η is given implicitly by

$$\int_0^{\varepsilon\eta(t)/H} \frac{d\sigma}{\sigma h(1/\sigma)} = \frac{R\varepsilon}{M}t \tag{1.23}$$

(note that η must be increasing, so $\eta' \geqslant 0$). Since $\tilde{h}$ is integrable, the initial value problem has a solution.

Now we choose $\varepsilon = (1-\beta_6)/2\beta_7$ and then H so that the function η given by (1.23) has the value 1 when $t = 3/4$, and define η by (1.23) for $0 \leqslant t \leqslant 3/4$, $\eta(t) = 0$ for $t \leqslant 0$, $\eta(t) = 1$ for $t \geqslant 3/4$. To see that this η is Lipschitz, we need only check that $\liminf_{t\to 0^+} \eta'(t)$ is finite. By virtue of (1.22), it suffices to show that $\lim_{t\to 0^+} \tilde{h}(t)$ is nonzero because $\eta'(t) = RH/M\tilde{h}(\eta\varepsilon/H)$. Simple calculations show that

$$d\{th(1/t)\}/dt = h(1/t) - h'(1/t)/t,$$

$$h(s) - sh'(s) = h(s)^2\psi'(h(s))/[\psi(h(s)) + h(s)\psi'(h(s))] \geqslant 0$$

and hence $\tilde{h}$ is decreasing. Since $\tilde{h}$ is nonnegative and somewhere positive, $\tilde{h}(0+)$ must be positive.

After noting that $\eta(1-|x|^2/R^2) = 1$ on B_ρ, we see that

$$\int_{\Omega'_\tau \cap B_\rho} Du\cdot A\,dx \leqslant \frac{2}{1-\beta_6}\,|B_\rho| \sup_{\substack{|Du|<\tau\\ x\in B_R}} \{u^i B_i - Du\cdot A\} +$$

$$+\frac{2}{1-\beta_6}C(n)\,\rho^n\,\frac{M}{R} \sup_{\substack{|Du|<\tau\\ x\in B_R}} |A| + \frac{2}{1-\beta_6}\beta_7\,|B_\rho|\,H.$$

Finally $\displaystyle\int_{\Omega'_{\tau_0}\cap B_\rho} Du\cdot A\,dx \leqslant \int_{\Omega'_\tau\cap B_\rho} Du\cdot A\,dx + c(n)\,\rho^n \sup_{|Du|\leqslant\tau} Du\cdot A.$ ■

Clearly $\tilde{h}$ is integrable if ψ grows no faster than polynomially. For example, if

$\psi(t) = t^m$, $m \geq 0$, then $h(t) = t^{1/(m+1)}$ and $\tilde{h}(t) = t^{-m/(m+1)}$. Additionally if $\psi(t) = (\exp t^k)/t$ for some $k > 0$, then $h(t) = (lnt)^{1/k}$ so $\tilde{h} = \{1/(-lnt)^{1/k} t\}$, which is integrable at zero if and only if $k < 1$.

2. - Nonconstant G and q.

When the matrices G and g depend on x and z, the argument of the previous section remain applicable upon taking proper account of the terms arising from the derivatives of these matrices. The details must be suitably modified but the results can be given the same general form as before.

The major difference is in the derivation of the energy inequality. If we note that now

$$D_\varepsilon v = v^{-1} G^{\alpha\beta} g_{ij} D_{\alpha\varepsilon} u^i D_\beta u^j + v^{-1} D_\varepsilon (G^{\alpha\beta} g_{ij}) D_\alpha u^i D_\beta u^j$$

and define

$$\text{(2.1}a) \qquad \xi^\alpha = G^{\alpha\beta} D_\beta (G^{\delta\varepsilon} g_{ij}) D_\delta u^j D_\varepsilon u^i (v^{-2} f - v^{-1} f') + G^{\alpha\beta} D_\beta (G^{\delta\varepsilon}) g_{ij} D_\alpha u^j D_\varepsilon u^i (v^{-1} f') - v^{-2} f D_\delta (G^{\alpha\beta} g_{ij}) D_\beta u^j D_\varepsilon u^i G^{\delta\varepsilon},$$

$$\text{(2.1}b) \qquad \varphi_i^{\alpha\beta} = G^{\delta\beta} g_{ij} D_\delta (g^{\alpha\varepsilon}) D_\varepsilon u^j + D_\delta (G^{\alpha\varepsilon} g_{ij}) D_\varepsilon u^j G^{\delta\beta},$$

$$\text{(2.1}c) \qquad \bar{\xi}^\alpha = v^{-1} f D_\delta (G^{\alpha\beta} g_{ij}) D_\beta u^j D_\varepsilon u^i G^{\delta\varepsilon},$$

$$\text{(2.1}d) \qquad \bar{f} = v^{-1} f D_\delta (G^{\alpha\beta} g_{ij}) D_\alpha (G^{\delta\varepsilon}) D_\beta u^j D_\varepsilon u^i,$$

we obtain (1.2) with the additional integrals

$$\int_\Omega \xi^\alpha D_\alpha v \zeta\, dx + \int_\Omega v^{-1} f \varphi_i^{\alpha\beta} D_{\alpha\beta} u^i \zeta\, dx + \int_\Omega \bar{\xi}^\alpha D_\alpha \zeta\, dx + \int_\Omega \bar{f} \zeta\, dx$$

on the right hand side. After taking $\zeta = (v - \tau)_+ \chi \eta^2$ as before, we estimate the first three of these new integrals via Cauchy's inequality. Because of the $\bar{\xi}^\alpha D_\alpha \zeta$ term, we must assume of χ that there is a constant c_χ such that

$$0 \leq \chi'(t)(t - \tau)/\chi(t) \leq c_\chi$$

for $t > \tau$. If we also assume in addition to (1.4), (1.5) that

$$\text{(2.2}a) \qquad v|\xi|^2 \leq \beta_1^2 \Lambda_1 f, \qquad |\varphi|^2 f \leq \beta_1^2 \Lambda_1 v, \qquad |\bar{\xi}|^2 \leq \beta_1^2 \Lambda_1 v f,$$

$$\text{(2.2}b) \qquad \bar{f} \leq \beta_1^2 \Lambda_1,$$

for some function $\Lambda_1 \geq \Lambda_0$, then our energy inequality has the form

$$(2.3)\qquad \int_\Omega \left[\mathcal{C}^2\left(1-\frac{\tau}{v}\right)_+ + \mathcal{E}\right]\chi\eta^2\,d\mu \leq c_1(\beta_2\mu, n)(1+c_\chi)\int_\Omega [\beta_1^2\Lambda_1\eta^2 + \bar{\mu}|D\eta|^2]\chi\,d\mu.$$

Before stating the new version of Lemma 1, let us note that when G and g are independent of z, conditions (2.2) hold for $\Lambda_1 = \lambda_0$, while if G and g depend on z, then $\Lambda_1 = v^2\Lambda_0$. This observation will be crucial for our examples. We also note that the appearance of c_χ in (2.3) entails the use of all the monotonicity conditions of [11].

LEMMA 2.1. – *Let τ_0 be a positive constant and let w, λ, Λ be C^1 increasing functions on $[\tau_0, \infty)$ satisfying* (1.6). *Suppose there are nonnegative constants $\mu < 1, \beta, \beta_1, \beta_2$ with $\tau_0\beta_2 > 1$ such that* (1.7)-(1.10) *are satisfied and also, for $\zeta, \varphi, \bar{\xi}, \bar{f}$ defined by* (2.1), *we have*

$$(2.4)\qquad v|\xi|^2 \leq \beta_1^2\Lambda f, \quad |\varphi|^2 f \leq \beta_1^2\Lambda v, \quad |\bar{\xi}|^2\beta_1^2\Lambda vf, \quad \bar{f} \leq \beta_1^2\Lambda,$$

for $v > \tau_0$, and

$$(2.5)\qquad t^{-\beta}(\Lambda(t)/\lambda(t))^{(n+2)/2}/\Lambda(t), \quad t^{-\beta}w(t) \text{ are decreasing}.$$

Then for any $\rho > 0$ and $x_0 \in \Omega$ with $B(x_0, \rho) \subset \Omega$, there is a constant $C_1(n, \beta, \beta_1\rho, \beta_2, \mu)$ such that (1.11) *is valid for any $\tau \geq \tau_0$ and any $W^{2,2} \cap W^{1,\infty}$ solution u of* (0.1). ■

The integral estimate on $w^q Du \cdot A$ is also a little trickier in this case.

LEMMA 2.2. – *Suppose in addition to the hypotheses on $\omega, \lambda, \Lambda, f, B, g, G$ in Lemma* 2.1 *that there are a positive constant β_5 and a positive decreasing function ε such that* (1.13) *holds and that*

$$(2.6)\qquad |G|\,|g_x| + |g|\,|G_x| \leq \beta_8.$$

Fix x_0, ρ as in Lemma 2.1 *and set $\sigma = \operatorname{osc}_{B(x_0,\rho)} u$. Then for any $q > 0$, there are positive constants $C_2(n, \beta, \mu, q)$ and $C_3(n, \beta, \mu, \beta_1\rho, \beta_2, \beta_5, \beta_8\rho)$ such that if τ_1 is so large that*

$$(2.7)\qquad C_2[\beta_5(\beta_1\sigma)^2\varepsilon(\tau_1) + \sup_{B(x_0,\rho)}\{|g|\,|G_z| + |g_z|\,|G|\}] \leq 1$$

and if $\tau \geq \tau_0$, then (1.15) *holds.*

PROOF. – As in the references listed before Lemma 1.2, we integrate the integral

$$\int_\Omega (w^q - w(\tau)^q)_+\eta^q D_\alpha u^i A_i^\alpha\,dx$$

by parts. We find rather easily that

$$\int_{\Omega_\tau} (w\eta)^q Du\cdot A\,dx \leq c(n, \beta, \mu, \beta_1\rho, \beta_2, \beta_5)\left[w(\tau_1)+\frac{\sigma}{\rho}\right]^q \int_{\Omega_\tau\cap B(x_0,\rho)} Du\cdot A\,dx +$$

$$+4\beta_8\sigma\int_\Omega w^q\eta^q|A|\,dx + c(\beta_2)\sigma\sup_{B(x_0,\rho)}\{|g|\,|G_z|+|g_z|\,|G|\}\int_\Omega (w\eta)^q Du\cdot A\,dx$$

for $\tau \geq \tau_1$. The last two integrals arise from the additional terms generated from $D_\alpha(A_i^\alpha)$ (written as $\operatorname{div} A$ in [6] and [7]), which have the form $fD_\alpha[G^{\alpha\beta}g_{ij}]D_\beta u^j(u^i - m^i)\eta^q[w^q - w(\tau)^q]_+$, where $m^i = \inf_{B(x_0,\rho)} u^i$. The integral of $w^q\eta^q|A|$ is estimated via (1.13*b*) and Young's inequality while the last integral is controlled by using (2.7) to guarantee that the expression multiplying it is sufficiently small. ∎

The estimates on the integral of $Du\cdot A$ in Lemmas 1.3A and 1.3B apply without change to the general case of nonconstant G and g.

The estimates in Lemmata 2.1 and 2.2 apply to equations in which B is not differentiable. We suppose (cf. [6, p. 220]) that $B = E + E'$ with E differentiable with respect to x, z, p. The proofs of the lemmata are only changed in the derivation of the energy estimate. First we redefine the quantities $B_j^{\alpha\delta}$ and $\overline{B}$, by replacing B by E in their definitions. Then we integrate by parts:

$$\int_\Omega G^{\delta\varepsilon}(D_\varepsilon u^i/v)\zeta D_\delta E_i'\,dx = -\int_\Omega D_\delta[G^{\delta\varepsilon}(D_\varepsilon ui/v)\zeta]E_i'\,dx.$$

We expand this derivative and then estimate the terms as we did for Lemma 2.1. The conditions on E' are similar to (2.2), namely,

$$v\sum_\alpha[v^{-2}G^{\alpha\varepsilon}D_\varepsilon u^iE_i']^2 \leq \beta_1^2\Lambda f,$$

$$f\sum_{\alpha,\beta,i}[f^{-1}G^{\alpha\beta}E_i']^2 \leq \beta_1^2\Lambda v,$$

$$\sum_\alpha[v^{-1}G^{\alpha\beta}D_\beta u^iE_i']^2 \leq B_1^2\Lambda vf,$$

$$v^{-1}D_\delta(G^{\delta\varepsilon})D_\varepsilon u^iE_i' \leq \beta_1^2\Lambda.$$

More simply, if we suppose that

$$|E'|^2 \leq \beta_1^2\Lambda fv, \tag{2.8}$$

then, except for a factor determined by β_2, the first three inequalities above are satisfied; the final one will be satisfied if we assume that

$$|G_x| + |G_z|\,|Du| \leqslant \beta_1 (\Lambda/fv)^{1/2}. \tag{2.9}$$

Of course when $G_z = 0$, this inequality follows if $|G_x| \leqslant \beta_1$.

The statements of Lemmata 2.1 and 2.2 under the decomposition $B = E + E'$ with (2.8) and (2.9) holding are omitted.

We close this section by obtaining an integral bound on $Du \cdot A$ under slightly different hypotheses than in Lemmata 1.3A and 1.3B.

LEMMA 2.3. – *Suppose that $B = -\,\partial F(v)/\partial z$ and that there is a constant m such that $tF'(t) \leqslant mF(t)$. If $u = \varphi$ on $\partial\Omega$ for some Lipschitz function φ, then*

$$\int_\Omega Du \cdot A\, dx \leqslant m \int_\Omega F(v(\varphi))\, dx. \tag{2.10}$$

PROOF. – The hypothesis on B guarantees that u minimizes the integral

$$\int_\Omega F(v)\, dx$$

over all functions agreeing with u on $\partial\Omega$. Hence

$$\int_\Omega F(v(u))\, dx \leqslant \int_\Omega F(v(\varphi))\, dx.$$

Now the hypothesis on F says that $t^2 f(t) \leqslant mF(t)$. But

$$Du \cdot A = v(u)^2 f(v(u)). \qquad \blacksquare$$

Note that Lemma 2.3 requires more of F than Lemmata 1.3A and 1.3B. On the other hand, if $B = -\,\partial F(v)/\partial z$, then

$$B = D_z (G^{\alpha\beta} g_{ij}) D_\alpha u^i D_\beta^j f(v),$$

which is $O(Du \cdot A)$ as $|Du| \to \infty$ if G or g depends on z. Hence Lemma 2.3 is not a consequence of the results in Section 1.

3. – Examples.

We now demonstrate our estimate with some examples.

EXAMPLE 1. – Suppose f (defined by $f(t) = F'(t)/t$) satisfies

$$\theta_1 \leqslant tf'/f \leqslant \theta_2$$

for some constants $\theta_2 \geq \theta_1 > -1$. Suppose also that

$$|B| \leq \theta_3 f(|Du|)|Du|.$$

If G and g are independent of z with

$$|G_x| + |g_x| \leq \theta_3,$$

we can use Lemma 2.1, with

$$\lambda = vf, \quad \Lambda = c(\theta_2) vf, \quad w = (vf)^{1/2}, \quad \beta = \max\{2, \theta_2\},$$
$$\mu = \max\{-\theta_1, 1/2\}, \quad \beta_1 = c(n, \beta_2)\theta_3, \quad \tau_0 = 1.$$

Hence

$$\sup_{B(x_0, \rho/4)} vf(v)\left(1 - \frac{1}{v}\right)_+^{(n+2)/2} \leq c(\theta_1, \theta_2, \theta_3\rho, \beta_2)\rho^{-n} \int_{\beta(x_0, \rho/2)\cap\Omega_\tau} Du \cdot A\, dx.$$

This integral is estimated via Lemma 1.3. First we note that, in this case, we can arrange $\beta_6 = 1/2$ by choosing $\tau_0 = 2\theta_3 M = 2(\theta_3\rho)(M/\rho)$. Then we can take $\psi(t) = tf(t)$ and $H = c(\beta_2)(M/\rho)^2 f(M/\rho)$. It follows that

$$\sup |Du| f(|Du|) \leq c(\theta_1, \theta_2, \theta_3\rho, \beta_2)[1 + (M/\rho)^2 f(M/\rho)].$$

More generally, if these is a decreasing function ε_1 with $\varepsilon_1(\infty) = 0$ such that

$$|B| \leq \theta_3 \varepsilon_1(|Du|) f(|Du|)(|Du|^2 + 1),$$

we can use Lemma 2.1 with

$$\lambda = vf, \quad \Lambda = C(\theta_2) v^2 f, \quad w = v, \quad \beta = c(\theta_2), \quad \mu = \max\{-\theta_1, 1/2\},$$
$$\beta_1 = c(n, \beta_2)\theta_3, \quad \tau_0 = 1,$$

Lemma 2.2 with

$$\Lambda = c(\theta_2)\varepsilon_1(v)^2 v^4 f,$$
$$q = (n+4)/2, \quad \beta_5 = c(\beta_2),$$
$$\varepsilon = \varepsilon_1^2, \quad \tau_1 = c(n, \theta_1, \theta_2, \theta_3\rho, \beta_2, M/\rho)$$

and $w, \beta, \mu, \beta_1, \tau_0$ as before, and then Lemma 1.3B with $\beta_6 = 1/2$, $\beta_7 = c(\beta_2)$, and $\tau = c(\theta_3\rho, M/r, \varepsilon)$. It follows that $\sup_{B_{\rho/2}} |Du| \leq c(n, \theta_1, \theta_2, \theta_3\rho, \beta_2, M/\rho, \varepsilon)$.

EXAMPLE 2. – Suppose f is as in Example 1 and that

$$|B| \leq \theta_3 f(|Du|)(1 + |Du|^2),$$
$$|G_x| + |g_x| \leq \theta_4, \quad |G_z| + |g_z| \leq \theta_3.$$

If θ_3 is sufficiently small (depending on $n, \theta_1, \theta_2, \theta_4\rho, \beta_2$)) we can imitate Example 1 with ε_1. More generally we see with these structure conditions Lemmata 2.1, 2.2 and 1.3B apply if $\theta_3 \operatorname*{osc}_{B(2\rho)} u$ is sufficiently small. Such is the case provided u is continuous at x_0, and ρ is chosen sufficiently small. Continuity at an arbitrary point x_0 can be obtained if the one-sided smallness condition (1.18) is satisfied and $\liminf_{t \to \infty} t^{2-n-\varepsilon} f(t) > 0$ for some $\varepsilon > 0$ because then (by Lemma 1.3B) $u \in W^{1, n+\varepsilon}$ and hence $u \in C^{0, \varepsilon/(n+\varepsilon)}$ by Morrey's imbedding theorem [4, Theorem 7.17].

Another case in which u is known to be continuous is when $F(t) = t^m$ for some $m \geq 2$ (or, equivalently, $f(t) = mt^{m-2}$) and

$$B_i = -\frac{F'(v)}{v} \frac{\partial G^{\alpha\beta}(x, z) g_{km}(x, z)}{\partial z^i} D_\alpha u^k D_\beta u^m .$$

Then [2, Theorem 6.2] tells us that $u \in C^{0,\alpha}(\Omega_0)$ for any $\alpha \in (0,1)$ and some open set $\Omega_0 \subset \Omega$ with $\Omega - \Omega_0$ having zero $(n-q)$-dimensional Hausdorff measure for some $q > m$. It follows that we can prove a gradient for any small enough ball centered at a point of Ω_0.

We note here that our results do not include the obvious extension to systems of the minimal surface equation, that is, the case $F(t) = (1 + t^2)^{1/2}$ and therefore $f'(t) = (1 + t^2)^{-1/2}$. For the single equation

$$D_\alpha((1 + |Du|^2)^{-1/2} \delta^{\alpha\beta} D_\beta u) = 0 ,$$

an interior gradient bound in more than two dimensions was first proved by BOMBIERI, DE GIORGI and MIRANDA [1], and this estimate formed the basis of Simon's approach in [11].

We also point out that when G and g are independent of u and $F(t) = t^m$, $m > 1$, our gradient bounds can be inferred from the $C^{1,\alpha}$ estimates of TOLKSDORF [12] (cf. [2, Theorem 7.1]).

On the other hand, our results apply to functions that grow faster than any poly-noòial, as we now show.

EXAMPLE 3. – Suppose

$$f(t) = t^{-2} \exp t^\theta, \qquad |G_x| + |g_x| + |G_z| + |g_z| \leq \theta_1$$

for some $\theta \in (0,1)$ and all $t > 1$. Suppose also that there is a decreasing function ε with $\varepsilon(1) = 1$, $\varepsilon(\infty) = 0$ such that

$$|B| \leq \theta_2 \varepsilon(v) \exp(v^\theta) .$$

We can the use Lemma 2.1 (in the form discussed at the end of Section 2) with

$$w = v^{1/2}, \qquad \Lambda = v^{2\theta+1} \exp(t^\theta), \qquad \lambda = t^{2\theta+\omega} \exp\left(\frac{n}{n+2} t^\theta\right), \qquad \omega = \frac{n+6}{n+2},$$

$$\beta = c(\theta, n), \qquad \beta_1 = c(\theta, n)(\theta_1 + \theta_2)$$

and τ_0 sufficiently large to conclude that

$$\sup v \leq c((\theta_1 + \theta_2)\rho, \theta, n, \beta_2)\left[1 + \rho^{-n}\int v(v^{1-\omega})^{(n+2)/2}\exp(v^\theta)\, v\, dx\right].$$

Now we note that $(v^{1-\omega})^{(n+2)/2} = v^{-2}$, and $\exp(v^\theta) \leq cDu\cdot A$ for v sufficiently large. Since $\varepsilon(\infty) = 0$, we can apply Lemma 1.3B to obtain a bound for $\sup v$ in terms of $(\theta_1 + \theta_2)\rho$, θ, n, β_2, and M/ρ. Note that we take $\varepsilon(v) \equiv 1$ if a modulus of continuity is known for u or if we assume the one-sided smallness condition (1.18).

EXAMPLE 4. – Suppose now that

$$f(t) = \exp t^2, \qquad |G_x| + |g_x| + |G_z| + |g_z| \leq \theta,$$

and that

$$|B| \leq \theta_2\, \varepsilon(v)\, v^2 \exp(v^2).$$

Now Lemma 2.1 applies with

$$w = v^{n/2}, \qquad \lambda = v^5 \exp\left(\frac{n}{n+2}v^2\right), \qquad \Lambda = 2v^5\exp(v^5),$$

$$\beta = 5, \qquad \beta_1 = C(n)(\theta_1 + \theta_2).$$

Thus the estimate on v is reduced to an integral estimate on $Du\cdot A$. It's not hard to see that the corresponding $\tilde{h}$ is not integrable in this case, so now interior estimates for v depend on the Lipschitz norm of the boundary values.

Acknowledgements. Portions of the research for this paper were carried out while the author was on Faculty Improvement Leave at the Centre for Mathematical Analysis at the Australian National University. The author is grateful to both Iowa State and the Centre for financial support and to NEIL TRUDINGER and JOHN HUTCHINSON for useful discussions.

REFERENCES

[1] E. BOMBIERI - E. DE GIORGI - M. MIRANDA, *Una maggiorazione a priori relativa alle ipersurfici minimali non parametriche*, Arch. Rational Mech. Anal., **32**(1969), pp. 255-267.

[2] N. FUSCO - J. HUTCHINSON, *Partial regularity for minimisers of certain functionals having non quadratic growth*, Ann. Mat. Pura Appl., **155** (1989), pp. 1-24.

[3] M. GIAQUINTA, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Equations*, Ann. Math. Study, **105**, Princeton University Press, Princeton, N.J. (1983).

[4] D. GILBARG - N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., Springer-Verlag, Berlin-Heidelberg-New York-Tokyo (1983).

[5] O. A. LADYZHENSKAYA - N. N. URAL'TSEVA, *Linear and Quasinlinear Elliptic Equations*, Izdat. «Nauka», Moscow (1964) (Russian). English translation: Academic Press, New York (1968).

[6] G. M. LIEBERMAN, *The conormal derivative problem for elliptic equations of variational type*, J. Diff. Eqs., **49** (1983), pp. 218-257.

[7] G. M. LIEBERMAN, *The conormal derivative problem for non-uniformly parabolic equations*, Indiana Univ. Math. J., **32** (1988), pp. 23-72.

[8] M. MEIER, *Boundedness and integrability properties of weak solutions of quasilinear elliptic systems*, J. Reine Angew. Math., **333** (1982), pp. 191-220.

[9] J. H. MICHAEL - L. M. SIMON, *Sobolev and mean-value inequalities on generalized submanifolds of* $\boldsymbol{R}^n$, Comm. Pure Appl. Math., **26** (1973), pp. 361-379.

[10] J. MOSER, *A new proof of de Giorgi's theorem concerning the regularity problem for elliptic differential equations*, Comm. Pure Appl. Math., **13** (1960), pp. 457-468.

[11] L. M. SIMON, *Interior gradient bounds for non-uniformly elliptic equations*, Indiana Univ. Math. J., **25** (1976), pp. 827-855.

[12] P. TOLKSDORF, *Everywhere-regularity for some quasilinear systems with a lack of ellipticity*, Ann. Mat. Pura Appl., **134** (1983), pp. 241-266.

[13] L. UHLENBECK, *Regularity for a class of non-linear elliptic systems*, Acta Math., **138** (1977), pp. 219-240.